Absolutely Rigid Systems and Absolutely Indecomposable Groups

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Abstract

We give a new proof that there are arbitrarily large indecomposable abelian groups; moreover, the groups constructed are absolutely indecomposable, that is, they remain indecomposable in any generic extension. However, any absolutely rigid family of groups has cardinality less than the partition cardinal $\kappa(\omega)$.

ADDED DECEMBER 2004: The proofs of Theorems 0.2 and 0.3 are not correct, and the claimed results remain open. (The "only if" part assertion in the last 3 lines before "Proof of (II)" on p. 266 is not correct.) However, Theorems 0.1 and 0.4, which give upper bounds to the size of rigid systems/groups, are valid. And the construction in the proof of Theorem 0.3 does yield an affirmative answer to Nadel's question whether there is a proper class of torsion-free abelian groups which are pairwise absolutely non-isomorphic.

0 Introduction

Mark Nadel [11] asked whether there is a proper class of torsion-free abelian groups $\{A_\nu : \nu \in Ord\}$ with the property that for any $\nu \neq \mu$, A_ν and A_μ are not $L_{\infty\omega}$-equivalent; this is the same as requiring that A_ν and A_μ do not become isomorphic in any generic extension of the universe. In that case we say that A_ν and A_μ are absolutely non-isomorphic . This is not hard to achieve for torsion abelian groups, since groups of different p-length are absolutely non-isomorphic. (See section 1 for more information.)

Nadel's approach to the question in [11] involved looking at known constructions of rigid systems $\{A_i : i \in I\}$ to see if they had the property that for $i \neq j$, $\mathrm{Hom}(A_i, A_j)$ remains zero in any generic extension of the universe. We call these *absolutely rigid systems*. Similarly we call a group *absolutely rigid* (resp. *absolutely indecomposable*) if it is rigid (resp. indecomposable) in any generic extension. Nadel showed that the Fuchs-Corner construction in [4, §89]

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constructs an absolutely rigid system $\{A_\nu : \nu < 2^\lambda\}$ of groups of cardinality λ, where λ is less than the first strongly inaccessible cardinal. But he pointed out that other constructions, such as Fuchs' construction [5] of a rigid system of groups of cardinality the first measurable or Shelah's [13] for an arbitrary cardinal involve non-absolute notions like direct products or stationary sets; so the rigid systems constructed may not remain rigid when the universe of sets is expanded. The same comment applies to any construction based on a version of the Black Box.

Here we show that there do *not* exist arbitrarily large absolutely rigid systems. The cardinal $\kappa(\omega)$ in the following theorem is defined in section 2; it is an inaccessible cardinal much larger than the first inaccessible, but small enough to be consistent with the Axiom of Constructibility.

Theorem 1 *If κ is a cardinal $\geq \kappa(\omega)$ and $\{A_\nu : \nu < \kappa\}$ is a family of non-zero abelian groups, then there are $\mu \neq \nu$ in κ such that in some generic extension $V[G]$ of the universe, V, there is a non-zero (even one-one) homomorphism $f : A_\nu \to A_\mu$.*

This cardinal $\kappa(\omega)$ (called the "first beautiful cardinal" by the second author in [14]) is the precise dividing line:

Theorem 2 *If κ is a cardinal $< \kappa(\omega)$ and λ is any cardinal $\geq \kappa(\omega)$, there is a family $\{A_\mu : \mu < \kappa\}$ of torsion-free groups of cardinality λ such that in any generic extension $V[G]$, for all $\mu \in \kappa$, A_μ is indecomposable and for $\nu \neq \mu$,* $\mathrm{Hom}(A_\nu, A_\mu) = 0$.

Despite the limitation imposed by Theorem 1, the construction used to prove Theorem 2 yields the existence of a proper class of absolutely different torsion-free groups, in the following strong form. This answers the question of Nadel in the affirmative, and also provides a new proof of the existence of arbitrarily large indecomposables.

Theorem 3 *For each uncountable cardinal λ, there exist 2^λ torsion-free absolutely indecomposable groups $\{H_{i,\lambda} : i < 2^\lambda\}$ of cardinality λ such that whenever $\lambda \neq \rho$ or $i \neq j$, $H_{i,\lambda}$ and $H_{j,\rho}$ are absolutely non-isomorphic.*

We show that the groups A_μ in Theorem 2 and the groups $H_{i,\lambda}$ in Theorem 3 are absolutely indecomposable by showing that in any generic extension the only automorphisms they have are 1 and -1. (The proof of Theorem 3 does not depend on results from [14].) However, we cannot make the groups absolutely rigid:

Theorem 4 *If κ is a cardinal $\geq \kappa(\omega)$ and A is a torsion-free abelian group of cardinality κ, then in some generic extension $V[G]$ of the universe, there is an endomorphism of A which is not multiplication by a rational number.*

1 Infinitary logic and generic extensions

We will confine ourselves to the language of abelian groups. Thus an *atomic formula* is one of the form $\sum_{i=0}^{n} c_i x_i = 0$ where the c_i are integers and the x_i are variables.

$L_{\omega\omega}$ consists of the closure of the atomic formula under negation ($\neg$), finite conjunctions ($\wedge$) and disjunctions ($\vee$), and existential ($\exists x$) and universal ($\forall x$) quantification (over a single variable — or equivalently over finitely many variables). $L_{\infty\omega}$ consists of the closure of the atomic formula under negation, *arbitrary* (possibly infinite) conjunction ($\bigwedge$) and disjunction ($\bigvee$), and under existential and universal quantification ($\exists x, \forall x$). Rather than give formal definitions of other model-theoretic concepts, we will illustrate them with examples. Thus the formula $\varphi(y)$:

$$\forall x \exists z (2z = x) \wedge (\neg \exists z (3z = y))$$

is a formula of $L_{\omega\omega}$ with one free variable, y, which "says" that every element is 2-divisible, but y is not divisible by 3. More formally, if A is an abelian group and $a \in A$ we write $A \models \varphi[a]$ and say "a satisfies φ in A", if and only if every element of A is divisible by 2 and there is no $b \in A$ such that $3b = a$.

Also, the formula $\psi(x)$:

$$\exists y (py = x) \wedge (\bigwedge_{n \geq 1} \exists z (p^n z = y)) \wedge (x \neq 0)$$

is a formula of $L_{\infty\omega}$ with free variable x such that $A \models \psi[a]$ if and only if $a \in p^{\omega+1}A - \{0\}$.

A *sentence* is a formula which has no free variables; if φ is a sentence of $L_{\infty\omega}$,we write $A \models \varphi$ if and only if φ is true in A. We write $A \equiv_{\infty\omega} B$ to mean that every sentence of $L_{\infty\omega}$ true in A is true in B (and conversely because $\neg\varphi$ true in A implies $\neg\varphi$ true in B.) Obviously, if there is an isomorphism $f : A \to B$, then $A \equiv_{\infty\omega} B$. A necessary and sufficient condition for $A \equiv_{\infty\omega} B$ is given by the following ([8], or see [1, pp. 13f]):

Lemma 5 *$A \equiv_{\infty\omega} B$ if and only if there is a set P of bijections $p : A_p \to B_p$ from a finite subset A_p of A onto a finite subset B_p of B with the following properties:*

(i)[the elements of P are partial isomorphisms] *for every atomic formula $\varphi(x_1, ..., x_m)$ and elements $a_1, ..., a_m$ of* $\mathrm{dom}(p)$*, $A \models \varphi[a_1, ..., a_m]$ if and only if $B \models \varphi[p(b_1), ..., p(b_m)]$;*

(ii)[the back-and-forth property] *for every $p \in P$ and every $a \in A$ (resp. $b \in B$), there is $p' \in P$ such that $p \subseteq p'$ and $a \in \mathrm{dom}(p')$ (resp. $b \in \mathrm{rge}(p')$).*

It is an easy consequence that if A and B are countable, then $A \equiv_{\infty\omega} B$ if and only if $A \cong B$. Also, this implies that if it is true in V that $A \equiv_{\infty\omega} B$, then $A \equiv_{\infty\omega} B$ remains true in $V[G]$. The converse is easy, and direct, since a sentence of $L_{\infty\omega}$ which, in V, holds true in A but false in B has the same status in $V[G]$, since no new elements are added to the groups.

(By a generic extension we mean an extension of the universe V of sets defined by the method of forcing. In general, more sets are added to the universe; possibly, for example, a bijection between an uncountable cardinal λ and the countable set ω. So cardinals of V may not be cardinals in $V[G]$; but the ordinals of $V[G]$ are the same as the ordinals of V. Also, the elements of any set in V are the same in V or $V[G]$.)

There exist non-isomorphic uncountable groups A and B (of cardinality $\aleph_1$ for example) such that $A \equiv_{\infty\omega} B$. (See for example [3].) However, for any groups A and B in the universe, V, there is a generic extension $V[G]$ of V in which A and B are both countable (cf. [7, Lemma 19.9, p. 182]). Therefore we can conclude that $A \equiv_{\infty\omega} B$ if and only if A and B are "potentially isomorphic", that is, there is a generic extension $V[G]$ of the universe in which they become isomorphic. Barwise argues in [1, p. 32] that potential isomorphism (that is, the relation $\equiv_{\infty\omega}$) is "a very natural notion of isomorphism, one of which mathematicians should be aware. If one proves that $A \not\cong B$ but leaves open the question [of whether A and B are potentially isomorphic] then one leaves the possibility that A and B are not isomorphic for trivial reasons of cardinality. Or to put it the other way round, a proof that [A is not potentially isomorphic to B] is a proof that $A \not\cong B$ for nontrivial reasons."

As an example, consider reduced p-groups A_ν (ν any ordinal) such that the length of A_ν is ν, that is, $p^\nu A_\nu = 0$ but for all $\mu < \nu$, $p^\mu A_\nu \neq 0$. Then for any $\nu_1 \neq \nu_2$, the groups A_{ν_1} and A_{ν_2} are not even potentially isomorphic: this is because for any ν there is a formula $\theta_\nu(x)$ such that $\exists x(\theta_\nu(x) \wedge x \neq 0)$ is true in a p-group A if and only if A has length $\geq \nu$. Indeed, we define, by induction, θ_ν to be

$$\exists y(py = x \wedge \theta_\mu(y))$$

if $\nu = \mu + 1$ and if ν is a limit ordinal define θ_ν to be

$$\bigwedge_{\mu<\nu} \exists y(py = x \wedge \theta_\mu(y)).$$

Thus there is a proper class of pairwise absolutely non-isomorphic p-groups. Although there is not available a standard group-theoretic notion which will serve the same purpose for torsion-free groups, we will prove in section 5 that there is a proper class of indecomposable torsion-free abelian groups $\{H_\lambda : \lambda$ a cardinal$\}$ such that for any $\lambda \neq \rho$, the groups H_λ and H_ρ are not $L_{\infty\omega}$-equivalent.

2 Quasi-well-orderings and beautiful cardinals

A *quasi-order* Q is a pair $(Q, \leq_Q)$ where $\leq_Q$ is a reflexive and transitive binary relation on Q. There is an extensive theory of well-orderings of quasi-orders developed by Higman, Kruskal, Nash-Williams and Laver among others (cf. [12], [9]). A generalization to uncountable cardinals is due to the second author ([14]). The key notion that we need is the following: for an infinite cardinal

κ, Q is called *κ–narrow* if there is no *antichain* in Q of size κ, i.e., for every $f : \kappa \to Q$ there exist $\nu \neq \mu$ such that $f(\nu) \leq_Q f(\mu)$. (Note that this use of the terminology "antichain" — in [10, p.32] for example — is different from its use in forcing theory.)

A *tree* is a partially-ordered set $(T, \leq)$ such that for all $t \in T$, pred$(t)= \{s \in T : s < t\}$ is a well-ordered set; moreover, there is only one element r of T, called the *root* of T, such that pred(r) is empty. The order-type of pred(t) is called the height of t, denoted ht(t); the height of T is $\sup\{\mathrm{ht}(t) + 1 : t \in T\}$.

If Q is a quasi-order, a *Q-labeled tree* is a pair (T, Φ_T) consisting of a tree T of height $\leq \omega$ and a function $\Phi_T : T \to Q$. On any set of Q-labeled trees we define a quasi-order by: $(T_1, \Phi_1) \preceq (T_2, \Phi_2)$ if and only if there is a function $\theta : T_1 \to T_2$ which preserves the tree-order (i.e. $t \leq_{T_1} t'$ implies $\theta(t) \leq_{T_2} \theta(t'))$ as well as the height of elements and also is such that for all $t \in T_1$, $\Phi_1(t) \leq_Q \Phi_2(\theta(t))$.

One result from [14] that we will use implies that for sufficiently large cardinals κ and sufficiently small Q, any set of Q-labeled trees is κ-narrow. In order to state the result precisely we need to define a certain (relatively small) large cardinal.

Let $\kappa(\omega)$ be the first ω-Erdös cardinal, i.e., the least cardinal such that $\kappa \longrightarrow (\omega)^{<\omega}$; in other words, the least cardinal such that for every function F from the finite subsets of κ to 2 there is an infinite subset X of κ such that there is a function $c : \omega \to 2$ such for every finite subset Y of X, $F(Y) = c(|Y|)$. It has been shown that this cardinal is strongly inaccessible (cf. [7, p. 392]). Thus it cannot be proved in ZFC that $\kappa(\omega)$ exists (or even that its existence is consistent). If it exists, there are many weakly compact cardinals below it, and, on the other hand, it is less than the first measurable cardinal (if such exists). Moreover, if it is consistent with ZFC that there is such a cardinal, then it is consistent with ZFC + V = L that there is such a cardinal ([15]). If $\kappa(\omega)$ does not exist, then Theorem 6 is uninteresting. On the other hand, Theorem 7 then applies to every cardinal κ, and its consequences, given in section 4, are still of interest.

The following is a consequence of results proved in [14] (cf. Theorem 5.3, p. 208 and Theorem 2.10, p. 197):

Theorem 6 *If Q is a quasi-order of cardinality $< \kappa(\omega)$, and $\mathcal{S}$ is a set of Q-labeled trees, then $\mathcal{S}$ is $\kappa(\omega)$-narrow.*

On the other hand, it follows from results in [14] that for any cardinal smaller than $\kappa(\omega)$, there is an absolute antichain of that size:

Theorem 7 *If $\kappa < \kappa(\omega)$, there is a family $\mathcal{T} = \{(T_\mu, \Phi_\mu) : \mu < \kappa\}$ of ω-labeled trees, each of cardinality $< \kappa(\omega)$, such that in any generic extension of V, for all $\mu \neq \nu$, $(T_\mu, \Phi_\mu) \not\preceq (T_\nu, \Phi_\nu)$.*

Some commentary is needed on the absoluteness of the antichain $\mathcal{T} = \{(T_\mu, \Phi_\mu) : \mu < \kappa\}$, since this is not dealt with directly in [14]. $\mathcal{T}$ is constructed in a concrete, absolute, way from a function F which is an example witnessing the fact that $\kappa < \kappa(\omega)$. First, a κ-D-barrier B and function $q : B \to \omega$ is constructed (B is a kind of elaborate indexing for an antichain cf. [14, proof of 2.5, p. 195]). This gives rise ([14, proof of 1.12, pp. 192f]) to an example showing that $\mathcal{P}_\beta(\omega)$ is not κ-narrow for some $\beta < \kappa(\omega)$; this example is embedded into the quasi-order of ω-labeled trees, giving rise to $\mathcal{T}$ ([14, p. 221]). The proof that $\mathcal{T}$ is an antichain reduces to the key property of F, a property which is absolute by an argument of Silver [15]. Using an equivalent definition of $\kappa(\omega)$, F is taken to be a function from the finite subsets of κ to ω such that there is no one-one function $\sigma : \omega \to \omega$ such that for all $n \in \omega$, $F(\{\sigma(0), ..., \sigma(n-1)\}) = F(\{\sigma(1), ..., \sigma(n)\})$; this property of F is preserved under generic extensions because it is equivalent to the well-foundedness of a certain tree; more precisely, the tree of finite partial attempts at σ has no infinite branch.

3 A bound on the size of absolutely rigid systems

In this section we will prove Theorems 1 and 4.. Suppose that $\{A_\nu : \nu < \kappa\}$ is a family of non-zero abelian groups, where we can assume that $\kappa = \kappa(\omega)$. For each $\nu < \kappa$, let T_ν be the tree of finite sequences of elements of A_ν; that is, the elements of T_ν are 1-1 functions $s : n_s \to A_\nu$ for some $n_s \in \omega$ and $s \le t$ if and only if $n_s \le n_t$ and $t \restriction n_s = s$.

Let Q_{ab} be the set of all quantifier-free n-types of abelian groups; that is, $Y \in Q_{ab}$ if and only if for some abelian group G, some $n \in \omega$, and some function $s : n \to G$, Y is the set $\mathrm{tp}_{qf}(s/G)$ of all quantifier-free formulas $\varphi(x_0, ..., x_{n-1})$ of $L_{\omega\omega}$ such that $G \vDash \varphi[s(0), ..., s(n-1)]$. Partially-order Q_{ab} by the relation of inclusion.

Define $\Phi_\nu : T_\nu \to Q_{ab}$ by letting $\Phi_\nu(s) = \mathrm{tp}_{qf}(s/A_\nu)$. Now we can apply Theorem 6 to the family of Q_{ab}-labeled trees $\mathcal{S} = \{(T_\nu, \Phi_\nu) : \nu < \kappa\}$. (Note that the cardinality of Q_{ab} is $2^{\aleph_0}$ which is $< \kappa(\omega)$ since $\kappa(\omega)$ is strongly-inaccessible.) Therefore there exists $\nu \ne \mu$ such that $(T_\nu, \Phi_\nu) \preceq (T_\mu, \Phi_\mu)$, say $\theta : T_\nu \to T_\mu$ is such that $s \le t$ implies $\theta(s) \le \theta(t)$ and for all $s \in T_\nu$, $\Phi_\nu(s) \subseteq \Phi_\mu(\theta(s))$.

Now move to a generic extension $V[G]$ in which A_ν is countable. In $V[G]$, let $\sigma : \omega \to A_\nu$ be a surjection. We will define an embedding $f : A_\nu \to A_\mu$ by letting $f(\sigma(n)) = \theta(\sigma \restriction n+1)(n)$ for all $n < \omega$. To see that f is an embedding, note that $f(\sigma(n)) = \theta(\sigma \restriction k)(n)$ for all $k > n$ since θ preserves the tree ordering; moreover, for any $a, b, c \in A_\nu$, there is a k such that $a, b, c \in \mathrm{rge}(\sigma \restriction k)$ so since

$$\Phi_\nu(\sigma \restriction k) \subseteq \Phi_\mu(\theta(\sigma \restriction k)) = \Phi_\mu(\langle f(0), ..., f(k-1)\rangle)$$

every quantifier-free formula satisfied by a, b, c in A_ν (e.g. $a \ne 0$, $a - b = c$, $ab = c$) is satisfied by $f(a), f(b), f(c)$ in A_μ. This completes the proof of Theorem 1

The argument is very general and could be applied to any family of structures (for example, to those in [6]). If we start with a torsion-free group A of cardinality $\kappa \ge \kappa(\omega)$, and apply the argument to the family of structures

$\{\langle A, a_v\rangle : \nu < \kappa(\omega)\}$ where $\{a_\nu : \nu < \kappa(\omega)\}$ is a linearly independent subset of A, then we obtain $\nu \neq \mu$ such that in a generic extension in which A becomes countable we have an embedding $f : A \to A$ taking a_ν to a_μ. This proves Theorem 4.

Ernest Schimmerling has pointed out that there is a "soft" proof of these results (not relying on Theorem 6) using a model of set theory (with $\{A_\nu : \nu < \kappa\}$ as additional predicate) and a set of indiscernibles given by the defining property of $\kappa = \kappa(\omega)$.

4 Existence theorem

In this section we will prove Theorem 2. So let $\kappa < \kappa(\omega)$ and let λ be a cardinal $\geq \kappa(\omega)$. Let $\{(T_\mu, \Phi_\mu) : \mu < \kappa\}$ be the family of ω-labeled trees as in Theorem 7. We can assume that each node of T_μ of height m is a sequence of length m and the tree-ordering is extension of sequences; so the root of the tree is the empty sequence $<>$.

Let $\langle p_{n,m,j} : n, m \in \omega, j \in \{0, 1\}\rangle$ and $\langle q_{n,m,\ell,j} : n, m, \ell \in \omega, j \in \{0, 1\}\rangle$ be two lists, with no overlap, of distinct primes.

For any ordinal α, Z_α be the tree of finite strictly decreasing non-empty sequences z of ordinals $\leq \alpha$ such that $z(0) = \alpha$. (Thus for some $n \in \omega$, $z : n \to \alpha$ such that $\alpha = z(0) > z(1) > \ldots > z(n-1)$.)

For $n \in \omega$ let $g_n : \lambda \to \mathcal{P}([\lambda n, \lambda(n+1)))$ such that for each $\nu < \lambda$, $g_n(\nu)$ is a subset of $[\lambda n, \lambda(n+1))$ which is cofinal in $\lambda(n+1) = \lambda n + \lambda$. (Here the operations are ordinal addition and multiplication, so, in particular, λn is less than λ^+, the cardinal successor of λ.) We also require that for $\mu \neq \nu$, $g_n(\mu) \cap g_n(\nu) = \emptyset$. For $n > 0$, let $Y_n = \bigcup \mathrm{rge}(g_n)$, and let $Y_0 = g_0(0)$.

For $\mu < \kappa$, let W_μ be the $\mathbb{Q}$-vector space with basis $\bigcup_{n\in\omega} \mathcal{A}_n \cup \mathcal{B}_{n,\mu}$ where for $n > 0$

$$\begin{aligned} \mathcal{A}_n &= \{a_z^\alpha : \alpha \in Y_{n-1}, z \in Z_\alpha\}, \\ \mathcal{B}_{n,\mu} &= \{b_{\eta,\mu}^\alpha : \alpha \in Y_{n-1}, \eta \in T_\mu - \{<>\}\} \end{aligned}$$

and $\mathcal{A}_0 = \{a_\mu^0\} = \mathcal{B}_{0,\mu}$. We are going to define A_μ to be a subgroup of W_μ. Since μ is fixed throughout the construction, we will usually omit the subscript μ from what follows (until we come to consider $\mathrm{Hom}(A_\nu, A_\mu)$).

For each $n > 0$, let h_n be a bijection from $\mathcal{A}_n \cup \mathcal{B}_n$ onto λ; let $h_0(a^0) = 0$. Then for any $w \in \mathcal{A}_n \cup \mathcal{B}_n$, and any $\alpha \in g_n(h_n(w))$, we will use $a_{<>}^\alpha$ or $b_{<>}^\alpha$ as a notation for w. (So $a_{<>}^\alpha = b_{<>}^\alpha$; moreover, $a_{<>}^\alpha = a_{<>}^\beta$ if and only if α and β belong to the same member of the range of g_n.) Now we can define A $(= A_\mu)$ to be the subgroup of W generated (as abelian group) by the union of

$$\bigcup_{n\geq 0} \{\frac{1}{p_{n,m,0}^k} a_z^\alpha : m, k \in \omega, z \in Z_\alpha \cup \{<>\}, \alpha \in Y_n, \mathrm{dom}(z) = m\} \tag{1}$$

$$\bigcup_{n\geq 0} \{\frac{1}{p_{n,m,1}^k}(a_z^\alpha + a_{z\restriction m-1}^\alpha) : m, k \in \omega - \{0\}, z \in Z_\alpha, \alpha \in Y_n, \mathrm{dom}(z) = m\} \tag{2}$$

and

$$\bigcup_{n\geq 0}\{\frac{1}{q^k_{n,m,\ell}}(b^\alpha_\eta+b^\alpha_{\eta\restriction m-1}):m,k\in\omega-\{0\},\eta\in T_\mu,\alpha\in Y_n,\mathrm{dom}(\eta)=m,\Phi_\mu(\eta)=\ell\}. \tag{3}$$

(where $b^\alpha_{\eta\restriction -1}=0$). We will use the sets (1) and (2) to prove that (I) A_μ is absolutely indecomposable and the set (3) to prove that (II) $\mathrm{Hom}(A_\mu, A_\nu)=0$ for all $\mu\neq\nu$.

If $x\in A_\mu$, we will write $p^\infty|x$ if for every $k\in\omega$, there exists $v\in A_\mu$ such that $p^k v=x$. For example, if $w\in\mathcal{A}_n\cup\mathcal{B}_n$ and $\alpha\in g_n(h_n(w))$ and $\Phi_\mu(<>)=\ell_0$, then $p^\infty_{n,0,0}|w$ and $q^\infty_{n,0,\ell_o}|w$. Assertions about divisibility in A_μ are easily checked by considering the coefficients of linear combinations over $\mathbb{Q}$ of elements of the basis $\bigcup_{n\in\omega}\mathcal{A}_n\cup\mathcal{B}_n$ of W_μ; for example, $p^\infty_{n,m,0}|x$ if and only if $x=\sum^r_{i=1}c_i a^{\alpha_i}_{z_i}$ for some $\alpha_1,...,\alpha_r$ in Y_n, z_i of length m, and $c_i\in\mathbb{Q}$ (with denominator a power of $p_{n,m,0}$).

Proof of (I)

We will show, in fact, that in any generic extension $V[G]$ the only automorphisms of A ($=A_\mu$) are the trivial ones, 1 and -1. This part of the proof does not use the trees in $\mathcal{T}$; the absoluteness is a consequence of an argument using formulas of $L_{\infty\omega}$, which therefore works in any generic extension. We will use the following claim:

> (1A) there are formulas $\psi_{n,\alpha}(x)$ of $L_{\infty\omega}$ ($n\in\omega$, $\alpha\in Y_n$) such that for any $u\in A$, $A\models\psi_{n,\alpha}[u]$ if and only if there are $w_1,...,w_r\in\mathcal{A}_n\cup\mathcal{B}_n$, and $c_1,...,c_r\in\mathbb{Z}-\{0\}$, such that $u=\sum^r_{i=1}c_iw_i$ and $\alpha\in\bigcup^r_{i=1}g_n(h_n(w_i))$.

Assuming the claim for now, suppose that in $V[G]$ there is an automorphism F of A. For any $n\in\omega$, consider any $w\in\mathcal{A}_n\cup\mathcal{B}_n$; since $w=a^\alpha_{<>}$ for $\alpha\in g_n(h_n(w))$, $p^\infty_{n,0,0}|w$; therefore $p^\infty_{n,0,0}|F(w)$, and hence $F(w)=\sum^r_{i=1}c_iw_i$ for some distinct $w_i\in\mathcal{A}_n\cup\mathcal{B}_n$. Moreover, by (1A), $A\models\psi_{n,\alpha}[w]$ if and only if $\alpha\in g_n(h_n(w))$ if and only if $A\models\psi_{n,\alpha}[F(w)]$. Thus, since the elements of the range of g_n are disjoint, we must have that $r=1$ and $w_1=w$, that is, $F(w)=cw$ for some $c=c(w)\in\mathbb{Q}$.

If we can show that $c(w)=c(a^0)$ for all $w\in\bigcup_{n\in\omega}\mathcal{A}_n\cup\mathcal{B}_n$, then F is multiplication by $c(a^0)$, and then it is easy to see that $c(a^0)$ must be ±1. It will be enough to show that if $w=a^\alpha_z$ (resp. $w=b^\alpha_\eta$) for some $\alpha\in Y_{n-1}$, then $c(w)=c(a^\alpha_{<>})$, for then $c(w)=c(w')$ for some $w'\in\mathcal{A}_{n-1}\cup\mathcal{B}_{n-1}$ (namely, the unique w' such that $\alpha\in g_{n-1}(h_{n-1}(w'))$) and by induction $c(w')=c(a^0)$.

So suppose $w=a^\alpha_z$; the proof will be by induction on the length of z that $c(a^\alpha_z)=c(a^\alpha_{<>})$. Suppose that the length of $z=m>0$. Let $c=c(a^\alpha_z)$

and $c' = c(a^\alpha_{z\restriction m-1})$. By induction it is enough to prove that $c = c'$. Since $p^\infty_{n,m,1}|(a^\alpha_z + a^\alpha_{z\restriction m-1})$, it is also the case that $p^\infty_{n,m,1}$ divides

$$F(a^\alpha_z) + F(a^\alpha_{z\restriction m-1}) = ca^\alpha_z + c'a^\alpha_{z\restriction m-1} = c(a^\alpha_z + a^\alpha_{z\restriction m-1}) + (c' - c)a^\alpha_{z\restriction m-1}$$

so $p^\infty_{n,m,1}|(c' - c)a^\alpha_{z\restriction m-1}$, which is impossible unless $c = c'$.

The proof is similar if $w = b^\alpha_\eta$, but uses the primes $q_{n,m,\ell}$. So it remains to prove (1A). We will begin by defining some auxiliary formulas of $L_{\infty\omega}$. First, we will use $p^\infty|x$ as an abbreviation for

$$\bigwedge_{k\in\omega} \exists v_k(p^k v_k = x).$$

Define $\varphi_{n,m,0}(y)$ to be $p^\infty_{n,m,0}|y$. Then for $u \in A$, $A \models \varphi_{n,m,0}[u]$ if and only if u is in the subgroup ($\mathbb{Z}$-submodule) generated by $\{\frac{1}{p^k_{n,m,0}} a^\alpha_z : \alpha \in Y_n, k \in \omega, z \in Z_\alpha, \mathrm{dom}(z) = m\}$. Define $\varphi_{n,m,\beta}(y)$ for each $m > 0$ by recursion on β: if $\beta = \gamma + 1$, $\varphi_{n,m,\beta}(y)$ is

$$\varphi_{n,m,\gamma}(y) \wedge \exists y'(\varphi_{n,m+1,\gamma}(y') \wedge (p^\infty_{n,m+1,1}|(y + y')).$$

If β is a limit ordinal, let $\varphi_{n,m,\beta}(y)$ be

$$\bigwedge_{\gamma<\beta} \varphi_{n,m,\gamma}(y).$$

Then for $u \in A$, $A \models \varphi_{n,m,\beta}[u]$ if and only if u is in the subgroup generated by

$$\{a^\alpha_z : \alpha \in Y_n, z \in Z_\alpha, \mathrm{dom}(z) = m \text{ and } z(m-1) \geq \beta\}.$$

In particular, for $m = 1$, recalling that $z \in Z_\alpha$ satisfies $z(0) = \alpha$, we have that $A \models \varphi_{n,1,\beta}[u]$ if and only if u is in the subgroup generated by

$$\{a^\alpha_{<\alpha>} : \alpha \in Y_n, \alpha \geq \beta\}.$$

Now define $\psi_{n,\alpha}(x)$ to be

$$\varphi_{n,0,0}(x) \wedge \exists y[p^\infty_{n,1,1}|(x + y) \wedge \varphi_{n,1,\alpha}(y) \wedge \neg\varphi_{n,1,\alpha+1}(y)].$$

If $u = \sum_{i=1}^r c_i w_i$, for some $w_i \in \mathcal{A}_n \cup \mathcal{B}_n$, then $p^\infty_{n,1,1}|(u+y)$ iff $y = \sum_{i=1}^r c_i a^{\alpha_i}_{<\alpha_i>}$ for some $\alpha_i \in g_n(h_n(w_i))$; using the cofinality of members of the range of g_n, it follows easily that $\psi_{n,\alpha}(x)$ has the desired property.

Proof of (II)

Suppose that there is a non-zero homomorphism $H : A_\nu \to A_\mu$. We are going to use H to define $\theta : T_\nu \to T_\mu$ showing that $(T_\nu, \Phi_\nu) \preceq (T_\mu, \Phi_\mu)$, contrary to the choice of the family of labeled trees. Now $H(w) \neq 0$ for some $w \in \mathcal{A}_n \cup \mathcal{B}_{n,\nu}$

for some $n \in \omega$. Thus for some $\alpha \in Y_n$, $H(b^\alpha_{<>,\nu}) \neq 0$. Fix such an α (which can in fact be any member of $g_n(h_n(w)))$. Let $\Phi_\nu(<>) = \ell_0$. Then $q^\infty_{n,0,\ell_0,0}|b^\alpha_{<>,\nu}$ so $q^\infty_{n,0,\ell_0,0}|H\ (b^\alpha_{<>,\nu})$, and hence $H(b^\alpha_{<>,\nu})$ must be of the form $\sum_{i=1}^r c_i b^{\alpha_i}_{<>,\mu}$ where $c_i \in \mathbb{Q} - \{0\}$, and $\Phi_\mu(<>) = \ell_0$. So letting $\theta(<>) = <>$ (as it must), we have confirmed that $\Phi_\mu(\theta(\eta)) = \Phi_\nu(\eta)$ for $\eta =<>$.

Now suppose that for some $m \geq 0$, $\theta(\eta)$ has been defined for all nodes η of T_ν of height $\leq m$ such that $\Phi_\mu(\theta(\eta)) = \Phi_\nu(\eta)$. Moreover, suppose that for every η of height $\leq m$, the coefficient of $b^\alpha_{\theta(\eta),\mu}$ in $H(b^\alpha_{\eta,\nu})$ is non-zero. Now consider any node ζ of T_ν of height $m+1$; let $\eta = \zeta \restriction m$. In A_ν, for $\ell = \Phi_\nu(\zeta)$, $q^\infty_{n,m+1,\ell,1}|b^\alpha_{\eta,\nu} + b^\alpha_{\zeta,\nu}$ so $q^\infty_{n,m+1,\ell,1}|H\ (b^\alpha_{\eta,\nu}) + H\ (b^\alpha_{\zeta,\nu})$ in A_μ. Since the coefficient — call it c — of $b^\alpha_{\theta(\eta),\mu}$ in $H(b^\alpha_{\eta,\nu})$ is non-zero, there must be a node ζ' in T_μ of height $m+1$ such that $\zeta' \restriction m = \theta(\eta)$ and the coefficient of $b^\alpha_{\zeta',\mu}$ in $H(b^\alpha_{\zeta,\nu})$ is c, and moreover such that $\Phi_\mu(\zeta') = \ell$. So we can let $\theta(\zeta) = \zeta'$. This completes the proof of Theorem 2.

5 Absolutely non-isomorphic indecomposables

In this section we sketch how to modify the construction in the preceding section in order to prove Theorem 3. (Note that this construction does not require the trees T_μ of Theorem 7.) Let $\langle p_{n,m,j} : n, m \in \omega, j \in \{0,1\}\rangle$ be a list of distinct primes. Fix an uncountable λ; for any $\alpha < \lambda\omega$, let Z_α be defined as before. Let $\langle S_{i,\lambda} : i < 2^\lambda \rangle$ be a list of 2^λ distinct subsets of λ, each of cardinality λ (and hence cofinal in λ).

For $n \neq 1$, let $g_n : \lambda \to \mathcal{P}([\lambda n, \lambda(n+1)))$ be defined as before. For $i < 2^\lambda$, define $g_{1,i} : \lambda \to \mathcal{P}([\lambda, \lambda + \lambda)))$ as before but with the additional stipulation that for all $\nu < \lambda$, $g_{1,i}(\nu) \subseteq \{\lambda + \gamma : \gamma \in S_{i,\lambda}\}$. (Here again the operation is ordinal addition.) Let $Y_{1,i} = \bigcup \mathrm{rge}(g_{1,i})$ we will also choose $g_{1,i}$ such that $Y_{1,i} = \{\lambda + \gamma : \gamma \in S_{i,\lambda}\}$. For convenience, for $n \neq 1$ we let $Y_{n,i}$ denote Y_n (independent of i).

For each $n > 0$, let $h_{n,i}$ be a bijection from $\{a^\alpha_z : \alpha \in Y_{n-1,i},\ z \in Z_\alpha\}$ onto λ; use these bijections to make identifications as in the previous construction.

Then $H_{i,\lambda}$ is defined to be the subgroup of the $\mathbb{Q}$-vector space with basis

$$\{a^0\} \cup \{a^\alpha_z : n > 0,\ \alpha \in Y_{n-1,i},\ z \in Z_\alpha\}$$

which is generated (as abelian group) by the union of

$$\bigcup_{n \geq 0} \{\frac{1}{p^k_{n,m,0}} a^\alpha_z : m, k \in \omega, z \in Z_\alpha \cup \{<>\}, \alpha \in Y_{n,i}, \mathrm{dom}(z) = m\}$$

and

$$\bigcup_{n \geq 0} \{\frac{1}{p^k_{n,m,1}} (a^\alpha_z + a^\alpha_{z \restriction m-1}) : m, k \in \omega - \{0\}, z \in Z_\alpha, \alpha \in Y_{n,i}, \mathrm{dom}(z) = m\}.$$

As before, the groups $H_{i,\lambda}$ are absolutely indecomposable. It remains to show that for $\lambda \neq \rho$ or $i \neq j$, $H_{i,\lambda}$ and $H_{j,\rho}$ are not $L_{\infty\omega}$-equivalent (and hence not isomorphic in any generic extension). For this we use the formulas $\psi_{1,\alpha}(x)$. If $\lambda = \rho$ and $i \neq j$, without loss of generality there exists $\gamma \in S_{i,\lambda} - S_{j,\lambda}$; let $\alpha = \lambda+\gamma$. If $\lambda < \rho$, let $\alpha = \lambda+\gamma$ for any γ in any $S_{i,\lambda}$. In either case, $\exists x \psi_{1,\alpha}(x)$ is true in $H_{i,\lambda}$ but not in $H_{j,\rho}$.

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