

ALMOST DISJOINT PURE SUBGROUPS OF THE BAER-SPECKER GROUP

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ABSTRACT. We prove in ZFC that the Baer-Specker group \mathbf{Z}^ω has 2^{\aleph_1} non-free pure subgroups of cardinality \aleph_1 which are almost disjoint: there is no non-free subgroup embeddable in any pair.

In this short paper we prove the following result.

Theorem 1. *There exists a family $\mathbf{G} = \{G_\alpha : \alpha < 2^{\aleph_1}\}$ of non-isomorphic non-free pure subgroups of the Baer-Specker group \mathbf{Z}^ω such that:*

- (1.1) *each G_α has cardinality \aleph_1 ;*
- (1.2) *if $\alpha < \beta$, then G_α and G_β are almost disjoint: if H is isomorphic to subgroups of G_α and G_β , then H is free. In particular, $G_\alpha \cap G_\beta$ is free.*

Recall that the Baer-Specker group \mathbf{Z}^ω is the abelian group of functions from the natural numbers into the integers (see [1] and [18]). It contains the canonical pure free subgroup $\mathbf{Z}_\omega = \bigoplus_{n < \omega} \mathbf{Z}$. The group \mathbf{Z}^ω is not κ -free for any cardinal $\kappa > \aleph_1$, but it is \aleph_1 -free, so the groups G_α in Theorem 1 are almost free.

Theorem 1 answers a question of the first author, and has its place in the line of recent research dealing with the lattice structure of the pure subgroups of \mathbf{Z}^ω (see [2], [3], and [5]–[8]). For example, Irwin asked whether there is a subgroup of \mathbf{Z}^ω with uncountable dual but no free summands of infinite rank. This problem was resolved recently by Corner and Goebel [5] who proved the following stronger fact.

Theorem 2. [5] *The Baer-Specker group \mathbf{Z}^ω contains a pure subgroup G whose endomorphism ring splits as $\text{End}(G) = \mathbf{Z} \oplus \text{Fin}(G)$, with $|G^*| = 2^{\aleph_0}$, where \mathbf{Z} is the scalar multiplication by integers and $\text{Fin}(G)$ is the ideal of all endomorphisms of G of finite rank.*

Quotient-equivalent and almost disjoint abelian groups have been studied by Eklof, Mekler and Shelah in [9]–[11], who showed that under various set-theoretic hypotheses, there exist families of maximal possible size of almost free abelian groups which are pairwise almost disjoint. Following [11], we say that two groups A and B are almost disjoint if whenever H is embeddable as a subgroup in both A and B , then H is free. Clearly if A and B are non-free and almost disjoint, then they are non-isomorphic in a very strong way. On the other hand, the intersection of two almost disjoint groups of size \aleph_1 need not necessarily be countable, so group-theoretic almost disjointness differs from its set-theoretic homonym. Theorem 1 establishes in ZFC that the Baer-Specker group contains large families of almost disjoint almost free non-free pure uncountable subgroups.

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Our group and set-theoretic notation is standard and can be found in [10] and [14]. For example, $\omega_1^{>2}$ is the set of partial functions from ω_1 into $\{0, 1\}$ whose domains are at most countable; $\omega_1 2$ is the set of all functions from ω_1 into $\{0, 1\}$; for a regular cardinal χ , $H(\chi)$ is the family of all sets of hereditary cardinality less than χ .

For a set $A \subseteq H(\chi)$ for χ large enough, we write $\text{dcl}_{(H(\chi), \in, <)}[A]$ for the Skolem closure (Skolem hull) of A in the structure $(H(\chi), \in, <)$, where $<$ is a well-ordering of $H(\chi)$ (for details, see [16], 400-402, or [15], 165-170).

In proving Theorem 1, we shall appeal to the well-known Engelking-Karłowicz theorem from set-theoretic topology:

Theorem 3. [13] *If $|Y| = \mu = \mu^{<\sigma} < \lambda = |X| \leq 2^\mu$, then there are functions $h_\alpha : X \rightarrow Y$ for $\alpha < \mu$ such that for every partial function f from X to Y of cardinality less than σ , for some $\alpha < \mu$, $f \subseteq h_\alpha$.*

A self-contained short proof can be found in [17], 422-423. We shall need just the case when $\mu = \sigma = \aleph_0$, and $\lambda = 2^\mu$. Since it may be less familiar to algebraists, for convenience we deduce the fact to which we appeal later on (although it also appears as Corollary 3.17 in [4]).

Lemma 4. *There exists a family $\{f_\eta : \eta \in \omega_1^{>2}\}$ such that $f_\eta : \omega \rightarrow \mathbf{Z}$, and whenever η_1, \dots, η_k are distinct and $a_1, \dots, a_k \in \mathbf{Z}$, then $\{i < \omega : (\forall l \leq k)(f_{\eta_l}(i) = a_l)\}$ is infinite.*

Proof. Take $\mu = \sigma = \aleph_0$, $\lambda = 2^\mu$, $X = \omega_1^{>2}$ and $Y = \mathbf{Z}$ in the Engelking-Karłowicz theorem. Since $|\omega_1^{>2}| = 2^{\aleph_0}$ and $|\mathbf{Z}| = \aleph_0$, we know that there exist functions $h_n : \omega_1^{>2} \rightarrow \mathbf{Z}$ for $n < \omega$ such that for every partial function f from $\omega_1^{>2}$ to \mathbf{Z} whose domain is finite, there is some $m < \omega$ such that $f \subseteq h_m$. Let $\{g_i : i < \omega\}$ be an enumeration with infinitely many repetitions of each h_n for $n < \omega$.

For each $\eta \in \omega_1^{>2}$, define $f_\eta : \omega \rightarrow \mathbf{Z}$ by $f_\eta(i) = g_i(\eta)$. The family $\{f_\eta : \eta \in \omega_1^{>2}\}$ is as required: for if η_1, \dots, η_k are distinct and $a_1, \dots, a_k \in \mathbf{Z}$ are given, then the set $f = \langle (\eta_1, a_1), \dots, (\eta_k, a_k) \rangle$ is a finite function, so there is some m such that $f \subseteq h_m$ and it is now easy to see that $\{i < \omega : (\forall l \leq k)(f_{\eta_l}(i) = a_l)\}$ is infinite. \square

A well-known algebraic fact will also be useful:

Lemma 5. *Let C be a closed unbounded subset of the regular uncountable cardinal κ . Suppose that H is an abelian group of cardinality κ , and $\langle H_\alpha : \alpha < \kappa \rangle$ is a κ -filtration of H (a continuous increasing chain of subgroups H_α , $|H_\alpha| < \kappa$, whose union is H). Let $S = \{\alpha \in C : H/H_\alpha \text{ is not } \kappa\text{-free}\}$. Then H is free if and only if S is non-stationary in κ .*

Proof. Well-known: see Proposition IV.1.7 in [10]. \square

We refer the reader to [14] for the definitions of the characteristic $\chi(g)$ and the type $\tau(g)$ of an element g in a group.

Now we prove Theorem 1.

Proof. Let \mathbf{P} be the set of prime numbers, and let $\{P_\eta : \eta \in \omega_1^{>2}\}$ be a family of almost disjoint (infinite) subsets of \mathbf{P} : $\eta \neq \nu \in \omega_1^{>2} \Rightarrow |P_\eta \cap P_\nu| < \aleph_0$. By Lemma 4, there exists $\{f_\eta : \eta \in \omega_1^{>2}\}$ such that $f_\eta : \omega \rightarrow \mathbf{Z}$, and if η_1, \dots, η_k are distinct and $a_1, \dots, a_k \in \mathbf{Z}$, then $\{i < \omega : (\forall l \leq k)(f_{\eta_l}(i) = a_l)\}$ is infinite.

Define functions x_η and $x_{\eta,j}$ in \mathbf{Z}^ω as follows. Let $x_\eta = \langle \pi_{\eta,i} \cdot f_\eta(i) : i < \omega \rangle$ where $\pi_{\eta,i} = \prod\{p \in P_\eta : p < i\}$, and let $x_{\eta,j} = \langle \pi_{\eta,i}^j \cdot f_\eta(i) : i < \omega \rangle$ where $\pi_{\eta,i}^j = \prod\{p \in P_\eta : j \leq p < i\}$ ($=0$ if $i \leq j$). Note that $x_\eta = x_{\eta,0}$.

For $\eta \in {}^{\omega_1}2$, let G_η be the subgroup of \mathbf{Z}^ω generated by $\mathbf{Z}_\omega \cup \{x_{\eta|\alpha,j} : \alpha < \omega_1, 0 \leq j < \omega\}$.

We show that the family $\mathbf{G} = \{G_\eta : \eta \in {}^{\omega_1}2\}$ satisfies the conclusions of Theorem 1.

Claim 1: G_η is pure in \mathbf{Z}^ω .

Proof of Claim 1: Suppose that $rx = g$ for some $x \in \mathbf{Z}^\omega$, $r \in \mathbf{N}$, and $g \in G_\eta$. Say $g = y + n_1 x_{\eta|\alpha_1, j_1} + \cdots + n_m x_{\eta|\alpha_m, j_m}$, $n_l \neq 0$, with $y \in \mathbf{Z}_\omega$. Without loss of generality (adding more elements from \mathbf{Z}_ω to the RHS if necessary), $(\forall l \leq m)(j_l = j)$ for some $j < \omega$, $j > r$, $y(i) = 0$ ($\forall i > j$), and $x(i) = 0$ ($\forall i \leq j$). Relabelling (if necessary), we may assume that $\alpha_1 < \cdots < \alpha_m < \omega_1$, and because $x_{\eta|\alpha_l, j}(i) = 0$ if $i \leq j$, we may write

$$rx = ry^* + n_1 x_{\eta|\alpha_1, j} + \cdots + n_m x_{\eta|\alpha_m, j}, \quad \text{for some } y^* \in \mathbf{Z}_\omega.$$

Fix $k \in \{1, \dots, m\}$. Since $\eta|\alpha_1, \dots, \eta|\alpha_m$ are distinct ($\alpha_1 < \cdots < \alpha_m$), letting $a_l = \delta_{kl}$ (Kronecker delta), we know that the set $N_k = \{i < \omega : (\forall l \neq k)(f_{\eta_l}(i) = 0, f_{\eta_k}(i) = 1)\}$ is infinite. For large enough i in this set (e.g. $i > \max_{1 \leq l \leq m} [\min(P_{\eta|\alpha_l} \setminus \{0, \dots, j\})]$), $x_{\eta|\alpha_l, j}(i)$ is zero if and only if $l \neq k$. So for infinitely many $i < \omega$, for $l \neq k$, $x_{\eta|\alpha_l, j}(i) = 0$, and $x_{\eta|\alpha_k, j}(i) \neq 0$.

Unfix k . For each $k \leq m$, for infinitely many $i \in (j, \omega) \cap N_k$, $rx(i) = n_k x_{\eta|\alpha_k, j}(i) = n_k \prod\{p \in P_{\eta|\alpha_k} : j \leq p < i\}$. Since $r < j$, we must have $rs_k = n_k$ for some s_k in \mathbf{Z} , and therefore $x = y^* + s_1 x_{\eta|\alpha_1, j} + \cdots + s_m x_{\eta|\alpha_m, j} \in G_\eta$ (G_η is torsion-free). Hence G_η is pure in \mathbf{Z}^ω , which establishes Claim 1.

Claim 2: G_η has cardinality \aleph_1 , so (1.1) holds.

Proof of Claim 2: If $\xi \neq \zeta \in {}^{\omega_1}2$, then for some $j < \omega$, $P_\xi \cap P_\zeta \subseteq j$. Pick $p, q > j$ with $p \in P_\xi$ and $q \in P_\zeta$; so the set $B = \{i < \omega : f_\xi(i) = p \text{ and } f_\zeta(i) = q\}$ is infinite, and if $i \in B$ is bigger than $\max\{j, p, q\}$, then $x_{\xi, j}(i) \neq x_{\zeta, j}(i)$, since $x_{\xi, j}(i)$ is non-zero and divisible by p^2 but by no prime in P_ζ , and $x_{\zeta, j}(i)$ is non-zero and divisible by q^2 but by no prime in P_ξ . It follows that G_η has cardinality \aleph_1 . After this observation, a second's reflection on the element types of G_η and G_ν (for $\eta \neq \nu$) should convince the reader that the groups are neither isomorphic nor free.

Claim 3: (1.2) holds: if $\eta_1 \neq \eta_2 \in {}^{\omega_1}2$, then G_{η_1} and G_{η_2} are almost disjoint.

Proof of Claim 3: Suppose (towards a contradiction) that for some $\eta_1 \neq \eta_2 \in {}^{\omega_1}2$, for some non-free abelian group H , there exist isomorphisms $\varphi_l : H \rightarrow \text{range}(\varphi_l) \leq G_{\eta_l}$, $l = 1, 2$. Since G_{η_l} is \aleph_1 -free, H must have cardinality \aleph_1 . Let $\langle H_i : i < \omega_1 \rangle$ be an ω_1 -filtration of H . Without loss of generality, we may assume that each H_i is pure in H , so that H/H_i is torsion-free.

Let $G_{\eta, i} = \langle \mathbf{Z}_\omega \cup \{x_{\eta|\beta, j} : j < \omega, \beta < i\} \rangle$ for $i < \omega_1$ and $\eta \in \{\eta_1, \eta_2\}$.

Note that $\langle G_{\eta, i} : i < \omega_1 \rangle$ is a ω_1 -filtration of G_η , since it is increasing and continuous with union G_η , and each $G_{\eta, i}$ is countable. For large enough χ , the set C defined by $\{\delta < \omega_1 : \text{dcl}_{(H(\chi), \epsilon, <)}[\delta \cup \{G_{\eta_1}, G_{\eta_2}, \{x_\nu, f_\nu : \nu \in {}^{\omega_1}2\}, \eta_1, \eta_2, \varphi_1, \varphi_2, \{H_i :$

$i < \omega_1\}} \cap \omega_1 = \delta\}$ is a club of ω_1 (well-known, or see [16], 401). Note that if $\delta \in C$, then φ_l maps H_δ into $G_{\eta_l, \delta}$. Since H is not free, it follows by Lemma 5 that $S = \{\delta \in C : H/H_\delta \text{ is not } \aleph_1\text{-free}\}$ is stationary. By Pontryagin's Criterion, for each $\delta \in S$, H/H_δ has a non-free (torsion-free) subgroup K_δ/H_δ of finite rank $n_\delta + 1$ such that every subgroup of K_δ/H_δ of rank less than $n_\delta + 1$ is free. Let H_δ^+/H_δ be a pure subgroup of K_δ/H_δ of rank n_δ . Then H_δ^+/H_δ is free with basis $y_0 + H_\delta, \dots, y_{n_\delta-1} + H_\delta$ say. So $K_\delta/H_\delta^+ \simeq (K_\delta/H_\delta)/(H_\delta^+/H_\delta)$ is a torsion-free rank-1 group which is not free, and hence there is a non-zero element $y_{n_\delta} + H_\delta^+$ which is divisible in K_δ/H_δ^+ by infinitely many natural numbers. Call this set of natural numbers A .

For $l = 1, 2$, for large enough $j_l(*) < \omega$, and $\beta^l_0 < \dots < \beta^l_{k_l} < \omega_1$, $\varphi_l(y_m)$ is an element of the subgroup of G_{η_l} generated by $G_{\eta_l, \delta} \cup \{x_{\eta_l|\beta^l_0, j_l(*)}, \dots, x_{\eta_l|\beta^l_{k_l}, j_l(*)}\}$ for all $m \leq n_\delta$.

Taking large enough $\delta \in S$, we may assume that $\min\{\alpha : \eta_1|\alpha \neq \eta_2|\alpha\} < \beta^l_0$, $l = 1, 2$. Since $\delta \in C$, we can show the following claims:

(*)₁: The set A does not contain infinitely many powers of one prime.

(*)₂: The set $Q = (\mathbf{P} \cap A) \subseteq P_{\eta_l|\beta^l_0} \cup \dots \cup P_{\eta_l|\beta^l_{k_l}}$.

Now (*)₁ is true because non-zero sums of elements in $G_{\eta_l, \delta} \cup \{x_{\eta_l|\beta^l_0, j_l(*)}, \dots, x_{\eta_l|\beta^l_{k_l}, j_l(*)}\}$ are divisible by at most finitely many powers of any given prime (by the definition of the elements $x_{\eta_l|\beta, j}$). Note that $\chi(y_{n_\delta} + H_\delta^+) = \cup_{\{y \in y_{n_\delta} + H_\delta^+\}} \chi(y) \leq \text{cup}_{\{y \in y_{n_\delta} + H_\delta^+\}} \chi(\varphi_l(y))$, where the characteristics are taken relative to K_δ/H_δ^+ , K_δ and $G_{\eta_l, \delta} \cup \{x_{\eta_l|\beta^l_0, j_l(*)}, \dots, x_{\eta_l|\beta^l_{k_l}, j_l(*)}\}$ respectively. Hence (*)₁ holds. By (*)₁, since A is infinite, the set $Q = \mathbf{P} \cap A$ is infinite.

Also, the same characteristic inequality implies that $Q \subseteq P_{\eta_l|\beta^l_0} \cup \dots \cup P_{\eta_l|\beta^l_{k_l}}$. So (*)₂ is true. Hence, $Q \subseteq \cap_{l=1,2} (\cup_{k \leq k_l} P_{\eta_l|\beta^l_k})$ which is finite (since the family $\{P_\eta : \eta \in \omega_1 > 2\}$ is almost disjoint). This is a contradiction, and so Claim 3 follows, completing the proof of Theorem 1. \square

Corollary 6. *Every non-slender \aleph_1 -free abelian group G has a family $\{G_\alpha : \alpha < 2^{\aleph_1}\}$ of non-free subgroups such that:*

1. *each G_α is almost free of cardinality \aleph_1 ;*
2. *if $\alpha < \beta$, then G_α and G_β are almost disjoint.*

Proof. By Nunke's characterisation of slender groups (see Corollary IX.2.5 in [10] for example), G must contain a copy of the Baer-Specker group. \square

Remark: For the same reason, the corollary is true for any non-slender cotorsion-free abelian group.

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