RELATIONS BETWEEN SOME CARDINALS IN THE ABSENCE OF THE AXIOM OF CHOICE

DEDICATED TO THE MEMORY OF PROF. HANS LÄUCHLI

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Abstract. If we assume the axiom of choice, then every two cardinal numbers are comparable. In the absence of the axiom of choice, this is no longer so. For a few cardinalities related to an arbitrary infinite set, we will give all the possible relationships between them, where possible means that the relationship is consistent with the axioms of set theory. Further we investigate the relationships between some other cardinal numbers in specific permutation models and give some results provable without using the axiom of choice.

§1. Introduction. Using the axiom of choice, Felix Hausdorff proved in 1914 that there exists a partition of the sphere into four parts, $S = A \dot\cup B \dot\cup C \dot\cup E$, such that E has Lebesgue measure 0, the sets A, B, C are pairwise congruent and A is congruent to $B \dot\cup C$ (*cf.* [9] or [10]). This theorem later became known as Hausdorff's paradox. If we want to avoid this paradox, we only have to reject the axiom of choice. But if we do so, we will run into other paradoxical situations. For example, without the aid of any form of infinite choice we cannot prove that a partition of a given set m has at most as many parts as m has elements. Moreover, it is consistent with set theory that the real line can be partitioned into a family of cardinality strictly bigger than the cardinality of the real numbers (see Fact 8.6).

Set theory without the axiom of choice has a long tradition and a lot of work was done by the Warsaw School between 1918 and 1940. Although, in 1938, Kurt Gödel proved in [5] the consistency of the axiom of

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choice with the other axioms of set theory, it is still interesting to investigate which results remain unprovable without using the axiom of choice (*cf.* [18]).

In 1963, Paul Cohen proved with his famous and sophisticated forcing technique, that it is also consistent with the other axioms of set theory that the axiom of choice fails (*cf.* [3]). Also with a forcing construction, Thomas Jech and Antonín Sochor could show in [15] that one can embed the permutation models (these are models of set theory with atoms) into well-founded models of set theory. So, to prove consistency results in set theory, it is enough to build a suitable permutation model.

We will investigate the relationships between some infinite cardinal numbers. For four cardinal numbers—which are related to an arbitrary given one—we will give all the possible relationships between two of them; where possible means that there exists a model of set theory in which the relationship holds. For example it is possible that there exists an infinite set m such that the cardinality of the set of all finite sequences of m is strictly smaller than the cardinality of the set of all finite subsets of m. On the other hand, it is also possible that there exists an infinite set m' such that the cardinality of the set of all finite sequences of m' is strictly bigger than the cardinality of the power-set of m'. In a few specific permutation models, like the basic Fraenkel model and the ordered Mostowski model, we will investigate also the relationships between some other cardinal numbers. Further we give some results provable without using the axiom of choice and show that some relations imply the axiom of choice.

§2. Definitions, notations and basic facts.

First we want to define the notion of a cardinal number and for this we have to give first the definition of ordinal numbers.

Definition: A set α is an **ordinal** if and only if every element of α is a subset of α and α is well-ordered by $\in$.

Now let V be a model for ZF (this is Zermelo-Fraenkel's set theory without the axiom of choice) and let $\mathrm{On} := \{\alpha \in V : \alpha \text{ is an ordinal}\}$; then On is a proper class in V. It is easy to see that if $\alpha \in \mathrm{On}$, then also $\alpha + 1 := \alpha \cup \{\alpha\} \in \mathrm{On}$. An ordinal α is called a **successor ordinal** if there exists an ordinal β such that $\alpha = \beta + 1$ and it is called a **limit ordinal** if it is neither a successor ordinal nor the empty-set.

By transfinite recursion on $\alpha \in \mathrm{On}$ we can define V_α as follows: $V_\emptyset := \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and $V_\alpha := \bigcup_{\beta\in\alpha} V_\beta$ when α is a limit ordinal. Note that by the axiom of power-set and the axiom of replacement, for each $\alpha \in \mathrm{On}$, V_α is a set in V. By the axiom of foundation we further get $V := \bigcup_{\alpha\in\mathrm{On}} V_\alpha$ (*cf.* [16, Theorem 4.1]).

Let m be a set in V, where V is a model of ZF, and let $\mathfrak{C}(m)$ denote the **class of all sets** x, such that there exists a **one-to-one mapping from** x **onto** m. We define the cardinality of m as follows.

DEFINITION: For a set m, let $\mathfrak{m} := \mathfrak{C}(m) \cap V_\alpha$, where α is the smallest ordinal such that $V_\alpha \cap \mathfrak{C}(m) \neq \emptyset$. The set $\mathfrak{m}$ is called the **cardinality** of m and a set $\mathfrak{n}$ is called a **cardinal number** (or simply a **cardinal**) if it is the cardinality of some set.

Note that a cardinal number is defined as a set.

A cardinal number $\mathfrak{m}$ is an **aleph** if it contains a well-ordered set. So, the cardinality of each ordinal is an aleph. Remember that the axiom of choice is equivalent to the statement that each set can be well-ordered. Hence, in ZFC (this is Zermelo-Fraenkel's set theory with the axiom of choice), every cardinal is an aleph; and vice versa, if every cardinal is an aleph, then the axiom of choice holds.

If we have a model V of ZF in which the axiom of choice fails, then we have more cardinals in V than in a model M of ZFC. This is because all the ordinals are in V and, hence, the alephs as well.

NOTATION: We will use fraktur-letters to denote cardinals and $\aleph$'s to denote the alephs. For finite sets m, we also use $|m|$ to denote the cardinality of m. Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ be the set of all natural numbers and let $\aleph_0$ denote its cardinality. We can consider $\mathbb{N}$ also as the set of finite ordinal numbers, where $n = \{0, 1, \ldots, n-1\}$ and $0 = \emptyset$. For a natural number $n \in \mathbb{N}$, we will not distinguish between n as an ordinal number and the cardinality of n. Further, the ordinal number ω denotes the order-type (with respect to $<$) of the set $\mathbb{N}$.

Now we define the order-relation between cardinals.

DEFINITION: We say that the cardinal number $\mathfrak{p}$ is **less than or equal** to the cardinal number $\mathfrak{q}$ if and only if for any $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ there is a **one-to-one mapping from** x **into** y.

NOTATION: If $\mathfrak{p}$ is less than or equal to the cardinal number $\mathfrak{q}$, we write $\mathfrak{p} \leq \mathfrak{q}$. We write $\mathfrak{p} < \mathfrak{q}$ for $\mathfrak{p} \leq \mathfrak{q}$ and $\mathfrak{p} \neq \mathfrak{q}$. If neither $\mathfrak{p} \leq \mathfrak{q}$ nor $\mathfrak{q} \leq \mathfrak{p}$ holds, then we say that $\mathfrak{p}$ and $\mathfrak{q}$ are **incomparable** and write $\mathfrak{p} \| \mathfrak{q}$. For $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ we write: $x \preccurlyeq y$ if $\mathfrak{p} \leq \mathfrak{q}$ and $x \not\preccurlyeq y$ if $\mathfrak{p} \not\leq \mathfrak{q}$ (*cf.* also [16, p. 27]). Notice that $x \preccurlyeq y$ iff there exists a one-to-one function from x into y.

Another order-relation which we will use at a few places and which was first introduced by Alfred Tarski (*cf.* [20]) is the following.

DEFINITION: For two cardinal numbers $\mathfrak{p}$ and $\mathfrak{q}$ we write $\mathfrak{p} \leq^* \mathfrak{q}$ if there are non-empty sets $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ and a function from y **onto** x.

Notice, that for infinite cardinals $\mathfrak{p}$ and $\mathfrak{q}$, we must use the axiom of choice to prove that $\mathfrak{p} \leq^* \mathfrak{q}$ implies $\mathfrak{p} \leq \mathfrak{q}$ (see *e.g.* [7]). In general, if we work in ZF, there are many relations between cardinals which do not exist if we assume the axiom of choice (*cf.* [7]); and non-trivial relations between cardinals become trivial with the axiom of choice (see also [17] or [23]).

The main tool in ZF to show that two cardinals are equal is the

Cantor-Bernstein Theorem: If $\mathfrak{p}$ and $\mathfrak{q}$ are cardinals with $\mathfrak{p} \leq \mathfrak{q}$ and $\mathfrak{q} \leq \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{q}$.

(For a proof see [14] or [1].)

Notice that for $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ we have $x \preccurlyeq y \not\preccurlyeq x$ is equivalent to $\mathfrak{p} < \mathfrak{q}$, and if $x \preccurlyeq y \preccurlyeq x$, then there exists a one-to-one mapping from x onto y.

A result which gives the connection between the cardinal numbers and the $\aleph$'s is

Hartogs Theorem: For every cardinal number $\mathfrak{m}$, there exists a least aleph, denoted by $\aleph(\mathfrak{m})$, such that $\aleph(\mathfrak{m}) \not\leq \mathfrak{m}$.

(This was proved by Friedrich Hartogs in [8], but a proof can also be found in [14] or in [1].)

Now we will define "infinity".

Definition: A cardinal number is called **finite** if it is the cardinality of a natural number, and it is called **infinite** if it is not finite.

There are some other degrees of infinity (*cf. e.g.* [6] or [26]), but we will use only "infinite" for "not finite" and as we will see, most of the infinite sets we will consider in the sequel will be Dedekind finite, where a cardinal number $\mathfrak{m}$ is called **Dedekind finite** if $\aleph_0 \not\leq \mathfrak{m}$.

There are also many weaker forms of the axiom of choice (we refer the reader to [12]). Concerning the notion of Dedekind finite we wish to mention five related statements.

AC: "The Axiom of Choice";

$2\mathfrak{m} = \mathfrak{m}$: "For every infinite cardinal $\mathfrak{m}$ we have $2\mathfrak{m} = \mathfrak{m}$";

$C(\aleph_0, \infty)$: "Every countable family of non-empty sets has a choice function";

$C(\aleph_0, <\aleph_0)$: "Every countable family of non-empty finite sets has a choice function";

$W_{\aleph_0}$: "Every Dedekind finite set is finite".

We have the following relations (for the references see [12]):

$$\text{AC} \Rightarrow 2\mathfrak{m} = \mathfrak{m} \Rightarrow W_{\aleph_0} \Rightarrow C(\aleph_0, <\aleph_0) \ \text{ and } \ \text{AC} \Rightarrow C(\aleph_0, \infty) \Rightarrow W_{\aleph_0},$$

but on the other hand have

$$\text{AC} \not\Leftarrow 2\mathfrak{m} = \mathfrak{m} \not\Leftarrow W_{\aleph_0} \not\Leftarrow C(\aleph_0, <\aleph_0) \ \text{ and } \ \text{AC} \not\Leftarrow C(\aleph_0, \infty) \not\Leftarrow W_{\aleph_0},$$

and further $2\mathfrak{m} = \mathfrak{m} \not\Rightarrow C(\aleph_0, \infty) \not\Rightarrow 2\mathfrak{m} = \mathfrak{m}$.

§3. Cardinals related to a given one. Let m be an arbitrary set and let $\mathfrak{m}$ denote the cardinality of m. In the following we will define some cardinalities which are related to the cardinal number $\mathfrak{m}$.

Let $[m]^2$ be the set of all 2-element subsets of m and let $[\mathfrak{m}]^2$ denote the cardinality of the set $[m]^2$.

Let $\operatorname{fin}(m)$ denote the set of all finite subsets of m and let $\operatorname{fin}(\mathfrak{m})$ denote the cardinality of the set $\operatorname{fin}(m)$.

For a natural number n, $\operatorname{fin}(m)^n$ denotes the set $\{\langle e_0, \dots, e_{n-1}\rangle : \forall i < n(e_i \in \operatorname{fin}(m))\}$ and $\operatorname{fin}(\mathfrak{m})^n$ denotes its cardinality.

For a natural number n, $\operatorname{fin}^{n+1}(m)$ denotes the set $\operatorname{fin}(\operatorname{fin}^n(m))$, where $\operatorname{fin}^0(m) := m$, and $\operatorname{fin}^{n+1}(\mathfrak{m})$ denotes its cardinality.

Let $m^2 := m \times m = \{\langle x_1, x_2\rangle : \forall i < 2(x_i \in m)\}$ and let $\mathfrak{m}^2 = \mathfrak{m} \cdot \mathfrak{m}$ denote the cardinality of the set m^2.

Let $\operatorname{seq}^{1\text{-}1}(m)$ denote the set of all finite one-to-one sequences of m, which is the set of all finite sequences of elements of m in which every element appears at most once, and let $\operatorname{seq}^{1\text{-}1}(\mathfrak{m})$ denote the cardinality of the set $\operatorname{seq}^{1\text{-}1}(m)$.

Let $\operatorname{seq}(m)$ denote the set of all finite sequences of m and let $\operatorname{seq}(\mathfrak{m})$ denote the cardinality of the set $\operatorname{seq}(m)$.

Finally, let $\mathcal{P}(m)$ denote the power-set of m, which is the set of all subsets of m, and let $2^{\mathfrak{m}}$ denote the cardinality of $\mathcal{P}(m)$.

In the sequel, we will investigate the relationships between these cardinal numbers.

§4. Cardinal relations which imply the axiom of choice. First we give some cardinal relations which are well-known to be equivalent to the axiom of choice. Then we show that also a weakening of one of these relations implies the axiom of choice.

The following equivalences are proved by Tarski in 1924. For the historical background we refer the reader to [21, 4.3].

Proposition 4.1. The following conditions are equivalent to the axiom of choice:

(1) $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{m} + \mathfrak{n}$ for every infinite cardinal $\mathfrak{m}$ and $\mathfrak{n}$
(2) $\mathfrak{m} = \mathfrak{m}^2$ for every infinite cardinal $\mathfrak{m}$
(3) If $\mathfrak{m}^2 = \mathfrak{n}^2$, then $\mathfrak{m} = \mathfrak{n}$
(4) If $\mathfrak{m} < \mathfrak{n}$ and $\mathfrak{p} < \mathfrak{q}$, then $\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{q}$
(5) If $\mathfrak{m} < \mathfrak{n}$ and $\mathfrak{p} < \mathfrak{q}$, then $\mathfrak{m} \cdot \mathfrak{p} < \mathfrak{n} \cdot \mathfrak{q}$
(6) If $\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{p}$, then $\mathfrak{m} < \mathfrak{n}$
(7) If $\mathfrak{m} \cdot \mathfrak{p} < \mathfrak{n} \cdot \mathfrak{p}$, then $\mathfrak{m} < \mathfrak{n}$

(The proofs can be found in [27], in [1] or in [23].)

As a matter of fact we wish to mention that Tarski observed that the statement

$$\text{If } 2\mathfrak{m} < \mathfrak{m} + \mathfrak{n}, \text{ then } \mathfrak{m} < \mathfrak{n}$$

is equivalent to the axiom of choice, while the proposition:

$$\text{If } 2\mathfrak{m} > \mathfrak{m} + \mathfrak{n}, \text{ then } \mathfrak{m} > \mathfrak{n}$$

can be proved without the aid of the axiom of choice (*cf.* [23, p. 421]).

To these cardinal equivalences mentioned above, we will now add two more:

PROPOSITION 4.2. The following conditions are equivalent to the axiom of choice:

(1) For every infinite cardinal $\mathfrak{m}$ we have $[\mathfrak{m}]^2 = \mathfrak{m}$
(2) For every infinite cardinal $\mathfrak{m}$ we have $[\mathfrak{m}]^2 = \mathfrak{m}$ or $\mathfrak{m}^2 = \mathfrak{m}$

PROOF. The proof is essentially the same as Tarski's proof that the axiom of choice follows if $\mathfrak{m}^2 = \mathfrak{m}$ for all infinite cardinals $\mathfrak{m}$ (*cf.* [27]).

Tarski proved in [27] (*cf.* also [23]) the following relation for infinite cardinals $\mathfrak{m}$:

$$\mathfrak{m} + \aleph(\mathfrak{m}) = \mathfrak{m} \cdot \aleph(\mathfrak{m}) \textit{ implies } \mathfrak{m} < \aleph(\mathfrak{m}).$$

Notice that $\mathfrak{m} < \aleph(\mathfrak{m})$ implies that every set $m \in \mathfrak{m}$ can be well-ordered. Therefore it is sufficient to show that (2), which is weaker than (1), implies that for every infinite cardinal number $\mathfrak{m}$ we have $\mathfrak{m} < \aleph(\mathfrak{m})$.

First we show that for two infinite cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ we have $\mathfrak{m} + \mathfrak{n} \leq \mathfrak{m} \cdot \mathfrak{n}$. For this, let $\mathfrak{m}_1$ and $\mathfrak{n}_1$ be such that $\mathfrak{m} = \mathfrak{m}_1 + 1$ and $\mathfrak{n} = \mathfrak{n}_1 + 1$. Now we get

$$\mathfrak{m} \cdot \mathfrak{n} = (\mathfrak{m}_1 + 1) \cdot (\mathfrak{n}_1 + 1) = \mathfrak{m}_1 \cdot \mathfrak{n}_1 + \mathfrak{m}_1 + \mathfrak{n}_1 + 1 \geq 1 + \mathfrak{m}_1 + \mathfrak{n}_1 + 1 = \mathfrak{m} + \mathfrak{n}.$$

It is easy to compute, that

$$[\mathfrak{m} + \aleph(\mathfrak{m})]^2 = [\mathfrak{m}]^2 + \mathfrak{m}\aleph(\mathfrak{m}) + [\aleph(\mathfrak{m})]^2,$$

and

$$(\mathfrak{m} + \aleph(\mathfrak{m}))^2 = \mathfrak{m}^2 + 2\mathfrak{m}\aleph(\mathfrak{m}) + \aleph(\mathfrak{m})^2.$$

Now we apply the assumption (2) to the cardinal $\mathfrak{m} + \aleph(\mathfrak{m})$. If $[\mathfrak{m} + \aleph(\mathfrak{m})]^2 = \mathfrak{m} + \aleph(\mathfrak{m})$, we get $\mathfrak{m}\aleph(\mathfrak{m}) \leq \mathfrak{m} + \aleph(\mathfrak{m})$ which implies (by the above, according to the Cantor-Bernstein Theorem) $\mathfrak{m}\aleph(\mathfrak{m}) = \mathfrak{m} + \aleph(\mathfrak{m})$. By the result of Tarski mentioned above we get $\mathfrak{m} < \aleph(\mathfrak{m})$. The case when $(\mathfrak{m} + \aleph(\mathfrak{m}))^2 = \mathfrak{m} + \aleph(\mathfrak{m})$ is similar. So, if the assumption (2) holds, then we get $\mathfrak{m} < \aleph(\mathfrak{m})$ for every cardinal number $\mathfrak{m}$ and therefore, each set m can be well-ordered, which is equivalent to the axiom of choice. ⊣

§5. A few relations provable in ZF. In this section we give some relationships between the cardinal numbers defined in section 3 which are provable without using the axiom of choice.

The most famous one is the

CANTOR THEOREM: For any cardinal number $\mathfrak{m}$ we have $\mathfrak{m} < 2^{\mathfrak{m}}$.

(This is proved by Georg Cantor in [2], but a proof can also be found in [14] or [1].)

Concerning the relationship between "$\leq^*$" and "$\leq$", it is obvious that $\mathfrak{p} \leq \mathfrak{q}$ implies $\mathfrak{p} \leq^* \mathfrak{q}$. The following fact gives a slightly more interesting relationship.

FACT 5.1. For two arbitrary cardinals $\mathfrak{n}$ and $\mathfrak{m}$ we have $\mathfrak{n} \leq^* \mathfrak{m} \rightarrow 2^{\mathfrak{n}} \leq 2^{\mathfrak{m}}$.

(For a proof see *e.g.* [23] or [1].)

The following two facts give a list of a few obvious relationships.

FACT 5.2. For every cardinal $\mathfrak{m}$ we have:
(1) $\mathfrak{m}^2 \leq \mathrm{fin}^2(\mathfrak{m})$
(2) $\mathrm{seq}^{\text{1-1}}(\mathfrak{m}) \leq \mathrm{fin}^2(\mathfrak{m})$
(3) $\mathrm{seq}^{\text{1-1}}(\mathfrak{m}) \leq \mathrm{seq}(\mathfrak{m})$
(4) If $\mathfrak{m}$ is infinite, then $2^{\aleph_0} \leq 2^{\mathrm{fin}(\mathfrak{m})}$

PROOF. First take an arbitrary set $m \in \mathfrak{m}$. For (1) note that a set $\langle x_1, x_2 \rangle \in m^2$ corresponds to the set $\{\{x_1\}, \{x_1, x_2\}\} \in \mathrm{fin}^2(m)$. For (2) note that a finite one-to-one sequence $\langle a_0, a_1, \ldots, a_n \rangle$ of m can always be written as $\{\{a_0\}, \{a_0, a_1\}, \ldots, \{a_0, \ldots, a_n\}\}$, which is an element of $\mathrm{fin}^2(m)$. The relation (3) is trivial. For (4) let $E_n := \{e \subseteq m : |e| = n\}$, where $n \in \mathbb{N}$. Because m is assumed to be infinite, every $x \subseteq \mathbb{N}$ corresponds to a set $F_x \in \mathcal{P}(\mathrm{fin}(m))$ defined by $F_x := \bigcup\{E_n : n \in x\}$. ⊣

FACT 5.3. $\aleph_0 = \aleph_0^2 = \mathrm{fin}(\aleph_0) = \mathrm{fin}^2(\aleph_0) = \mathrm{seq}^{\text{1-1}}(\aleph_0) = \mathrm{seq}(\aleph_0) < 2^{\aleph_0}$

PROOF. The only non-trivial part is $\aleph_0 < 2^{\aleph_0}$, which follows by the Cantor Theorem. ⊣

Three non-trivial relationships are given in the following

PROPOSITION 5.4. For any infinite cardinal $\mathfrak{m}$ we have:
(1) $\mathrm{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$
(2) $\mathrm{seq}^{\text{1-1}}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$
(3) $\mathrm{seq}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$

(These three relationships are proved in [7].)

§6. Permutation models. In this section we give the definition of permutation models (*cf.* also [13]). We will use permutation models to derive relative consistency results. But first we have to introduce models of ZFA, which is set theory with atoms (*cf.* [13]). Set theory with atoms is characterized by the fact that it admits objects other than sets, namely **atoms**, (also called **urelements**). Atoms are objects which do not have any elements but which are distinct from the empty-set. The development of the theory ZFA is very much the same as that of ZF (except for the definition of ordinals, where we have to require that an ordinal does not have atoms among its elements). Let S be a set, then by transfinite recursion on $\alpha \in \mathrm{On}$ we can define $\mathcal{P}^{\alpha}(S)$ as follows: $\mathcal{P}^{\emptyset}(S) := S$,

$\mathcal{P}^{\alpha+1}(S) := \mathcal{P}^{\alpha}(S) \cup \mathcal{P}(\mathcal{P}^{\alpha}(S))$ and $\mathcal{P}^{\alpha}(S) := \bigcup_{\beta\in\alpha} \mathcal{P}^{\alpha}(S)$ when α is a limit ordinal. Further let $\mathcal{P}^{\infty}(S) := \bigcup_{\alpha\in\mathrm{On}} \mathcal{P}^{\alpha}(S)$. If $\mathcal{M}$ is a model of ZFA and A is the set of atoms of $\mathcal{M}$, then we have $\mathcal{M} := \mathcal{P}^{\infty}(A)$. The class $M_0 := \mathcal{P}^{\infty}(\emptyset)$ is a model of ZF and is called the **kernel**. Note that all the ordinals are in the kernel.

The underlying idea of permutation models, which are models of ZFA, is the fact that the axioms of ZFA do not distinguish between the atoms, and so a permutation of the set of atoms induces an automorphism of the universe. The method of permutation models was introduced by Adolf Fraenkel and, in a precise version (with supports), by Andrzej Mostowski. The version with filters is due to Ernst Specker in [25].

In the permutation models we have a set of atoms A and a group $\mathcal{G}$ of permutations (or automorphisms) of A (where a permutation of A is a one-to-one mapping from A onto A). We say that a set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a **normal filter** on $\mathcal{G}$ if for all subgroups H, K of $\mathcal{G}$ we have:

(A) $\mathcal{G} \in \mathcal{F}$;
(B) if $H \in \mathcal{F}$ and $H \subseteq K$, then $K \in \mathcal{F}$;
(C) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$;
(D) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$;
(E) for each $a \in A$, $\{\pi \in \mathcal{G} : \pi a = a\} \in \mathcal{F}$.

Let $\mathcal{F}$ be a normal filter on $\mathcal{G}$. We say that x is **symmetric** if the group

$$\mathrm{sym}_{\mathcal{G}}(x) := \{\pi \in \mathcal{G} : \pi x = x\}$$

belongs to $\mathcal{F}$. By (E) we have that every $a \in A$ is symmetric.

Let $\mathcal{V}$ be the class of all hereditarily symmetric objects, then $\mathcal{V}$ is a transitive model of ZFA. We call $\mathcal{V}$ a permutation model. Because every $a \in A$ is symmetric, we get that the set of atoms A belongs to $\mathcal{V}$.

Now every $\pi \in \mathcal{G}$ induces an $\in$-automorphism of the universe $\mathcal{V}$, which we denote by $\hat{\pi}$ or just π.

Because $\emptyset$ is hereditarily symmetric and for all ordinals α the set $\mathcal{P}^{\alpha}(\emptyset)$ is hereditarily symmetric too, the class $V := \mathcal{P}^{\infty}(\emptyset)$ is a class in $\mathcal{V}$ which is equal to the kernel M_0.

Fact 6.1. For any ordinal α and any $\pi \in \mathcal{G}$ we have $\pi\alpha = \alpha$.

(This one can see by induction on α, where $\pi\emptyset = \emptyset$ is obvious.)

Since the atoms $x \in A$ do not contain any elements, but are distinct from the empty-set, the permutation models are models of ZF without the axiom of foundation. However, with the Jech-Sochor Embedding Theorem (*cf.* [15], [13] or [14]) one can embed arbitrarily large fragments of a permutation model in a well-founded model of ZF:

Jech-Sochor Embedding Theorem: Let $\mathcal{M}$ be a model of ZFA + AC, let A be the set of all atoms of $\mathcal{M}$, let M_0 be the kernel of $\mathcal{M}$ and let α be an ordinal in $\mathcal{M}$. For every permutation model $\mathcal{V}\subseteq\mathcal{M}$ (a model of ZFA) there exists a symmetric extension $V \supseteq M_0$ (a model of ZF) and an embedding $x \mapsto \tilde{x}$ of $\mathcal{V}$ in V such that

$$(\mathcal{P}^\alpha(A))^{\mathcal{V}} \text{ is } \in\text{-isomorphic to } (\mathcal{P}^\alpha(\tilde{A}))^{V}.$$

Most of the well-known permutation models are of the following simple type: Let $\mathcal{G}$ be a group of permutations of A. A family I of subsets of A is a **normal ideal** if for all subsets E, F of A we have:
(a) $\emptyset \in I$;
(b) if $E \in I$ and $F\subseteq E$, then $F \in I$;
(c) if $E \in I$ and $F \in I$, then $E \cup F \in I$;
(d) if $\pi \in \mathcal{G}$ and $E \in I$, then $\pi E \in I$;
(e) for each $a \in A$, $\{a\} \in I$.

For each set $S\subseteq A$, let

$$\text{fix}_{\mathcal{G}}(S) := \{\pi \in \mathcal{G} : \pi s = s \text{ for all } s \in S\};$$

and let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by the subgroups $\{\text{fix}_{\mathcal{G}}(E) : E \in I\}$. Then $\mathcal{F}$ is a normal filter. Further, x is symmetric if and only if there exists a set of atoms $E_x \in I$ such that

$$\text{fix}_{\mathcal{G}}(E_x)\subseteq \text{sym}_{\mathcal{G}}(x).$$

We say that E_x is a **support** of x.

§7. Consistency results derived from a few permutation models. In this section we will give some relationships between the cardinals defined in section 3 which are consistent with ZF. We will do this by investigating the relations between certain sets in a few permutation models. Let $\mathcal{V}$ be a permutation model with the set of atoms A and let m be a set in $\mathcal{V}$. Let $\mathfrak{C}(m) := \{x \in \mathcal{V} : \mathcal{V} \models x \preccurlyeq m \preccurlyeq x\}$, then $\mathfrak{C}(m)$ is a class in $\mathcal{V}$. The cardinality of m in the model $\mathcal{V}$ (denoted by $\mathfrak{m}$) is defined by $\mathfrak{m} := \mathfrak{C}(m) \cap \mathcal{P}^\alpha(A) \cap \mathcal{V}$, where α is the smallest ordinal such that $\mathfrak{C}(m) \cap \mathcal{P}^\alpha(A) \cap \mathcal{V} \neq \emptyset$. Note that if m and n are two arbitrary sets in a permutation model $\mathcal{V}$ and we have for example $\mathcal{V} \models m \preccurlyeq n \not\preccurlyeq m$ (and therefore $\mathcal{V} \models \mathfrak{m} < \mathfrak{n}$), then by the Jech-Sochor Embedding Theorem there exists a well-founded model V of ZF such that $V \models \tilde{m} \preccurlyeq \tilde{n} \not\preccurlyeq \tilde{m}$ and therefore $V \models \mathfrak{m} < \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the cardinalities of the sets $\tilde{m}$ and $\tilde{n}$. Hence, since every relation between sets in a permutation model can be translated to a well-founded model, to prove that a relation between some cardinals is consistent with ZF, it is enough to find a permutation model in which the desired relation holds between the corresponding sets. In the sequel we will frequently make use of this method without always mention it.

7.1. The basic Fraenkel model. First we present the basic Fraenkel model (*cf.* [13]).

Let A be a countable infinite set (the atoms), let $\mathcal{G}$ be the group of all permutations of A and let I_{fin} be the set of all finite subsets of A. Obviously, I_{fin} is a normal ideal.

Let $\mathcal{V}_F$ (F for Fraenkel) be the corresponding permutation model, the so called **basic Fraenkel model**. Note that a set x is in $\mathcal{V}_F$ iff x is symmetric and each $y \in x$ belongs to $\mathcal{V}_F$, too.

Now we will give two basic facts involving subsets of A.

Lemma 7.1.1. Let $E \in I_{\text{fin}}$, then each $S \subseteq A$ with support E is either finite or co-finite (which means $A \setminus S$ is finite). Further, if S is finite, then $S \subseteq E$; and if S is co-finite, then $A \setminus S \subseteq E$.

Proof. Let $S \subseteq A$ with support E. Because E is a support of S, for all $\pi \in \text{fix}(E)$ and every $a \in A$ we have $\pi a \in S$ if and only if $a \in S$. If S is neither finite nor co-finite, the sets $(A \setminus E) \setminus S$ and $(A \setminus E) \cap S$ are both infinite and hence we find a $\pi \in \text{fix}(E)$ such that for some $s \in S$, $\pi s \notin S$. Now, if S is finite, then S must be a subset of E because otherwise we have $S \setminus E \neq \emptyset$ and we find again a $\pi \in \text{fix}(E)$ such that for some $s \in S$, $\pi s \notin S$. The case when S is co-finite is similar. ⊣

Lemma 7.1.2. Let A be the set of atoms of the basic Fraenkel model and let $\mathfrak{m}$ denote its cardinality, then $\mathcal{V}_F \models \aleph_0 \not\leq 2^{\mathfrak{m}}$.

Proof. Assume there exists a one-to-one function $f : \mathbb{N} \to \mathcal{P}(A)$ which belongs to $\mathcal{V}_F$. Then, because f is symmetric, there exists a finite set $E_f \subseteq A$ (a support of f) such that $\text{fix}_{\mathcal{G}}(E_f) \subseteq \text{sym}_{\mathcal{G}}(f)$. Now let $n \in \mathbb{N}$ be such that $\text{fix}_{\mathcal{G}}(f(n)) \nsubseteq \text{fix}_{\mathcal{G}}(E_f)$ and let $\pi \in \text{fix}_{\mathcal{G}}(E_f)$ be such that $\pi f(n) \neq f(n)$. With the fact 6.1 we get that $\pi n = n$ and therefore $f(\pi n) = f(n)$. So, E_f cannot be a support of f, which implies that the function f does not belong to $\mathcal{V}_F$. ⊣

The following proposition gives the relationships in the basic Fraenkel model between some of the cardinals defined in section 3, where $\mathfrak{m}$ denotes the cardinality of the set of atoms of $\mathcal{V}_F$.

Proposition 7.1.3. Let $\mathfrak{m}$ denote the cardinality of the set of atoms A of $\mathcal{V}_F$. Then the in the model $\mathcal{V}_F$ we have the following:

(1) $\text{fin}(\mathfrak{m}) \| \text{seq}^{1\text{-}1}(\mathfrak{m})$
(2) $\text{fin}(\mathfrak{m}) \| \text{seq}(\mathfrak{m})$
(3) $\text{seq}^{1\text{-}1}(\mathfrak{m}) \| 2^{\mathfrak{m}}$
(4) $\text{seq}(\mathfrak{m}) \| 2^{\mathfrak{m}}$

Proof. (1) Assume first that there exists a function $f \in \mathcal{V}_F$ from $\text{fin}(A)$ into $\text{seq}^{1\text{-}1}(A)$ and let $E_f \in I_{\text{fin}}$ be a support of f. Choose two arbitrary distinct elements a_0 and a_1 of $A \setminus E_f$ such that $U := \{x \in A :$

x occurs in $f(\{a_0,a_1\}\cup E_f)\}\nsubseteq E_f$ and put $E_f^*:=\{a_0,a_1\}\cup E_f$. Choose a $y\in U\setminus E_f$ and a permutation $\pi\in\text{fix}_{\mathcal{G}}(E_f)$ such that $\pi y\neq y$ and $\pi a_i=a_{1-i}$ (for $i\in\{0,1\}$). Now, $\pi E_f^*=E_f^*$ but $\pi f(E_f^*)\neq f(E_f^*)$, which implies either that f is not a function or that E_f is not a support of f. In both cases we get a contradiction to our assumption.
The fact that $\text{seq}^{\text{1-1}}(\mathfrak{m})\nleq\text{fin}(\mathfrak{m})$ we get by $\text{fin}(\mathfrak{m})<2^{\mathfrak{m}}$ (see Proposition 5.4 (1)) and by $\text{seq}^{\text{1-1}}(\mathfrak{m})\nleq 2^{\mathfrak{m}}$ (which will be shown in (3)).

(2) Because $\text{seq}^{\text{1-1}}(\mathfrak{m})\leq\text{seq}(\mathfrak{m})$, by (1) it remains to show that $\text{fin}(A)\npreccurlyeq\text{seq}(A)$. Assume there exists a function $g\in\mathcal{V}$ from $\text{fin}(A)$ into $\text{seq}(A)$ and let $E_g\in I_{\text{fin}}$ be a support of it.
– If for each $p\in[A\setminus E_g]^2$ we have $\text{fix}_{\mathcal{G}}(E_g)\subseteq\text{sym}_{\mathcal{G}}(g(p))$, then we find $\{a_0,a_1\}$ and $\{b_0,b_1\}$ in $[A\setminus E_g]^2$ with $\{a_0,a_1\}\cap\{b_0,b_1\}=\emptyset$, and a permutation $\pi\in\text{fix}_{\mathcal{G}}(E_g)$ such that $\pi a_i=b_i$ and $\pi b_i=a_i$ (for $i\in\{0,1\}$). Now we get $\pi g(\{a_0,a_1\})=g(\{a_0,a_1\})$ and $\pi\{a_0,a_1\}=\{b_0,b_1\}$, which contradicts our assumption.
– Otherwise, there exists a set $\{a_0,a_1\}\in[A\setminus E_g]^2$ with $\text{fix}_{\mathcal{G}}(E_g)\nsubseteq\text{sym}_{\mathcal{G}}(g(\{a_0,a_1\}))$, hence we find in the sequence $g(\{a_0,a_1\})$ an element $y\in A$ which does not belong to E_g. Now let $\pi\in\text{fix}_{\mathcal{G}}(E_g)$ be such that $\pi a_i=a_{1-i}$ (for $i\in\{0,1\}$) and $\pi y\neq y$, then $\pi g(\{a_0,a_1\})\neq g(\{a_0,a_1\})$ and $\pi\{a_0,a_1\}=\{a_0,a_1\}$, which contradicts again our assumption.

(3) Because $\mathfrak{m}$ is infinite we have (by Proposition 5.4 (1)) $\text{fin}(\mathfrak{m})<2^{\mathfrak{m}}$, which implies (by (1)) that $2^{\mathfrak{m}}\nleq\text{seq}^{\text{1-1}}(\mathfrak{m})$ and it remains to show that $\text{seq}^{\text{1-1}}(A)\npreccurlyeq\mathcal{P}(A)$. Assume there exists a function $h\in\mathcal{V}_F$ from $\text{seq}^{\text{1-1}}(A)$ into $\mathcal{P}(A)$ and let $E_h\in I_{\text{fin}}$ be a support of h with $|E_h|\geq 4$. Consider $\text{seq}^{\text{1-1}}(E_h)$, then, because $|E_h|\geq 4$, it is easy to compute that $|\text{seq}^{\text{1-1}}(E_h)|>2\cdot 2^{|E_h|}$, which implies (by Lemma 7.1.1) that there exists an $s_0\in\text{seq}^{\text{1-1}}(E_h)$ such that E_h is not a support of $h(s_0)$. Let $E_0:=\bigcap\{E\in I_{\text{fin}}:E\text{ is a support of }h(s_0)\}$, then E_0 is a support of $h(s_0)$, too. Choose a $y\in E_0\setminus E_h$ and a permutation $\pi\in\text{fix}_{\mathcal{G}}(E_h)$ such that $\pi y\neq y$. Now, because $\pi\in\text{fix}_{\mathcal{G}}(E_h)$ and $s_0\in\text{seq}^{\text{1-1}}(E_h)$ we have $\pi s_0=s_0$, and by construction we get $\pi h(s_0)\neq h(s_0)$. This implies either that h is not a function or that E_h is not a support of h and in both cases we get a contradiction to our assumption.

(4) By $\text{fin}(\mathfrak{m})<2^{\mathfrak{m}}$ and $\text{fin}(\mathfrak{m})\nleq\text{seq}(\mathfrak{m})$ we get $2^{\mathfrak{m}}\nleq\text{seq}(\mathfrak{m})$, and the inequality $\text{seq}(\mathfrak{m})\nleq 2^{\mathfrak{m}}$ follows from $\text{seq}^{\text{1-1}}(\mathfrak{m})\nleq 2^{\mathfrak{m}}$ and $\text{seq}^{\text{1-1}}(\mathfrak{m})\leq\text{seq}(\mathfrak{m})$. ⊣

7.2. The ordered Mostowski model. Now we shall construct the ordered Mostowski model (*cf.* also [13]).

Let the infinite set of atoms A be countable, and let $<^M$ be a linear order on A such that A is densely ordered and does not have a smallest or greatest element (thus A is isomorphic to the rational numbers). Let $\mathcal{G}$ be the group of all order-preserving permutations of A, and let again I_{fin} be the ideal of the finite subsets of A.

Let $\mathcal{V}_M$ (M for Mostowski) be the corresponding permutation model (given by $\mathcal{G}$ and I_{fin}), the so called **ordered Mostowski model**.

Because all the sets in the ordered Mostowski model are symmetric, each subset of A has a finite support. By similar arguments as in the proof of Lemma 7.1.2 one can show

Lemma 7.2.1. Let A be the set of atoms of the ordered Mostowski model and let $\mathfrak{m}$ denote its cardinality, then $\mathcal{V}_M \models \aleph_0 \not\leq 2^{\mathfrak{m}}$.

For a finite set $E \subseteq A$, one can give a complete description of the subsets of A with support E and one gets the following

Fact 7.2.2. If $E \subseteq A$ is a finite set of cardinality n, then there are 2^{2n+1} sets $S \subseteq A$ (in $\mathcal{V}_M$) such that E is a support of S.

(For a proof see [7, p. 32].)

In the following we investigate the relationships between some of the cardinals defined in section 3 in the ordered Mostowski model, where $\mathfrak{m}$ will be cardinality of the set of atoms of $\mathcal{V}_M$.

Let $\mathfrak{m}$ denote the cardinality of the set of atoms A (of the ordered Mostowski model). In Theorem 1 of [7] it is shown that $2^{\mathfrak{m}} \leq^* \text{fin}(\mathfrak{m})$. Now, by Fact 5.1, we get $2^{2^{\mathfrak{m}}} \leq 2^{\text{fin}(\mathfrak{m})}$ which implies (by the Cantor-Bernstein Theorem, as $\text{fin}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$) that the equation $2^{2^{\mathfrak{m}}} = 2^{\text{fin}(\mathfrak{m})}$ holds in the ordered Mostowski model.

Unlike in the basic Fraenkel model, all the simple cardinalities defined in section 3 are comparable in the ordered Mostowski model:

Proposition 7.2.3. Let $\mathfrak{m}$ denote the cardinality of the set of atoms of $\mathcal{V}_M$. Then the following holds in $\mathcal{V}_M$:

$$\mathfrak{m} < \text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}} < \text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{seq}(\mathfrak{m})\,.$$

Proof. Let A be the set of atoms A of the ordered Mostowski model.

$\mathfrak{m} < \text{fin}(\mathfrak{m})$: It is obvious that the function $f : A \to \text{fin}(A)$, defined by $f(a) := \{a\}$, is a one-to-one function from A into $\text{fin}(A)$. Now assume that there exists also a one-to-one function g from $\text{fin}(A)$ into A. Let $a_0 := g(\emptyset)$ and $a_{n+1} := g(\{a_0, \dots, a_n\})$ (for $n \in \mathbb{N}$). The ω-sequence $\langle a_0, a_1, \dots, a_n, \dots\rangle$ is a one-to-one sequence of A, which implies that $\aleph_0 \leq \mathfrak{m}$, but this is a contradiction to Lemma 7.2.1.

$\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$: Because A is infinite, by Proposition 5.4 (1) we have $\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

$2^{\mathfrak{m}} < \text{seq}^{1\text{-}1}(\mathfrak{m})$: For a set $S \subseteq A$, let $\text{supp}(S) := \bigcap\{E \in I_{\text{fin}} : E \text{ is a support of } S\}$, then $\text{supp}(S)$ is a support of S, too; in fact, it is the smallest support of S. Using the order-relation "$<^M$" on the set of atoms A, we can define an ordering on the set of finite subsets of A as follows. For two finite sets $\{a_0, \dots, a_n\}$ and $\{b_0, \dots, b_m\}$ of A, where $a_i <^M a_{i+1}$ and $b_j <^M b_{j+1}$ (for $i < n$ and $j < m$), let $\{a_0, \dots, a_n\} <_{\text{fin}} \{b_0, \dots, b_m\}$

iff either $n < m$ or for $n = m$ we have $\exists i \leq n \forall j < i(a_j = b_j \wedge a_i <^M b_i)$). The ordering "$<_{\text{fin}}$" on the finite subsets of A induces an ordering on the power-set of A (because every subset of A has a well-defined smallest finite support). Further, the order-relation "$<^M$" induces in a natural way an ordering on the set of all permutations of a given finite subset of A and we identify a permutation τ of a finite subset $\{c_0 <^M \ldots <^M c_{n-1}\}$ with $\langle \tau(c_0), \tau(c_1), \ldots, \tau(c_{n-1})\rangle \in \text{seq}^{1\text{-}1}(A)$. Now we choose 20 distinct atoms $c_0 <^M c_1 <^M \ldots <^M c_{19}$ of A and define a function f from $\mathcal{P}(A)$ into $\text{seq}^{1\text{-}1}(A)$ as follows. For $S \subseteq A$ with $|\operatorname{supp}(S)| \geq 11$, let $f(S)$ be the kth permutation of $\operatorname{supp}(S)$, where S is the kth subset of A with smallest support $\operatorname{supp}(S)$ (this we can do because for $|\operatorname{supp}(S)| \geq 11$ we have $|\operatorname{supp}(S)|! \geq 2^{2|\operatorname{supp}(S)|+1}$). If $\operatorname{supp}(S) = \{a_0, \ldots, a_l\}$ for $l \leq 9$ (where $a_i < a_{i+1}$), then we choose the first 10 elements (with respect to $<^M$) of $\{c_0, \ldots, c_{19}\}$ which are not in $\operatorname{supp}(S)$, say $\{d_0, \ldots, d_9\}$ and put $f(S) = \langle a_0, \ldots, a_l, d_{\iota_0}, \ldots, d_{\iota_9}\rangle$, where $d_{\iota_0} \ldots d_{\iota_9}$ is the $(10! - k)$th permutation of $d_0 \ldots d_9$ and S is the kth subset of A with smallest support $\operatorname{supp}(S)$. By Lemma 7.2.2, the function f is a well-defined one-to-one function from $\mathcal{P}(A)$ into $\text{seq}^{1\text{-}1}(A)$. If there exists a one-to-one function from $\text{seq}^{1\text{-}1}(A)$ into $\mathcal{P}(A)$, then, because $n! > 2^{2n+1} + 2$ for $n \geq 10$, we can build an one-to-one ω-sequence of A, which is a contradiction to Lemma 7.2.1.

$\text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{seq}(\mathfrak{m})$: Because each one-to-one sequence of A is a sequence of A, we have $\text{seq}^{1\text{-}1}(A) \preccurlyeq \text{seq}(A)$. Now assume that there exists also a one-to-one function g from $\text{seq}(A)$ into $\text{seq}^{1\text{-}1}(A)$. Choose an arbitrary atom $a \in A$ and let $s_n := g(\langle a, a, \ldots, a\rangle_n)$, where $\langle a, a, \ldots, a\rangle_n$ denotes the sequence of $\{a\}$ of length n. Because for every $n \in \mathbb{N}$, the sequence s_n is a one-to-one sequence of A, for every $n \in \mathbb{N}$ there exists a $k > n$ and a $b \in A$ such that b occurs in s_k but for $i \leq n$, b does not occur in s_i. Because a sequence is an ordered set, with the function g we can build an one-to-one ω-sequence of A, which contradicts Lemma 7.2.1. ⊣

Let again $\mathfrak{m}$ denote the cardinality of the set of atoms of the ordered Mostowski model. Using some former facts and some arithmetical calculations, by similar arguments as is the proof of Proposition 7.2.3 one can show that the following sequence of inequalities holds in the ordered Mostowski model:

$$\mathfrak{m} < [\mathfrak{m}]^2 < \mathfrak{m}^2 < \text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}} < \text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{fin}^2(\mathfrak{m}) < \text{seq}^{1\text{-}1}(\text{fin}(\mathfrak{m})) <$$
$$< \text{fin}(2^{\mathfrak{m}}) < \text{fin}^3(\mathfrak{m}) < \text{fin}^4(\mathfrak{m}) < \ldots < \text{fin}^n(\mathfrak{m}) < \text{seq}(\mathfrak{m}) < 2^{\text{fin}(\mathfrak{m})} = 2^{2^{\mathfrak{m}}}$$

7.3. A custom-built permutation model. In the proof of Theorem 2 of [7], a permutation model $\mathcal{V}_s$ (s for sequences) is constructed in which there exists a cardinal number $\mathfrak{m}$ such that $\mathcal{V}_s \models \text{seq}(\mathfrak{m}) < \text{fin}(\mathfrak{m})$

and hence, $\mathcal{V}_s \models \text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{fin}(\mathfrak{m})$. Specifically, $\mathfrak{m}$ is the cardinality of the set of atoms of $\mathcal{V}_s$.

The set of atoms of $\mathcal{V}_s$ is built by induction, where every atom contains a finite sequence of atoms on a lower level. We will follow this idea, but instead of finite sequences we will put ordered pairs in the atoms. The model we finally get will be a model in which there exists a cardinal $\mathfrak{m}$, such that $\mathfrak{m}^2 < [\mathfrak{m}]^2$ (this is in fact a finite version of Theorem 2 of [7]).

We construct by induction on $n \in \mathbb{N}$ the following:

(α) A_0 is an arbitrary countable infinite set.

(β) $\mathcal{G}_0$ is the group of all permutations of A_0.

(γ) $A_{n+1} := A_n \dot\cup \big\{(n+1, p, \varepsilon) : p \in A_n \times A_n \wedge \varepsilon \in \{0,1\}\big\}$.

(δ) $\mathcal{G}_{n+1}$ is the subgroup of the group of permutations of A_{n+1} containing all permutations h such that for some $g_h \in \mathcal{G}_n$ and $\varepsilon_h \in \{0,1\}$ we have

$$h(x) = \begin{cases} g_h(x) & \text{if } x \in A_n, \\ (n+1, g_h(p), \varepsilon_h +_2 \varepsilon_x) & \text{if } x = (n+1, p, \varepsilon_x), \end{cases}$$

where $g_h(p) = \langle g_h(p_1), g_h(p_2)\rangle$ for $p = \langle p_1, p_2\rangle$ and $+_2$ is the addition modulo 2.

Let $A := \bigcup\{A_n : n \in \mathbb{N}\}$ and let $\text{Aut}(A)$ be the group of all permutations of A; then

$$\mathcal{G} := \{H \in \text{Aut}(A) : \forall n \in \mathbb{N}(H|_{A_n} \in \mathcal{G}_n\}$$

is a group of permutations of A. Let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \subseteq A \text{ is finite}\}$, and let $\mathcal{V}_p$ (p for pairs) be the class of all hereditarily symmetric objects.

Now we get the following

Proposition 7.3.1. Let $\mathfrak{m}$ denote the cardinality of the set of atoms A of $\mathcal{V}_p$. Then we have $\mathcal{V}_p \models \mathfrak{m}^2 < [\mathfrak{m}]^2$.

Proof. First we show that $\mathcal{V}_p \models \mathfrak{m}^2 \leq [\mathfrak{m}]^2$. For this it is sufficient to find a one-to-one function $f \in \mathcal{V}_p$ from A^2 into $[A]^2$. We define such a function as follows. For $x, y \in A$ where $x = (n, p_x, \varepsilon_x)$ and $y = (m, p_y, \varepsilon_y)$ let

$$f(\langle x, y\rangle) := \big\{(n+m+1, \langle x, y\rangle, 0), (n+m+1, \langle x, y\rangle, 1)\big\}\,.$$

For any $\pi \in \mathcal{G}$ and $x, y \in A$ we have $\pi f(\langle x, y\rangle) = f(\langle \pi x, \pi y\rangle)$ and therefore, the function f is as desired and belongs to $\mathcal{V}_p$.

Now assume that there exists a one-to-one function $g \in \mathcal{V}_p$ from $[A]^2$ into A^2 and let E_g be a finite support of g. Without loss of generality we may assume that if $(n+1, \langle x, y\rangle, \varepsilon) \in E_g$, then also $x, y \in E_g$. Let $k := |E_g|$ and for $x, y \in A$ let $g(\{x, y\}) = \langle t^0_{\{x,y\}}, t^1_{\{x,y\}}\rangle$. Let $r := k + 4$ and let $N := \text{Ramsey}(2, r^2, 3)$, where $\text{Ramsey}(2, r^2, 3)$ is the least natural number such that for every coloring $\tau : [\text{Ramsey}(2, r^2, 3)]^2 \to r^2$ we find a 3-element subset $H \subseteq \text{Ramsey}(2, r^2, 3)$ such that $\tau|_{[H]^2}$ is constant. (If p, r, m are natural numbers such that $p \leq m$ and $r > 0$, then by the Ramsey Theorem (*cf.* [22, Theorem B]), Ramsey(p, r, m) is well-defined.) Choose N distinct elements $x_0, \ldots, x_{N-1} \in A_0 \setminus E_g$, let $X = \{x_0, \ldots, x_{N-1}\}$ and let c_h $(h < k)$ be an enumeration of E_g. We define a coloring $\tau : [X]^2 \to r \times r$ as follows. For $\{x_i, x_j\} \in [X]^2$ such that $i < j$ let $\tau(\{x_i, x_j\}) = \langle \tau_0(\{x_i, x_j\}), \tau_1(\{x_i, x_j\}) \rangle$ where for $l \in \{0, 1\}$ we define

$$\tau_l(\{x_i, x_j\}) := \begin{cases} h & \text{if } t^l_{\{x_i, x_j\}} = c_h, \\ k & \text{if } t^l_{\{x_i, x_j\}} = x_i, \\ k+1 & \text{if } t^l_{\{x_i, x_j\}} = x_j, \\ k+2 & \text{if } t^l_{\{x_i, x_j\}} \in A_0 \setminus (\{x_i, x_j\} \cup E_g), \\ k+3 & \text{if } t^l_{\{x_i, x_j\}} \in A \setminus (A_0 \cup E_g). \end{cases}$$

By the definition of N we find 3 elements $x_{\iota_0}, x_{\iota_1}, x_{\iota_2} \in X$ with $\iota_0 < \iota_1 < \iota_2$ such that for $l \in \{0, 1\}$, τ_l is constant on $[\{x_{\iota_0}, x_{\iota_1}, x_{\iota_2}\}]^2$. So, for $\{x_{\iota_i}, x_{\iota_j}\} \in [\{x_{\iota_0}, x_{\iota_1}, x_{\iota_2}\}]^2$ with $i < j$ and for $l \in \{0, 1\}$, we are at least in one of the following cases:

(1) $t^l_{\{x_{\iota_i}, x_{\iota_j}\}} = c_{h_0}$ and $t^{1-l}_{\{x_{\iota_i}, x_{\iota_j}\}} = c_{h_1}$,
(2) $t^l_{\{x_{\iota_i}, x_{\iota_j}\}} = c_h$ and $t^{1-l}_{\{x_{\iota_i}, x_{\iota_j}\}} = x_{\iota_i}$,
(3) $t^l_{\{x_{\iota_i}, x_{\iota_j}\}} = c_h$ and $t^{1-l}_{\{x_{\iota_i}, x_{\iota_j}\}} = x_{\iota_j}$,
(4) $t^l_{\{x_{\iota_i}, x_{\iota_j}\}} = x_{\iota_i}$ and $t^{1-l}_{\{x_{\iota_i}, x_{\iota_j}\}} = x_{\iota_j}$,
(5) $t^l_{\{x_{\iota_i}, x_{\iota_j}\}} \in A_0 \setminus (E_g \cup \{x_{\iota_i}, x_{\iota_j}\})$,
(6) $t^l_{\{x_{\iota_i}, x_{\iota_j}\}} \in A \setminus (E_g \cup A_0)$.

If we are in case (1) or (2), then $g(\{x_{\iota_0}, x_{\iota_1}\}) = g(\{x_{\iota_0}, x_{\iota_2}\})$, and therefore g is not a one-to-one function. If we are in case (3), then g is also not a one-to-one function because $g(\{x_{\iota_0}, x_{\iota_2}\}) = g(\{x_{\iota_1}, x_{\iota_2}\})$.

If we are in case (4), let $\pi \in \text{fix}(E_g)$ be such that $\pi x_{\iota_0} = x_{\iota_1}$ and $\pi x_{\iota_1} = x_{\iota_0}$. Assume $g(\{x_{\iota_0}, x_{\iota_1}\}) = \langle x_{\iota_0}, x_{\iota_1} \rangle$ (the case when $g(\{x_{\iota_0}, x_{\iota_1}\}) = \langle x_{\iota_1}, x_{\iota_0} \rangle$ is symmetric). Then we have $\pi\{x_{\iota_0}, x_{\iota_1}\} = \{x_{\iota_0}, x_{\iota_1}\}$, but $\pi g(\{x_{\iota_0}, x_{\iota_1}\}) = \langle x_{\iota_1}, x_{\iota_2} \rangle \neq \langle x_{\iota_0}, x_{\iota_1} \rangle$, and therefore g is not a function in $\mathcal{V}_p$.

If we are in case (5), let $l \in \{0, 1\}$ be such that $t^l_{\{x_{\iota_0}, x_{\iota_1}\}} \in A_0 \setminus (E_g \cup \{x_{\iota_0}, x_{\iota_1}\})$ and let $a := t^l_{\{x_{\iota_0}, x_{\iota_1}\}}$. Take an arbitrary $a' \in A_0 \setminus (E_g \cup \{a, x_{\iota_0}, x_{\iota_1}\})$ and let $\pi \in \text{fix}(E_g \cup \{x_{\iota_0}, x_{\iota_1}\})$ be such that $\pi a = a'$ and $\pi a' = a$. Then we get $\pi\{x_{\iota_0}, x_{\iota_1}\} = \{x_{\iota_0}, x_{\iota_1}\}$ but $\pi g(\{x_{\iota_0}, x_{\iota_1}\}) \neq g(\{x_{\iota_0}, x_{\iota_1}\})$, and therefore g is not a function in $\mathcal{V}_p$.

If we are in case (6), let $l \in \{0,1\}$ be such that $t^l_{\{x_{\iota_0},x_{\iota_1}\}} \in A \setminus (E_g \cup A_0)$, thus $t^l_{\{x_{\iota_0},x_{\iota_1}\}} = (n+1,p,\varepsilon)$ for some $(n+1,p,\varepsilon) \in A$. Let $\pi \in \text{fix}(E_g \cup \{x_{\iota_0},x_{\iota_1}\})$ be such that $\pi(n+1,p,\varepsilon) = (n+1,p,1-\varepsilon)$. Then we have $\pi\{x_{\iota_0},x_{\iota_1}\} = \{x_{\iota_0},x_{\iota_1}\}$ but $\pi g(\{x_{\iota_0},x_{\iota_1}\}) \neq g(\{x_{\iota_0},x_{\iota_1}\})$, and therefore g is not a function in $\mathcal{V}_p$.

So, in all the cases, g is either not a function or it is not one-to-one, which contradicts our assumption and completes the proof. ⊣

7.4. On sequences and the power-set. The Theorem 2 of [7] states that the relation $\text{seq}(\mathfrak{m}) < \text{fin}(\mathfrak{m})$ is consistent with ZF. If we consider the permutation model $\mathcal{V}_s$ (s for sequences) constructed in the proof of this theorem, we see that even more is consistent with ZF, namely

Proposition 7.4.1. It is consistent with ZF that there exists a cardinal number $\mathfrak{m}$, such that $\text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{seq}(\mathfrak{m}) < \text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

Proof. Let $\mathfrak{m}$ denote the cardinality of the set of atoms of the permutation model $\mathcal{V}_s$ constructed in the proof of Theorem 2 of [7]. Then in $\mathcal{V}_s$ we have $\text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{seq}(\mathfrak{m}) < \text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$:

The inequality $\text{seq}(\mathfrak{m}) < \text{fin}(\mathfrak{m})$ is Theorem 2 of [7] and because $\mathfrak{m}$ is infinite, by Proposition 5.4 (1), we also get $\text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

To see that also $\text{seq}^{1\text{-}1}(\mathfrak{m}) < \text{seq}(\mathfrak{m})$ holds in $\mathcal{V}_s$, assume that there exists (in $\mathcal{V}$) a one-to-one function from $\text{seq}(\mathfrak{m})$ into $\text{seq}^{1\text{-}1}(\mathfrak{m})$. Such a one-to-one function would generate a function $f \in \mathcal{V}$ from $\aleph_0$ into $\mathfrak{m}$, but because f—as an element of $\mathcal{V}$—has a finite support, this is impossible. ⊣

In the remainder of this section we show that it is consistent with ZF that there exists a cardinal number $\mathfrak{m}$ such that $\text{seq}^{1\text{-}1}(\mathfrak{m}) < 2^{\mathfrak{m}} < \text{seq}(\mathfrak{m})$. For this we construct a permutation model $\mathcal{V}_c$ (c for categorical) where $\mathfrak{m}$ will be the cardinality of the set of atoms of $\mathcal{V}_c$.

Let L be the signature containing the binary relation symbol "$<$" and for each $n \in \mathbb{N}$ an $n+1$-ary relation symbol R_n. Let T_0 be the following theory:

(α) $<$ is a linear order,

(β) for each $n \in \mathbb{N}$: $R_n(z_0,\dots,z_n) \to \bigwedge_{l \neq m} (z_l \neq z_m)$.

Let $K = \{N : N \text{ is a finitely generated structure of } T_0\}$, then $K \neq \emptyset$ and further we have the following fact (*cf.* also [11, p. 325]).

Fact 7.4.2. K has the amalgamation property.

Proof. If $N_0 \subseteq N_1 \in K$, $N_0 \subseteq N_2 \in K$ and $N_1 \cap N_2 = N_0$, then we can define an $N \in K$ such that $\text{dom}(N) = \text{dom}(N_1) \cup \text{dom}(N_2)$, $N_1 \subseteq N$, $N_2 \subseteq N$, $<^{N_1} \cup <^{N_2} \subseteq <^N$ and for any $n \in \mathbb{N}$ we have $R_n^{N_1} \cup R_n^{N_2} = R_n^N$. ⊣

As a consequence of Fact 7.4.2 we get the

Lemma 7.4.3. There exists (up to isomorphism) a unique structure M of T_0 such that the cardinality of $\text{dom}(M)$ is $\aleph_0$, each structure $N \in K$

can be embedded in M and every isomorphism between finitely generated substructures of M (between two structures of K) extends to an automorphism of M.

PROOF. For a proof see *e.g.* Theorem 7.1.2 of [11]. ⊣

Therefore, $\mathrm{Th}(M)$ is $\aleph_0$-categorical and, because every isomorphism between finitely generated substructures of M extends to an automorphism of M, the structure M has non-trivial automorphisms.

Now we construct the permutation model $\mathcal{V}_c$ as follows. The set $\mathrm{dom}(M)$ constitutes the set of atoms A of $\mathcal{V}_c$ and $\mathcal{G}$ is the group of all permutations π of A such that: $M \models x <^M y$ iff $M \models \pi x <^M \pi y$ and for each $n \in \mathbb{N}$, $M \models R_n(z_0, \dots, z_n)$ iff $M \models R_n(\pi z_0, \dots, \pi z_n)$. In fact, the group $\mathcal{G}$ is the group of all automorphisms of M. Further, let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by $\{\mathrm{fix}(E)_{\mathcal{G}} : E \subseteq A \text{ is finite}\}$ and let $\mathcal{V}_c$ be the class of all hereditarily symmetric objects.

NOTATION: If $i+k = n$ and $\bar{y} = \langle y_0, \dots, y_{n-1}\rangle$, then we write $R_{i,k}(x, \bar{y})$ instead of $R_n(y_0 \dots, y_{i-1}, x, y_i, \dots y_{i+k-1})$. If $i = 0$, we write just $R_n(x, \bar{y})$.

The following lemma follows from the fact that every isomorphism between two structures of K extends to an automorphism of M and from the fact that $\mathrm{Th}(M)$ is $\aleph_0$-categorical (*cf.* also [11, Theorem 7.3.1]).

LEMMA 7.4.4. For every set $S \subseteq A$ in $\mathcal{V}_c$ there exists a unique smallest support $\mathrm{supp}(S)$ and for each finite set $E \subseteq A$, the set $\{S \subseteq A : S \in \mathcal{V}_c \wedge \mathrm{supp}(S) = E\}$ is finite.

PROOF. For $n \in \mathbb{N}$ let $E = \{e_0, \dots, e_{n-1}\} \subseteq A$ be a finite set of atoms. Further, let Θ_E the set of all atomic L-formulas $\varphi_i(x)$ such that we have $\varphi_i(x)$ is *either* the formula $x = e_j$ (for some $j < n$) *or* $x <^M e_j$ (for some $j < n$) *or* $R_{i,k}(x, \bar{e})$ (for $i + k \leq n$ and $\bar{e} \in \mathrm{seq}^{1\text{-}1}(E)$). For an atom $a \in A$ let

$$\vartheta_E(a) := \{\varphi_i(x) \in \Theta_E : M \models \varphi_i(a)\}\,;$$

thus, $\vartheta_E(a)$ is the set of all atomic formulas in Θ_E such that $\varphi_i(a)$ holds in M.

Take an arbitrary $S \subseteq A$ in $\mathcal{V}_c$ and let E be a support of S. If $s, t \in A$ are such that $\vartheta_E(t) = \vartheta_E(s)$, then we find (by construction of M and $\mathcal{G}$) a permutation $\pi \in \mathrm{fix}_{\mathcal{G}}(E)$ such that $\pi s = t$ and therefore we have $s \in S$ if and only if $t \in S$. Hence, the set S is determined by $\{\vartheta_E(s) : s \in S\}$, which is a finite set of finite sets of atomic formulas.

Now we show that if E_1 and E_2 are two distinct supports of a set $S \subseteq A$, then $E_1 \cap E_2$ is also a support of S. If $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$, then it is obvious that $E_1 \cap E_2$ is a support of S. So, assume that $E_1 \setminus E_2$ and $E_2 \setminus E_1$ are both non-empty and let $E_0 := E_1 \cap E_2$. Take an arbitrary $s_0 \in S$ and let $\vartheta_0 := \vartheta_{E_0}(s_0)$. Let $t \in A$ be any atom such that $\vartheta_{E_0}(t) = \vartheta_0$. We have to show that also $t \in S$. If $x = e_j$ belongs to ϑ_0 (and thus $e_j \in E_0$), then $s_0 = t = e_j$ and we have $t \in S$. So, assume that $x = e_j$ does not belong to ϑ_0 (for any $e_j \in E_0$). If $x = e_i$ does not belong to $\vartheta_{E_1}(t)$, let $t' := t$. Otherwise, if $x = e_i$ belongs to $\vartheta_{E_1}(t)$, because $\vartheta_{E_0}(t) = \vartheta_0$ and $E_0 = E_1 \cap E_2$ we have $e_i \in E_1 \setminus E_2$. By construction of M we find a $t' \in A$ such that $t' \notin E_1$ and $\vartheta_{E_2}(t') = \vartheta_{E_2}(t)$, hence, $t' \in S \Leftrightarrow t \in S$. Now let

$$\Xi_2 := \{\vartheta_{E_2}(s) : s \in S \wedge \vartheta_{E_2}(s) \cap \vartheta_{E_0}(s) = \vartheta_0\}.$$

Because $t' \notin E_1$ we find (again by construction of M) a $t'' \in A$ such that $\vartheta_{E_1}(t'') = \vartheta_{E_1}(t')$ and $\vartheta_{E_2}(t'') \in \Xi_2$. Now, by $\vartheta_{E_2}(t'') \in \Xi_2$ we have $t'' \in S$, by $\vartheta_{E_1}(t'') = \vartheta_{E_1}(t')$ we have $t'' \in S \Leftrightarrow t' \in S$, and because $t' \in S \Leftrightarrow t \in S$ we finally get $t \in S$.

Hence, $\text{supp}(S) := \bigcap\{E : E \text{ is a support of } S\}$ is a support of S and by construction it is unique. ⊣

Now we are ready to prove the

Proposition 7.4.5. Let $\mathfrak{m}$ denote the cardinality of the set of atoms of $\mathcal{V}_c$, then we have $\mathcal{V}_c \models \text{seq}^{\text{1-1}}(\mathfrak{m}) < 2^{\mathfrak{m}} < \text{seq}(\mathfrak{m})$.

Proof. First we show $\mathcal{V}_c \models \text{seq}^{\text{1-1}}(\mathfrak{m}) < 2^{\mathfrak{m}}$. For $\bar{y} = \langle y_0, \ldots, y_{n-1}\rangle \in \text{seq}^{\text{1-1}}(A)$ let

$$\Phi(\bar{y}) : \{x \in A : M \models R_n(x, \bar{y})\}.$$

By the construction of $\mathcal{V}_c$, the function Φ belongs to $\mathcal{V}_c$ and is a one-to-one mapping from $\text{seq}^{\text{1-1}}(A)$ to $\mathcal{P}(A)$. Hence, $\mathcal{V}_c \models \text{seq}^{\text{1-1}}(\mathfrak{m}) \leq 2^{\mathfrak{m}}$ and because (by Proposition 5.4 (2)) $\text{seq}^{\text{1-1}}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$ is provable in ZF, we get $\mathcal{V}_c \models \text{seq}^{\text{1-1}}(\mathfrak{m}) < 2^{\mathfrak{m}}$.

To see that $\mathcal{V}_c \models 2^{\mathfrak{m}} < \text{seq}(\mathfrak{m})$, notice first that by Proposition 5.4 (3), the inequality $\text{seq}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$ is provable in ZF, and therefore it is enough to find a one-to-one function from $\mathcal{P}(A)$ into $\text{seq}(A)$ which lies in $\mathcal{V}_c$. For each finite set $E \subseteq A$, let η_E be an enumeration of the set Θ_E. (The function $E \mapsto \eta_E$ exists as $<^M$ is a linear order on the finite set E.) Then by the Lemma 7.4.4 and its proof, for each finite set $E \subseteq A$, η_E induces a mapping from $\{S \subseteq A : \text{supp}(S) = E\}$ into k, for some $k \in \mathbb{N}$. Now fix two distinct atoms $a, b \in A$ and let

$$\begin{array}{rccc} \Psi : & \mathcal{P}(A) & \longrightarrow & \text{seq}(A) \\ & S & \longmapsto & \langle e_0, \ldots, e_{n-1}, a, b, \ldots, b\rangle \end{array}$$

be defined as follows: $E = \{e_0, \ldots, e_{n-1}\} := \text{supp}(S)$ such that $e_0 <^M \ldots <^M e_{n-1}$ and the length of the sequence $\Psi(S)$ is equal to $n + 1 + l$, where η_E maps S to l. The function Ψ is as desired, because it is a one-to-one function from $\mathcal{P}(A)$ into $\text{seq}(A)$ which lies in $\mathcal{V}_c$. ⊣

Remark: Because the relation $<^M$ is a dense linear order on the set of atoms of $\mathcal{V}_c$, with similar arguments as in the proof of the Proposition 7.2.3 one can show that $\mathcal{V}_c \models \text{fin}(\mathfrak{m}) < \text{seq}^{\text{1-1}}(\mathfrak{m})$ (where $\mathfrak{m}$ denotes the cardinality of the atoms of $\mathcal{V}_c$).

§8. Cardinals related to the power-set. In this section we compare the cardinalities of some sets which are related to the power-set. First we consider the power-set itself and afterwards we give some results involving the set of partitions.

The following fact can be found also in [20] or [23, VIII 2 Ex. 9]. However, we want to give here a combinatorial proof of this fact.

FACT 8.1. If $\aleph_0 \leq 2^{\mathfrak{m}}$, then $2^{\aleph_0} \leq 2^{\mathfrak{m}}$.

PROOF. Take an arbitrary $m \in \mathfrak{m}$. Because $\aleph_0 \leq 2^{\mathfrak{m}}$, we find an one-to-one ω-sequence $\langle p_0, p_1, \dots, p_n, \dots \rangle$ of $\mathcal{P}(m)$. Define an equivalence relation on m by

$$x \sim y \text{ if and only if } \forall n \in \mathbb{N}(x \in p_n \leftrightarrow y \in p_n)\,,$$

and let $[x] := \{y \in m : y \sim x\}$. For $x \in m$ let $g[x] := \{n \in \mathbb{N} : x \in p_n\}$, then, for every $x \in m$, we have $g[x] \subseteq \mathbb{N}$ and $g[x] = g[y]$ if and only if $[x] = [y]$. We can consider $g[x]$ as an ω-sequence of $\{0, 1\}$ by stipulating $g[x](n) = 0$ if $x \in p_n$ and $g[x](n) = 1$ if $x \notin p_n$. Now we define an ordering on the set $\{g[x] : x \in m\}$ as follows:

$$g[x] <_g g[y] \quad \textit{if and only if} \quad \exists n \in \mathbb{N}\big(g[x](n) < g[y](n) \wedge \forall k < n(g[x](k) = g[y](k))\big)\,.$$

This is a total order on the set $\{g[x] : x \in m\}$. Let $P_n^0 := \{g[x] : g[x](n) = 0\}$, then for each $n \in \mathbb{N}$ the set P_n^0 is a set of ω-sequences of $\{0, 1\}$. The order relation $<_g$ defines an ordering on each P_n^0 and we must have one of the following two cases:

Case 1: For each $n \in \mathbb{N}$, P_n^0 is well-ordered by $<_g$.

Case 2: There exists a least $n \in \mathbb{N}$ such that P_n^0 is not well-ordered by the relation $<_g$.

If we are in case 1, then we find a well-ordering on $\bigcup_{n \in \mathbb{N}} P_n^0$. Let the ordinal α denote its order-type, then $\alpha \geq \omega$ (otherwise the ω-sequence $\langle p_0, p_1, \dots \rangle$ would not be one-to-one) and therefore we can build a one-to-one ω-sequence $\langle g[x_0], g[x_1], \dots \rangle$ of $\{g[x] : x \in m\}$. If we define $q_i := \{x \in m : g[x] = g[x_i]\}$, then the set $Q := \{q_i : i \in \mathbb{N}\}$ is a set of pairwise disjoint subsets of m of cardinality $\aleph_0$. Therefore, the cardinality of $\mathcal{P}(Q)$ is $2^{\aleph_0}$ and because for $q \subseteq Q$ the function $\varphi(q) := \bigcup q \subseteq m$ is a one-to-one function, we get $2^{\aleph_0} \leq 2^{\mathfrak{m}}$.

If we are in case 2, let n be the least natural number such that P_n^0 is not well-ordered by $<_g$. Let $S_0 := \bigcup\{s \subseteq P_n^0 : s \text{ has no smallest element}\}$. Then $S_0 \subseteq P_n^0$ has no smallest element, too. For $k \in \mathbb{N}$ we define S_{k+1} as follows. If $S_k \cap P_{n+k+1}^0 = \emptyset$, then $S_{k+1} := S_k$; otherwise, $S_{k+1} := S_k \cap P_{n+k+1}^0$. By construction, for every $k \in \mathbb{N}$, the set S_k is not the empty set and it is not well-ordered by $<_g$. Thus, for every $k \in \mathbb{N}$ there exists an $l > k$ such that S_l is a proper subset of S_k. Now let $\langle S_{k_0}, S_{k_1}, \dots \rangle$

be such that for all $i < j$ we have $S_{k_i} \setminus S_{k_j} \neq \emptyset$ and let $q_i := \{x \in m : g[x] \in (S_{k_i} \setminus S_{k_{i+1}}\}$. Then the set $Q := \{q_i : i \in \mathbb{N}\}$ is again a set of pairwise disjoint subsets of m of cardinality $\aleph_0$ and we can proceed as above. ⊣

FACT 8.2. If $\aleph_0 \not\leq 2^{\mathfrak{m}}$, then for every natural number n we have $n \cdot 2^{\mathfrak{m}} < (n+1) \cdot 2^{\mathfrak{m}}$ and if $\emptyset \notin \mathfrak{m}$ we also have $2^{n \cdot \mathfrak{m}} < 2^{(n+1) \cdot \mathfrak{m}}$.

PROOF. We will give the proof only for the former case, since the proof of the latter case is similar. Let n be an arbitrary natural number. It is obvious that we have $n \cdot 2^{\mathfrak{m}} \leq (n+1) \cdot 2^{\mathfrak{m}}$. So, for an $m \in \mathfrak{m}$, let us assume that we also have a one-to-one function f from $(n+1) \times \mathcal{P}(m)$ into $n \times \mathcal{P}(m)$. For $k \geq 1$ let $\langle s_0, \dots, s_{k-1} \rangle_k$ be a one-to-one k-sequence of $\mathcal{P}(m)$ and let $U_k := \{s_i : i < k\}$. We can order the set $(n+1) \times U_k$ as follows: $\langle l_i, s_i \rangle <_U \langle l_j, s_j \rangle$ iff either $i < j$ or $i = j \to l_i < l_j$. Because $|\,(n+1) \times U_k\,| = (n+1) \cdot k$ and $k \geq 1$, we have $(n+1) \cdot k > n \cdot k$ and hence there exists a first $\langle l_i, s_i \rangle$ (w.r.t. $<_U$), such that the second component of $f(\langle l_i, s_i \rangle)$ does not belong to U_k. Now we define $s_k := f(\langle l_i, s_i \rangle)$ and the $(k+1)$-sequence $\langle s_0, \dots, s_k \rangle_{k+1}$ is a one-to-one sequence of $\mathcal{P}(m)$. Repeating this construction, we finally get an one-to-one ω-sequence of $\mathcal{P}(m)$. But this is a contradiction to $\aleph_0 \not\leq 2^{\mathfrak{m}}$. So, our assumption was wrong and we must have $n \cdot 2^{\mathfrak{m}} < (n+1) \cdot 2^{\mathfrak{m}}$. ⊣

Because it is consistent with ZF that there exists an infinite cardinal number $\mathfrak{m}$ such that $\aleph_0 \not\leq 2^{\mathfrak{m}}$ (see Lemma 7.1.2), it is also consistent with ZF that there exists an infinite cardinal $\mathfrak{m}$ such that $2^{\mathfrak{m}} < 2^{\mathfrak{m}} + 2^{\mathfrak{m}}$. Concerning $2^{2^{\mathfrak{m}}}$, Hans Läuchli proved in ZF that for every infinite cardinal number $\mathfrak{m}$ we have $2^{2^{\mathfrak{m}}} + 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}$ (see [17]). In particular, he got this result as a corollary of the following: It is provable in ZF that for any infinite cardinal $\mathfrak{m}$ we have $\left(2^{\mathrm{fin}(\mathfrak{m})}\right)^{\aleph_0} = 2^{\mathrm{fin}(\mathfrak{m})}$ (*cf.* [17]). Now, because $2^{\mathfrak{m}} = \mathrm{fin}(\mathfrak{m}) + \mathfrak{q}$ (for some $\mathfrak{q}$), we have $2^{2^{\mathfrak{m}}} = 2^{\mathrm{fin}(\mathfrak{m})+\mathfrak{q}} = 2^{\mathrm{fin}(\mathfrak{m})} \cdot 2^{\mathfrak{q}}$, and therefore, $2^{2^{\mathfrak{m}}} = \left(2^{\mathrm{fin}(\mathfrak{m})}\right)^{\aleph_0} \cdot 2^{\mathfrak{q}} \geq 2^{\mathrm{fin}(\mathfrak{m})} \cdot 2^{\mathrm{fin}(\mathfrak{m})} \cdot 2^{\mathfrak{q}} \geq 2 \cdot 2^{\mathrm{fin}(\mathfrak{m})+\mathfrak{q}} = 2^{2^{\mathfrak{m}}} + 2^{2^{\mathfrak{m}}}$, and the equation $2^{2^{\mathfrak{m}}} + 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}$ follows by the Cantor-Bernstein Theorem.

Now we give some results concerning the set of partitions of a given set.

A set $p \subseteq \mathcal{P}(m)$ is a partition of m if p is a set of pairwise disjoint, non-empty sets such that $\bigcup p = m$. We denote the set of all partitions of a set m by $\mathrm{part}(m)$ and the cardinality of $\mathrm{part}(m)$ by $\mathrm{part}(\mathfrak{m})$. Because each partition of m is a subset of the power set of m, we obviously have $\mathrm{part}(\mathfrak{m}) \leq 2^{2^{\mathfrak{m}}}$. It is also easy to see that if m has more than 4 elements, then $2^{\mathfrak{m}} \leq \mathrm{part}(\mathfrak{m})$. If we assume the axiom of choice, then for every infinite set m we have $2^{\mathfrak{m}} = \mathrm{part}(\mathfrak{m})$ (*cf.* [23, XVII.4 Ex. 3]). But it is consistent with ZF that there exists an infinite set m such that $2^{\mathfrak{m}} < \mathrm{part}(\mathfrak{m})$. Moreover we have the following

Proposition 8.3. If $\mathfrak{m} \geq 5$ and $\aleph_0 \not\leq 2^{\mathfrak{m}}$, then $2^{\mathfrak{m}} < \operatorname{part}(\mathfrak{m})$.

Proof. For a finite $\mathfrak{m} \geq 5$, it is easy to compute that $2^{\mathfrak{m}} < \operatorname{part}(\mathfrak{m})$. So, let us assume that $\mathfrak{m}$ is infinite and take $m \in \mathfrak{m}$. Because m has more than 4 elements we have $\mathcal{P}(m) \preccurlyeq \operatorname{part}(m)$. Now let us further assume that there exists a one-to-one function f from $\operatorname{part}(m)$ into $\mathcal{P}(m)$. First we choose 4 distinct elements a_0, a_1, a_2, a_3 from m. Let $c_i := \{a_i\}$ (for $i < 4$) and $c_4 := m \setminus \bigcup\{c_i : i < 4\}$, then $C_5 := \{c_i : i \leq 4\}$ is a set of pairwise disjoint, non-empty subsets of m such that $\bigcup C_5 = m$. Let $S_k = \langle X_0, \dots, X_{k-1}\rangle_k$ be a one-to-one sequence of $\mathcal{P}(m)$ of length k. With respect to the sequence S_k we define an equivalence relation on m as follows. $x \sim y$ if and only if for all $i < k$: $x \in X_i \Leftrightarrow y \in X_i$. For $x \in m$ let $[x] := \{y \in m : y \sim x\}$ and let $\chi_x : k \to \{0,1\}$ be such that $\chi_x(i) = 0$ if and only if $x \in X_i$. Notice that we have $\chi_x = \chi_y$ if and only if $x \sim y$. We define an ordering on the set of equivalence classes by stipulating $[x] <_\chi [y]$ if there exists an $i < k$ such that $\chi_x(i) < \chi_y(i)$ and for all $j < i$ we have $\chi_x(j) = \chi_y(j)$. Further let $C_k = \mathcal{C}(S_k) := \{[x] : x \in m\}$, then C_k is a set of pairwise disjoint, non-empty subsets of m such that $\bigcup C_k = m$ and $|C_k| \geq k$.

Let us assume that we already have constructed a set $C_k = \{c_0, \dots, c_{k-1}\}$ (for some $k \geq 5$) where $C_k = \mathcal{C}(S_k)$ and $S_k = \langle X_0, \dots, X_{k-1}\rangle_k \in \operatorname{seq}^{1\text{-}1}(\mathcal{P}(m))$ for some $k \geq 5$. Every partition of $l = |\mathcal{C}_k|$ induces a partition of m (this is because of the properties of C_k) and hence we get a one-to-one mapping ι from $\operatorname{part}(l)$ into $\operatorname{part}(m)$. Notice that the ordering $<_\chi$ on C_k induces an ordering on $\operatorname{part}(l)$. Because $l \geq k \geq 5$ we have $|\operatorname{part}(l)| > |\mathcal{P}(l)|$ and therefore we find a first partition q of l (first in the sense of the ordering on $\operatorname{part}(l)$) such that the set $f(\iota(q))$ is not the union of elements of C_k. We define $X_k := f(\iota(q))$, $S_{k+1} := \langle X_0, \dots, X_k\rangle_{k+1}$ and $C_{k+1} := \mathcal{C}(S_{k+1})$. Repeating this construction, we finally get an one-to-one ω-sequence of $\mathcal{P}(m)$. But this is a contradiction to $\aleph_0 \not\leq 2^{\mathfrak{m}}$ and therefore we have $\operatorname{part}(m) \not\preccurlyeq \mathcal{P}(m)$ and by $\mathcal{P}(m) \preccurlyeq \operatorname{part}(m)$ we get $2^{\mathfrak{m}} < \operatorname{part}(\mathfrak{m})$. ⊣

One can consider a partition of a set m also as a subset of $[m]^2$. To see this, let $f : \operatorname{part}(m) \to \mathcal{P}([m]^2)$ be such that for $p \in \operatorname{part}(m)$ we have $\{i,j\} \in f(p)$ if and only if $\exists b \in p(\{i,j\} \subseteq b)$. Therefore, for any cardinal $\mathfrak{m}$ we have $\operatorname{part}(\mathfrak{m}) \leq 2^{[\mathfrak{m}]^2}$ and as a consequence we get

Fact 8.4. If $\mathfrak{m} \geq 4$ and $\aleph_0 \not\leq 2^{\mathfrak{m}}$, then $2^{\mathfrak{m}} < 2^{[\mathfrak{m}]^2}$.

Proof. This follows from the Fact 8.3 and the fact that $\operatorname{part}(\mathfrak{m}) \leq 2^{[\mathfrak{m}]^2}$. ⊣

Let $\operatorname{CH}(\mathfrak{m})$ be the following statement: If $\mathfrak{n}$ is a cardinal number such that $\mathfrak{m} \leq \mathfrak{n} \leq 2^{\mathfrak{m}}$, then $\mathfrak{n} = \mathfrak{m}$ or $\mathfrak{n} = 2^{\mathfrak{m}}$. Specker showed in [24] that if $\operatorname{CH}(\mathfrak{m})$ holds for every infinite cardinal $\mathfrak{m}$, then we have the axiom of choice.

Concerning the set of partitions we get the following easy

FACT 8.5. If $\mathfrak{m}$ is infinite and CH($\mathfrak{m}$) holds, then $\mathrm{part}(\mathfrak{m}) = 2^{\mathfrak{m}}$.

PROOF. Note that $\mathfrak{m} \leq [\mathfrak{m}]^2 \leq \mathrm{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$ and therefore, by CH($\mathfrak{m}$), we must have $\mathfrak{m} = [\mathfrak{m}]^2$, and by $2^{\mathfrak{m}} \leq \mathrm{part}(\mathfrak{m}) \leq 2^{[\mathfrak{m}]^2}$ we get $\mathrm{part}(\mathfrak{m}) = 2^{\mathfrak{m}}$. ⊣

The assumption in Fact 8.5 is of course very strong. For example it is also consistent with ZF that there exists an infinite set m such that $[\mathfrak{m}]^2 > \mathfrak{m}$ (*e.g.*, let m be the set of atoms in the basic Fraenkel model or in the model $\mathcal{V}_p$). Moreover, in the second Cohen model constructed in [13, 5.4]—which is a symmetric model—there exists a set m such that $\aleph_0 \leq [\mathfrak{m}]^2$, $\aleph_0 \not\leq \mathfrak{m}$ and $\aleph_0 \leq^* \mathfrak{m}$.

One cannot expect that the cardinality of a partition $p \in \mathrm{part}(m)$ is very large: If $\mathfrak{p}$ is a partition of $\mathfrak{m}$, then $\mathfrak{p} \leq^* \mathfrak{m}$ and (by Fact 5.1) we get $2^{\mathfrak{p}} \leq 2^{\mathfrak{m}}$, which implies $\mathfrak{p} < 2^{\mathfrak{m}}$. On the other hand, for $p \in \mathrm{part}(m)$ we can have $\mathfrak{p} > \mathfrak{m}$. To see this, take any two cardinal numbers $\mathfrak{n}$ and $\mathfrak{m}$ such that $\mathfrak{n} < \mathfrak{m}$ and $\mathfrak{m} \leq^* \mathfrak{n}$ (examples for such cardinals can be found *e.g.* in [7]). Now take $m \in \mathfrak{m}$ and $n \in \mathfrak{n}$, then by the definition of $\leq^*$ there exists a function f from n onto m and the set $p := \big\{\{x \in n : f(x) = y\} : y \in m\big\}$ is a partition of n of cardinality $\mathfrak{m}$. Moreover, this can also happen even if we partition the real line:

FACT 8.6. It is consistent with ZF that the real line can be partitioned into a family p, such that $\mathfrak{p} > 2^{\aleph_0}$, where $2^{\aleph_0}$ is the cardinality of the set of the real numbers.

PROOF. Specker showed in [25, II 3.32] that if the real numbers are the countable union of countable sets, then $\aleph_1$ and $2^{\aleph_0}$ are incomparable. Furthermore, Henri Lebesgue gave in [19] a proof that $\aleph_1 \leq^* 2^{\aleph_0}$ (see also [23, XV 2]). Therefore we can decompose effectively the interval $(0,1)$ into $\aleph_1$ disjoint non-empty sets and obtain a decomposition of the real line into $\aleph_1+2^{\aleph_0}$ disjoint non-empty sets. If $\aleph_1 \not\leq 2^{\aleph_0}$, then $2^{\aleph_0} < \aleph_1+2^{\aleph_0}$. Hence, in the model of Solomon Feferman and Azriel Levy (*cf.* [4])—in which the real numbers are the countable union of countable sets—we find a decomposition of the real line into more than $2^{\aleph_0}$ disjoint non-empty sets (see also [23, p. 372]). ⊣

§9. Summary. First we summarize the results we got in the sections 5 and 7 by listing all the possible relationships between the cardinal numbers $\mathfrak{m}$, $\mathrm{fin}(\mathfrak{m})$, $\mathrm{seq}^{1\text{-}1}(\mathfrak{m})$, $\mathrm{seq}(\mathfrak{m})$ and $2^{\mathfrak{m}}$, where the cardinal number $\mathfrak{m}$ is infinite.

	$\mathfrak{m}$	$\mathrm{fin}(\mathfrak{m})$	$\mathrm{seq}^{1\text{-}1}(\mathfrak{m})$	$\mathrm{seq}(\mathfrak{m})$	$2^{\mathfrak{m}}$
$\mathfrak{m}$	$=$	$\overset{5}{=}\ \overset{7.2}{<}$	$\overset{5}{=}\ \overset{7.2}{<}$	$\overset{5}{=}\ \overset{7.2}{<}$	$\overset{5}{<}$
$\mathrm{fin}(\mathfrak{m})$		$=$	$\overset{7.3}{>}\ \overset{5}{=}\ \overset{7.2}{<}\ \overset{7.1}{\parallel}$	$\overset{7.3}{>}\ \overset{5}{=}\ \overset{7.2}{<}\ \overset{7.1}{\parallel}$	$\overset{5}{<}$
$\mathrm{seq}^{1\text{-}1}(\mathfrak{m})$			$=$	$\overset{5}{=}\ \overset{7.2}{<}$	$\overset{7.2}{>}\ \overset{5}{\neq}\ \overset{5}{<}\ \overset{7.1}{\parallel}$
$\mathrm{seq}(\mathfrak{m})$				$=$	$\overset{7.2}{>}\ \overset{5}{\neq}\ \overset{5}{<}\ \overset{7.1}{\parallel}$
$2^{\mathfrak{m}}$					$=$

One has to read the table from the left to the right and upwards. The number over a relation refers to the section where the relation was mentioned.

For any infinite cardinal number $\mathfrak{m}$, if $\mathrm{seq}^{1\text{-}1}(\mathfrak{m})$, $\mathrm{seq}(\mathfrak{m})$ and $2^{\mathfrak{m}}$ are all comparable; the only relations between these three cardinals which are consistent with ZF are the following:

(i) $\mathrm{seq}^{1\text{-}1}(\mathfrak{m}) = \mathrm{seq}(\mathfrak{m}) < 2^{\mathfrak{m}}$
(this is true for $\mathfrak{m} = \aleph_0$)

(ii) $\mathrm{seq}^{1\text{-}1}(\mathfrak{m}) < \mathrm{seq}(\mathfrak{m}) < 2^{\mathfrak{m}}$
(see section 7.4)

(iii) $\mathrm{seq}^{1\text{-}1}(\mathfrak{m}) < 2^{\mathfrak{m}} < \mathrm{seq}(\mathfrak{m})$
(see section 7.4)

(iv) $2^{\mathfrak{m}} < \mathrm{seq}^{1\text{-}1}(\mathfrak{m}) < \mathrm{seq}(\mathfrak{m})$
(see section 7.2)

To see this, remember that by Proposition 5.4 (2) and (3), the inequalities $\mathrm{seq}^{1\text{-}1}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$ and $\mathrm{seq}(\mathfrak{m}) \neq 2^{\mathfrak{m}}$ are both provable in ZF, and further notice that $\mathrm{seq}^{1\text{-}1}(\mathfrak{m}) = \mathrm{seq}(\mathfrak{m})$ implies $\aleph_0 \leq \mathfrak{m}$ which implies $2^{\mathfrak{m}} \nleq \mathrm{seq}^{1\text{-}1}(\mathfrak{m})$ (*cf.* [7, Lemma]). So, in ZF it is provable that there exists no cardinal $\mathfrak{m}$ such that $2^{\mathfrak{m}} \leq \mathrm{seq}^{1\text{-}1}(\mathfrak{m}) = \mathrm{seq}(\mathfrak{m})$.

Some other relationships which are provable without the axiom of choice are the following.

1. $\mathfrak{m}^2 > \aleph_0 \rightarrow \mathfrak{m} > \aleph_0$
   (see [23, VIII 2 Ex. 5])
2. $2^{\mathfrak{m}} < 2^{\aleph_0} \rightarrow \mathfrak{m} < \aleph_0$ (this means that $\mathfrak{m}$ is finite)
   (see [23, VIII 2 Ex. 3])
3. $(\mathfrak{m} \nless \aleph_0 \wedge \mathfrak{m} \leq 2^{\aleph_0}) \rightarrow 2^{\aleph_0} \leq 2^{\mathfrak{m}}$
   (see [23, VIII 2 Ex. 2])

4. $\aleph_0 \leq 2^{\mathfrak{m}} \rightarrow 2^{\aleph_0} \leq 2^{\mathfrak{m}}$
(see [23, VIII 2 Ex. 9] or Fact 8.1)
5. $\aleph_0 \leq 2^{\mathfrak{m}} \rightarrow 2^{\mathfrak{m}} \not\leq \text{fin}(\mathfrak{m})^n$ (where $n \in \mathbb{N}$)
(see [7, p. 36])
6. $\aleph_0 \leq 2^{\mathfrak{m}} \rightarrow 2^{\mathfrak{m}} \not\leq \text{fin}^n(\mathfrak{m})$
(the proof is similar to the proof of the previous fact 5)
7. $n \times \text{fin}(\mathfrak{m}) = 2^{\mathfrak{m}} \rightarrow n = 2^k$ (where $n, k \in \mathbb{N}$)
(see [7, p. 36])
8. $\aleph_0 \leq 2^{\mathfrak{m}} \rightarrow 2^{\mathfrak{m}} \not\leq \text{seq}^{\text{1-1}}(\mathfrak{m})$
(see [7, Lemma])
9. $\aleph_0 \leq \mathfrak{m} \rightarrow 2^{\mathfrak{m}} \not\leq \text{seq}(\mathfrak{m})$
(the proof is similar to the proof of the Lemma of [7])
10. $2^{2^{\mathfrak{m}}} \neq 2^{\aleph_0}$
(see [23, VIII 2 Ex. 7])
11. $\left(2^{\text{fin}(\mathfrak{m})}\right)^{\aleph_0} = 2^{\text{fin}(\mathfrak{m})}$
(see [17])
12. For every $n \in \mathbb{N}$ we have $\aleph_0 \not\leq 2^{\mathfrak{m}} \rightarrow n \cdot 2^{\mathfrak{m}} < (n+1) \cdot 2^{\mathfrak{m}}$
(see section 8)
13. $2^{2^{\mathfrak{m}}} + 2^{2^{\mathfrak{m}}} = 2^{2^{\mathfrak{m}}}$
(see [17] or section 8)

For each of the following statements we find a permutation model in which there exists an infinite set m witnessing the corresponding result, and therefore, by the Jech-Sochor Embedding Theorem, the following statements are consistent with ZF.

14. $n \times \text{fin}(\mathfrak{m}) = 2^{\mathfrak{m}}$ (for any $n \in \mathbb{N}$ of the form $n = 2^{k+1}$)
(see [7])
15. $\aleph_0 \leq 2^{2^{\mathfrak{m}}} = 2^{\text{fin}(\mathfrak{m})}$
(see [7, Theorem 1])
16. $\mathfrak{m}^2 < [\mathfrak{m}]^2$
(see section 7.3)
17. $\text{fin}(\mathfrak{m}) < \text{seq}^{\text{1-1}}(\mathfrak{m}) < 2^{\mathfrak{m}} < \text{seq}(\mathfrak{m})$
(see section 7.4)
18. $\text{seq}^{\text{1-1}}(\mathfrak{m}) < \text{seq}(\mathfrak{m}) < \text{fin}(\mathfrak{m}) < 2^{\mathfrak{m}}$
(see section 7.4)

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