

# DECOMPOSITIONS OF REFLEXIVE MODULES

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## Abstract

In a recent paper [11] we answered to the negative a question raised in the book by Eklof and Mekler [8, p. 455, Problem 12] under the set theoretical hypothesis of  $\diamond_{\aleph_1}$  which holds in many models of set theory. The Problem 12 in [8] reads as follows: *If  $A$  is a dual (abelian) group of infinite rank, is  $A \cong A \oplus \mathbb{Z}$ ?* The set theoretic hypothesis we made is the axiom  $\diamond_{\aleph_1}$  which holds in particular in Gödel's universe as shown by R. Jensen, see [8]. Here we want to prove a stronger result under the special continuum hypothesis (CH).

The question in [8] relates to dual abelian groups. We want to find a particular example of a dual group, which will provide a negative answer to the question in [8]. In order to derive a stronger and also more general result we will concentrate on reflexive modules over countable principal ideal domains  $R$ . Following H. Bass [1] an  $R$ -module  $G$  is reflexive if the evaluation map  $\sigma : G \rightarrow G^{**}$  is an isomorphism. Here  $G^* = \text{Hom}(G, R)$  denotes the dual group of  $G$ . Guided by classical results the question about the existence of a reflexive  $R$ -module  $G$  of infinite rank with  $G \not\cong G \oplus R$  is natural, see [8, p. 455]. We will use a theory of bilinear forms on free  $R$ -modules which strengthens our algebraic results in [11]. Moreover we want to apply a model theoretic combinatorial theorem from [14] which allows us to avoid the weak diamond principle. This has the great advantage that the used prediction principle is still similar to  $\diamond_{\aleph_1}$  but holds under CH. This will simplify algebraic argument for  $G \not\cong G \oplus R$ .

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# 1 Introduction

Let  $R$  be a countable principal ideal domain with  $1 \neq 0$  and  $S$  a multiplicative closed subset of  $R \setminus \{0\}$  containing 1. If  $S = \{s_n : n \in \omega\}$ ,  $s_0 = 1$  and  $q_n = \prod_{i < n} s_i$ , then

$$q_{n+1} = q_n s_n \text{ and } \bigcap_{n \in \omega} q_n R = 0. \quad (1.1)$$

The condition (1.1) requires that  $R$  is not a field, but then we may choose  $S = R \setminus \{0\}$  or any other ‘classical example’. The right ideals  $q_n R$  form a basis of the  $S$ -topology on  $R$  which is Hausdorff by (1.1). The  $S$ -adic completion of  $R$  under the  $S$ -topology is the ring  $\widehat{R}$  which has the size  $2^{\aleph_0}$ , see Göbel, May [10] for properties on  $\widehat{R}$ . Similar consideration carry over to (free)  $R$ -modules which we will use in the next sections. If  $F$  is a free  $R$ -module then the  $S$ -topology is generated by  $Fq_n$  ( $n \in \omega$ ) and if  $\widehat{F}$  denotes the  $S$ -adic completion of  $F$  then  $F \subset \widehat{F}$  and  $F$  is pure and dense in  $\widehat{F}$ . Recall that  $F \subseteq_* \widehat{F}$  is pure (w.r.t.  $S$ ) if and only if

$$\widehat{F}s \cap F \subseteq Fs \text{ for all } s \in S.$$

Also  $F$  is dense in  $\widehat{F}$  if and only if  $\widehat{F}/F$  is  $S$ -divisible (in the obvious sense). Note that an element  $e$  of an  $R$ -module  $G$  is pure, we write  $e \in_* G$  if  $eR \subseteq_* G$ . If  $X \leq G$ , then  $\langle X \rangle$  denotes the submodule generated by  $X$  and  $\langle X \rangle_*$  denotes the submodule purely generated by  $X$ . The following observation is well-known and can be looked up in [10].

**Observation 1.1** *If  $0 \neq r_n \in R$  infinitely often ( $n \in \omega$ ), then we can find  $\epsilon_n \in \{0, 1\}$  such that*

$$\sum_{n \in \omega} r_n q_n \epsilon_n \in \widehat{R} \setminus R.$$

If  $G$  is any  $R$ -module then  $G^* = \text{Hom}(G, R)$  denotes its dual module, and  $G$  is a dual module if  $G \cong D^*$  for some  $R$ -module  $D$ . Particular dual modules are reflexive modules  $D$  introduced by Bass [1, p. 476]. We need the evaluation map

$$\sigma = \sigma_D : D \longrightarrow D^{**} \quad (d \longrightarrow \sigma(d))$$

where  $\sigma(d) \in D^{**}$  is defined by evaluation

$$\sigma(d) : D^* \longrightarrow R \quad (\varphi \longrightarrow \varphi(d)).$$

The module  $D$  is *reflexive* if the evaluation map  $\sigma_D$  is an isomorphism. Obviously  $D \cong (D^*)^*$  is a dual module if  $D$  is reflexive. However, there are very many  $\aleph_1$ -free

modules  $G$  with  $G^* = 0$ , see [2]. An  $R$ -module is  $\aleph_1$ -free if all countable submodules are free. Using the special continuum hypothesis CH the existence of many reflexive modules (Section 3) will follow from considerations of free modules with bilinear form in Section 2. Classical examples are due to Specker and Łos and it follows from Specker's theorem that reflexive modules must be  $\aleph_1$ -free, see Fuchs [9]. Hence it is not too surprising that we will deal with free  $R$ -modules first in Section 2. The bilinear form is needed to control their duals when passing from  $\aleph_0$  to size  $\aleph_1$  in Section 3.

In order to find reflexive groups  $G$  of cardinality  $\aleph_1$  with  $G \not\cong R \oplus G$  we must discard all possible monomorphisms

$$\varphi : G \hookrightarrow G \text{ with } G\varphi \oplus eR = G. \quad (1.2)$$

This will be established with the help of an  $\aleph_1$ -filtration

$$G = \bigcup_{\alpha \in \omega_1} G_\alpha \text{ of countable, pure, free submodules } G_\alpha$$

such that  $G_\alpha$  is a summand with  $R$ -free complement of any  $G_\beta$  for  $\alpha < \beta$  if  $\alpha$  does not belong to a fixed stationary subset  $E$  of  $\omega_1$ . Given  $\varphi$  as in (1.2) by a back and forth argument there is a cub  $C \subseteq \omega_1$  such that  $G_\alpha\varphi \subseteq G_\alpha$  and  $e \in G_\alpha$  for all  $\alpha \in C$ .

If  $\alpha \in E \cap C$  then

$$G_\alpha\varphi \oplus eR = G_\alpha. \quad (1.3)$$

And if  $\varphi \upharpoonright G_\alpha$  is predicted by a function  $\varphi_\alpha$  as under the assumption of the diamond principle  $\diamond_{\aleph_1}$ , then we construct  $G_{\alpha+1}$  by a Step-Lemma from  $G_\alpha$  and  $\varphi_\alpha$  such that  $\varphi_\alpha$  does not extend to  $G_{\alpha+1}$ . Moreover  $G_{\alpha+1}$  is the  $S$ -adic closure of  $G_\alpha$  in  $G$  by the summand property mentioned above. Hence  $\varphi$  coincides with  $\varphi_\alpha$  on  $G_\alpha$  and maps  $G_{\alpha+1}$  into itself, a contradiction. However assuming CH only weaker prediction principles like weak diamond  $\Phi_{\aleph_1}$  are available, see Devlin and Shelah [3] or Eklof and Mekler [8, p. 143, Lemma 1.7]. If we discard  $\varphi_\alpha$  in the construction as before, then  $\varphi \upharpoonright G_\alpha$  might extend because it does not entirely agree with  $\varphi_\alpha$ . In this case  $\varphi$  will show up at some  $\alpha < \beta \in E \cap C$  and we have a new chance to discard  $\varphi \upharpoonright G_{\alpha+1}$  at level  $\beta$  - provided we know  $\varphi \upharpoonright G_{\alpha+1}$  when constructing  $G_{\beta+1}$ .

This time we need a stronger algebraic Step-Lemma: Note that  $G_{\alpha+1} \oplus F_\beta = G_\beta$ . Now we have a partial map  $\varphi_\beta := \varphi \upharpoonright G_{\alpha+1}$  with domain  $\text{Dom } \varphi_\beta = G_{\alpha+1}$  a summand of  $G_\beta$  and the splitting property (1.3) for the new map  $\varphi_\beta$ :

$$(G_{\alpha+1}\varphi_\beta \oplus eR) \oplus F_\beta = G_\beta. \quad (1.4)$$

In order to proceed we must discard these partial maps and indeed we are able to prove a generalized Step-Lemma taking care of partial maps (1.4) in Section 2. Moreover, a

counting argument also shows that a list of such partial maps

$$\{\varphi_\beta : \beta \in E\}$$

exists which predict any given map  $\varphi : G \rightarrow G$  such that the following holds.

**Lemma 1.2** [14] *Let  $G = \bigcup_{\alpha \in \omega_1} G_\alpha$  be an  $\aleph_1$ -filtration of  $G$  and  $E \subseteq \omega_1$  be a stationary subset of  $\omega_1$ . Then there is a list of predicting partial maps*

$$\{\varphi_\alpha : G_\alpha \rightarrow G_\alpha : \alpha \in E\}$$

*such that for any countable subset  $A$  of  $G$  there is an ordinal  $\beta \in E$  (in fact an unbounded set of such ordinals) with*

$$\varphi \upharpoonright A = \varphi_\beta \upharpoonright (G_\beta \cap A).$$

For a suitable  $A = G_{\alpha+1}$  and  $\varphi \upharpoonright A$  we choose  $\beta \in E$  by Lemma 1.2 with  $\varphi \upharpoonright G_{\alpha+1} = \varphi_\beta \upharpoonright G_{\alpha+1}$  and  $G_\beta = G_{\alpha+1} \oplus F_\beta$ . Then we construct  $G_{\beta+1}$  from  $G_\beta$  and the given  $\varphi_\beta \upharpoonright G_{\alpha+1}$  such that  $\varphi_\beta \upharpoonright G_{\alpha+1}$  does not extend to  $G_{\beta+1}$ . This contradiction will provide the

**Main Theorem 1.3** (ZFC + CH) *If  $R$  is a countable domain but not a field, then there is a family of  $2^{\aleph_1}$  pair-wise non-isomorphic reflexive  $R$ -modules  $G$  of cardinality  $\aleph_1$  such that  $G \not\cong R \oplus G$ .*

We also would like to draw attention to a slight modification of the proof of the Main Theorem 1.3. In addition we may assume that  $G$  in the Main Theorem 1.3 is essentially indecomposable, that any decomposition into two summands has one summand free of finite rank. This follows from a split realization result Corollary 4.3 with  $\text{End } G \cong A \oplus \text{Fin}(G)$  where  $A$  is any  $R$ -algebra which is free of countable rank. Recall that  $\text{Fin}(G)$  is the ideal of all endomorphisms of  $G$  of finite rank.

The formal proof for the prediction Lemma 1.2 is not complicated and uses repeatedly often the weak diamond prediction  $\Phi_{\aleph_1}$ . It is also clear from what we said that the underlying module theory is not essential for proving Lemma 1.2 and that it should be possible to replace modules by many other categories like non-commutative groups, fields or Boolean algebras. In order to cover all these possibilities, the prediction principle is formulated in terms of model theory and will appear in this setting in a forthcoming book by Shelah [14, Chapter IX, Claim 1.5].

We close this introduction with some historical remarks. Using a theorem of Lo  (see Fuchs [9]) on slender groups, the first ‘large’ reflexive abelian groups are free groups or

(cartesian) products of  $\mathbb{Z}$  - assuming for a moment that all cardinals under consideration are  $< \aleph_m$ , the first measurable cardinal. Also the members of the class of groups generated by  $\mathbb{Z}$  and taking direct sums and products alternatively are reflexive, called Reid groups. Using a generalized ‘Chase Lemma’, which controls homomorphisms from products of modules into direct sums of modules, Dugas and Zimmermann-Huisgen [5] showed that the class of Reid groups is ‘really large’. Nevertheless there are more reflexive groups - Eda and Otha [7] applied their ‘theory of continuous functions on 0-dimensional topological spaces’ to find reflexive groups not Reid-groups. As a by-product we also get dual groups which are not reflexive, see also [8]. All these groups  $G$  have the property that they are either free of finite rank or

$$\mathbb{Z} \oplus G \cong G. \tag{1.5}$$

As indicated in the abstract, we applied  $\diamond_{\aleph_1}$  to find examples  $G$  of size  $\aleph_1$ , where (1.5) is violated. The obvious question to replace  $\diamond_{\aleph_1}$  by CH was the main goal of this paper. The question whether the Main Theorem 1.3 holds in any model of ZFC remains open. On the other hand we are able to show that the conclusion of the Main Theorem 1.3 also follows in models of ZFC and Martin’s axiom MA, see [12]. Hence CH is not necessary to derive the existence of these reflexive modules.

## 2 Free modules with bilinear form and partial dual maps

**Definition 2.1** *Let  $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1)$  be a triple of a bilinear map  $\Phi : F_0 \oplus F_1 \longrightarrow R$  for some countable, free  $R$ -modules  $F_i$  of infinite rank,  $\mathfrak{F}_i \subseteq F_i^*$  ( $i = 0, 1$ ) families of dual maps subject to the following conditions*

- (i)  $\Phi$  is not degenerated. This is to say if  $\Phi(e, \ ) \in F_1^*$  or  $\Phi(\ , f) \in F_0^*$  is the trivial map then  $e = 0$  or  $f = 0$ , respectively.
- (ii)  $\Phi$  preserves purity, that is  $\Phi(e, \ ) \in_* F_1^*$  if  $e \in_* F_0$  and dually  $\Phi(\ , f) \in_* F_0^*$  if  $f \in_* F_1$ .
- (iii)  $\mathfrak{F}_i$  is a countable, non-empty family of homomorphisms  $\varphi : F_\varphi \longrightarrow R$  such that  $\text{Dom } \varphi = F_\varphi \subseteq_* F_i$ . The set  $\text{Dom } \mathfrak{F}_i = \{F_\varphi : \varphi \in \mathfrak{F}_i\}$  is well-ordered by inclusion, for  $i \in \{0, 1\}$ .
- (iv) For any  $0 \neq x \in F_1$  and any finite subset  $E \subset \mathfrak{F}_0$  we have  $\ker E \not\subseteq \ker \Phi(\ , x)$ , and dually for any  $0 \neq y \in F_0$  and any finite subset  $E \subset \mathfrak{F}_1$ , we have  $\ker E \not\subseteq \ker \Phi(y, \ )$ .

Here, and in the subsequent parts we use the following

**Notation 2.2** (i)  $\mathfrak{F}$  is the collection of all triples  $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1)$  as in Definition 2.1.

(ii) If  $E \leq \text{Hom}(G, H)$  then  $\ker E = \bigcap_{\varphi \in E} \ker \varphi$ .

(iii) Similarly  $\ker \Phi(\cdot, E) = \bigcap_{e \in E} \ker \Phi(\cdot, e)$  and dually.

Next we define a partial order on  $\mathfrak{F}$ .

**Definition 2.3** (i)  $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \subseteq (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1) \iff$

(ii)  $\left\{ \begin{array}{l} (a) \ \Phi \subseteq \Phi' \text{ and } \text{Dom } \Phi \subseteq_* \text{Dom } \Phi' \text{ is a pure submodule.} \\ (b) \ \text{If } \varphi \in \mathfrak{F}_i \text{ then there is a unique } \varphi' \in \mathfrak{F}'_i \text{ such that } \varphi \subseteq \varphi' \text{ and} \\ \text{Dom } \varphi \subseteq_* \text{Dom } \varphi'. \end{array} \right.$

We will construct the reflexive modules of size  $\aleph_1$  by using an order preserving continuous map and let

$$p : {}^{\omega_1}2 \longrightarrow \mathfrak{F} \quad (\eta \longrightarrow p_\eta) \tag{2.1}$$

from the tree  $\mathbf{T}$  of all branches of length  $\text{lg}(\eta) = \alpha$

$$\eta : \alpha \longrightarrow 2 = \{0, 1\} \text{ for all } \alpha < \omega_1.$$

The order on  $\mathbf{T}$  is defined naturally by extensions, i.e. if  $\eta, \eta' \in \mathbf{T}$ , then  $\eta \leq \eta'$  if and only if  $\eta \subseteq \eta'$  as maps. Hence  $\text{lg}(\eta) \leq \text{lg}(\eta')$  and  $\eta' \upharpoonright \text{lg}(\eta) = \eta$ , and we will require that  $p_\eta \subseteq p_{\eta'}$  by the ordering of  $\mathfrak{F}$  as defined in Definition 2.3. If  $\eta \in {}^{\omega_1}2$ , then  $p_{\eta \upharpoonright \alpha}$  ( $\alpha \in \omega_1$ ) is linearly ordered and the triple

$$p_\eta = \bigcup_{\alpha \in \omega_1} p_{\eta \upharpoonright \alpha} \tag{2.2}$$

is well-defined. In details we have  $\Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R$  is defined by continuity such that  $\Phi_\eta = \bigcup_{\alpha \in \omega_1} \Phi_{\eta \upharpoonright \alpha}$  and  $F_{i\eta} = \bigcup_{\alpha \in \omega_1} F_{i\eta \upharpoonright \alpha}$ . The bilinear forms  $\Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R$  will be our candidates for modules  $G$  as in the Main Theorem 1.3. First we will show that  $\mathfrak{F} \neq \emptyset$  and the arguments will be refined for Lemma 2.5.

**Lemma 2.4** *The partially ordered set  $\mathfrak{F}$  is non-empty.*

**Proof.** Choose  $F_0 = F_1 = \bigoplus_{n \in \omega} e_n R$  and extend

$$\Phi(e_i, e_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

linearly to get a bilinear map  $\Phi : F_0 \oplus F_1 \rightarrow R$ . The map  $\Phi$  satisfies Definition 2.1 for  $\mathfrak{F}_0 = \mathfrak{F}_1 = \emptyset$ . Next we want find  $\mathfrak{F}_0 = \{\varphi_0\}$  and  $\mathfrak{F}_1 = \{\varphi_1\}$ . If  $\varphi_i$  ( $i \in 2$ ) satisfy Definition 2.1 (iv) and  $\text{Dom } \varphi_i = F_i$  then obviously

$$(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}.$$

We will work for  $\varphi = \varphi_0 : F_0 \rightarrow R$  and enumerate  $F_1 \setminus \{0\} = \{x_i : i \in \omega\}$ . If  $\varphi_i = \Phi(\cdot, x_i)$  then  $\varphi_i \neq 0$  by (i) and for (iv) we must show that

$$\ker \varphi \not\subseteq \ker \varphi_i \text{ for all } i \in \omega \quad (2.3)$$

Write  $F_0 = \bigoplus_{i \in \omega} L_i$  such that each  $L_i = e_i R \oplus e'_i R$  is free of rank 2 and let  $L'_i = \bigoplus_{j \neq i} L_j$  be a complement of  $L_i$ . If  $L_i \varphi_i = 0$  then we use  $\varphi_i \neq 0$  to find some  $0 \neq y \in {}_* L'_i$  such that  $y \varphi_i \neq 0$ . Choose a new complement of  $L'_i$  and rename it  $L_i = (e_i + y)R \oplus e'_i R$ .

Hence  $L_i \varphi_i \neq 0$  and there is a pure element  $y_i \in L_i$  with  $y_i \varphi_i \neq 0$ . We found an independent family  $\{y_i : i \in \omega\}$  with  $F_0 = \bigoplus_{i \in \omega} y_i R \oplus C$  for some  $0 \neq C \subseteq F_0$ . Choose  $\varphi \in \text{Hom}(F_0, R)$  such that  $y_i \varphi = 0$  for all  $i \in \omega$  and  $\varphi \upharpoonright C \neq 0$ . Hence

$$y_i \in \ker \varphi \setminus \ker \varphi_i \text{ for all } i \in \omega$$

and (2.3) holds. Hence Definition 2.1 holds and  $\mathfrak{F}_1$  can be chosen dually.  $\square$

The crucial step in proving the next result is again verification of Definition 2.1 (iv), this time for  $\Phi'(\cdot, x)$ . The proof is similar to the last one, hence we can be less explicit and just refine the old arguments.

**Lemma 2.5** *Let  $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  be with  $\text{Dom } \Phi = F_0 \oplus F_1$  and  $\varphi \in L^*$  for some pure submodule  $L$  of finite rank in  $F_0$ . Then we find  $p \subseteq p' = (\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  with  $\text{Dom } \Phi' = F_0 \oplus F'_1$ ,  $F'_1 = F_1 \oplus xR$  and  $\varphi \subseteq \Phi'(\cdot, x)$ .*

**Proof.** First we want to extend  $\varphi$  to  $\varphi' : F_0 \rightarrow R$  such that

$$\ker E \not\subseteq \ker \varphi' \text{ for all finite } E \leq \mathfrak{F}_0. \quad (2.4)$$

Enumerate all  $\ker E$  for  $E \leq \mathfrak{F}_0$  finite by  $\{K_i : 0 \neq i \in \omega\}$ . Hence  $K_i = \bigcap_{\varphi \in E_i} \ker \varphi$  is a pure submodule of  $F_0$  with  $F_0/K_i$  free of rank  $\leq |E_i|$ . As in the proof of (2.5) we can

choose inductively  $\bigoplus_{i \in \omega} L_i = F_0$  and  $0 \neq y_i \in {}^* L_i \cap K_i$  for  $i > 0$  and  $L_0 = L$ . For some  $C \subseteq F_0$  we have  $F_0 = \bigoplus_{i \in \omega} y_i R \oplus L \oplus C$ . Now we extend  $\varphi \in L^*$  such that  $y_i \varphi' = 1$  and  $\varphi \upharpoonright C = 0$ . Hence

$$y_i \in K_i \setminus \ker \varphi' \text{ for all } 0 \neq i \in \omega$$

and (2.4) holds. Finally we extend  $\Phi$  to  $\Phi' : F_0 \oplus (F_1 \oplus xR) \rightarrow R$  by taking  $\Phi'(\cdot, x) = \varphi'$ . Condition (2.4) carries over to

$$\ker E \not\subseteq \ker \Phi(\cdot, x) \text{ for all finite } E \leq \mathfrak{F}_0.$$

From this it is immediate that  $\ker E \not\subseteq \ker \Phi(\cdot, y)$  for all  $0 \neq y \in F_1'$  and finite sets  $E \leq \mathfrak{F}_0$ . Hence Definition 2.1(iv) holds,  $p \leq (\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  and  $\varphi \subseteq \Phi(\cdot, x)$ .  $\square$

Next we move elements from  $\text{Hom}(F_\varphi, R)$  for some  $\varphi \in \mathfrak{F}_i$  to  $\mathfrak{F}_i$ .

**Lemma 2.6** *Let  $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  be with  $\text{Dom } \Phi = F_0 \oplus F_1$ . If  $\psi \in \text{Hom}(F_\varphi, R)$  for some  $\varphi \in \mathfrak{F}_0$  such that*

$$\ker \psi \cap \ker E \not\subseteq \ker \Phi(\cdot, x) \text{ for all finite } E \leq \mathfrak{F}_0 \text{ and } x \in F_1,$$

then

$$p \leq p' = (\Phi, \mathfrak{F}'_0, \mathfrak{F}_1) \in \mathfrak{F} \text{ for } \mathfrak{F}'_0 = \mathfrak{F}_0 \cup \{\psi\}$$

**Proof.** We only have to check Definition 2.1(iv) for finite subsets of  $\mathfrak{F}'_0 = \mathfrak{F}_0 \cup \{\psi\}$ . But this follows by hypothesis on  $\psi$ .

The proof of the following observation is obvious.

**Observation 2.7** *If  $p_n = (\Phi_n, \mathfrak{F}_{0n}, \mathfrak{F}_{1n})$  ( $n \in \omega$ ) is an ascending chain of elements  $p_n \in \mathfrak{F}$  and elements in  $\mathfrak{F}_i$  are unions of extensions in  $\mathfrak{F}_{in} \subseteq \mathfrak{F}_{i(n+1)}$  ( $n \in \omega$ ), then  $p = (\bigcup_{n \in \omega} \Phi_n, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ .*

**Definition 2.8** *If  $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  with  $\text{Dom } \Phi = F_0 \oplus F_1$  then  $\varphi \in F_0^*$  is essential for  $\Phi$  if for any finite rank summand  $L$  of  $F_0$  and any finite subset  $E$  of  $F_1$  there is  $g \in F_0 \setminus L$  with  $g\varphi \neq 0$  and  $\Phi(g, e) = 0$  for all  $e \in E$ .*

The notion ' $\varphi \in F_1^*$  is essential for  $\Phi$ ' is dual.

If  $g\varphi = 0 = \Phi(g, e)$  for  $e \in E$  and some  $E \leq \mathfrak{F}_1$  then  $g \in \langle \Phi(\cdot, e) : e \in E \rangle \subseteq F_0^*$  by induction on  $|E|$ . Hence  $\varphi \in F_0^*$  is essential for  $\Phi$  is equivalent to say that  $\varphi$  is not in  $\Phi(\cdot, F_1)$  modulo summands of finite rank. This leads to the following



**Observation 2.9** Let  $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  with  $\text{Dom } \Phi = F_0 \oplus F_1$ . If  $E \leq F_1$  is a finite subset and  $\varphi \in F_0^*$  inessential for  $\Phi$  with  $\ker \varphi \subseteq \ker E$  i.e.

$$(\Phi(x, e) = 0 \text{ for all } e \in E) \implies x\varphi = 0,$$

then there is  $e_0 \in \langle E \rangle \subseteq F_1$  such that  $\varphi = \Phi(\cdot, e_0)$ .

**Proof.** By induction on  $|E|$ .

**First Killing-Lemma 2.10** Suppose  $\varphi \in F_0^*$  is essential for  $\Phi$  with  $\text{Dom } \Phi = F_0 \oplus F_1$  and  $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ . Then we find  $p \leq p' \in \mathfrak{F}$  with  $p' = (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1)$ ,  $\text{Dom } \Phi' = F'_0 \oplus F_1$  and  $F'_0 = \langle F_0, y \rangle \subseteq_* \widehat{F}_0$  for some  $y \in \widehat{F}_0$  such that  $\varphi$  does not extend to  $\varphi' : F'_0 \rightarrow R$ .

**Remark** A dual lemma holds for  $\varphi \in F_1^*$ .

**Proof.** Let  $F_0 = \bigoplus_{i \in \omega} e_i R$  and  $F_1 = \bigoplus_{n \in \omega} f_n R$ . First we apply that  $\varphi$  is essential. It is easy to find inductively elements  $g_n \in F_0 \setminus F^n$  with  $F^n = \langle e_i, g_i : i < n \rangle_* \subseteq F_0$  such that the following holds

(i)  $\Phi(g_n, f_i) = 0$  for all  $i < n$ .

(ii)  $\bigoplus_{i < n} g_i R$  is a direct summand - also  $\bigoplus_{i \in \omega} g_i R$  is a summand of  $G$ .

(iii)  $g_n \varphi \neq 0$  for all  $n \in \omega$ .

Decompose  $\omega$  into a disjoint union of infinite subsets  $S_i$  ( $i \in \omega$ ) and let  $\{E_i : 0 \neq i \in \omega\}$  be an enumeration of all finite subsets of  $\mathfrak{F}_1$  and write  $K_i = \ker E_i$  for all  $i > 0$ . In order to get  $K_i \not\subseteq \ker \Phi(x, \cdot)$  for all  $i > 0, x \in F'_0 \setminus F$ , we choose for each  $n \in S_i$  an element  $k_n \in K_i$  such that  $\Phi(g_n, k_n) \neq 0$ . This is possible as  $F_1/K_i$  is free of finite rank  $\leq |E_i|$ , hence  $K_i$  is ‘quite large’.

Now we use  $g_n \varphi \neq 0$  ( $n \in \omega$ ) and apply Observation 1.1 (for  $S_0$ ) and choose a sequence  $\epsilon_n \in \{0, 1\}$  ( $n \in \omega$ ) and suitable  $q_n \in S$  as in (1.1) such that

$$r = \sum_{n \in \omega} (g_n \varphi) q_n \epsilon_n \in \widehat{R} \setminus R \tag{2.5}$$

$$\text{If } n \in S_i, \text{ then } \sum_{j=0}^n \Phi(g_j, k_i) q_j \epsilon_j \not\equiv 0 \pmod{q_{n+1}} \tag{2.6}$$

If  $s \in \omega$  and  $\epsilon_s = 1$ , then let

$$y_s = \sum_{s \leq n \in \omega} (q_s)^{-1} q_n g_n \epsilon_n \in \widehat{F}_0$$

and consider the  $R$ -module  $F'_0 = \langle F_0, y_0 \rangle_* \subset \widehat{F}_0$ , which can be generated by

$$F'_0 = \langle F_0, y_s R : s \in \omega, \epsilon_s = 1 \rangle.$$

Note that  $F'_0$  is a countable  $R$ -module. It is easy to see that  $F'_0$  is free - either apply Pontryagin's theorem (Fuchs [9, p. 93]) or determine a free basis. The bilinear form  $\Phi : F_0 \oplus F_1 \rightarrow R$  extends uniquely to  $\Phi' : F'_0 \oplus F_1 \rightarrow \widehat{R}$  by continuity and density. We want to show that

$$(\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}. \quad (2.7)$$

First we claim that  $\text{Im } \Phi' \subseteq R$ . By (i) we have

$$\Phi'(y_0, f_j) = \Phi'(\sum_{n \in \omega} \epsilon_n q_n g_n, f_j) = \sum_{n \in \omega} \Phi(g_n, f_j) q_n \epsilon_n = \sum_{n < j} \Phi(g_n, f_j) q_n \epsilon_n \in R$$

and  $\text{Im } \Phi' \subseteq R$  follows.

We also must check (iv) from Definition 2.1 for the new elements  $y \in F'_0 \setminus F_0$ . It is enough to consider

$$\ker E \not\subseteq \ker \Phi(y_s, \ ) \text{ for all } s \in \omega \text{ with } \epsilon_s = 1.$$

By definitions and enumerations this is equivalent to say that

$$K_i \not\subseteq \ker \Phi(y_s, \ ) \text{ for each } i > 0.$$

If  $n \in S_i$  and  $n > s$  by (2.6) we have that

$$\Phi(y_s, k_i) = \sum_{j \in \omega} \Phi(g_j, k_i) q_j \epsilon_j \equiv \sum_{j=0}^n \Phi(g_j, k_i) q_j \epsilon_j \not\equiv 0 \pmod{q_{n+1}}$$

Hence  $k_i \notin \ker \Phi(y_s, \ )$  but  $k_i \in K_i$  and (2.7) follows. Finally we must show that  $\varphi \in F_0^*$  does not extend to  $(F'_0)^*$ . By continuity  $\varphi : F_0 \rightarrow R$  extends uniquely to  $\varphi' : F'_0 \rightarrow \widehat{R}$ . However

$$y_0 \varphi = (\sum_{n \in \omega} g_n q_n \epsilon_n) \varphi' = \sum_{n \in \omega} (g_n \varphi) q_n \epsilon_n = r \in \widehat{R} \setminus R$$

by (2.5), hence  $\varphi$  does not extend to  $F'_0 \rightarrow R$ . □

**Second Killing-Lemma 2.11** *Let  $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  and  $\eta : F_\varphi \rightarrow F_\varphi$  be some monomorphism with  $F_\varphi = x_0R \oplus F_\varphi\eta$  for some  $\varphi \in \mathfrak{F}_0$  with  $x_0 \in F_\varphi = \ker \varphi$ . Then there is  $p \leq p' = (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1) \in \mathfrak{F}$  with  $\text{Dom } \Phi' = F'_0 \oplus F'_1, \varphi \subseteq \varphi' \in \mathfrak{F}'_0$  such that  $\eta$  does not extend to a monomorphism*

$$\eta'' : F_{\varphi'} \rightarrow F_{\varphi'} \subset_* F''_0 \text{ with } F''_0 = x_0R \oplus F''_0\eta''$$

where  $p' \leq p'' := (\Phi'', \mathfrak{F}''_0, \mathfrak{F}''_1)$  and  $\text{Dom } \Phi'' = F''_0 \oplus F''_1$ .

**Proof.** In order to satisfy Definition 2.1 (iv) for the new  $p' \in \mathfrak{F}$  we argue similar to the First Killing Lemma 2.10. Let  $\omega = \bigcup_{i \in \omega} S_i$  be a decomposition into infinite subsets  $S_i$  and  $\{K_i : 0 \neq i \in \omega\}$  be all kernels  $K_i = \ker E_i$  for an enumeration of finite subsets  $E_i$  of  $\mathfrak{F}_1$ . The set  $S_0$  will be used for killing  $\eta$  and the  $S_i$  ( $i > 0$ ) are in charge of  $K_i$  and (iv) above. Extending  $\Phi \subset \Phi'$  we must ensure  $\text{Im } \Phi' \subseteq R$ . Hence we construct an increasing sequence  $s_n \in \omega$  ( $n \in \omega$ ) and pose more conditions on  $s_n$  later on. If  $F' = \text{Dom } \varphi$  then  $F' = x_0R \oplus F'\eta$  and if  $x_i = x_0\eta^i$  we get

$$F' = \bigoplus_{i < n} x_iR \oplus F'\eta^n \text{ for all } n \in \omega. \quad (2.8)$$

Note that  $\bigoplus_{i < n} x_iR$  is pure of finite rank, hence  $F_0 = \bigoplus_{i < n} x_iR \oplus C_n$  for some  $C_n \subseteq F_0$ . Now we construct  $T_n = \bigoplus_{s_n \leq i < s_{n+1}} e_iR$  from  $F_0 = \bigoplus_{i \in \omega} e_iR$  and refine an argument from Göbel, Shelah [11]. Obviously  $F_0 = \bigoplus_{n \in \omega} T_n$  and let  $e^n \in \text{Hom}(F_0, R)$  ( $n \in \omega$ ) be defined by

$$e_i e^n = \delta_{i,n} = \begin{cases} 0 & \text{if } i \neq n \\ 1 & \text{if } i = n \end{cases}$$

for all  $i \in \omega$ . If  $i < n$  then  $\pi_i : F_0 \rightarrow R$  denotes the projection modulo  $\bigoplus_{i \neq j < n} x_jR \oplus C_n$ , moreover let  $F_1 = \bigoplus_{j \in \omega} f_jR$  be as before. We now seek for elements  $w_n \in F_0$  ( $n \in \omega$ ) subject to the following four conditions

- (i)  $0 \neq w_n \in T_n$  and  $w_n\eta \in T_n$ .
- (ii)  $\Phi(w_n, f_n) = w_n e^k = w_n \eta e^k = 0$  for all  $k < s_n$
- (iii) If  $n \in S_0$ , then let  $\pi_n^*$  be the projection  $\pi_i$  with  $i$  maximal such that

$$w_n \pi_i = 0 \neq w_n \eta \pi_i, \quad s_n \leq i < s_{n+1}$$

(iv) If  $n \in S_i, i > 0$ , there is  $y_n \in K_i$  such that  $\Phi(w_n, y_n) \neq 0$ .

Suppose  $s_0, \dots, s_n, w_0, \dots, w_{n-1}$  are constructed and we want to choose  $w_n, s_{n+1}$ . Then pick  $s_n < s_{n+1}$  such that

$$\{x_{s_n}, \dots, x_{4s_{n+1}}\} \subseteq \bigoplus_{i \leq s_{n+1}} e_i R.$$

We want to choose  $w_n = \sum_{i=s_n}^{4s_n} x_i a_i^n$  for some  $a_i^n \in R$ . If  $w_n e^k = 0$  for  $k < s_n$ , then  $w_n \in T_n$ . Moreover

$$w_n \eta = \sum_{i=s_n}^{4s_n} x_i a_i^n \eta = \sum_{i=s_n}^{4s_n} x_{i+1} a_i^n = \sum_{i=s_n+1}^{4s_n+1} x_i a_{i-1}^n$$

and  $x_n \eta \in T$  because  $x_{4s_{n+1}} \in \bigoplus_{i \leq s_{n+1}} e_i R$  as well.

Hence (i) follows provided  $w_n \neq 0$  is generated by those  $x'_i s$ . The conditions (ii) can be viewed as a system of  $3s_n$  homogeneous linear equations in  $4s_n + 1 - s_n = 3s_n + 1$  unknowns  $a_i^n \in R$ . We find a non-trivial solution  $w_n \neq 0$  by linear algebra. Hence (i) and (ii) hold. Condition (iii) follows by hypothesis on  $\mathfrak{F}_1$  for  $E_i$  and  $K_i = \ker E_i$ . Condition (iii) finally follows by the action of  $\eta$  on  $w_n \neq 0$  and the maximality of  $i$  with  $s_n < i \leq 4s_n$  and  $w_n \pi_i = 0$  for  $\pi_n^* = \pi_i$ . Hence (i),  $\dots$ , (iv) follow.

As in the proof of the First Killing Lemma 2.10 inductively we choose a strictly increasing sequence  $m_j \in \omega$  ( $j \in \omega$ ). If  $m_j$  is defined up to  $j \leq n$  we must choose  $m_{n+1}$  large enough such that

$$\sum_{j \leq n} q_{m_j} \Phi(w_j, y_n) \not\equiv 0 \pmod{q_{m_{n+1}}}.$$

This needs inductively the hypothesis that  $\sum_{j \leq n} q_{m_j} \Phi(w_j, y_n) \neq 0$ . If  $n+1 \in S_i$ , then  $\Phi(w_{n+1}, y_{n+1}) \neq 0$  by (iv) and we may assume

$$\sum_{j \leq n+1} p_j^m \Phi(w_j, y_{n+1}) \neq 0$$

hence the inductive hypothesis follows and we can proceed. By Observation 1.1 and (iii) we also find  $\epsilon_j \in \{0, 1\}$  ( $j \in S_0$ ) such that

$$\sum_{j \in S_0} (w_j \eta) \pi_j^* q_{m_j} \epsilon_j \in \widehat{R} \setminus R. \tag{2.9}$$

Now we are ready to extend  $\Phi$ . Choose new elements

$$z_k = \sum_{j \geq k} w_j q_{m_j} (q_{m_k})^{-1} \in \widehat{F}_0$$

for all  $k \in \omega$ . Hence the submodule  $F'_0 = \langle F_0, z_0 \rangle_* \subseteq \widehat{F}_0$  purely generated by adding  $z = z_0$  is generated by

$$F'_0 = \langle F_0, z_k : k \in \omega \rangle.$$

Again we see that  $F'_0$  is a countable, free  $R$ -module. The map  $\Phi : F_0 \oplus F_1 \rightarrow R$  by continuity extends uniquely to

$$\Phi' : F'_0 \oplus F_1 \rightarrow \widehat{R}.$$

Recall that  $F'_0/F_0$  is S-divisible, hence  $F_0$  is S-dense in  $F'_0$  in the S-adic topology. First we must show that  $\text{Im } \Phi' \subseteq R$ . We apply (ii) and continuity to see that

$$\Phi'(z, f_k) = \Phi' \left( \sum_{n \in \omega} q_{m_n} w_n, f_k \right) = \sum_{n \in \omega} q_{m_n} \Phi(w_n, f_k) = \sum_{n \leq k} q_{m_n} \Phi(w_n, f_k) \in R,$$

hence  $\Phi' : F'_0 \oplus F_1 \rightarrow R$ .

We also must show Definition 2.1 (iv) for the new elements  $z_t \in F'_0$ . We have  $K_i = \ker E_i$  and  $S_i$  is unbounded. Hence we find  $n \in S_i, n > t$  and  $y_n \in K_i$  such that

$$\Phi(w_n, y_n) \neq 0.$$

We apply  $\Phi'$  to  $(z_t, y_n)$  and get

$$\Phi'(z_t, y_n) \equiv \sum_{t \leq j \leq n} q_{m_j} (q_{m_t}^{-1} \Phi(w_j, y_n)) \not\equiv 0 \pmod{q_{m_{n+1}} q_{n_t}^{-1}}$$

hence  $\Phi(z_t, y_n) \neq 0$  and  $y_n \in K_i$ . This is equivalent to say that  $\ker E_i \not\subseteq \ker \Phi(z_t, \cdot)$  and Definition 2.1 (iv) follows. Hence  $(\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$  with  $\text{Dom } \Phi' = F'_0 \oplus F_1$ .

Next we extend  $\Phi'$  under the name  $\Phi'$ . Let  $F'_1 = F_1 \oplus fR$  be a free rank-1 extension. We want  $\Phi' : F'_0 \oplus F'_1 \rightarrow R$  and must define  $\Phi'(\cdot, f) : F'_0 \rightarrow R$ . Put  $\Phi'(\cdot, f) \upharpoonright T_n = \epsilon_n \pi_n^* \upharpoonright T_n$  if  $n \in S_0$  and  $\Phi'(\cdot, f) \upharpoonright T_n = 0$  otherwise. By linear extension and  $F_0 = \bigoplus_{n \in \omega} T_n$  the map  $\Phi'(\cdot, f) : F_0 \rightarrow R$  is well-defined. It extends further by continuity to

$$\Phi' : F'_0 \oplus F'_1 \rightarrow \widehat{R}.$$

Again we must show that  $\text{Im } \Phi' \subseteq R$ . Note that  $\Phi'(w_n, f) = w_n \epsilon_n \pi_n^* = 0$  for  $n \in S_0$  from (iii), and  $\Phi'(w_n, f) = 0$  for  $n \in \omega \setminus S_0$  by the above, hence

$$\Phi'(z, f) = \sum_{n \in \omega} q_{m_n} \Phi'(w_n, f) = 0$$

and  $\text{Im } \Phi' \subseteq R$  follows.

We also must check condition (iv) of Definition 2.1 for the new element  $f \in F'_1$ . Recall  $K_i$  ( $i \in \omega$ ) is a list of all kernels  $\ker E_i$  for finite sets in  $\mathfrak{F}_0$ . Also enumerate all  $T_n$ 's with  $n \in S_0$  and  $r_n = 1$  as, say  $T^i = T_{n_i}$  ( $i \in \omega$ ). Choosing the ranks of the  $T^i$ 's large enough we can find (as before)  $y'_i \in K_i \cap T^i$  with  $y'_i \pi_{n_i}^* \neq 0$ . Then

$$y'_i \in \ker E_i \text{ but } \Phi'(y'_i, f) = y'_i \pi_{n_i}^* \neq 0,$$

hence  $y'_i \notin \ker \Phi(\cdot, f)$  as desired for (iv) above.

Finally we must get rid of  $\eta$  by showing that there is no extension  $\eta'' \supset \eta$  as stated in the Lemma. Otherwise we have  $\Phi' \subset \Phi'' : F'' \oplus R \rightarrow R$  and

$$x_0 R \oplus F'' \eta'' = F''$$

for  $\text{Dom } \eta'' = F'' \subset_* F''_0$  with  $z \in \text{Dom } \eta' \subseteq F''$ . Hence  $r = \Phi''(z \eta'', f) \in R$ . On the other hand

$$r = \Phi''(z \eta'', f) = \Phi''\left(\sum_{j \in \omega} q_{m_j}(w_j \eta'), f\right) = \sum_{j \in \omega} q_{m_j} \Phi'(w_j \eta', f) = \sum_{j \in S_0} q_{m_j} \epsilon_j (w_j \eta \pi_j^*) \in \widehat{R} \setminus R$$

is a contradiction. The lemma follows.  $\square$

### 3 Construction of reflexive modules assuming CH

Let  $\{S_0, \dots, S_5\}$  be a decomposition of the set of all limit ordinals in  $\omega_1$  into stationary sets. Using CH we can enumerate sets of cardinality  $2^{\aleph_0}$  by any of these stationary sets of size  $\aleph_1$ .

Let

$$\mathbf{T} = {}^{\omega_1} 2 = \{\eta_\alpha : \alpha \in \omega_1\} \tag{3.1}$$

be an enumeration of the tree with countable branches such that

$$\eta_\alpha \subseteq \eta_\beta \implies \alpha < \beta \tag{3.2}$$

We are working in the ‘universe’  $\omega_1$  and let

$$\text{Map}(\omega_1, R) = \{\varphi_i : i \in S_0\} = \{\varphi_i : i \in S_1\} = \{\varphi_i : i \in S_2\} = \{\varphi_i : i \in S_3\} \quad (3.3)$$

be four lists of all partial functions  $\varphi : \delta \rightarrow R$  for  $\delta \in \omega_1$  any limit ordinal with  $\aleph_1$  repetitions  $\varphi_i$  for each  $\varphi$ . Similarly let

$$\text{Map}(\omega_1, \omega_1) = \{\mu_i : i \in S_4\} = \{\mu_i : i \in S_5\}$$

be two lists of all partial maps  $\mu : X \rightarrow X \subseteq \omega_1$  with countable domains  $X$  given by the prediction Lemma 1.2 for  $E \in \{S_4, S_5\}$ .

Inductively we want to construct an order preserving continuous map

$$p : {}^{\omega_1}>2 = \mathbf{T} \rightarrow \mathfrak{F}, (\eta \rightarrow p_\eta)$$

subject to certain conditions (a), . . . , (d) dictated by the proof of the main theorem and stated below. Some preliminary words are in order. Using the enumeration (3.1) we may write

$$p_{\eta_\alpha} = p_\alpha = (\Phi_\alpha, \mathfrak{F}_{0\alpha}, \mathfrak{F}_{1\alpha}) \text{ and } \text{Dom } \Phi_\alpha = F_{0\alpha} \oplus F_{1\alpha} \quad (3.4)$$

If  $p$  is order preserving then  $(\eta_\alpha \subseteq \eta_\beta \implies p_\alpha \subseteq p_\beta)$  and continuity applies. If  $\eta \in {}^\alpha 2$  has length  $\text{lg}(\eta) = \alpha$  and  $\alpha < \omega_1$  is a limit ordinal, then  $\bigcup_{\beta < \alpha} \eta \upharpoonright \beta = \eta$  and  $\eta \upharpoonright \beta \subseteq \eta$ ,

hence

$$\eta \in {}^\alpha 2 \implies p_\eta = \bigcup_{\beta < \alpha} p_{\eta \upharpoonright \beta} \quad (3.5)$$

If all  $p_{\eta \upharpoonright \beta} \in \mathfrak{F}$  then  $p_\eta \in \mathfrak{F}$  by Observation 2.7. Hence we have no problem in defining  $p$  at limit stages by continuity. It remains to consider inductive steps for  $p$ . If  $\eta, \eta' \in {}^{\omega_1}>2$  then  $\gamma = \text{br}(\eta, \eta')$  denotes the branching point of  $\eta, \eta'$ , this is to say that

(i)  $\gamma < \min\{\text{lg}(\eta), \text{lg}(\eta')\}$

(ii)  $\eta \upharpoonright \gamma = \eta' \upharpoonright \gamma$

(iii)  $\eta(\gamma) \neq \eta'(\gamma)$ .

If  $\eta$  is a branch of length  $\alpha$  and  $i \in \{0, 1\}$  then  $\eta' = \eta \wedge \{i\}$  is a branch of length  $\alpha + 1$  with  $\eta' \upharpoonright \alpha = \eta$  and  $\eta'(\alpha) = i$ . Now we continue defining  $p : {}^{\omega_1}>2 \rightarrow \mathfrak{F}$ . Generally we require

(a) If  $\varphi \in \mathfrak{F}_{i\alpha}$  then  $\text{Dom } \varphi = F_{i\beta}$  for some  $\eta_\beta \subseteq \eta_\alpha$ .

- (b) The set  $\{\text{Dom } \varphi : \varphi \in \mathfrak{F}_{i\alpha}\}$  is a well ordered set of pure submodule of  $F_{i\alpha}$ .
- (c) If  $\gamma = \text{br}(\eta_\alpha, \eta_\beta)$  and  $\eta_\alpha \upharpoonright \gamma = \eta_\beta \upharpoonright \gamma = \eta_\epsilon$  for some  $\epsilon < \omega_1$ , and  $y \in F_{1\beta} \setminus F_{1\epsilon}$  then  $\Phi^\beta(y, ) \upharpoonright F_{0\epsilon} \in \mathfrak{F}_{0\alpha}$ . Dually, if  $y \in F_{0\beta} \setminus F_{1\beta}$  then  $\Phi^\beta(y, ) \upharpoonright F_{1\epsilon} \in \mathfrak{F}_{1\alpha}$ .
- (d) If  $\nu \in {}^\gamma 2$  is a branch of length  $\gamma$ ,  $i \in \{0, 1\}$  and  $\eta = \nu \wedge \{i\}$  then we want to define  $p_\eta$  depending on  $\gamma$ .

(i) For  $\nu = \emptyset$  choose  $p_{\{i\}}$  by Lemma 2.5.

(ii) If  $\gamma \in S_0$  then we want to enlarge  $\Phi_\nu$  to make the evaluation map injective: If  $\varphi_\gamma \upharpoonright F_\varphi$  for some  $\varphi \in \mathfrak{F}_{0\nu}$  is a partial  $R$ -homomorphism  $F_\varphi \rightarrow R$  with  $\text{Dom}(\varphi_\gamma \upharpoonright F_\varphi)$  a pure  $R$ -submodule of finite rank of  $F_\varphi$ , then we apply Lemma 2.6 such that  $p_\nu \subset p_\eta$  with  $F_{1\eta} = F_{1\nu} \oplus x_\eta R$ ,  $F_{0\eta} = F_{0\nu}$ ,  $\Phi_\nu \subset \Phi_\eta$  and  $\varphi_\gamma \subset \Phi_\eta(\cdot, x_\eta)$ . If  $\varphi_\gamma$  does not satisfy the requirements, then we choose any  $\Phi_\nu \subset \Phi_\eta$ .

(iii) If  $\gamma \in S_1$  then we argue as in (ii) but dually. A dual version of Lemma 2.6 provides  $\Phi_\nu \subset \Phi_\eta$  and  $\varphi_\gamma \subset \Phi_\eta(x_\eta, \cdot)$  if  $\varphi_\gamma$  meets the requirements.

(iv) If  $\gamma \in S_2$  then we want to kill bad dual maps to make the evaluation map surjective. If  $\varphi_\gamma \upharpoonright F_{0\nu}$  is an  $R$ -homomorphism  $F_{0\nu} \rightarrow R$  which is essential for  $\Phi_\nu$  then we apply the First Killing Lemma 2.10 to find  $p_\gamma \subset p_\eta$  such that  $F_{0\eta} = \langle F_{0\nu}, y \rangle \subseteq_* \widehat{F}_{\nu 0}$  for some  $y \in \widehat{F}_{\nu 0}$  and  $\varphi_\gamma \upharpoonright F_{0\nu}$  does not extend to  $F_{0\eta} \rightarrow R$ .

(v) If  $\gamma \in S_3$  and  $\varphi_\gamma \upharpoonright F_{1\nu}$  is an essential  $R$ -homomorphism for  $\Phi_\nu$  then we argue as in (iv) but dually.

(vi) If  $\gamma \in S_4$  then we want to get rid of potential monomorphisms  $\eta$  of the final module  $G$  with  $G\eta \oplus xR = G$ . If  $\varphi \in \mathfrak{F}_{0\nu}$ ,  $\mu = \varphi_\gamma \upharpoonright F_\varphi$  and  $\mu : F_\varphi \rightarrow F_\varphi$  is an  $R$ -monomorphism such that  $F_\varphi = x_\varphi R \oplus F_\varphi \mu$ , then we apply the Second Killing Lemma 2.11 to find  $p_\nu \leq p_\eta$  such that  $\mu$  does not extend to a monomorphism  $\mu'$  of an extension  $F'_0$  of  $F_{0\eta}$  with

$$p_\eta \subseteq p' = (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1), \text{Dom } \Phi' = F'_0 \oplus F'_1 \text{ and } F'_0 = x_\varphi R \oplus F'_0 \mu'.$$

(vii) If  $\gamma \in S_5$ , then we argue dually for some partial monomorphism  $\mu$  with domain and range some  $F_\varphi \subseteq_* F_{1\nu}$ .

(viii) If  $\gamma \in \omega_1$  is not a limit ordinal, then we are free to choose any 'trivial' extension  $p_\nu \subset p_\eta$ .



## 4 Proof of the Main Theorem

Recall from Section 3 that we are given an order preserving continuous map

$$p : \mathbf{T} \longrightarrow \mathfrak{F}.$$

By continuity we may extend  $p$  to

$$p_\eta = (\Phi_\eta, \mathfrak{F}_{0\eta}, \mathfrak{F}_{1\eta}) \text{ for all } \eta \in {}^{\omega_1}2.$$

It is immediate that

$$F_{i\eta} \text{ is } \aleph_1\text{-free and } \Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R \text{ is a bilinear form,} \quad (4.1)$$

where  $i \in \{0, 1\}$ ,  $\eta \in {}^{\omega_1}2$ . Moreover,  $\Phi_\eta$  preserves purity and is not degenerated in the sense of Definition 2.1.  $\Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R$  is our candidate for a reflexive modules, expressed in an unusual way.

To see that  $\Phi_\eta$  preserves purity we must show that each pure element  $e \in {}_{*}F_{0\eta}$  induces  $\Phi_\eta(e, \cdot) \in {}_{*}F_{1\eta}^*$  (and dually). We may restrict to the first case. If  $e \in {}_{*}F_{0\eta}$  then  $e \in {}_{*}F_{0\eta|\alpha}$  for any  $\alpha \in \omega_1$  large enough, and  $\Phi_{\eta|\alpha}(e, \cdot) \in {}_{*}F_{1\eta|\alpha}^*$  by  $(\Phi_{\eta|\alpha}, \mathfrak{F}_{0\eta|\alpha}, \mathfrak{F}_{1\eta|\alpha}) \in \mathfrak{F}$  and Definition 2.1 (ii). It follows that  $\bigcup_{\alpha \in \omega_1} \Phi_{\eta|\alpha}(e, \cdot) \in {}_{*}F_{1\eta}^*$ .

To see that  $\Phi_\eta$  is not degenerated we consider  $0 \neq e \in F_{0\eta}$ . Hence  $e \in e'R \subseteq {}_{*}F_{0\eta|\alpha}$  for a pure element  $e'$  and any  $\alpha \in \omega_1$  large enough. The partial homomorphism  $\varphi : e'R \longrightarrow R$  defined by  $e'\varphi = 1$  has a number  $i \in S_0$  in the list and  $\varphi = \varphi_i$ . By construction there is  $y \in F_{1\eta|\gamma}$  with

$$\varphi \subset \Phi_{\eta|\gamma}(\cdot, y).$$

Hence  $0 \neq e\varphi = \Phi_{\eta|\gamma}(e, y) = \Phi_\eta(e, y)$  and  $\Phi_\eta$  is not degenerated.  $\square$

**Definition 4.1** *We will say that  $(\Phi, F_0, F_1)$  with  $\text{Dom } \Phi = F_0 \oplus F_1$  is fully represented if  $\Phi_\eta(F_{0\eta}, \cdot) = F_{1\eta}^*$  and  $\Phi_\eta(\cdot, F_{1\eta}) = F_{0\eta}^*$ .*

We claim that

$$(\Phi_\eta, F_{0\eta}, F_{1\eta}) \text{ is fully represented for almost all } \eta \in {}^{\omega_1}2 \quad (4.2)$$

There are at most  $\eta \in W \subseteq {}^{\omega_1}2$  exceptions with  $|W| < 2^{\aleph_1}$ .

Suppose for contradiction that  $|W| = 2^{\aleph_1}$  and

$$\varphi_\eta \in F_{1\eta}^* \setminus \Phi_\eta(F_{0\eta}, \cdot) \text{ for all } \eta \in W \quad (4.3)$$

By a pigeon hole argument there are  $\eta, \eta' \in W$  with  $\text{br}(\eta, \eta') = \alpha$  and

- (a)  $\varphi_\eta \upharpoonright F_{1\alpha} = \varphi_{\eta'} \upharpoonright F_{1\alpha}$
- (b)  $\varphi_\eta : F_{1\eta} \rightarrow R$ , and  $\varphi_{\eta'} : F_{1\eta'} \rightarrow R$  are not represented.

Let  $\psi = \varphi_\eta \upharpoonright F_{1\alpha} = \varphi_{\eta'} \upharpoonright F_{1\alpha} : F_{1\alpha} \rightarrow R$  and recall that there is some  $\gamma \in S_3$  with  $\psi = \varphi_\gamma$  and (d)(v) of the construction applies. Hence  $\psi$  is inessential for  $\Phi_\gamma$ . There is a finite set  $E \leq F_{0\gamma}$  such that

$$\Phi_\gamma(e, x) = 0 \text{ for all } e \in E, x \in \text{Dom } \psi = F_{1\alpha} \text{ implies } x\psi = 0.$$

By Observation 2.9 we have some  $e \in \langle E \rangle \subseteq F_{0\gamma}$  such that  $\psi = \Phi_\gamma(, e) \upharpoonright F_{1\alpha}$ . The same argument applies for  $\varphi_{\eta'}$  and there are  $\gamma' \in S_3$  and  $e' \in F_{0\gamma'}$  such that  $\gamma < \gamma'$  and  $\psi = \Phi_{\gamma'}(, e') \upharpoonright F_{1\alpha}$ . Hence  $\psi = \Phi_{\gamma'}(, e') \upharpoonright F_\alpha = \Phi_\gamma(, e) \upharpoonright F_\alpha$ .

Finally we apply (c) of the construction to get  $\psi \in \mathfrak{F}_{0\gamma'}$ . Now Definition 2.1 (iv) applies and  $\ker \psi \not\subseteq \ker \Phi_{\gamma'}(, e')$  is a contradiction because  $\psi = \Phi_{\gamma'}(, e')$ . The claim (4.2) follows.  $\square$

Note that we did not use (vi) so far. Hence without the Second Killing Lemma 2.11 we are able to derive reflexivity of the modules  $G_\eta$ , which we will do next.

We will use the following notations.

$$\text{Let } I' = \{\eta \in {}^{\omega_1}2 \text{ such that } p_\eta \text{ is not fully represented}\} \text{ and } I = {}^{\omega_1}2 \setminus I'$$

From (4.2) we see that  $|I'| < 2^{\aleph_1}$ , hence  $|I| = 2^{\aleph_1}$ . If  $\eta \in I$  and  $i \in \{0, 1\}$  then we also fix the evaluation map

$$\sigma_{i\eta} : F_{i\eta} \rightarrow F_{i\eta}^{**}$$

and claim that

$$\sigma_{i\eta} : F_{i\eta} \rightarrow F_{i\eta}^{**} \text{ is injective for all } \eta \in I, i \in \{0, 1\}. \quad (4.4)$$

**Proof.** We consider  $\sigma = \sigma_{0\eta}$  and apply that  $\Phi_\eta$  is not degenerated. If  $0 \neq x \in F_{0\eta}$  there is  $y \in F_{1\eta}$  such that  $\Phi_\eta(x, y) \neq 0$ . Hence  $\varphi := \Phi_\eta(, y) \in F_{0\eta}^*$  and  $x\varphi = \Phi_\eta(x, y) \neq 0$ , thus  $x\sigma \neq 0$  and  $\sigma$  is injective. The case  $\sigma_{1\eta}$  is similar.  $\square$

Next we show that

$$\sigma_{i\eta} : F_{i\eta} \rightarrow F_{i\eta}^{**} \text{ is surjective for all } \eta \in I, i \in \{0, 1\}. \quad (4.5)$$

**Proof.** First note that

$$\Phi_\eta^\bullet : F_{0\eta} \rightarrow F_{1\eta}^* \text{ ( } x \rightarrow \Phi_\eta(x, ) \text{ ) is bijective} \quad (4.6)$$

and

$$\bullet\Phi_\eta : F_{1\eta} \longrightarrow F_{0\eta}^*(y \longrightarrow \Phi_\eta(\cdot, y)) \quad \text{is bijective} \quad (4.7)$$

because  $\Phi_\eta$  is not degenerated and  $\eta \in I$ . Hence we can identify  $F_{1\eta}$  and  $F_{0\eta}^*$  by  $\bullet\Phi_\eta$  and  $F_{0\eta}^* = \text{Im}(\bullet\Phi_\eta) = \Phi_\eta(\cdot, F_{1\eta})$ . Moreover

$$F_{0\eta}^{**} = (F_{0\eta}^*)^* = (F_{1\eta})^* = \text{Im} \Phi_\eta^\bullet = \Phi_\eta(F_{0\eta}, \cdot)$$

and for any  $\varphi \in F_{0\eta}^{**}$  we find  $f \in F_{0\eta}$  with  $\varphi = \Phi_\eta(f, \cdot)$ . We consider the case  $\sigma = \sigma_{0\eta}$ , and get

$$\Phi_\eta(\cdot, x)\sigma(f) = \Phi_\eta(f, x) = \Phi_\eta(\cdot, x)\Phi_\eta(f, \cdot)$$

for all  $x \in F_{1\eta}$  and  $\Phi_\eta(\cdot, x)$  runs through all of  $F_{0\eta}^*$ . We derive  $\sigma(f) = \Phi_\eta(f, \cdot) = \varphi$  and  $\sigma$  is surjective. The case  $\sigma_{1\eta}$  is similar.  $\square$

We have an immediate corollary from (4.4) and (4.5).

**Corollary 4.2** *If  $\eta \in I$  and  $i \in \{0, 1\}$  then  $F_{i\eta}$  is a reflexive  $R$ -module. Moreover  $I$  is a subset of  ${}^{\omega_1}2$  of cardinality  $2^{\aleph_1}$ .*

For the **Proof of the Main Theorem 1.3** we finally must show that

$$F_{i\eta} \not\cong R \oplus F_{i\eta} \text{ for any } \eta \in I, i \in \{0, 1\}. \quad (4.8)$$

We consider  $F_{i\eta}$  with some monomorphism  $\xi : F_{i\eta} \longrightarrow F_{i\eta}$  such that  $F_{i\eta} = F_{i\eta}\xi \oplus xR$  for any  $\eta \in I, i \in \{0, 1\}$ . By a back and forth argument there is an  $\alpha < \omega_1$  such that  $F_{i\eta|\alpha} = F_{i\eta|\alpha}\xi \oplus xR$  and we take  $\psi = \xi \upharpoonright F_{i\eta|\alpha}$  into consideration. There is some  $\gamma \in S_4$  such that  $\psi = \varphi_\gamma$  and  $\varphi_\gamma$  is discarded by the construction. Hence  $\xi$  does not exist.  $\square$

We would like to add a modification of our main result which can be shown using one more stationary subset  $S_6$  after introducing inessential endomorphisms for our category of reflexive modules. Let  $\text{Fin}(G)$  be the ideal of all endomorphisms

$$\{\sigma \in \text{End}(G) : G\sigma \text{ has finite rank}\}$$

for some torsion-free  $R$ -module  $G$ . Then we can find a ‘Killing-Lemma’ in Dugas, Göbel [4], see also [2], which ‘takes care’ of all endomorphisms which are not in  $\text{Fin}(F_{i\eta})$  with  $i, \eta$  as above. Hence we can strengthen our Main Theorem 1.3 and get with slight modification from known results the following

**Corollary 4.3** *(ZFC + CH) Let  $R$  is a countable domain but not a field and  $A$  be a countable  $R$ -algebra with free additive structure  $A_R$ . Then there is a family of  $2^{\aleph_1}$  pair-wise non-isomorphic reflexive  $R$ -modules  $G$  of cardinality  $\aleph_1$  such that  $G \not\cong R \oplus G$  and  $\text{End}(G) = A \oplus \text{Fin}(G)$  a split extension.*

In particular  $\text{End}(G)/\text{Fin}(G) \cong A$  and if  $A$  has only trivial idempotents like  $A = R$  then  $G$  in the Corollary 4.3 is reflexive, essentially indecomposable of size  $\aleph_1$  and does not decompose into  $G \cong R \oplus G$ .

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