# ON NEEDED REALS

HEIKE MILDENBERGER AND SAHARON SHELAH

ABSTRACT. Given a binary relation $R$, we call a subset $A$ of the range of $R$ $R$-adequate if for every $x$ in the domain there is some $y \in A$ such that $(x, y) \in R$. Following Blass [4], we call a real $\eta$ "needed" for $R$ if in every $R$-adequate set we find an element from which $\eta$ is Turing computable. We show that every real needed for inclusion on the Lebesgue null sets, $\mathbf{Cof}(\mathcal{N})$, is hyperarithmetic. Replacing "$R$-adequate" by "$R$-adequate with minimal cardinality" we get the related notion of being "weakly needed". We show that it is consistent that the two notions do not coincide for the reaping relation. (They coincide in many models.) We show that not all hyperarithmetic reals are needed for the reaping relation. This answers some questions asked by Blass at the Oberwolfach conference in December 1999 and in [4].

## 0. Introduction

We consider some aspects of the following notions:

**Definition 0.1.** *(1) (Needed reals). Suppose that we have a cardinal characteristic $\mathfrak{x}$ of the reals of the following form: There are (in most cases: Borel) sets $A_-, A_+ \subseteq \mathbb{R}$ and there is a (in most cases: Borel) relation $R \subseteq A_- \times A_+$ such that*

$$\mathfrak{x} = ||R|| := \min\{|Y| \, : \, Y \subseteq A_+ \wedge (\forall x \in A_-)(\exists y \in Y) R(x, y)\}.$$

*We call $||R||$ the norm of $R$. A set $Y \subseteq A_+$ is called $R$-adequate if $(\forall x \in \mathrm{dom}(R))\ (\exists y \in Y) R(x, y)$. We say that $\eta \in {}^{\omega}2$ is needed for $R$ if for every $R$-adequate set $Y$ there is some $y \in Y$ such that $\eta$ is Turing reducible to $y$, abbreviated $\eta \leq_{Tur} y$.*

*If $A_+ \not\subseteq \mathbb{R}$ but can be mapped continuously, injectively into $\mathbb{R}$ by a mapping $c$ which is, as a function on the digits, computable in both directions, then we call the real $\eta$ needed for $R$ and $c$ if for any $R$-adequate set $Y \subseteq A_+$ there is some $y \in Y$ such that $\eta \leq_{Tur} c(y)$. We*

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*call such a function $c$ a coding. In this situation, a real $\eta$ is called needed for $R$, if it is needed for $R$ and $c$ for any coding $c$.*

(2) *(Weakly needed Reals). We call a real $\eta$ weakly needed for $R$ if for any $R$-adequate set $Y$* of minimal cardinality *there is some $y \in Y$ such that $\eta \leq_{Tur} y$.*

Every needed real is weakly needed. Sections 4 to 7 will give some information on the reverse direction. A very good motivation for the investigation of needed reals is given in [4].

In the rest of this introduction, we describe briefly what will be proved in the sections. In Section 1 we prove that only hyperarithmetic reals are needed for the cofinality relation on the ideal of Lebesgue null sets. In the second section we prove the analogous statement for the slalom relation. In the third section we extract from these two results sufficient conditions for the property "every needed real for $R$ is hyperarithmetic". In the fourth section we construct a forcing extension such that all hyperarithmetic reals are weakly needed for the reaping relation in the extension. This quite difficult model is further used in the fifth section, where we build a composed relation for which there are more weakly needed reals than needed reals. In Section 6 we prove that all needed reals for the reaping relation are in complexity less than $\mathbf{0}^\omega = \{\langle x, y\rangle : x, y \in \omega, x \in \mathbf{0}^y\}$, where $\mathbf{0}^y$ is the $y$th jump of the degree $\mathbf{0}$ of all recursive sets. Moreover, using the model of Section 4 once more, we get that it is consistent that for the reaping relation weakly needed and needed do not coincide. In the final section we give a sufficient criterion for a relation $R$ such that the two notions "needed for $R$" and "weakly needed for $R$" coincide. From the proof in Section 1, we derive one example of a relation for which the criterion is true. The definitions of the mentioned relations will be recalled at their first appearance.

## 1. Needed reals for $\mathbf{Cof}(\mathcal{N})$

In this section we answer affirmatively Blass' question whether only hyperarithmetic reals are needed for the cofinality relation on the ideal of Lebesgue null sets.

In this section we work with two particular relations on the reals: For functions $f, g \colon \omega \to \omega$ we write $f \leq^* g$ and say $g$ eventually dominates $f$ if $(\exists n < \omega)(\forall k \geq n)(f(k) \leq g(k))$. The dominating relation is

$$\mathbf{D} = \{(f, g) : f, g \in {}^\omega\omega \ \wedge \ f \leq^* g\},$$

and the cofinality relation for the ideal of sets of Lebesgue measure zero is

$$\mathbf{Cof}(\mathcal{N}) = \{(F, G) : F, G \text{ are } G_\delta\text{-sets of Lebesgue measure 0 and } F \subseteq G\}.$$

We write $\mathbf{cof}(\mathcal{N})$ for $||\mathbf{Cof}(\mathcal{N})||$.

Before stating our first theorem, we review some notation: For $s \in {}^{\omega>}2 = \{r : (\exists m \in \omega)(r \colon m \to 2)\}$, we write $\lg(s) = \mathrm{dom}(s)$. If $r \in {}^{\omega\geq}2$ and $s \in {}^{\omega\geq}2$, we write $r \trianglelefteq s$ if $r = s \restriction \lg(r)$. Let $r \vartriangleleft s$ denote that $r \trianglelefteq s$ and $r \neq s$. A subset

$T \subseteq {}^{\omega>}2$ is called a tree if it is downward closed, i.e., if for all $s \in T$ for all $r \trianglelefteq s$, we have that $r \in T$. An element $r \in T$ is a leaf if there is no $s \in T$ such that $r \triangleleft s$. For a tree $T \subseteq {}^{\omega>}2$ and some $n \in \omega$ we set $T \restriction n = T \cap {}^{n>}2$. For $t \subseteq {}^{n}2$ set $\bar{t} = \{f \in {}^{\omega}2 \,:\, f \restriction n \in t\}$. The set of infinite branches of $T$ is denoted by $\lim(T) = \{f \in {}^{\omega}2 \,:\, (\forall n)(f \restriction n \in T)\}$. The same notation applies to trees on ${}^{\omega>}H$ for an arbitrary set $H$. We will consider trees whose nodes are not finite sequences but finite sets of finite sequences.

Leb denotes the Lebesgue measure on the measurable subsets of ${}^{\omega}2$, the product space of $\omega$ copies of the space $\{0,1\}$ where each point has measure $\frac{1}{2}$.

We work with the Amoeba forcing in the representation of [2, 3.4B]:

$$\mathbb{Q} = \left\{ p \,:\, p \subseteq {}^{\omega>}2, p \text{ is a tree without leaves and } \mathrm{Leb}(\lim(p)) > \frac{1}{2} \right\}.$$

For trees $p, q \in \mathbb{Q}$, $q$ is a stronger forcing condition than $p$, abbreviated $q \geq p$, if $q \subseteq p$. In addition to the Jerusalem convention, that stronger conditions are the greater or equal ones, we also follow the alphabetical convention [6]: letters later in the alphabet or carrying more primes or stars are used for stronger conditions. The weakest element in $\mathbb{Q}$ is ${}^{\omega>}2$, and we write 1 for it. We write $G$ for some $\mathbb{Q}$-generic filter and $T = \bigcap G$ for the generic tree. For $x \in V[G]$, let $\underset{\sim}{x}$ denote a name of $x$.

**Definition 1.1.** $\mathbb{Q}^1$ *is the set of conditions* $p \in \mathbb{Q}$ *that fulfill:*

(1.1) *for all* $n < \omega$, $s \subseteq p \cap {}^{n}2$, $\mathrm{Leb}(\overline{s} \cap \lim(p)) \neq \frac{1}{2}$.

**Claim 1.2.** $\mathbb{Q}^1$ *is dense in* $\mathbb{Q}$.

*Proof.* Let $p \in \mathbb{Q}$ and $\mathrm{Leb}(\lim(p)) = \frac{1}{2} + \varepsilon$. Let $s_n$, $n < \omega$, be an enumeration of $\bigcup_{i<\omega} \mathcal{P}({}^{i}2)$. Now choose by induction on $n$ $p = p_0 \supseteq p_1 \supseteq p_2 \dots$ in $\mathbb{Q}$ such that $\mathrm{Leb}(\lim(p_n)) \geq \frac{1}{2} + \varepsilon(1 - \sum_{j<n} \frac{1}{2^{j+2}})$. We set $\varepsilon_0 = \varepsilon$. In step $n$, we set $\varepsilon_n := \min(\{\mathrm{Leb}(\overline{s_i} \cap \lim(p_n)) - \frac{1}{2} \,:\, i < n \;\wedge\; \mathrm{Leb}(\overline{s_i} \cap \lim(p_n)) - \frac{1}{2} > 0\} \cup \{\varepsilon_{n-1}\})$ and choose $p_{n+1} \subseteq p_n$ such that $\mathrm{Leb}(\overline{s_n} \cap \lim(p_{n+1})) \neq \frac{1}{2}$ and such that $\mathrm{Leb}(\lim(p_{n+1})) \geq \mathrm{Leb}(\lim(p_n)) - \frac{\varepsilon_n}{2^{n+2}}$. Then automatically also $\mathrm{Leb}(\overline{s_i} \cap \lim(p_{n+1})) \neq \frac{1}{2}$ for $i < n$ and once property (1.1) is true for a condition $p_n$ and $s_i$, $i < n$, it holds also for all later $p_k$ because we chose the $p_k$'s such that the differences in their measures are sufficiently small. Then by the choices, $q = \bigcap_{n<\omega} p_n \in \mathbb{Q}^1$. □

The following definition is crucial for building an algorithm that uses the oracle $T$ already in $V$. For this purpose we require: incompatibility of a finite part of $T$ with a condition $p$ can be read off a finite part of $p$ (this is (b)), that measure $\frac{1}{2}$ is forbidden in a preciser way than in equation (1.1) (this is (c)), and that the convergence from above of $\langle \frac{|p \cap {}^{k}2|}{2^k} \,:\, k \in \omega \rangle$ to $\mathrm{Leb}(\lim(p))$ is sufficiently fast (this is (d)).

**Definition 1.3.** *We say* $p$ *obeys* $g$ *if the following holds:*

(a) $p \in \mathbb{Q}^1$, $g \in {}^{\omega}\omega$, $(\forall n)(n < g(n))$.

(b) *If* $n < \omega$, $s \subseteq p \cap {}^n2$ *and* $\mathrm{Leb}(\overline{s} \cap \lim(p)) < \frac{1}{2}$, *then* $\frac{|\{\rho \in {}^{g(n)}2 : \rho \restriction n \in s, \rho \in p\}|}{2^{g(n)}} < \frac{1}{2}$.

(c) *If* $n < \omega$, $s \subseteq p \cap {}^n2$ *and* $\mathrm{Leb}(\overline{s} \cap \lim(p)) \geq \frac{1}{2}$, *then* $\mathrm{Leb}(\overline{s} \cap \lim(p)) \geq \frac{1}{2}\left(1 + \frac{1}{g(n)}\right)$.

(d) *If* $n < \omega$, *then* $\mathrm{Leb}(\lim(p)) \geq \frac{|p \cap {}^{g(n)}2|}{2^{g(n)}}\left(1 - \frac{1}{2^{2^n}}\right)$.

The main part of the section will be the proof of

**Theorem 1.4.** *Suppose that* $\eta \in V \cap {}^\omega\omega$, *that* $M \in V$ *is a Turing machine and that* $p$ *obeys* $g$ *and*

$$p \Vdash_{\mathbb{Q}} \text{“}M \text{ computes } \eta \text{ from } \underset{\sim}{T}\text{”}. \tag{1.2}$$

*Then* $\eta$ *is computable from* $g$.

*Proof.* We fix such an $\eta$.

Fix for a while $j \in \omega$. Since the statement “for every $j$ there is some $j'$ such that $M$ computes $\eta(j)$ using $T \restriction j'$” is forced, there is some stronger condition $r$ that forces a value for $j'$ for the fixed $j$. So $[r \restriction j']$ could serve as a condition that describes enough of the oracle $T$ in order to give the right computation of $\eta(j)$.

Now the assigment $j \mapsto j'$ (say the minimal one), is an element of $V^{\mathbb{Q}}$, and in general not in $V$. But since our computation is not allowed to use additional information except for $g$, we will look, given $j$, at all possible $r$'s and $j'$'s simultaneously. The procedure to give a computation in $V$ will be built upon guessing finite parts of conditions $r$ and finite parts of $T$ that are already determined by the same finite part of $r$. But, such an approximation, starting with trials of size zero and successively increasing the size, could give a unique (and, of course, halting) computation that gives the same outcome on all possibilities within the guessed part and still be not the right guess because a too small part of $r$ is used and only a larger approximation would mirror correctly what happens in the forcing process. However, fortunately from some approximation size onwards, the outcome will not change any more. So we can remedy the problem of wrong guesses by first choosing a suitable $n(*)$ and then looking into densely many forcing conditions above $\overline{p \cap {}^{n(*)}2}$ simultaneously, and search increasing in $m$ for an aproximation of size $m$. Starting from some $m$, all larger approximations will give the same result. The search will be based upon $g$. And, from the definition of “$p$ obeys $g$” it follows that any eventually larger function could serve as well.

We assign some structure to the collection of finite initial segments of members of $\mathbb{Q}^1$, that will allow us to work with finitely branching trees. These will be the trees $(\mathbf{T}^{n(*)}_{p,g}, \trianglelefteq)$ from Definition 1.7. The procedure that computes $\eta$ relative to $g$ will first search for a sufficiently large finite approximation of $r$, and then argue that this approximation already determines the run of $M$ with oracle $T$ on the given input $j$. The height of this approximation in $\mathbf{T}^{n(*)}_{p,g}$ will be an appropriate measure for being a sufficiently large approximation of $r$.

**Definition 1.5.** *A set $t \subseteq {}^{m\geq}2$ is a subtree of ${}^{m\geq}2$ iff $t$ is not empty and closed under initial segments and $(\forall \nu \in t \cap {}^{m>}2)(\exists e \in 2)(\nu\char`\^\langle e\rangle \in t)$.*

**Definition 1.6.** *We say that a subtree $t$ of ${}^{m\geq}2$ obeys $g$ if the following holds:*

(a) $t$ is a subtree of ${}^{m\geq}2$, $g \in {}^{\omega}\omega$, $(\forall n)(n < g(n))$.

(b) If $n \leq m$, $g(n) \leq m$, $s \subseteq t \cap {}^{n}2$, and if $\mathrm{Leb}(\bar{s} \cap \bar{t}) < \frac{1}{2}$, then $\frac{|\{\rho \in {}^{g(n)}2 : \rho\restriction n \in s, \rho \in t\}|}{2^{g(n)}} < \frac{1}{2}$.

(c) If $n \leq m$, $g(n) \leq m$, $s \subseteq t \cap {}^{n}2$, and if $\mathrm{Leb}(\bar{s} \cap \bar{t}) \geq \frac{1}{2}$, then $\mathrm{Leb}(\bar{s} \cap \bar{t}) \geq \frac{1}{2}\left(1 + \frac{1}{g(n)}\right)$.

(d) If $g(n) \leq m$, then

$$\frac{|t \cap {}^{g(n)}2|}{2^{g(n)}} \geq \frac{|t \cap {}^{m}2|}{2^{m}} \geq \frac{|t \cap {}^{g(n)}2|}{2^{g(n)}}\left(1 - \frac{1}{2^{2^n}}\right)$$

*(note that the first inequality holds trivially).*

The properties "$t$ obeys $g$" and "$p$ obeys $g$" look similar. In order to decribe the features of the similarities that will be useful later we make another definition:

**Definition 1.7.** *Assume that $p$ obeys $g$ and $n(*) < \omega$ and that $m \geq n(*)$.*

(1) $Nb^{n(*)}_{p,g} = \{q \in \mathbb{Q}^1 : q \restriction (n(*)+1) = p \restriction (n(*)+1)$ and $q$ obeys $g\}$.

(2) ${}^{m}\mathbf{T}^{n(*)}_{p,g} = \{\emptyset \neq t \subseteq {}^{m\geq}2 :$ $t$ is a subtree of ${}^{m\geq}2$ and $t$ obeys $g$ and $t \restriction (n(*)+1) = p \restriction (n(*)+1)\}$.

(3) We let $\mathbf{T}^{n(*)}_{p,g} = \bigcup_{0 \leq m < \omega} {}^{m}\mathbf{T}^{n(*)}_{p,g}$.

(4) If $t \in \mathbf{T}^{n(*)}_{p,g}$, then let $m(t)$ be the minimal $m$ such that $t \subseteq {}^{m\geq}2$.

(5) The partial order $\trianglelefteq$ on $\mathbf{T}^{n(*)}_{p,g}$ is defined it as follows: $s \trianglelefteq t$ iff $t \restriction (m(s)+1) = s$.

Our first claim about the trees $\mathbf{T}^{n(*)}_{p,g}$ and the neighborhoods $\mathrm{N}b^{n(*)}_{p,g}$ concerns the easier inclusion: members of the neighborhoods can be seen as branches of the trees:

**Claim 1.8.** (1) *$\mathbf{T}^{n(*)}_{p,g}$ is a tree whose $m$-th level is ${}^{m}\mathbf{T}^{n(*)}_{p,g}$. If $0 \leq m \leq m(t)$ and $t \in \mathbf{T}^{n(*)}_{p,g}$ then $t \restriction (m+1) \trianglelefteq t$ and $t \restriction (m+1) \in {}^{m}\mathbf{T}^{n(*)}_{p,g}$.*

(2) *If $q \in Nb^{n(*)}_{p,g}$ and $m < \omega$ then $q \restriction (m+1) \in {}^{m}\mathbf{T}^{n(*)}_{p,g}$.*

In the next claim some easy and useful facts are listed.

**Claim 1.9.** (1) *Every $p \in \mathbb{Q}^1$ obeys some $g$.*

(2) $q \in \lim(\mathbf{T}^{n(*)}_{p,g})$ iff $(\forall m) q \restriction (m+1) \in {}^{m}\mathbf{T}^{n(*)}_{p,g}$.

(3) If $p \in \mathbb{Q}^1$ obeys $g$ and $n(*) \in \omega$ then $Nb^{n(*)}_{p,g} \subseteq \mathbb{Q}^1$.

(4) If $g_1 \leq^* g_2$ then there is $g^1_2$ recursive in $g_2$ such that $g_1 \leq g^1_2$, where $\leq$ is the pointwise order.

(5) *If $g_1 \leq g_2$ and $p$ obeys $g_1$, then $p$ obeys $g_2$.*

(6) *Fixing $p$, $n(*)$, the function $m \mapsto {}^m\mathbf{T}^{n(*)}_{p,g}$ is recursive in $g$.*

The next claim will allow us to apply König's Lemma at an important step in the proof of Claim 1.13.

**Claim 1.10.** *Suppose that*

$$(\exists k)(g(k) \leq n(*) \ \wedge \ \mathrm{Leb}(\lim(p))\left(1 - \frac{1}{2^{2^{k-1}}}\right) > \frac{1}{2}). \tag{1.3}$$

*Then $Nb^{n(*)}_{p,g}$ is the set of unions of $\omega$-branches of the tree $\mathbf{T}^{n(*)}_{p,g}$.*

*Proof.* Suppose that for all $m \in \omega$, $q \restriction (m+1) \in {}^m\mathbf{T}^{n(*)}_{p,g}$. We prove that $q \in \mathrm{Nb}^{n(*)}_{p,g}$. We first prove that $q \in \mathbb{Q}$. From the definition of "subtree of ${}^{m\geq}2$" it follows that $q$ has no leaves.

The main point is: Why is $\mathrm{Leb}(\lim(q)) > \frac{1}{2}$? For every $m \geq g(k)$, we have by clause (d) of Definition 1.6

$$\frac{|q \cap {}^{g(k)}2|}{2^{g(k)}} \geq \frac{|q \cap {}^m 2|}{2^m} \geq \frac{|q \cap {}^{g(k)}2|}{2^{g(k))}}\left(1 - \frac{1}{2^{2^k}}\right).$$

But as $g(k) \leq n(*)$ clearly $q \cap {}^{g(k)}2 = p \cap {}^{g(k)}2$. As $p$ obeys $g$ we have

$$\mathrm{Leb}(\lim(p)) \geq \frac{|p \cap \ {}^{g(k)}2|}{2^{g(k)}}\left(1 - \frac{1}{2^{2^k}}\right).$$

Since the quotients are approaching the measure from above, the right side is greater than or equal to $\mathrm{Leb}(\lim(p))\left(1 - \frac{1}{2^{2^k}}\right)$. So $\frac{|q \cap {}^m 2|}{2^m} \geq \mathrm{Leb}(\lim(p))\left(1 - \frac{1}{2^{2^k}}\right)$ and this holds for every $m \in \omega$. Hence $\mathrm{Leb}(\lim(q)) \geq \mathrm{Leb}(\lim(p))\left(1 - \frac{1}{2^{2^k}}\right)$ and the right hand side is strictly larger than $\frac{1}{2}$ by equation (1.3).

Now we have to prove that $q \in \mathbb{Q}^1$. So let $n \in \omega$ and $s \subseteq q \cap {}^n2$. Suppose that $\mathrm{Leb}(\overline{s} \cap \lim(q)) = \frac{1}{2}$. Then for all $m$, $\mathrm{Leb}(\overline{s} \cap \overline{q \cap {}^m 2}) \geq \frac{1}{2}$. So by 1.6(b) for all $m$, $\mathrm{Leb}(\overline{s} \cap \overline{q \cap {}^m 2}) \geq \frac{1}{2}\left(1 + \frac{1}{g(n)}\right)$. Hence also the limit is greater than or equal to $\frac{1}{2}\left(1 + \frac{1}{g(n)}\right)$.

Now we have to prove that $q$ obeys $g$. This follows from Definition 1.6 and the nature of the limit process.

The reverse inclusion is Claim 1.8(2).

□

**Definition 1.11.** *Assume that $T$ is a Turing machine. We let $\Gamma_{M,k}$ be the set of finite partial characteristic functions $h$, $\mathrm{dom}(h) \subset {}^{\omega>}2$ such that if $M$ runs with the input $k$ it uses only $h$ as an oracle. So it answers $h(\rho) =?$ for $\rho \in \mathrm{dom}(h)$ according to $h$ and does not ask questions of the kind $h(\rho) =?$ for $\rho \notin \mathrm{dom}(h)$. We let $e_{M,h}(k)$ be the result of such a run.*

**Definition 1.12.** *For $p \in \mathbb{Q}$ set*

$$\triangle(p) = \{h : (\exists m)(h \colon {}^{m\geq}2 \to 2 \ \wedge \ p \not\Vdash h \not\subseteq ch_{\underset{\sim}{T}})\}.$$

*For $t \in {}^m\mathbf{T}^{n(*)}_{p,g}$ let*

$$\triangle(t) = \Bigg\{h : (\exists n)\Bigg(h \colon {}^{n\geq}2 \to 2 \ \wedge g(n) \leq m \ \wedge \ h^{-1}(\{1\}) \text{ is a tree} \\ \wedge \ h^{-1}\{1\} \subseteq t \wedge \ \frac{|\{\rho \in {}^{g(n)}2 \, : \, h(\rho \restriction n) = 1 \wedge \rho \in t\}|}{2^{g(n)}} > \frac{1}{2}\Bigg)\Bigg\}.$$

**Claim 1.13.** *(1) For every $p \in \mathbb{Q}$ and finite $u \subseteq {}^{\omega>}2$ there is some $h \in \triangle(p)$ whose domain is a superset of $u$.*

*(2) If $p \in \mathbb{Q}^1$ obeys $g$ and $h \colon {}^{n\geq}2 \to 2$ then $p \cap {}^{g(n)}2$ determines the truth value of $h \in \triangle(p)$, in fact $h \in \triangle(p)$ iff $h \in \triangle(p \restriction (g(n)+1))$.*

*Proof.* (2) By looking at the relation $\leq_{\mathbb{Q}}$ we see: For $h \colon {}^{n\geq}2 \to 2$, $p \Vdash h \not\subseteq ch_{\underset{\sim}{T}}$ iff $(h^{-1}(\{1\}) \not\subseteq p$ or $(h^{-1}(\{1\}) \subseteq p$ and $\mathrm{Leb}(\overline{h^{-1}(\{1\})} \cap \lim(p)) \leq \frac{1}{2}))$. But the right hand side of the iff-clause can be read off $p \restriction (g(n)+1)$ by clause (b) of the Definition of "$p$ obeys $g$". □

Now we are ready to finish the proof of Theorem 1.4. So assume that $p \in \mathbb{Q}^1$, $\eta \in {}^\omega 2$, $p \Vdash$ "$M$ computes $\eta$ from the oracle $\underset{\sim}{T}$" and $p$ obeys $g$. We show that $\eta$ is computable from $g$.

Step a) Let $k > 0$ be such that

$$\frac{1}{2} < \mathrm{Leb}(\lim(p)) \left(1 - \frac{1}{2^{2^k - 1}}\right).$$

Step b) Let $n(*) \geq g(k)$.

Step c) Now we have for every $q \in \mathrm{Nb}^{n(*)}_{p,g}$ and $j < \omega$ that $\triangle(q) \cap \triangle(p) \cap \Gamma_{M,j} \neq \emptyset$. Why?

First, by the choice of $n(*)$, $\mathrm{Leb}(\lim(p) \cap \lim(q)) > \frac{1}{2}$. This follows from the computation: $\lim(p) \subseteq \overline{p \cap {}^{n(*)}2} \subseteq \overline{p \cap {}^{g(k)}2}$ and the same holds for $q$. Since both obey $g$ we have $\lim(p) > \frac{|p \cap {}^{g(k)}2|}{2^{g(k)}}\left(1 - \frac{1}{2^{2^k}}\right)$ and the same holds for $q$. Hence $\lim(p)$ and $\lim(q)$ each are missing only a part of $\overline{p \cap {}^{g(k)}2}$ of measure less than $\frac{|p \cap {}^{g(k)}2|}{2^{g(k)+2^k}}$. The missing part of $\lim(p) \cap \lim(q)$ in $\overline{p \cap {}^{g(k)}2}$ is at most twice this. Hence $\mathrm{Leb}(\lim(p) \cap \lim(q)) \geq \frac{|p \cap {}^{g(k)}2|}{2^{g(k)}}\left(1 - \frac{1}{2^{2^k - 1}}\right) \geq \mathrm{Leb}(\lim(p))\left(1 - \frac{1}{2^{2^k-1}}\right) > \frac{1}{2}$.

So there is some $r \in \mathbb{Q}$ that is above both. As $r \geq p$, it forces that $M$ running on $j$ and oracle $\underset{\sim}{T}$ gives $\eta(j)$ and the run uses only $h = \underset{\sim}{T} \cap {}^{n\geq}2$ for some $n$.

Step d) If $e_{M,h}(j)$ is well-defined and $h \in \triangle(p)$, then $e_{M,h}(j) = \eta(j)$. This is because $\eta \in V$.

Step e) For every $j$ there is some $m$, such that for all $t \in {}^m\mathbf{T}^{n(*)}_{p,g}$ we have that $\triangle(t) \cap \triangle(p) \cap \Gamma_{M,j} \neq \emptyset$. Why? This follows by Claim 1.10 and step c) and König's lemma applied to the finitely branching tree $\mathbf{T}^{n(*)}_{p,g}$.

Step f) For every $j$ there are $m$ and $t_1 \in {}^m\mathbf{T}^{n(*)}_{p,g}$ such that for every $t_2 \in {}^m\mathbf{T}^{n(*)}_{p,g}$ we have $\triangle(t_1) \cap \triangle(t_2) \cap \Gamma_{M,j} \neq \emptyset$. This holds by e): We can take $t_1 = p \restriction (m+1)$.

Step g) For every $j < \omega$ and $e \in 2$ the following are equivalent

(i) $\eta(j) = e$.

(ii) For some $m < \omega$ and $t_1 \in {}^m\mathbf{T}^{n(*)}_{p,g}$ for every $t_2 \in {}^m\mathbf{T}^{n(*)}_{p,g}$ there is $h \in \triangle(t_1) \cap \triangle(t_2) \cap \Gamma_{M,j}$ such that $e_{M,h}(j) = e$.

(i) $\Rightarrow$ (ii) is step f).

(ii) $\Rightarrow$ (i): Let $m$, $t_1$ be as in (ii) Let $t_2 = p \restriction (m+1)$. By (ii) there is some $h \in \triangle(t_1) \cap \triangle(t_2) \cap \Gamma_{M,j}$, $e_{M,h}(j) = e$. By step b) $h \in \triangle(p)$. So by step d) we are done.

By Claim 1.9(6), the procedure indicated in (ii) of step g) is recursive.

So finally Theorem 1.4 is proved. □

**Corollary 1.14.** *Suppose that $p \in \mathbb{Q}$ and $p \Vdash_{\mathbb{Q}} \eta$ is computable from $\underset{\sim}{T}$. Then $\eta$ is needed for the dominating relation.*

*Proof.* Choose some $q \geq p$, $q \in \mathbb{Q}^1$ and some machine $M$ such that $q \Vdash_{\mathbb{Q}}$ $M$ computes $\eta$ from $\underset{\sim}{T}$. Then choose $g$ such that $q$ obeys $g$. By Theorem 1.4, $\eta$ is computable from $g$. Since $q$ obeys also every $g' \geq g$ and since finite changes in the oracle may only change the algorithm but not the fact whether a real is computable from the oracle, $\eta$ is also computable from any $g' \geq^* g$. Hence $\eta$ is needed for the dominating relation. □

The following equivalent formulation of neededness is useful to show that in a generic extension that contains a real $\xi$ such that for all reals $\eta$ in the ground model $\eta R \xi$, all needed reals in the ground model can be computed from such a $\xi$.

**Fact 1.15.** *(Blass* [3, following Definition 1]*) An equivalent condition for "$\eta$ is needed for $R$" is*

$$(\exists x \in \mathrm{dom}(R))(\forall y \in \mathrm{range}(R))(xRy \rightarrow \eta \leq_{Tur} y). \tag{1.4}$$

*Proof.* Suppose that $\eta$ is needed for $R$ and that there is no $x$ as in (1). Then $(\forall x \in \mathrm{dom}(R))$ $(\exists y \in \mathrm{range}(R))(xRy \wedge \eta \not\leq_{Tur} y)$. So we can build a $R$-adequate set from all these $y$'s, that shows that $\eta$ is not needed for $R$. For the other implication: Fix $x$ as in (1). Every $R$-adequate set has to contain some $y$ such that $xRy$ and hence $\eta \leq_{Tur} y$. □

If $R$ is a transitive relation and $\mathcal{R}$ is a given $R$-adequate set, then $x$ with the property of equation (1.4) can be found in $\mathcal{R}$.

**Theorem 1.16.** *Every needed real for* $\mathbf{Cof}(\mathcal{N})$ *is needed for the dominating relation.*

*Proof.* : Let $\{A_i : i < \kappa\}$ be a $\mathbf{Cof}(\mathcal{N})$-adequate set, such that each $A_i$ is a $F_\sigma$ set. Let $\eta \in {}^\omega 2$.

For each $i$ choose a countable elementary submodel $N_i$ of $(\mathcal{H}(\beth_3), \in)$ to which $\eta$ and $A_i$ belong. We let $G_i$ be a subset of $\mathbb{Q}^{N_i}$ that is generic over $N_i$ and let $T_i = \underset{\sim}{T}[G_i]$. Now let $A_i^*$ be

$$\begin{aligned} A_i^* = \{\rho \in {}^\omega 2 : \ & \text{no } \rho' \in {}^\omega 2 \text{ which is almost equal to } \rho \\ & \text{(i.e. } \rho(n) = \rho'(n) \text{ for every large enough } n\text{) belongs to } \lim(T_i)\} \end{aligned}$$

$A_i^*$ is a null set: Since it is a tail set, by the zero-one law it can only have measure zero or one. Since it is disjoint from the set $\lim(T_i)$, that has measure one half, it is a null set.

By genericity of $T_i$ and because $A_i \in N_i$ and because $A_i$ is a nullset in $N_i$ we have that $A_i \subseteq \lim(T_i)^c$. The same argument shows that for all $s \in {}^{\omega >}2$ we have that $\{s\hat{\ }f : \exists s' \, (|s'| = |s| \wedge s'\hat{\ }f \in A_i)\}$ is a subset of $(\lim(T_i))^c$. Hence we have that $A_i \subseteq A_i^*$. Therefore also $\{A_i^* : i < \kappa\}$ is a $\mathbf{Cof}(\mathcal{N})$-adequate set. We choose $i$ such that $\eta$ is recursive in $A_i$ and in all its supersets. Since $\mathbf{Cof}(\mathcal{N})$ is transitive, such an $A_i$ exists by Fact 1.15 and the remark thereafter. Then $\eta$ is also recursive in $A_i^*$, because $A_i^* \supseteq A_i$. If $\eta$ is recursive in $A_i^*$ it is also recursive in $T_i$. Since this holds for arbitrary $G_i$, by Theorem 1.4, applied to some $p \in \mathbb{Q}^1$ that obeys some $g$, the real $\eta$ is needed for dominating. $\square$

**Fact 1.17.** *a) (Solovay* [11]*) Every real that is needed for the dominating relation is hyperarithmetic.*
*b) (Jockusch,* [8]*) Every hyperarithmetic real is needed for the dominating relation.*

Blass [4, Theorem 6, Corollary 8] showed that every real that is needed for **D** is also needed for $\mathbf{Cof}(\mathcal{N})$ and hence that all hyperarithmetic reals are needed for $\mathbf{Cof}(\mathcal{N})$. So this gives the other inclusion in the following corollary:

**Corollary 1.18.** *Exactly the hyperarithmetic reals are needed for the* $\mathbf{Cof}(\mathcal{N})$*-relation.*

## 2. Needed reals for the slalom relation

In this section we deal with a forcing $\mathbb{L}$ which is closely related to the localization forcing from [2, page 106]. Theorem 2.4 is analogous to Theorem 1.4, but for the forcing $\mathbb{L}$. Theorem 2.16 is analogous to Theorem 1.16 together with Corollary 1.18.

For $\bar{u} = \langle u_\ell : \ell \in \omega \rangle$, $m < \omega$, let $\bar{u} \restriction n = \langle u_\ell : \ell < n \rangle$.

**Definition 2.1.**

$$\mathbb{L} = \{p : p = (n, \bar{u}) = (n^p, \bar{u}^p), \bar{u} = \langle u_\ell : \ell \in \omega\rangle, u_\ell \in [\omega]^{\leq \ell},$$
$$\langle |u_\ell| : \ell \in \omega\rangle \text{ is bounded}\},$$
$$p \leq q \leftrightarrow \left(\bigwedge_{\ell \in \omega} u_\ell^p \subseteq u_\ell^q \wedge \bar{u}^q \restriction n^p = \bar{u}^p \restriction n^p \wedge n^p \leq n^q\right).$$

*Again we denote the weakest element* $(0, \langle \emptyset : i < \omega\rangle)$ *of* $\mathbb{L}$ *by 1. For* $p \in \mathbb{L}$*, we write* $b(p) = \max\{|u_\ell| : \ell \in \omega\}$.

**Notation 2.2.** *Let* $\underset{\sim}{G}$ *be a name for an* $\mathbb{L}$*-generic element. Let* $S = S_G = \bigcup\{\bar{u}^p \restriction n^p : p = (n^p, \bar{u}^p) \in G\}$. *We write* $\underset{\sim}{S}$ *for a* $\mathbb{L}$*-name for* $S$. *We think of* $S$ *as a subset of* $\omega \times \omega$ *and have its characteristic function* $ch_S(\ell, m) = 1$ *iff* $m \in u_\ell$.

**Definition 2.3.** *We say* $p = (n, \bar{u}) \in \mathbb{L}$ *obeys* $(g, b)$ *if* $(\forall \ell < \omega)(\ell < g(\ell))$, $b \in \omega$, $(\forall \ell)(u_\ell^p \subseteq g(\ell))$ *and* $(\forall \ell)(|u_\ell^p| \leq b)$. *Note that the condition 1 obeys every* $(g, b)$.

**Theorem 2.4.** *Assume that* $M$ *is a Turing machine and that* $\eta \in {}^\omega 2$. *Suppose that* $p \in \mathbb{L}$ *obeys* $(g, b)$ *and*

$$p \Vdash_{\mathbb{L}} M \text{ computes } \eta \text{ from } \underset{\sim}{S}. \tag{2.1}$$

*Then* $\eta$ *is computable from* $g$.

*Proof.* As in the proof of Theorem 1.4, we use (2.1) and the fact that $p$ obeys $(g, b)$ in order to find densely many conditions above $p$ and finite approximations of $ch_S$ and of the respective condition. We will keep the numbering of the claims and of the definitions used in the proof of Theorem 2.4 parallel to the numbering in the proof of Theorem 1.4, though many of them are much easier for $\mathbb{L}$ and will be almost obsolete or empty. But this procedure will help to establish a general scheme.

**Definition 2.5.** *A tuple* $(n, \langle u_\ell : \ell < m\rangle)$ *is a finite part of a condition iff* $n \leq m$ *and for all* $\ell < m$, $u_\ell$ *is a non-empty finite set.*

**Definition 2.6.** *A finite part of a condition* $(n, \langle u_\ell : \ell < m\rangle)$ *obeys* $(g, b)$ *iff* $(\forall \ell < m)(u_\ell \subseteq g(\ell) \wedge |u_\ell| \leq b)$.

Now, in the following we do not only number analogously but also use similar names $\mathrm{Nb}^{n(*)}_{p,g,b}$ and ${}^m\mathbf{T}^{n(*)}_{p,g,b}$ for the corresponding objects. We use $g$ so that the $\mathbf{T}^{n(*)}_{p,g,b}$ will be finitely branching, and we use $b$ to get the boundedness clause in the definition of a condition.

**Definition 2.7.** *Let* $p = (n(*), \bar{u}^p)$ *be a condition that obeys* $(g, b)$.

(1) $Nb^{n(*)}_{p,g,b} = \{q = (n^q, \bar{u}^q) \in \mathbb{L} : n^q \geq n(*) \wedge (\forall i < n(*)) u_i^p = u_i^q \wedge (\forall i) u_i^q \subseteq g(i) \wedge (\forall i)|u_i^q| \leq b\}$. *As the trees and neighborhoods in Definition 1.7 used only* $p \restriction n(*)$, *also here the part of* $u^p$ *above* $n(*)$ *is ignored. The algorithm will depend on* $g$ *and on a finite part of* $p$.

(2) *If* $m \geq n(*)$, *let* ${}^m\mathbf{T}^{n(*)}_{p,g,b} = \{(n, \langle u_i : i < m\rangle) : n \geq n(*), (\forall i < n(*))(u_i = u^p_i) \wedge (\forall i < m)(u_i \subseteq g(i) \wedge |u_i| \leq b)\}$. *If* $m < n(*)$, *let* ${}^m\mathbf{T}^{n(*)}_{p,g,b} = \{(m, \bar{u}^p \restriction m)\}$.

(3) $\mathbf{T}^{n(*)}_{p,g,b} = \bigcup_{m<\omega} {}^m\mathbf{T}^{n(*)}_{p,g,b}$.

(4) *For* $(n, \langle u_i : i < m\rangle) \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$ *we let* $m(n, \langle u_i : i < m\rangle) = m$.

(5) *For* $(n, \bar{u}), (n', \bar{v}) \in \mathbf{T}^{n(*)}_{p,g,b}$ *we write* $(n, \bar{u}) \trianglelefteq (n', \bar{v})$ *iff* $\bar{u}$ *is an initial segment of* $\bar{v}$ *and* $\bar{u} \restriction n = \bar{v} \restriction n$.

**Claim 2.8.** (1) $\mathbf{T}^{n(*)}_{p,g,b}$ *is a tree whose* $m$*-th level is* ${}^m\mathbf{T}^{n(*)}_{p,g,b}$. *If* $0 \leq m \leq m(t)$ *and* $t \in \mathbf{T}^{n(*)}_{p,g,b}$ *then* $t \restriction (m+1) \trianglelefteq t$ *and* $t \restriction (m+1) \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$.

(2) *If* $q = (n, \bar{u}) \in Nb^{n(*)}_{p,g,b}$ *and* $n \leq m < \omega$ *then* $(n, \bar{u} \restriction m) \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$.

**Claim 2.9.** (1) *Every* $p \in \mathbb{L}$ *obeys some* $(g, b)$.

(2) $q = (n, \bar{u}) \in \lim(\mathbf{T}^{n(*)}_{p,g,b})$ *iff* $(\forall m)(\min(m, n), \bar{u} \restriction m) \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$.

(3) *If* $p = (n(*), \bar{u}) \in \mathbb{L}$ *obeys* $(g, b)$ *and* $n(*) \in \omega$ *then* $Nb^{n(*)}_{p,g,b} \subseteq \mathbb{L}$.

(4) *If* $g_1 \leq^* g_2$ *then there is* $g^1_2$ *recursive in* $g_2$ *such that* $g_1 \leq g^1_2$, *where* $\leq$ *is the pointwise order.*

(5) *If* $g_1 \leq g_2$ *and* $p$ *obeys* $(g_1, b)$, *then* $p$ *obeys* $(g_2, b)$.

(6) *Fixing* $p$, $n(*)$, *the function* $m \mapsto {}^m\mathbf{T}^{n(*)}_{p,g,b}$ *is recursive in* $g$.

**Claim 2.10.** *Suppose that* $p \in \mathbb{L}$ *obeys* $(g, b)$. *Then* $Nb^{n(*)}_{p,g,b}$ *is compact as a subset of* $\mathcal{P}({}^{\omega>}2)$, *and is the set of unions of* $\omega$*-branches of the tree* $\mathbf{T}^{n(*)}_{p,g,b}$.

*Proof.* This is obvious.

**Definition 2.11.** *Assume that* $T$ *is a Turing machine. We let* $\Gamma_{M,k}$ *be the set of finite partial characteristic functions* $h$, $\mathrm{dom}(h) \subset {}^{\omega\times\omega}2$ *such that if* $M$ *runs with the input* $k$ *it uses only* $h$ *as an oracle. So it does not ask questions of the kind* $h(\rho) =?$ *for* $\rho \notin \mathrm{dom}(h)$. *We let* $e_{M,h}(k)$ *be the result of such a run.*

**Definition 2.12.** *For* $p \in \mathbb{L}$ *set*

$$\triangle(p) = \{h : (\exists m)(h \colon m \times m \to 2 \ \wedge \ p \not\Vdash h \not\subseteq ch_{\underset{\sim}{S}})\}.$$

*For* $t = (n(*), \langle u_i : i < m\rangle) \in {}^m\mathbf{T}^{n(*)}_{p,g}$ *let*

$$\triangle(t) = \{h : h \colon m \times m \to 2 \ \wedge \ (\forall i < n(*))h^{-1}(\{1\}) \cap (\{i\} \times \omega) = \{i\} \times u_i \\ \wedge \ (\forall i \in [n(*), m))h^{-1}\{1\} \cap (\{i\} \times \omega) \supseteq \{i\} \times u_i\}.$$

**Claim 2.13.** (1) *For every* $p \in \mathbb{L}$ *and finite* $z \subseteq \omega \times \omega$ *there is some* $h \in \triangle(p)$ *whose domain is a superset of* $z$.

(2) *If $p = (n(*), \bar{u})) \in \mathbb{L}$ obeys $(g, b)$, $n(*) \leq m$, and $h \colon m \times m \to 2$ then $t = (n(*), \bar{u} \restriction m)$ determines the truth value of $h \in \triangle(p)$, in fact $h \in \triangle(p)$ iff $h \in \triangle(t)$.*

*Proof.* (2) By looking at $\leq_{\mathbb{L}}$ we see: For $h \colon m \times m \to 2$, $p \Vdash h \not\subseteq ch_{\underset{\sim}{S}}$ iff $(h^{-1}(\{1\}) \not\supseteq \bigcup_{i<m} \{i\} \times u_i$ or $(\exists i < n(*))(h^{-1}(\{1\}) \cap (\{i\} \times \omega) \neq \{i\} \times u_i))$. □

Now we finish the proof of Theorem 2.4: Assume $p = (n(*), \langle u_i : i \in \omega \rangle) \in \mathbb{L}$, $\eta \in {}^\omega 2$, $p \Vdash$ "$M$ computes $\eta$ from the oracle $\underset{\sim}{S}$" and $p$ obeys $(g, b)$. We show that $\eta$ is recursive from $g$.

We name the steps in parallel to the steps in the end of the proof of Theorem 1.4. Since $n(*)$ is already given, the first two steps are empty.

Step c) Now we have that for every $q \in \mathrm{Nb}^{n(*)}_{p,g,b}$ and $j < \omega$ that $\triangle(q) \cap \triangle(p) \cap \Gamma_{M,j} \neq \emptyset$. Why? It is easy to see that $p$ and $q$ are compatible in $\mathbb{L}$. So there is some $r \in \mathbb{Q}$ that is above both. As $r \geq p$, it forces that $M$ running on $j$ and oracle $\underset{\sim}{S}$ gives $\eta(j)$ and the run uses only $h := ch_S \cap {}^{m \times m}2$ for some $m$.

Step d) If $e_{M,h}(j)$ is well-defined and $h \in \triangle(p)$, then $e_{M,h}(j) = \eta^*(j)$.

Step e) For every $j$ there is some $m$, such that for all $t \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$ we have such that $\triangle(t) \cap \triangle(p) \cap \Gamma_{M,j} \neq \emptyset$. Why? this follows by c) and König's lemma applied to the finitely branching tree $\mathbf{T}^{n(*)}_{p,g,b}$.

Step f) For every $j$ there are $m$ and $t_1 \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$ such that for every $t_2 \in {}^m\mathbf{T}^{n(*)}_{p,g}$ we have $\triangle(t_1) \cap \triangle(t_2) \cap \Gamma_{M,j} \neq \emptyset$. This holds by e): We can take $t_1 = (n(*), \langle u_i : i < m \rangle)$.

Step g) For every $j < \omega$ and $e \in 2$ the following are equivalent

(i) $\eta(j) = e$.

(ii) For some $m < \omega$ and $t_1 \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$ for every $t_2 \in {}^m\mathbf{T}^{n(*)}_{p,g,b}$ there is $h \in \triangle(t_1) \cap \triangle(t_2) \cap \Gamma_{M,j}$ such that $e_{M,h}(j) = e$.

(i) ⇒ (ii) is step f).

(ii) ⇒ (i): Let $m$, $t_1$ be as in (ii) Let $t_2 = (n(*), \bar{u} \restriction m)$. By (ii) there is some $h \in \triangle(t_1) \cap \triangle(t_2) \cap \Gamma_{M,j}$, $e_{M,h}(j) = e$. By step b) $h \in \triangle(p)$. So by step d) we are done.

By Claim 2.9(6), the procedure in (ii) of step g) is recursive. □

**Corollary 2.14.** *Suppose that $p \in \mathbb{L}$ and $p \Vdash_{\mathbb{Q}}$ "$\eta$ is computable from $\underset{\sim}{S}$". Then $\eta$ is needed for the dominating relation.*

*Proof.* Choose some $q \geq p$, $q \in \mathbb{L}$ and some machine $M$ such that $q \Vdash_{\mathbb{Q}}$ "$M$ computes $\eta$ from $\underset{\sim}{S}$". Then choose $(g, b)$ such that $q$ obeys $(g, b)$. By Theorem 2.4, $\eta$ is computable from $g$. Since $q$ obeys also every $(g', b)$ for $g' \geq g$ and since finite changes in the oracle may only change the algorithm but not the fact whether a real is computable from the oracle $\eta$ is also computable from any $g' \geq^* g$. Hence $\eta$ is needed for the dominating relation. □

**Definition 2.15.** $S \in {}^{\omega}([\omega]^{<\omega})$ *is called a slalom iff for all* $n$, $|S(n)| \leq n$.

**Theorem 2.16.** *Exactly the hyperarithmetic reals are needed for the slalom relation*

$$\mathbf{SL} = \{(f, S) : f \in {}^{\omega}\omega \wedge S \text{ is a slalom and } (\forall n \in \omega)(f(n) \in S(n))\}.$$

*Proof.* First we show that only hyperarithmetic reals are needed for **SL**: Let $\{S_i : i < \kappa\}$ be an **SL**-adequate set, $\eta \in {}^{\omega}2 \cap V$ be needed for **SL**. We take $N_i \prec (H(\beth_3), \in)$ such that $\eta, S_i \in N_i$. Then we let $G_i$ be $\mathbb{L}$-generic over $N_i$. Now we set $S_i^* = S_{G_i} = \bigcup\{\bar{u}^p \restriction n^p : p \in G_i\}$. Then we have that for all but finitely many $n$, $S_i(n) \subseteq S_i^*(n)$. Let $\eta$ be computable from $S_i$. Then for all wider slaloms $S_i'$ than $S_i$ there is a (possibly even) wider slalom from which $\eta$ is computable as well, because that collection of slaloms wider than $S_i'$ is **SL**-adequate. Then by density $\eta \leq_{Tur} S_{G_i}$ for all generic $G_i$. Hence we may choose $p$ and $M$ and $(g, b)$ such that Theorem 2.4 applies.

All hyperarithmetic reals are needed for **SL**, because all of then are needed for **D**. Suppose that $\{\langle S_n^i : n \in \omega\rangle : i < \kappa\}$ is **SL**-adequate and that $\eta \in {}^{\omega}2$ is hyperarithmetic. Then $\mathcal{D} = \{\langle \max S_n^i : n \in \omega\rangle : i < \kappa\}$ is **D**-adequate and hence there is some element $\langle \max S_n^i : n \in \omega\rangle \in \mathcal{D}$ from which $\eta$ is computable. But then of course $\eta$ is also computable in $\langle S_n^i : n \in \omega\rangle$. □

## 3. A general connection

In this section, we collect sufficient conditions and give a general scheme for the proofs of "every real needed for $R$ is hyperarithmetic". As in Theorems 1.4 and 2.4 we use a forcing that adds an $R$-dominating real $\rho$. The first step is to prove that "being computable in $\rho$ and being in $V$ implies being hyperarithmetic". A form of this step will be given in Theorem 3.1. The second step is to show that every needed real for $R$ is computable from any generic $\rho$. We write a general version of this step in Theorem 3.6.

Now we take $(Q, R)$ instead of our two examples $(\mathbb{Q}, \mathbf{Cof}(\mathcal{N}))$ and $(\mathbb{L}, \mathbf{SL})$. $R$ is a Borel binary relation on the reals, and $Q$ is a notion of forcing adding some element in the range of the extension of $R$. Since $R$ is Borel, we can use its code and thus get a unique extension of $R$ to a larger model of ZFC. From the work in the previous two sections we collect the following scheme:

**Theorem 3.1.** *Assume that*

(a) *There is a notion "$p$ obeys $g$" such that if $g' \geq g$ and $p$ obeys $g$ then $p$ obeys $g'$. For a dense subset $Q'$ of $Q$ we have $(\forall p \in Q')(\exists g)(p$ obeys $g)$.*

(b) *$\mathbf{T}_{p,g}$ is a recursive finitely branching tree whose nodes are finite functions.*

(c) *$Q$ is a forcing notion, $p \in Q$, $p$ obeys $g$, $\rho$ is a $Q$-name, and*

$$p \Vdash_Q \rho \in \lim(\mathbf{T}_{p,g}).$$

(d) *For a dense set of $p_0 \in Q'$ there is some $p \geq p_0$ such that the following conditions are fulfilled:*

*($\alpha$)* *Let* $\triangle(p) = \{\nu \in \mathbf{T}_{p,g} : p \not\Vdash \nu \not\subseteq \rho\}$. *This is a subtree of* $\mathbf{T}_{p,g}$.

*($\beta$)* *Let* $S^*_{p,g} = \Big\{t :$ *for some leafless subtree* $T'$ *of* $\mathbf{T}_{p,g}$ *and some* $k$, $t = \{\nu \in T' : level_{\mathbf{T}_{p,g}}(\nu) \leq k\}\Big\}$, *and order* $S^*_{p,g}$ *naturally.*

*($\gamma$)* $S_{p,g}$ *is a recursive subtree of* $S^*_{p,g}$ *such that*

- *(i)* $\mathbf{T}_{p,g}$ *is the union of an* $\omega$*-branch of* $S_{p,g}$,
- *(ii)* *for every branch* $\bar{t} = \langle t_\ell : \ell \in \omega\rangle$ *of* $S_{p,g}$ *there is* $q \in Q$ *such that* $q$ *is compatible with* $p$ *and* $\mathbf{T}_{q,g} = \bigcup_{\ell\in\omega} t_\ell$.

*(d)* $\eta \in {}^\omega 2$ *or* ${}^\omega\omega$

*Then the following holds: if* $p \Vdash_Q$ *"*$\eta$ *is recursive in* $\underset{\sim}{\rho}$*" then* $\eta$ *is hyperarithmetic.*

*Proof.* For some $p$ as in (c) and Turing machine $M$

$$p \Vdash_Q \text{“}M \text{ computes } \eta \text{ from } \underset{\sim}{\rho}\text{”}.$$

Now we prove some intermediate facts, and the proof of 3.1 will be finished with 3.4.

**Fact 3.2.** *For every* $\omega$*-branch* $\langle t_k : k \in \omega\rangle$ *of* $S_{p,g}$ *and* $j \in \omega$ *for some (equivalently every) large enough* $m \in \omega$ *for every* $\nu \in t_m \cap level_k(\mathbf{T}_{p,g})$ *if* $M$ *runs on input* $j$ *and oracle* $\nu$ *it finishes (so we do not ask oracle questions outside the domain) and gives the result* $\eta(j) = k$.

*Proof.* There is $q$ such that $\mathbf{T}_{q,g} \subseteq \bigcup_{n\in\omega} t_n$ . Let $r \geq q$, and let $G \subseteq Q$ be generic with $r \in G$. If $M$ runs with $\underset{\sim}{\rho}[G] \restriction m$ it gives $\eta(j)$, so for some $\nu \in \mathbf{T}_{p,g}$, $\nu \subseteq \underset{\sim}{\rho}[G]$. Now we proceed as in 1.4. If there is some $\rho$ of height $k$ that gives another computation result, then it is incompatible with $r$. But then this is witnessed by some initial segment of $r$. Take $m$ larger than all these initial segments.

**Fact 3.3.** *For* $j \in \omega$, *for every large enough* $m$, *for every* $t \in level_m(S_{p,g})$ *there is* $\nu \in t \cap level_m(\mathbf{T}_{p,g})$ *such that if* $M$ *runs with* $\nu$ *as an oracle then it computes* $\eta(j)$.

*Proof.* By the previous fact and König's lemma applied to $S_{p,g}$.

**Crucial Fact 3.4.** *For* $j \in \omega$, $k \in 2$, *the following are equivalent:*

- *(i)* $\eta(j) = k$.
- *(ii)* *There are some* $m$ *and some* $t^0 \in level_m(S_{p,g})$ *such that for every* $t^1 \in level_m(S_{p,g})$ *there is* $\nu \in t^0 \cap t^1$ *such that if we let* $M$ *run with input* $j$ *and oracle* $\nu$ *then the run finishes and there are no questions to the oracle that do not have an answer, and it gives answer* $k$.

*Proof.* Analogous to the end of the proof of Theorem 1.4. □

So we have proved 3.1.

**Remark 3.5.** *Usually, $S_{p,g}$ depends only on a finite part of $p$, so that we have that $Q = \bigcup_{k\in\omega} Q_k$, and for all $k \in \omega$ we have $S_{p,g}$ as above being the same for each $p \in Q_k$.*

**Theorem 3.6.** *Suppose $Q$ is a notion of forcing and $\rho$ is a $Q$-name and $1 \Vdash (\forall x)(xR\rho)$. Then $1 \Vdash$ "every real in $V$ that is needed for $R$ is recursive in $\underset{\sim}{\rho}$".*

*Proof.* Let $p \in Q$ and $\eta \in V$. Since $\eta$ is needed for $R$, by Fact 1.15 there is some $x$ in $\mathrm{dom}(R)$ that for any $y$ such that $xRy$, $\eta \leq_{Tur} y$. Now if $p \Vdash xR\rho$, then $p \Vdash \eta \leq_{Tur} \rho$. □

## 4. Weakly needed reals for the reaping relation

In this section we show that for any ground model $V$ there is a forcing extension $V[G]$ such that all hyperarithmetic reals from $V$ are weakly needed in $V[G]$ for the reaping relation. The extension is necessarily a model where weakly needed and needed are different and the CH fails, because of the following: In Section 6 we shall prove in ZFC that not all hyperarithmetic reals are needed for the reaping relation, answering another question from Blass' work [4]. In a model of CH, the notions "needed real" and "weakly needed real" coincide, and thus in such a model not all hyperarithmetic reals in any submodel are weakly needed for the reaping relation. If we take a ground model $V$ with CH then from the coincidence of needed and weakly needed and from the fact that there are so few needed reals, we see that there are hyperarithmetic reals in $V$ that are weakly needed in $V[G]$ but not weakly needed in $V$. So the model of this section, together with the result from Section 6, gives an example for the fact that in contrast to the notion of "being needed", the notion of "being weakly needed" is not absolute.

**Theorem 4.1.** *For any ground model $V$ there is a generic extension $V[G]$ by some c.c.c. forcing such that in $V[G]$ every hyperarithmetic real in $V$ is weakly needed for the reaping relation.*

The proof of this theorem will occupy the whole section. First we recall the definition of the reaping relation:

**Definition 4.2.** *The relation*

$$\mathbf{R} = \{(f, X) : f \in {}^{\omega}2, X \in {}^{\omega}[\omega] \wedge f \restriction X \text{ is constant}\}$$

*is called the reaping or the refining or the unsplitting relation. We say "$X$ refines $f$" if $f \restriction X$ is constant. We say "$\mathcal{R}$ refines $f$" if there is some $X \in \mathcal{R}$ that refines $f$. Finally we say "$\mathcal{R}$ refines $F$" if for every $f \in F$ we have that $\mathcal{R}$ refines $f$.*

*The norm of this relation is called $\mathfrak{r}$, the reaping number or the refining number or the unsplitting number.*

In this section we often use (finite) boolean combinations. For any finite set $u$ and $\eta \in {}^u 2$ and $A_i$, $i \in u$, we set

$$A_i^\ell = \begin{cases} A_i, & \text{if } \ell = 1, \\ \omega \setminus A_i, & \text{if } \ell = 0; \end{cases}$$

and

$$\bar{A}^{[\eta]} = \bigcap_{i \in u} A_i^{\eta(i)}.$$

**Definition 4.3.** *Let $g \in {}^\omega\omega$ be strictly increasing and $g(n) > n$.*

(1) *We say $A \in [\omega]^\omega$ is $g$-slow if $(\exists^\infty n)|A \cap g(n)| \geq n$.*

(2)

$$\mathcal{F}_g = \{f : \mathrm{dom}(f) \in [\omega]^\omega, \text{ for } i \in \mathrm{dom}(f) \text{ we have that } f(i) = (f^1(i), f^2(i)) \text{ and } f^2(i) \in [g(f^1(i))]^{\geq f^1(i)} \text{ and } \limsup\langle f^1(i) : i \in \mathrm{dom}(f)\rangle = \omega\}.$$

(3) *We say that a sequence $\bar{A} = \langle A_i : i < \kappa\rangle$ of infinite subsets of $\omega$ is $(g,\kappa)$-o.k. if*

$$\text{if } k < \omega, f_0, \dots, f_{k-1} \in \mathcal{F}_g, \bigcap_{\ell \in k} \mathrm{dom}(f_\ell) = B \in [\omega]^\omega$$

$$\text{and } \limsup\langle \min\{f_\ell^1(i) : \ell \in k\} : i \in B\rangle = \omega,$$

*then for some $\alpha = \alpha(\langle f_\ell : \ell < k\rangle)$ we have that:*

*For every $u_\ell \in [\kappa \setminus \alpha]^{<\omega}$ and $\eta_\ell \in {}^{u_\ell}2$ the set*

$$\{n \in B : (\forall \ell < k)(f_\ell^2(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset)\} \text{ is infinite.}$$

Remarks: $f \in \mathcal{F}_g$ implies that $\bigcup_{i \in \mathrm{dom}(f)} f^2(i)$ is $g$-slow. If $g \leq g'$ then $\mathcal{F}_g \subseteq \mathcal{F}_{g'}$.

**Claim 4.4.** *We get an equivalent notion to "$\bar{A}$ is $(g,\kappa)$-o.k.", if we modify the Definition 4.3(3) as in (a) and/or as in (b), where*

(a) *We demand 4.3(3) only for $f_\ell \in \mathcal{F}_g$ that additionally satisfy $\mathrm{dom}(f_0) = \dots = \mathrm{dom}(f_{k-1}) = \omega$.*

(b) *We demand 4.3(3) only for $f_0, \dots, f_{k-1} \in \mathcal{F}_g$ such that $\langle \min\{f_\ell^1(i) : i < k\} : i < B\rangle$ is strictly increasing (we can even demand, increasing faster than any given $h$), and for $i \in B$, $\max\{f_\ell^1(i) : \ell < k\} < \min\{f_\ell^1(i+1) : \ell < k\}$.*

*Proof.* (a) Suppose the $f_0, \dots, f_{k-1} \in \mathcal{F}_g$ in the original sense, and that we have required the analogue of 4.3(3) only for $\mathcal{F}_g$ in the restricted sense. We suppose that $\bigcap_{\ell<k} \mathrm{dom}(f_\ell) = B$ and take a strictly increasing enumeration $\{b_r : r \in \omega\}$ of $B$. Then we take $\tilde{f}_\ell \colon \omega \to [\omega]^{<\omega}$, $\tilde{f}_\ell(r) = f_\ell(b_r)$ for $r \in \omega$. The analogue of 4.3(3) for the $\mathcal{F}_g$ in the restricted sense gives $\alpha \in \kappa$ and infinite intersections in 4.3(3) for the $\tilde{f}_\ell$. The intersections are also infinite for the original $f_\ell$.

(b) Suppose that $k < \omega$, $f_0, \dots, f_{k-1} \in \mathcal{F}_g$, $\bigcap_{\ell \in \omega} \mathrm{dom}(f_\ell) = B \in [\omega]^\omega$ and $\limsup \langle \min\{f^1_\ell(i) : \ell \in k\} : i \in B\rangle = \omega$. Then we can thin out the domain $B$ to some infinite $B'$, inductively on $i$ such that the $f_\ell \restriction B'$ fulfil all the requirements from 4.4(b).

The following lemma describes the combinatorics that is used in the final model:

**Lemma 4.5.** *Let $g \in {}^\omega\omega$. If $\mathfrak{r} < \kappa = \mathrm{cf}(\kappa)$ and if there is some $\bar{A}$ that is $(g, \kappa)$-o.k., then every real that is computable in every function $g' \geq^* g$ is weakly needed for the refining relation.*

*Proof.* Let $\mathcal{R} = \{B_\alpha : \alpha < |\mathcal{R}|\}$ be a refining family of size $\mathfrak{r} < \kappa$. Since the family $\bar{A}$ is refined by $\mathcal{R}$, for every $i < \kappa$ there are some $\alpha_i < |\mathcal{R}|$ and $\nu(i) \in \{0, 1\}$ such that $B_{\alpha_i} \subseteq A_i^{\nu(i)}$. Since $\kappa$ is regular and since $\mathfrak{r} < \kappa$, there are some $\ell < 2$ and some $\beta < |\mathcal{R}|$ such that

$$Y = \{i < \kappa : \nu(i) = \ell \wedge \alpha_i = \beta\}$$

is unbounded. So $B_\beta \subseteq \bigcap_{i \in Y} A_i^\ell$. We claim that $B_\beta$ is not $g$-slow. Why? Otherwise we have $C = \{n < \omega : |B_\beta \cap g(n)| \geq n\} \in [\omega]^\omega$. We take a partial function $f = (f^1, f^2)$ with $C = \mathrm{dom}(f)$, $f^1(n) = n$ and $f^2(n) = B_\beta \cap g(n)$. Then $f \in \mathcal{F}_g$. Now let $\alpha \in \kappa$ be given. Then we take $u_0$ such that $u_0 = \{\gamma\}$, $\gamma \in Y$, $\gamma > \alpha$ and $\eta_0 = \{(\gamma, 0)\}$ and $\eta'_0 = \{(\gamma, 1)\}$. Then we do not have $(\exists^\infty n) f^2(n) \cap A^0_\gamma \neq \emptyset$ and $(\exists^\infty n) f^2(n) \cap A^1_\gamma \neq \emptyset$ at the same time, because $B_\beta$ is refining $A^\ell_\gamma$. So $\bar{A}$ is not $(g, \kappa)$-o.k., in contrast to our assumption.

But now we can compute recursively from $B_\beta$ some $g' \geq^* g$, for example we may take $g'(n) =$(the $n$th element of $B_\beta$) $+1$. Hence every real that is computable in every function $g' \geq^* g$ is recursive in $B_\beta$. □

Now we show that there is a version of Lemma 4.5 that works simultaneously for all hyperarithmetic reals in $V$.

**Lemma 4.6.** *There is some $g \colon \omega \to \omega$ such that every hyperarithmetic real is computable in any $g' \geq g$.*

*Proof.* For any number $e \in \omega$ for a Turing machine take a real $r_e$ and a lower bound $g_e \in {}^\omega\omega$ such that for all $g' \geq g_e$, $e$ computes $r_e$ with the oracle $g'$, if there are such $r_e, g_e$. Now take $g$ eventually dominating all the $g_e$, $e \in \omega$. □

We will find $\bar{A}$ that is $(g, \kappa)$-o.k. in a forcing extension. However, the construction works only for $g \in V$. So the constellation in which we use Lemmata 4.5 and 4.6 is as follows:

**Corollary 4.7.** *Let $g \in V$ be as in Lemma 4.6 in $V$. If in $V[G]$, $\mathfrak{r} < \kappa = \mathrm{cf}(\kappa)$ and there is some $\bar{A}$ that is $(g, \kappa)$-o.k., then every hyperarithmetic real in $V$ is in $V[G]$ weakly needed for the refining relation.*

So, how do we get a c.c.c. forcing extension in which $\mathfrak{r} < \kappa = \mathrm{cf}(\kappa)$ and in which there is some $\bar{A}$ that is $(g, \kappa)$-o.k.? The rest of this section will be devoted to this issue. We consider the case $\kappa = \mathrm{cf}(\kappa) > \aleph_1$ and intend to show that for every $g$ it is consistent that "$\mathfrak{r} = \aleph_1$ and there is some $\bar{A}$ that is $(g, \kappa)$-o.k." The construction works for any fixed $g \in V$. It is open whether a statement like "for all $g \in V[G]$, there is some $\bar{A}_g$ that is $(g, \kappa)$-o.k. and $\mathfrak{r} < \kappa = \mathrm{cf}(\kappa)$" is consistent.

We give a sketch of the construction in the consistency proof. We first add $\kappa$ Cohen reals to some ground model where there are at most $\kappa$ reals. We show that from these we get some $\bar{A}$ that is $(g, \kappa)$-o.k. for all $g$ simultaneously. The next step is to extend further, in $\aleph_1$ steps, so that along this iteration a refining family of size $\aleph_1$ is added. The lengthy work is to show that we can find an extension such that $\bar{A}$ stays $(g, \kappa)$-o.k. for one chosen $g$. This is not trivial because $\mathcal{F}_g$ is enlarged.

**Definition 4.8.** *(1)* $K_g = K = \{(P, \underset{\sim}{\bar{A}}) : P$ *is a ccc forcing and* $\Vdash_P$ *"*$\underset{\sim}{\bar{A}}$ *is* $(g, \kappa)$*-o.k."*$\}$. *For a fixed g, we often leave out the subscript.*

*(2)* $(P_1, \underset{\sim}{\bar{A}}_1) \leq_K (P_2, \underset{\sim}{\bar{A}}_2)$ *iff* $P_1 \lessdot P_2$ *and* $\underset{\sim}{\bar{A}}_1 = \underset{\sim}{\bar{A}}_2$.

We really mean the same names $\underset{\sim}{\bar{A}}_i$ not just the same interpretations. Indeed we think of a finite support iteration $\langle P_\alpha, Q_\beta : \beta < \aleph_1, \alpha \leq \aleph_1\rangle$ giving the $P_\alpha$'s. But we formulated 4.8 a bit more general, because also in the next claim the $Q_\beta$'s do not appear.

**Claim 4.9.** *(1)* *We have that* $K \neq \emptyset$. *In fact, if* $P$ *is the forcing adding* $\kappa$ *Cohen reals and* $\underset{\sim}{\bar{A}}$ *is the enumeration of the* $\kappa$ *Cohen reals, then* $(P, \underset{\sim}{\bar{A}}) \in K_g$ *for any function* $g \in V$.

*(2)* *If* $(P_\alpha, \underset{\sim}{\bar{A}}) \in K_g$ *for* $\alpha < \delta$, $\delta$ *a limit cardinal, and* $\langle P_\alpha : \alpha < \delta\rangle$ *is increasing and continuous w.r.t. the complete embedding relation, and* $P = \bigcup_{\alpha<\delta} P_\alpha$, *and* $P$ *has the c.c.c., then* $(P, \underset{\sim}{\bar{A}}) \in K_g$ *and* $\alpha < \delta \Rightarrow (P_\alpha, \underset{\sim}{\bar{A}}) \leq_K (P, \underset{\sim}{\bar{A}})$.

*Proof.* (1) Suppose that $f_0, \dots, f_{k-1} \in V[G_\kappa]$ are injective functions. We take $\alpha$ such that $f_0, \dots, f_{k-1} \in V[G_\alpha]$ where $G_\alpha$ is a generic filter for the first $\alpha$ Cohen reals. Suppose that $\eta_\ell \in {}^{u_\ell}2$, $u_\ell \subseteq \kappa \setminus \alpha$. Now a density argument gives that these $\bar{A}^{[\eta_\ell]}$ "flip for infinitely many $n \in B$" to 0 or to 1 within $f^2_\ell(n)$ for every $\ell < k$.

(2) Now we show that $\Vdash_P$ "$\underset{\sim}{\bar{A}}$ is $(g, \kappa)$-o.k.". Only the case of $\mathrm{cf}(\delta) = \omega$ is not so easy. We suppose that $\delta = \bigcup_{n\in\omega} \alpha(n)$, $0 < \alpha(n) < \alpha(n + 1)$. Towards a contradiction we assume that $p^* \in P_\delta$, and

$$p^* \Vdash_P \text{"}\underset{\sim}{B}, \langle \underset{\sim}{f_\ell} : \ell < k\rangle \text{ form a counterexample to } \underset{\sim}{\bar{A}} \text{ being } (g, \kappa)\text{-o.k."}$$

For each $n \in \omega$ we find $\langle q_{n,i} : i \in \omega\rangle$ such that

($\alpha$) $q_{n,i} \in P$,

($\beta$) $q_{n,0} = p^*$,

($\gamma$) $P \models q_{n,i} \leq q_{n,i+1}$,

($\delta$) for some $\underset{\sim}{b}_{n,i}$, $\underset{\sim}{f}^1_{n,\ell,i}$, $\underset{\sim}{f}^2_{n,\ell,i}$ $P_{\alpha(n)}$-names we have

$$q_{n,i} \Vdash \text{"}\underset{\sim}{b}_{n,i} \text{ is the } i\text{-th member of } \underset{\sim}{B}, \underset{\sim}{f_\ell}(\underset{\sim}{b}_{n,i}) = (\underset{\sim}{f}^1_{n,\ell,i}, \underset{\sim}{f}^2_{n,\ell,i})\text{"},$$

($\varepsilon$) $q_{n,i} \restriction \alpha(n) = q_{n,0} \restriction \alpha(n) = p^* \restriction \alpha(n)$.

How do we choose these? Let $n$ and $\alpha(n)$ be given. Then we choose $q'_{n,i}$ increasing in $i$ such that $q'_{n,i} \in P$ and $b'_{n,i}$, $(f^1)'_{n,i}$, $(f^2)'_{n,\ell,i}$ in $V$ and

$$q'_{n,i} \Vdash \bigwedge_{\ell<k} \text{ the } i\text{th element of } \underset{\sim}{B} = \check{b'_{n,i}} \wedge \underset{\sim}{f}_\ell(\check{b'_{n,i}}) = ((\check{f^1)'_{n,\ell,i}}, (\check{f^2)'_{n,\ell,i}}).$$

Then we take

$$\begin{aligned} \underset{\sim}{b}_{n,i} &= (b'_{n,i}, q'_{n,i} \restriction P_{\alpha(n)}), \\ \underset{\sim}{f}^1_{n,\ell,i} &= ((f^1)'_{n,\ell,i}, q'_{n,i} \restriction P_{\alpha(n)}), \\ \underset{\sim}{f}^2_{n,\ell,i} &= ((f^2)'_{n,\ell,i}, q'_{n,i} \restriction P_{\alpha(n)}), \\ q_{n,i} &= p^* \restriction \alpha(n) \cup q'_{n,i} \restriction [\alpha(n), \delta). \end{aligned}$$

Here, the restriction $\restriction \alpha$ is any reduction function witnessing $P_\alpha \lessdot P$ (see [1]), and in the general case, if $P_\alpha$ is not the initial segment of length $\alpha$ of some iteration, the term $q'_{n,i} \restriction [\alpha(n), \delta)$ has to be interpreted as some element from a quotient forcing algebra.

Now for every $n$ we define $P_{\alpha(n)}$-names

$$\begin{aligned} &\underset{\sim}{B}'_n = \{\underset{\sim}{b}_{n,i} : i < \omega\}, \\ &\underset{\sim}{f}_{\ell,n} \colon \underset{\sim}{B}'_n \to V, \\ &\underset{\sim}{f}_{\ell,n}(\underset{\sim}{b}_{n,i}) = (\underset{\sim}{f}^1_{\ell,n}(\underset{\sim}{b}_{n,i}), \underset{\sim}{f}^2_{\ell,n}(\underset{\sim}{b}_{n,i})) = (\underset{\sim}{f}^1_{\ell,n,i}, \underset{\sim}{f}^2_{\ell,n,i}). \end{aligned}$$

Now we have that

$p^* \Vdash$ "$\underset{\sim}{B}'_n \in [\omega]^{\aleph_0}$, $\underset{\sim}{f}_{\ell,n}$ is a function with domain $\underset{\sim}{B}'_n$ and
$\limsup \langle \underset{\sim}{f}^1_{\ell,n}(b) : b \in \underset{\sim}{B}'_n \rangle = \omega$ and
$\underset{\sim}{f}^2_{\ell,n,i}$ when defined is a subset of $[0, g(\underset{\sim}{f}^1_{\ell,n,i}))$ of cardinality $\geq \underset{\sim}{f}^1_{\ell,n,i}$".

As $(P_{\alpha(n)}, \underset{\sim}{\bar{A}})$ is in $K$ we have for every $n$

$$p^* \restriction \alpha(n) \Vdash_{P_{\alpha(n)}} \text{" for some } \underset{\sim}{\beta} < \kappa \text{ for every } u_\ell \subseteq [\kappa \setminus \underset{\sim}{\beta}]^{\aleph_0} \text{ for every } \eta_\ell \in {}^{u_\ell}2$$
$$\left\{ b \in \underset{\sim}{B}'_n : \bigwedge_{\ell<k} \underset{\sim}{f}^2_{\ell,n}(b) \cap \underset{\sim}{\bar{A}}^{[\eta_\ell]} \neq \emptyset \right\} \text{ is infinite."}$$

Let $\underset{\sim}{\beta}_n < \kappa$ be such a $P_{\alpha(n)}$-name. Since $P_{\alpha(n)}$ has the ccc, there is some $\beta^*_n < \kappa$ such that $\Vdash_{P_{\alpha(n)}} \underset{\sim}{\beta}_n < \beta^*_n < \kappa$. Since $\kappa$ is regular we have that $\beta^* = \bigcup_{n\in\omega} \beta^*_n < \kappa$.

It suffices to prove that

$$p^* \Vdash \text{"}\beta^* \text{ is as required in the definition of } (g, \kappa)\text{-o.k."}$$

If not, then there are counterexamples $u_\ell \in [\kappa \setminus \beta^*]^{<\aleph_0}$, $\eta_\ell \in {}^{u_\ell}2$, $q$ and $b^*$ such that

$$\begin{aligned}&p^* \le q \in P = P_\delta\\ &q \Vdash \text{“}\Big\{b \in \underset{\sim}{B} \,:\, (\forall \ell < k)(\underset{\sim}{f}{}^2_\ell(b) \cap \underset{\sim}{\bar{A}}{}^{[\eta_\ell]} \neq \emptyset)\Big\} \subseteq [0, b^*]\text{”}.\end{aligned} \tag{4.1}$$

For some $n(*) < \omega$ we have that $q \in P_{\alpha(n(*))}$. Let $G \subseteq P$ be generic over $V$, and let $q \in G_{\alpha(n(*))}$. So by the choice of $\beta_{n(*)} < \beta^*$ we have that

$$q \Vdash_{P_{\alpha(n(*))}} C = \{b \in \underset{\sim}{B}{}'_{n(*)} \,:\, (\forall \ell < k)(\underset{\sim}{f}{}^2_{\ell,n(*)}(b) \cap \underset{\sim}{\bar{A}}{}^{[\eta_\ell]} \neq \emptyset)\} \text{ is infinite”}.$$

Recall that $\underset{\sim}{B}{}'_{n(*)}$ and $\underset{\sim}{f}_{\ell,n(*)}(b)$ are $P_{\alpha(n(*))}$-names and that $\underset{\sim}{\bar{A}}{}^{[\eta_\ell]}$ is a $P_0$-name.

Now $\underset{\sim}{B}{}'_{n(*)} = \{\underset{\sim}{b}_{n(*),i} \,:\, i < \omega\}$, so for some $i$ we have that $\underset{\sim}{b}_{n(*),i}[G] > b^*$. So $q_{n(*),i} \in \tilde{G} \cap P_{\alpha(n(*))}$ forces “the $i$-th member of $\underset{\sim}{B}$ is $\underset{\sim}{b}_{n(*),i}$ and $\underset{\sim}{f}_\ell(\underset{\sim}{b}_{n(*),i}) = \underset{\sim}{f}_{\ell,n(*)}(\underset{\sim}{b}_{n(*),i}) = (\underset{\sim}{f}{}^1_{\ell,n(*),i}, \underset{\sim}{f}{}^2_{\ell,n(*),i})$. Note that $q_{n(*),i} \restriction \alpha(n(*)) = p^* \restriction \alpha(n(*))$ according to $\varepsilon$), and hence $q_{n(*),i} \not\perp q$. So there is some $r \ge q$ and $r \ge q_{n(*),i}$. Such an $r$ forces the contrary of the property forced in (4.1), and finally we reached a contradiction. □

The conclusion of the next claim is a strengthening of 4.3(3). Let $D$ be an ultrafilter on $\omega$. Instead of “infinite” we require “being in $D$”. Since ultrafilters are closed under finite intersections we need to mention only one function in $\mathcal{F}_g$.

Claims 4.10 and 4.11 are like [10]. For $h\colon \omega \to \omega$ we write $\lim_D \langle h(i) \,:\, i \in \omega\rangle = \omega$ if for all $m < \omega$ we have that $\{i \,:\, h(i) > m\} \in D$.

**Claim 4.10.** *Assume that in $V$:*

*(a)* $\bar{A}$ *is $(g,\kappa)$-o.k.*

*(b)* $\kappa = 2^{\aleph_0}$ *is regular.*

*Then there is an ultrafilter $D$ on $\omega$ such that*

$$\begin{aligned}&\textit{if } f \in \mathcal{F}_g \textit{ and } \operatorname{dom}(f) \in D \textit{ and } \lim_D\langle f^1(n) \,:\, n \in \operatorname{dom}(f)\rangle = \omega\\ &\textit{then for some } \alpha_f < \kappa \textit{ for every } u \in [\kappa \setminus \alpha_f]^{<\aleph_0} \textit{ and } \eta \in {}^u2\\ &\textit{we have that } \{n \in \operatorname{dom}(f) \,:\, f^2(n) \cap \bar{A}^{[\eta]} \neq \emptyset\} \in D.\end{aligned} \tag{4.2}$$

*Proof.* The following is a mock forcing argument. We work with the partial order $\mathcal{AP}$, which is $< \kappa$-closed. We have to meet only $\kappa$ dense sets. So, by taking one union over $\kappa$ conditions in the end, we find a generic in $V$. Let $\mathcal{F}_g = \{f_j \,:\, j < \kappa\}$. Let $\mathcal{AP}$ be the set of tuples $(D, i, \alpha)$ such that

(i) $D$ is a filter on $\omega$ containing the co-finite subsets, $\emptyset \notin D$, $i, \alpha < \kappa$,

(ii) $D$ is generated by $< \kappa$ members,

(iii) if $k < \omega$ and for $\ell < k$, $j_\ell < i$, and $\operatorname{dom}(f_{j_\ell}) \in D$ and $\lim_D\langle f^1_{j_\ell}(n) \,:\, n \in \operatorname{dom}(f_{j_\ell})\rangle = \omega$ and $u_\ell \in [\kappa \setminus \alpha]^{<\aleph_0}$, $\eta_\ell \in {}^{u_\ell}2$, then

$$\Big\{n \in \bigcap_{\ell<k} \operatorname{dom}(f_{j_\ell}) \,:\, \bigwedge_{\ell<k} \Big(f^2_{j_\ell}(n) \cap \bar{A}^{[\eta]} \neq \emptyset\Big)\Big\} \neq \emptyset \bmod D.$$

Let $(D_1, i_1, \alpha_1) \leq_{\mathcal{AP}} (D_2, i_2, \alpha_2)$ if both tuples are in $\mathcal{AP}$ and

($\alpha$) $D_1 \subseteq D_2$, $i_1 \leq i_2$, $\alpha_1 \leq \alpha_2$, and

($\beta$) if $k < \omega$ and $\{j_0, \dots, j_{k-1}\} \subseteq i_1$, $\mathrm{dom}(f_{j_\ell}) \in D_2$ and $\lim_{D_2}\langle f^1_{j_\ell}(i) : i \in \mathrm{dom}(f_{j_\ell})\rangle = \omega$ and $u_\ell \subseteq [\alpha_1, \alpha_2)$ is finite and $\eta_\ell \in {}^{u_\ell}2$ then

$$\left\{ n \in \bigcap_{\ell<k} \mathrm{dom}(f_{j_\ell}) : \bigwedge_{\ell<k} f^2_{j_\ell(n)} \cap \bar{A}^{[\eta_\ell]} \neq \emptyset \right\} \in D_2.$$

Now we have that $(\mathcal{AP}, \leq_{\mathcal{AP}})$ is a non-empty partial order. Take $i = \alpha = 0$ and $D$ the filter of all cofinite subsets of $\omega$. In $(\mathcal{AP}, \leq_{\mathcal{AP}})$ every increasing sequence of length $< \kappa$ has an upper bound, namely, take the filter generated by the union in the first coordinate and take the supremum in the second and in the third coordinate.

Now we come to the first kind of sets we want to meet: If $B \subseteq \omega$ and $(D, i, \alpha) \in \mathcal{AP}$ then there are some $D'$, $i'$, $\alpha'$ such that $(D', i', \alpha') \geq_{\mathcal{AP}} (D, i, \alpha)$ and that $B \in D'$ or that $\omega \setminus B \in D'$. Why? Try $D'$ = the filter generated by $D \cup \{B\}$ and the same $i$ and $\alpha$. If this fails then we can find $k < \omega$, such that for $\ell < k$ we have $j_\ell < i$, such that $\mathrm{dom}(f_{j_\ell}) \in D'$ and $\lim_{D'}\langle f^1_{j_\ell}(i) : i \in \mathrm{dom}(f_{j_\ell})\rangle = \omega$, $u_\ell \in [\kappa \setminus \alpha]^{<\aleph_0}$, $\eta_\ell \in {}^{u_\ell}2$ and such that

$$\left\{ n \in \bigcap_{\ell<k} \mathrm{dom}(f_{j_\ell}) : f^2_{j_\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset \right\} \cap B = \emptyset \bmod D.$$

Let $\alpha' < \kappa$ be such that $\alpha \leq \alpha'$ and $\bigwedge_{\ell<k} u_\ell \subseteq \alpha'$. Let $D'$ be the filter generated by

$$D \cup \left\{ \left\{ n \in \bigcap_{\ell<k} \mathrm{dom}(f_{j_\ell}) : f^2_{j_\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset \right\} : k < \omega, j_\ell < i, u_\ell \in [\alpha' \setminus \alpha]^{<\aleph_0}, \eta_\ell \in {}^{u_\ell}2 \right\}.$$

Then $\omega \setminus B \in D'$, and $(D', i, \alpha') \in \mathcal{AP}$.

Finally, there is a second useful kind of dense sets: If $(D, i, \alpha) \in \mathcal{AP}$ then for some $D'$, $\alpha'$ we have that $(D', i+1, \alpha') \in \mathcal{AP}$.

*Proof.* Let $M \prec (H(\chi), \in)$ such that $M \cap \kappa \in \kappa$, $(D, i, \alpha) \in M$, $\mathcal{F}_g \in M$, and $|M| < \kappa$. Suppose that $\mathrm{dom}(f_i) \in D$ and that $\lim_D\langle f^1_i(k) : k \in \mathrm{dom}(f_i)\rangle = \omega$. Let $\alpha' = M \cap \kappa$. Let $D_1$ be the filter in the boolean algebra in $\mathcal{P}(\omega) \cap M$ generated by

$$(D \cap M) \cup \left\{ \left\{ n \in \bigcap_{\ell<k} \mathrm{dom}(f_{j_\ell}) : f^2_{j_\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset \right\} : k < \omega, j_\ell \leq i, u_\ell \in [\alpha' \setminus \alpha]^{<\aleph_0}, \eta_\ell \in {}^{u_\ell}2 \right\}.$$

Since in $M$, $\bar{A}$ is $(g,\kappa)$-o.k., this has the infinite intersection property. Let $D_2'$ be an ultrafilter in $M$ extending $D_1$. Let $D'$ be the filter on $\omega$ in $V$ that $D_2'$ generates.

Now we take an increasing chain $\langle (D^j, i^j, \alpha^j) : j < \kappa\rangle$ in the partial order $(\mathcal{AP}, \leq_{\mathcal{AP}})$ such that $i^j$ is unbounded in $\kappa$ and $D := \bigcup_{j<\kappa} D^j$ is an ultrafilter. Then $D$ fulfills (4.2). □

Now we use equation (4.2) of 4.7, which implies that $\bar{A}$ is $(g,\kappa)$-o.k., to construct an extension in which $\bar{A}$ is still $(g,\kappa)$-o.k. The following preservation theorem is a bit more general: it works also when the $D_\eta$'s do not coicide. In our application, however, they will coincide.

**Claim 4.11.** *Assume that*

(a) $\bar{A}$ *is* $(g,\kappa)$*-o.k.*

(b) $\bar{D} = \langle D_\eta : \eta \in {}^{<\omega}\omega\rangle$, $D_\eta = D$, $D$ *is an ultrafilter on* $\omega$ *as in 4.7.*

(c) $Q_{\bar{D}} = \{T : T \subseteq {}^{<\omega}\omega$ *is a subtree with exactly one* $\trianglelefteq$*-minimal element, and for some* $\eta \in T, \eta \trianglelefteq \nu \in T \Rightarrow \{k : \nu\hat{\ }k \in T\} \in D_\nu\}$*, ordered by inverse inclusion. (The* $\trianglelefteq$*-minimal* $\eta$ *of this sort is called the trunk of* $T$, $\mathrm{tr}(T)$*.)*

*Then* $\Vdash_{Q_{\bar{D}}}$ *"*$\bar{A}$ *is* $(g,\kappa)$*-o.k.".*

*Proof.* We use the fact [10] that $Q_{\bar{D}}$ has the pure decision property: Let $\varphi_i$, $i \in \omega$, be countably many sentences of the $Q_{\bar{D}}$-forcing language. We think of names $\underset{\sim}{f}_\ell$, $\ell < k$, for some elements of $\mathcal{F}_g$ and $\varphi_i =$ "$\Big($the $i$-th element of $\underset{\sim}{B} = \bigcap_{\ell<k} \mathrm{dom}(\underset{\sim}{f}_\ell)\Big) = \check{b}_i$ and $\bigwedge_{\ell<k} \underset{\sim}{f}_\ell(\check{b}_i) = (\check{f}^1_{\ell,i}, \check{f}^2_{\ell,i})$". The pure decision property says:

$$\forall p \in Q_{\bar{D}}\ \exists q \geq_{tr} p\ \forall r \geq q\ \forall i\ \Big(r \Vdash \varphi_i \rightarrow (\exists s_i \in r) q^{[s_i]} \Vdash \varphi_i\Big),$$

where we write $\geq_{tr}$ for the pure extension: $q \leq_{tr} r$ if $r \subseteq q$ and $\mathrm{tr}(q) = \mathrm{tr}(r)$, and $q^{[s_i]} = \{\eta \in q : s_i \trianglelefteq \eta\}$.

Towards a contradiction we assume that there is a counterexample. By Claim 4.4 (first (b) and then (a)) we may assume that it is of the following form

$$\begin{aligned} p^* \Vdash \text{"}&\langle \underset{\sim}{f}_\ell : \ell < k\rangle \text{ are functions from } \omega \text{ to } \omega \\ &\text{and for } i \in \omega, \max\{\underset{\sim}{f}^1_\ell(i) : \ell < k\} < \min\{\underset{\sim}{f}^1_\ell(i+1) : \ell < k\} \\ &\text{and there is no } \alpha < \kappa \text{ such that the statement} \\ &\text{Definition 4.3(3) holds."} \end{aligned} \tag{4.3}$$

We find $q$ such that

($\alpha$) $q \in P$

($\beta$) $q \geq_{tr} p^*$,

($\gamma$) for all $i \in \omega$ for all $f^1_{\ell,i} \in \omega$, $f^2_{\ell,i} \subseteq [0, g(f^1_{\ell,i}))$ of size bigger than $f^1_{\ell,i}$ we have that

if $r \geq q, r \Vdash$ "$\underset{\sim}{f}_\ell(\check{i}) = (\check{f}^1_{\ell,i}, \check{f}^2_{\ell,i})$",

then also for some $s_i \in r$, the condition $q^{[s_i]}$ forces the same."

We fix such a $q$.

Now we set for $\nu \in q$ and $\ell < k$

$$B^1_{\nu,\ell} = \{i \in \omega \,:\, \text{ some pure extension of } q^{[\nu]} \text{ decides } \underset{\sim}{f}_\ell(i)\}.$$

We say $(\nu, \ell)$ is 1-good if $B^1_{\nu,\ell} \in D$. Let for $i \in B^1_{\nu,\ell}$, $h_{\nu,\ell}(i) = (h^1_{\nu\ell}, h^2_{\nu,\ell})$ the value of $\underset{\sim}{f}_\ell(i)$ that is given by the pure decision. This is well-defined because any two pure extensions are compatible. Of course, by the requirements we had put on the counterexample, we have that $\lim_D \langle h^1_{\nu,\ell}(i) \,:\, i \in B^1_{\nu,\ell}\rangle = \omega$.

We say that $(\nu, \ell) \in q \times k$ is 2-good, if it is not 1-good and we have for all $m \in \omega$ that

$$M_{\nu,\ell,m} = \{j \in \omega \,:\, (\exists i \in \omega)(h_{\nu\hat{\ }j,\ell}(i)) \text{ is well-defined}, \\ \text{and } h^1_{\nu\hat{\ }j,\ell}(i) > m)\big\} \in D.$$

So, for 2-good but not 1-good $(\nu, \ell)$ we may define for $j \in M_{\nu,\ell,m}$,

$g_{\nu,\ell}(j) = h_{\nu\hat{\ }j,\ell}(i_{\nu\hat{\ }j,\ell})$,

where $i_{\nu\hat{\ }j,\ell}$ is such that $h_{\nu\hat{\ }j,\ell}(i_{\nu\hat{\ }j,\ell})$ is defined in $h^1_{\nu\hat{\ }j,\ell}(i_{\nu\hat{\ }j,\ell}) > m$ and if there is a maximal such $i$, then take this as $i_{\nu\hat{\ }j,\ell}$.

We show that there is $M'_{\nu,\ell,m} \in D$, $M_{\nu,\ell,m} \supseteq M'_{\nu,\ell,m}$ such that for $j \in M'_{\nu,\ell,m}$ there is a maximal such $i$: If $h_{\nu\hat{\ }j,\ell}(i)$ is defined and $i' < i$ then there is some pure extension deciding $h_{\nu\hat{\ }j,\ell}(i')$ since there are only finitely many possibilities for its values, by the third line of (4.3). Hence some pure extension decides the value. Hence also $h_{\nu\hat{\ }j,\ell}(i')$ is defined. If $h_{\nu\hat{\ }j,\ell}(i)$ is defined for all $i$, then $(\nu\hat{\ }j, \ell)$ is 1-good. Hence , if $(\nu\hat{\ }j, \ell)$ is 2-good but not 1-good, then there is a maximal $i$ witnessing $j \in M_{\nu,\ell,m}$. If $\{j \,:\, (\nu\hat{\ }j, \ell)$ is 1-good $\} \in D$, then by gluing together suitable pure extensions $r_j$ of $q^{[\nu\hat{\ }j]}$ together we get a pure extension of $q^{[\nu]}$ that shows that $(\nu, \ell)$ is 1-good. Hence $X = \{j \,:\, (\nu\hat{\ }j$ is 2-good and not 1-good $\} \in D$. So we may take $M'_{\nu,\ell,m} = M_{\nu,\ell,m} \cap X$. In order to simplify notation, we assume that $M'_{\nu,\ell,m} = M_{\nu,\ell,m}$.

Also from the third line of equation (4.3) we get that for every $\nu \in q$ either for all $\ell < k$, $(\nu, \ell)$ is 1-good or no $(\nu, \ell)$ is 1-good. In the latter case there is some $i_\nu$, such that for all $\ell < k$, $\mathrm{dom}(h_{\nu,\ell}) = i_\nu$ or $\mathrm{dom}(h_{\nu,\ell}) = i_\nu + 1$. Moreover, also by (4.3) we get that if for some $\ell < k$, for all $m$, $M_{\nu,\ell,m} \in D$, then for all $\ell < k$, for all $m$, $M_{\nu,\ell,m} \in D$. So if $(\nu, \ell)$ is 2-good, then all $(\nu, \ell')$ are 2-good. We call $\nu$ $i$-good if there is some $\ell$ such that $(\nu, \ell)$ is $i$-good. We set $M_{\nu,m} = \bigcap_{\ell<k} M_{\nu,\ell,m}$.

We fix some pseudo-intersection $M_\nu$ of $\langle M_{\nu,m} \,:\, m \in \omega\rangle$, such that $\lim\langle i_{\nu\hat{\ }j} \,:\, j \in M_\nu\rangle = \omega$.

Then we also have that $\lim_D\langle\min\{g^1_{\nu,\ell}(j) : \ell < k\} : j \in M_\nu\rangle = \omega$, because for each $z < \omega$, $\{j : \min\{g^1_{\nu,\ell}(j) : \ell < k\} < z\}$ is a finite set. Hence $g_{\nu,\ell} \in \mathcal{F}_g$. By combining with an enumeration of $M_\nu$, we may assume that $\mathrm{dom}(g_{\nu,\ell}) = \omega \in D$. We will not write this enumeration, in order to prevent too clumsy notation, but we shall later apply that $D$ is as in 4.7 for $\mathcal{F}_g$, and therefore we need that the domains are in $D$.

Now we take $\chi$ sufficiently large and $N \prec (H(\chi), \in)$ such that $\langle \underset{\sim}{f}_\ell : \ell < k\rangle \in N$, $\langle B^1_{\nu,\ell}, h_{\nu,\ell}, g_{\nu,\ell} : \nu \in q, \ell < k\rangle \in N$, $q, D \in N$. We take $\alpha^* = \sup(N \cap \kappa)$. We claim that $q$ forces that $\alpha^*$ is as in the Definition 4.3(3).

If not, then there are counterexamples $u_\ell \in [\kappa \setminus \alpha^*]^{<\aleph_0}$ and $\eta_\ell \in {}^{u_\ell}2$ and $r \in Q_D$, $r \geq q$, and $b^*$ such that

$$
\begin{aligned}
&r \geq q, \text{ and}\\
&r \Vdash_{Q_D} \text{“} \bigcap_{\ell<k} \mathrm{dom}(\underset{\sim}{f}_\ell) = \omega \text{ and}\\
&(\forall i \in \omega) \max\{\underset{\sim}{f}^1_\ell(i) : \ell < k\} < \min\{\underset{\sim}{f}^1_\ell(i+1) : \ell < k\}\\
&\text{and } \Big\{b \in \omega : (\forall \ell < k)(\underset{\sim}{f}^2_\ell(b) \cap \underset{\sim}{\bar{A}}^{[\eta_\ell]} \neq \emptyset)\Big\} \subseteq [0, b^*]\text{”}.
\end{aligned} \tag{4.4}
$$

First case: There is some $\nu \in r$ with $\mathrm{tr}(r) \trianglelefteq \nu$ such that all $(\nu, \ell)$ are 1-good. Now we take for each $t \in \omega$, some pure extension of $q^{[\nu]}_t$ of $r^{[\nu]}$ such that it forces $\bigwedge_{\ell<k}(h_{\nu,\ell} \restriction t = \underset{\sim}{f}_\ell \restriction t)$. Since $\bar{A}$ is $(g,\kappa)$-o.k., and since all is reflected to $N$, and by the choice of $\alpha^*$, we have that $I = \{n \in \omega : (\forall \ell < k)(h^2_{\nu,\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\}$ is infinite. So we take $t \in I$ such that $t > b^*$. Now $q^{[\nu]}_t$ contradicts (4.4).

Second case. There is some $\nu \in r$ such that all $\nu$, $\ell < k$ are 2-good but not 1-good. We set $g_{\nu,\ell}(j) = h_{\nu\hat{\ }j,\ell}(i_{\nu\hat{\ }j,\ell})$ as purely decided above $q^{[\nu\hat{\ }j]}$. Fact: Now $\langle g_{\nu,\ell} : \ell < k\rangle$ is as required in the definition of $\bar{A}$ being $(g,\kappa)$-o.k., because $\omega = \lim_D\langle g^1(i_{\nu\hat{\ }j}) : j \in \omega\rangle$.

Now we take for each $t \in \omega$, some pure extension $q^{[\nu\hat{\ }j]}_t$ of $r^{[\nu\hat{\ }j]}$ such that it determines $\bigwedge_{\ell<k} g_{\nu,\ell} \restriction t$. Since $\bar{A}$ is $(g,\kappa)$-o.k., and since all is reflected to $N$, and by the choice of $\alpha^*$, we have that $J = \{n \in \omega : (\forall \ell < k)(g^2_{\nu,\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\}$ is infinite. Then also $\hat{J} = \{i_{\nu\hat{\ }n} : n \in J\}$ is infinite. So we take $t > b^*$, $t \in \hat{J}$. Now the gluing together of $q^{[\nu\hat{\ }j]}_t$, $j \in \bigcap_{\ell<k} M_{\nu,\ell,t}$, contradicts (4.4) because we have $g_{\nu,\ell}(j) = h_{\nu\hat{\ }j,\ell}(i_{\nu\hat{\ }j,\ell}) = f_\ell(i_{\nu\hat{\ }j})$, if $q^{[\nu]}_t \in G$. Here we write $f_\ell$ for $\underset{\sim}{f}_\ell[G]$.

Third case: All $\nu \in r$ are neither 1-good nor 2-good. We shall prove something stronger:

> An end-segment of the generic $\bigcup\{\eta :$ there is some element $q \in G$ with trunk $\eta\}$ can be thinned out (so that still infinitely many points are left) and injected into an infinite subset of $\{n \in \omega : \bigwedge_{\ell<k} \underset{\sim}{f}^2_\ell[G](n) \cap A^{[\eta_\ell]} \neq \emptyset\}$.

This is more than enough.

Let $i_{\nu,\ell} = \max(B^1_{\nu,\ell}) < \omega$, because $(\nu, \ell)$ is neither 1-good nor 2-good. Let $i^*_\nu = \mathrm{dom}(h_{\nu,\ell})$ such that $i^*_\nu = i^*_{\nu,\ell}$ or $i^*_\nu = i^*_{\nu,\ell} + 1$. By the premise (4.3), there are such $i^*_\nu$. There is $r \geq q$ with no $\nu \in r$ being 1-good or 2-good in $N$. Without loss of generality, we take $q$ like that. Now we try to shrink $q$ purely. Let $\nu_0 = \mathrm{tr}(q)$.

First: We have that $\underset{\sim}{f_\ell} \restriction i^*_\nu$ is decided by $q$. The range of $\langle i^*_{\nu\hat{}j} : \nu\hat{}j \in q\rangle$ is bounded modulo $D$ because $\nu$ is not 2-good. Hence we may assume that there is just one value $i^{**}_\nu$. So say (after shrinking $q$) that it is constant with value $i^{**}_\nu \geq i^*_\nu$.

Second we have that $\nu_0 \trianglelefteq \nu \in q$ implies that $q^{[\nu]}$ decides $\underset{\sim}{f_\ell} \restriction i^{**}_\nu$.

Third we have that if $i \in [i^*_\nu, i^{**}_\nu]$ then $\lim_D \langle f^1_{\nu\hat{}j,\ell}(i) : j \in \omega\rangle = \omega$ by the definition of $i^*_\nu$ and $i^{**}_\nu$. So define $g_{\nu,\ell,i}$ by $g_{\nu,\ell,i}(j) = h_{\nu\hat{}j,\ell}(i)$. So $g_{\nu,\ell,i} \in N$ is a function of the right form.

We have by the definition of $\alpha^*$, for all $i \in [i^*_\nu, i^{**}_\nu)$ for all $\nu \in q$ for all $u_\ell$, $\eta_\ell$ that

$$A := \{b : (\forall \ell < k) g^2_{\nu,\ell,i}(b) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\} \in D.$$

Since the range of $\bigcup\{\eta$ : there is some element $q \in G$ with trunk $\eta\} =: \eta_\omega$ is eventually contained in every set in $D$, we now find the following infinite set: We take $\langle \eta_n : n \in \omega\rangle$ such that $\eta_n \in \mathrm{range}(\eta_\omega) \cap A$ and such that $i^{**}_{\eta_n} < i^*_{\eta_{n+1}}$. We set $\xi_n = \eta_n \restriction |\eta_n - 1|$. Then we have for almost all $n$ such that $\xi_n \in A$ and hence for all $i \in [i^*_{\xi_n}, i^{**}_{\xi_n})$: $g_{\xi_n,\ell,i}(\eta_n(|\eta_n - 1|)) = h_{\xi_n\hat{}\eta_n(|\eta_n-1|),\ell}(i) = h_{\eta_n,\ell}(i) = f_\ell(i)$. So $\bigcup_{n\in\omega}[i^*_{\xi_n}, i^{**}_{\xi_n}] \subseteq^* \{b : (\forall \ell < k) f^2_\ell(b) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\}$ is infinite. □

**Claim 4.12.** *Let $\kappa = \mathrm{cf}(\kappa) > \omega_1$. Let $V \models 2^\omega \leq \kappa$ and let $P_0 = \mathbb{C}_\kappa$ be the forcing adding $\kappa$ Cohen reals. We fix some function $g \in V$ such that every hyperarithmetic function is computable in every $g' \geq g$. We let $G_0$ be $P_0$-generic over $V$ and set $V_1 = V[G_0]$. Let in $V_1$, $\bar{A}$ be the enumeration of the $\kappa$ Cohen reals.*

(1) *In $V_1$, there is $(P, \bar{A}) \in K_g$ such that $\Vdash_P$ "$\mathfrak{r} = \aleph_1$".*

(2) *For $(P, \bar{A})$ as in (1), we have that in $V_1$, $\Vdash_P$ "every hyperarithmetic real in $V$ is weakly needed for the reaping relation".*

*Proof.* (1) By 4.7 we have that $\bar{A}$ is $(g, \kappa)$-o.k. in $V_1$. According to 4.9, we may choose in $V_1$ a $\lessdot$-increasing sequence such that $(P_i, \bar{A}) \in K$, $P_{i+1} = P_i * Q_{\underset{\sim}{\bar{D}}^i}$ and $\langle P_j, Q_i : i < \aleph_1, j \leq \aleph_1\rangle$ is a finite support iteration and $\underset{\sim}{D}^i = \langle \underset{\sim}{\bar{D}}^i_\eta : \eta \in {}^{<\omega}\omega\rangle$ $\bar{D}^i_\eta = D^i \in V^{P_i}$ enjoy the properties required there.

Then $P$ forces that $\mathfrak{r} = \aleph_1$: $P$ consecutively adds ("shoots") $\aleph_1$ reals through the ultrafilters $D^\alpha$ in the intermediate models $\mathbf{V}[G_\alpha]$, $\alpha < \omega_1$. Let $f \in V^P$ be a real. Then, by the countable chain condition, there is some $\alpha < \omega_1$ such that $f \in V^{P_\alpha}$. Then $f^{-1}[\{0\}] \in D^\alpha$ or $f^{-1}[\{1\}] \in D^\alpha$. Since $Q_{\bar{D}^\alpha}$ adds a real $r_\alpha$ that is almost a subset of every member of $D^\alpha$, we have that $r_\alpha$ refines $f$. Hence in $V^P$, $\{r_\alpha : \alpha < \omega_1\}$ witnesses $\mathfrak{r} = \aleph_1$.

(2) Now by part (1) and by Lemma 4.6 the proof of (2) follows. □

Finally, taking $P = P_0 * \underset{\sim}{P}$ and $G$ $P$-generic over $V$ and $\bar{A}$ as in 4.11, statement (2) yields that in $V[G]$ every hyperarithmetic real from $V$ in weakly needed for the reaping relation, and thus the proof of Theorem 4.1 is finished.

## 5. There may be more weakly needed reals than needed reals

Under CH, or if $||R|| = 2^{\aleph_0}$, the notions "needed for $R$" and "weakly needed for $R$" coincide. In this section, we show that there is some quite simply defined relation $R$ and that there is some model of ZFC in which there are more weakly needed reals for $R$ than needed reals for $R$. The idea is to use the forcing model from the previous section.

**Claim 5.1.** *There is a simply defined relation $R$ for which it is consistent that the notions "weakly needed" and "needed" do not coincide. In fact, in the forcing models $V$, $V[G]$ from the previous section, in $V[G]$ every needed real for $R$ is recursive (hence in $V$), and all the hyperarithmetic (and possibly more) reals are weakly needed for $R$.*

*Proof.* Let $R_0$ be the ordinary reaping relation, which we write for functions on ${}^\omega 2 \times {}^\omega 2$:

$$\eta R_0 \nu \Leftrightarrow \eta, \nu \in {}^\omega 2 \ \wedge \ (\exists^\infty n)\nu(n) = 1 \ \wedge \ \eta \restriction \nu^{-1}\{1\} \text{ is almost constant.}$$

Let $R_1$ be as follows:

$$\eta R_1 \nu \Leftrightarrow \eta, \nu \in {}^\omega 2 \ \wedge \ (\exists^\infty n)\eta(n) = 1 \ \wedge \ (\exists^\infty n)\nu(n) = 1 \ \wedge$$
$$\left\langle \frac{|\nu^{-1}\{1\} \cap \eta^{-1}\{1\} \cap n|}{|\eta^{-1}\{1\} \cap n|} : n \in \omega \right\rangle \text{ converges to } \frac{1}{2}.$$

We set $R = R_0 \cup R_1$ and use $V^P$ from the previous section. There we have that $P = P_0 * \underset{\sim}{Q}$, $P_0$ is the forcing adding $\kappa$ Cohen reals, and $\underset{\sim}{\bar{A}}$ is an enumeration of the names of these Cohen reals, and $Q$ is the iteration described in 4.9. Then in $V^P$ we have that $||R|| \le ||R_0|| = \aleph_1$.

We first show that every hyperarithmetic real $\eta \in V$ is weakly needed for $R$ in this model. We take some $R$-adequate set $\mathcal{R}$ in $V^P$ of power $\aleph_1$. We let

$$Y_\ell = \{i < \kappa : (\exists x \in \mathcal{R})(A_i R_\ell x)\}.$$

So, by the definition of adequate we have that $Y_0 \cup Y_1 = \kappa$. If $|Y_0| = \kappa$, then by the proof of 4.5, we get some $x \in \mathcal{R}$ whose enumeration $f$ with $f(n) = m$ if $m$ is the $n$th element of $x$ is so large in the eventual domination order that the real $\eta$ is computable from it.

We now show that $|Y_1| < \kappa$. Then it follows that $|Y_0| = \kappa$. Towards a contradiction, we assume that $|Y_1| = \kappa$. In the model from the previous section we have that $P = \bigcup_{i<\omega_1} P_i$, $P_0$ adds $\kappa$ Cohen reals $A_\alpha$, $\alpha < \kappa$, $P_i$ increasing and continuous, $P_{i+1} = P_i * Q_{\underset{\sim}{D}_i}$ as there, $P = P_0 * \underset{\sim}{Q}$. We work in $V^{P_0}$. We have that for some $p^* \in Q$ and some $Q$-names $\underset{\sim}{\nu}_i$, $i < \omega_1$,

$$p^* \Vdash_{Q/P_0} |\underset{\sim}{Y}_1| = \kappa \ \wedge \ \underset{\sim}{\mathcal{R}} = \{\underset{\sim}{\nu}_i \,:\, i < \omega_1\}.$$

Let $Y^* = \{\alpha \,:\, \exists p_\alpha \geq p^*, p_\alpha \Vdash_{Q/P_0} \alpha \in \underset{\sim}{Y}_1\}$. By the ccc of $Q/P_0$, we have that $Y^* \in [\kappa]^\kappa$, and for $\alpha \in Y^*$ we choose $p_\alpha$ such that $p^* \leq p_\alpha \Vdash_{Q/P_0}$ "$\alpha \in \underset{\sim}{Y}_1$". So for $\alpha \in \kappa$ we have that $A_\alpha R_1 \nu_{i(\alpha)}$ and hence for a large enough $n^*$ for $\kappa$ many $\alpha \in Y^*$ (without loss of generality, for all $\alpha \in Y^*$) we have $(\forall n \geq n^*)(\frac{|\nu_{i(\alpha)}^{-1}\{1\} \cap A_\alpha \cap n|}{|A_\alpha \cap n|} \in \left[\frac{1}{4}, \frac{3}{4}\right])$. Moreover, there is a $\Delta$-system for the $\operatorname{dom} p_\alpha \in [\kappa \setminus \{0\}]^{<\omega}$ whose root is $u^*$, and we assume that for $\alpha \in Y^*$, $i(\alpha) = i^*$.

So we may assume that for $j \in u^*$ we have that $p_\alpha(j)$ is an object with trunk $\rho_j$ and not just a $P_0$-name. By pure decidability for some $\nu^* \in V^{P_0}$ we have: For every $\alpha \in Y^*$ and $m$ for some pure extension $q$ of $p_\alpha$ with the same domain $q \Vdash \nu_{i^*} \restriction m = \nu^* \restriction m$. By the choice of $n^*$ we get an easy contradiction: Suppose $p \in \tilde{P}_0$ and

$$p \Vdash_{P_0} \text{“}\forall \alpha \in Y^* \, \forall n \geq n^* \, \exists q_\alpha \geq_{tr} p_\alpha,$$
$$q_\alpha \Vdash_{Q/P_0} \text{“}\frac{|\underset{\sim}{\nu}^{*-1}\{1\} \cap A_\alpha \cap n(*)|}{|A_\alpha \cap n(*)|} \in \left[\frac{1}{4}, \frac{3}{4}\right]\text{” ”}.$$

This is impossible, because we may assume that $\nu^* \in V$ (it needs only countably many of the $\kappa$ Cohen reals) and we may arrange all other $A_\alpha$'s so that the quotient will be arbitrary. The forcing $P/P_0$ does not change the value of the quotient.

Now we show that if a real is not recursive then it is not needed for $R$. If $\eta$ is not recursive and $x \in {}^\omega 2$, let $\{x, \eta\} \in N \prec (H(\chi), \in)$, $N$ countable. Let $\nu = \nu(x, \eta)$ be random over $N$, and we claim

$$\eta \not\leq_{Tur} \nu. \tag{5.1}$$

Proof of (5.1): Otherwise we would have that $\eta$ is recursive in the ground model. This is proved in [5] and in [4, Proposition 14]. Since the proof is short, we repeat it here: Suppose

$$p \Vdash_{\text{Random}} \text{“}M \text{ computes } \eta \text{ from the oracle } \underset{\sim}{\nu}\text{”}. \tag{5.2}$$

Then by the Lebesgue density theorem we find $s \in {}^{<\omega}2$ such that above $s$, $p$ has Lebesgue measure $> \frac{99}{100} \cdot \text{Leb}(\{\rho \,:\, s \triangleleft \rho\}$. The we set

$$B_n = \{\nu' \in {}^\omega 2 \,:\, s \triangleleft \nu' \text{ and from } \nu' \ M \text{ computes } \eta(n) \text{ correctly}\}.$$

From (5.2) we get that $\text{Leb}(B_n) \geq \frac{99}{100} \cdot \text{Leb}(\{\rho \,:\, s \triangleleft \rho\}$. So for every sufficiently large $m \in \omega$ we have that

$$2^{m - \lg(s)} \leq |\{\nu' \in {}^m 2 \,:\, s \triangleleft \nu' \text{ and from } \nu' \ M \text{ computes } \eta(n) \text{ correctly}\}|. \tag{5.3}$$

So we can run a machine, that has $s$ as a fixed ingredient, and which, given input $n$, increases $m$ successively, and then computes $\eta(n)$ with all possible oracles above $s$ of length $m \geq \lg(s)$ and decides with (5.3), when it is true for $m$ (and hence for all later $m$), which is the right value. So equation (5.1) is proved.

Thus the collection $\{\nu(x,\eta) : x \in {}^\omega 2\}$ is an $R_1$-adequate family and hence an $R$-adequate family. So, if $\eta$ is needed for $R$, there is some $\nu$ such that $\eta \leq_{Tur} \nu$, and hence by equation (5.1) $\eta$ is recursive. □

## 6. Needed reals for the reaping relation

In this section we prove in ZFC that for any submodel not all hyperarithmetic reals in it are needed for the reaping relation. Since in the model $V[G]$ from Section 4 all hyperarithmetic reals in $V$ are weakly needed for the reaping relation, the model $V[G]$ shows that also for the reaping relation it is consistent that weakly needed and needed do not coincide. In contrast to the result on the relation $R$ from the previous section, we do not prove that only recursive reals are needed for the reaping relation. It is open whether our result here is sharp.

**Theorem 6.1.** *If $\eta$ is needed for the reaping relation, then $\eta$ is arithmetical.*

So, applied to the pairs of models from Section 4, we get:

**Conclusion 6.2.** *For any $V$, $V[G]$, not all hyperarithmetic reals are needed for the reaping relation in $V[G]$.*

Since there are non-arihmetical hyperarithmetic reals, 6.2 follows from 6.1.

Proof of 6.1: Suppose that $\eta$ is needed for the reaping relation. Then by Fact 1.15 there is some $B^* \subseteq \omega$ such that:

(6.1) For all $X$, if $X \subseteq B_* = B_*^1$ or $X \subseteq \omega \setminus B_* = B_*^2$ then $\eta$ is recursive in $X$.

For all $X$ that refine $B^*$, we have that $\eta$ is recursive in $X$. Note that equation (6.1) is similar to $\eta$ being hyperarithmetic: the difference is that $\eta$ is computable also in every infinite subset of the complement of $B_*$. Unless $\eta$ is recursive, we have that $B_*$ in equation (6.1) is infinite and coinfinite.

**Notation 6.3.** *Let $\langle (M_1^n, M_2^n, a_1^n, a_2^n) : n < \omega\rangle$ be a recursive list of the quadruples $(M_1, M_2, a_1, a_2)$ such that*

*(i)* *$M_1, M_2$ are Turing machines (with reference to an oracle),*

*(2)* *$a_1, a_2$ are finite disjoint sets.*

*Without loss of generality, $a_1^n \cup a_2^n \subseteq n$ and each quadruple appears infinitely often. (This will be used in 6.10.)*

**Definition 6.4.** *We say $\bar{E} = \langle E_n : n \in \omega\rangle$ is special if*

*(i)* *$E_n$ is an equivalence relation on $\omega \setminus n$, and*

*(ii)* *for $m < n$, $E_n$ refines $E_m \restriction (\omega \setminus n)$, i.e., every $E_m$-class is the union of some $E_n$-classes plus some subset of $n$,*

*(iii)* *if $A$ is an $E_n$-equivalence class, then $A \setminus (n+1)$ is divided by $E_{n+1}$ in at most two equivalence classes, and $E_0$ has finitely many classes,*

*(iv)* *if*

  *($\alpha$)* *$A$ is an $E_n$-equivalence class and*

> *($\beta$) there is a partition $X_1, X_2$ of $A \setminus (n+1)$ such that for all $j < \omega$, $Y_i \subseteq \omega$, $i = 1, 2$, (if $a_i^n \subseteq Y_i \subseteq X_i \cup a_i^n$, $h_i < \omega$, and if the machine $M_i^n$ running with input $j$ and oracle $Y_i$ finishes its run giving $h_i$, then $h_1 = h_2$),*
>
> *then $E_{n+1}$ induces such a partition of $A$. (But note: it need not be the same partition.)*

**Lemma 6.5.** *There is a special $\bar{E}$ that has as a three place relation $\{\langle n, x, y\rangle : xE_ny\}$ Turing degree $\leq \mathbf{0}^\omega$ and such that if $A$ is an $E_n$-equivalence class then $A \leq_{Tur} \mathbf{0}^{n+1}$.*

*Proof.* We choose $E_n$ by induction on $n$.

$n = 0$. If for every $m$ there is a partition $(c_0, c_1)$ of $m$ such that for $i \in \{1, 2\}$ for every $b_i \subseteq c_i$ and $j < n$, if $M_i^0$ running with input $j$ and oracle $b_i \restriction m$ and gives the results $k_i$ then $k_0 = k_1$, then we choose among these pairs $(c_1^m, c_2^m)$ such that $c_1^m$ is minimal in the lexicographical order. If $(c_1^m, c_2^m)$ are defined for every $m$, then we have that $m^1 \leq m^2 \leq m^3 \Rightarrow c_1^{m_2} \cap m_1 \leq_{lex} c_1^{m_3} \cap m_1$. So $\langle c_1^m : m \in \omega\rangle$ converges to some $c_1$. Now we define $E_0$, having two classes: $c_1$ and $\omega \setminus c_1$. The relation $E_0$ is computable in $\mathbf{0}^1$.

In the step from $n$ to $n+1$, the relation $E_{n+1}$ is defined similarly, with the modification that we use the description of $E_n$ as a parameter and take partitions $(c_0, c_1)$ of $(m \setminus n) \cap C$ for each $E_n$-class $C$ and oracles $b_i \cup a_i^n$. Clearly using $\mathbf{0}^{n+1}$ we can define $E_{n+1}$ and it is in the degree $\mathbf{0}^{n+2}$.

Note different successful computations have the same outcome. $\square_{6.5}$

**Lemma 6.6.** *If $\eta$ is needed for reaping and $\bar{E}$ that is special, then there are $n \in \omega$ and an $E_n$-class $A$ such that $\eta$ is recursive in $A$.*

*Proof.* We assume the contrary. Based on this assumption, forcing with $Q_{\bar{E},B_*}$ and absoluteness we will lead to a contradiction. The proof will be finished with Claim 6.10.

**Definition 6.7.** *For a special $\bar{E}$ we define $Q = Q_{\bar{E},B_*}$ as the following notion of forcing:*

*(1) $p \in Q$ has the form $p = (n, A, b_1, b_2) = (n^p, A^p, b_1^p, b_2^p)$ such that*
  - *(i) $n < \omega$,*
  - *(ii) $A$ is an $E_n$-equivalence class,*
  - *(iii) $A$ is infinite,*
  - *(iv) $b_1, b_2$ are disjoint subsets of $n$,*
  - *(v) $b_1 \subseteq B^*$, $b_2 \subseteq \omega \setminus B^*$.*

*(2) $p \leq q$ iff*
  - *(i) $n^p \leq n^q$, $A^p \supseteq A^q$, $b_i^p \subseteq b_i^q$, for $i = 1, 2$,*
  - *(ii) $(b_1^q \cup b_2^q) \setminus (b_1^p \cup b_2^p) \subseteq A^p$.*

(3) $\underset{\sim}{B}_i = \bigcup\{b_i^p : p \in \underset{\sim}{G}_Q\}$ *is a* $Q$*-name of a subset* $B_i \in V[G]$ *of* $B_i^*$ *for* $i = 1, 2$.

$Q$ is equivalent to Cohen forcing and independent of $\bar{E}$ and $B_*$. Nevertheless we keep the complicated conditions, because they fit better to the investigation of the $\eta$'s needed for the reaping relation. We shall show: $\eta$ is computable in the generic (hence it is recursive) or it is computable in some $E_n$-class $A$. So, for the following three claims we assume that $\eta$ is not computable in any $E_n$-class $A$.

**Claim 6.8.** *For* $i = 1, 2$ *we have*

(1) $\Vdash_Q$ "$\underset{\sim}{b}_i$ *is an infinite subset of* $B_i^*$".

(2) *For densely many* $p^*$, $p^* \Vdash_Q$ "$\bigwedge_{i=1,2}(M_i^{n^{p^*}}$ *computes* $\eta$ *with the oracle* $\underset{\sim}{b}_i)$".

*Proof.* (1) It is enough, to find for a given $p \in Q$ some $q \geq p$, $q \in Q$ such that for $i = 1, 2$, $b_i^p \neq b_i^q$. Now $A^p \cap B_i^*$ is infinite, because of the hypothesis on $B^*$ and because $\eta$ is not recursive in $A^p$ by the assumption. We may choose $h \in A^p \cap B_i^*$, $h \geq \max(\max(b_1^p), \max(b_2^p)) + 2$ and an infinite $E_h$-class $A \subseteq A^p$, which exists because $A^p$ is infinite and because $E_h$ has finitely many equivalence classes. We define $q$ as $n^q = h + 2$, $A^q = A$, $b_1^q = b_1^p \cup \{h\}$, $b_{3-i}^q = b_{3-i}^p \cup \{h + 1\}$.

(2) The statement made in equation (6.1) on $B^*$ and on $\eta$ is $\Pi_1^1$ and holds in $V$; hence it holds in $V[G]$ as well by Shoenfield's absoluteness theorem [7, Theorem 98, p. 530]. We arrange by possibly increasing $n^{p^*}$ that the machines witnessing the recursiveness are $M_i^{n^{p^*}}$. Now we apply it in $V[G]$ to part (1) of this claim. □

We fix $p^*$, $M_1^{n^{p^*}}$, $M_2^{n^{p^*}}$ as in part (2) of Claim 6.7.

**Fact 6.9.** *There is some* $q \geq p^*$ *such that for* $i = 1, 2$, $M_i^{n^q} = M_i^{n^{p^*}}$ *and such that* $b_i^q = a_i^{n^q}$.

*Proof.* For some $n^q \geq n^{p^*}$ the quadruple $(M_1^{n^q}, M_2^{n^q}, a_1^{n^q}, a_2^{n^q})$ is equal to $(M_1^{n^{p^*}}, M_2^{n^{p^*}}, b_1^{p^*}, b_2^{p^*})$. Let $A$ be an infinite $E_{n^{p^*}}$-class which is a subset of $A^{p^*}$. So we take $q = (n^*, A, a_1^{n^q}, a_2^{n^q})$. □

**Claim 6.10.** *For* $n^*$, $A$ *the demands* $(\alpha) + (\beta)$ *of clause (iv) of 6.4 hold, hence the conclusion.*

*Proof.* We work first in $V[G]$. There, by 6.8, $X_i = A \cap B_i^*$ and $A$ exemplify 6.4(iv). But 6.4(iv) is a $\Sigma_2^1$-statement of the parameters $(A, a_1^n, a_2^n)$, and therefore it holds in $V$ as well by Shoenfield's absoluteness theorem. □

Finally we finish the proof of Theorem 6.1: Let $A_1 \neq A_2$ be the $E_{n^*+1}$-equivalence classes which are $\subseteq A$, with $A_i$ for $M_i$ as in 6.4(iv). So by 6.4(iv), $\eta$ is computable in $E_{n^*+1}$. □

## 7. Coincidence

In this section we give a condition on a relation $R$ under which the notions "needed for $R$" and "weakly needed for $R$" coincide and show that the condition is fulfilled for the relation $R_{random}$ defined below.

**Definition 7.1.** *The domain of the relation $R_{random}$ is $\{T \subseteq {}^{<\omega}2$ with no leaves and $\mathrm{Leb}(\lim(T)) > \frac{1}{2}\}$, i.e., the domain of the notion of forcing from Section 1. The range of $R_{random}$ is ${}^{\omega}2$. We set $T R_{random} \nu$ iff $\nu \in A_T := \{\rho \in {}^{\omega}2$ : for some $\rho' \in \lim(T)$ we have that $\rho =^* \rho'\}$.*

**Definition 7.2.** *$R$ is boring if*

*(a) $R$ is a 2-place Borel relation on ${}^{\omega}2$ and $(\forall x \in {}^{\omega}2)(\exists y \in {}^{\omega}2)(xRy)$ , and*

*(b) for every $x_1, x_2 \in {}^{\omega}2$, if $x_2$ is not recursive, there is $x \in 2^{\omega}$ such that*

$$(\forall \nu)\Big(xR\nu \to (x_1 R\nu \wedge \neg(x_2 \leq_{Tur} \nu))\Big).$$

**Claim 7.3.** *(1) Assume that $R$ is boring. Then the notions of being needed for $R$ and being weakly needed for $R$ coincide and coincide with being recursive.*

*(2) The relation $R_{random}$ is boring.*

*Proof.* (1) We have show that every weakly needed real for $R$ is recursive. Since every recursive real is needed for $R$, and since weakly needed reals are needed, this will complete the cycle of implications.

Suppose that $x^* \in {}^{\omega}2$ is not recursive. We show that $x^*$ is not weakly needed for $R$. Let $Y$ be an $R$-adequate set of minimal cardinality. Let $Y^* = \{\nu \in Y : \neg x^* \leq_{Tur} \nu\}$. $Y^* \subseteq Y$, and hence $|Y^*| \leq |Y| = ||R||$. We show that $Y^*$ is also $R$-adequate. Then, by the definition of $Y^*$, $x^*$ is not needed for $R$, and the proof is finished.

Let $x_1 \in {}^{\omega}2$ be given. We take $x_2 = x^*$, and apply (b) of the definition of "boring". So we get $x$ as there. Since $Y$ is $R$-adequate we find some $\nu \in Y$ such that $xR\nu$. Hence by $R$'s boringness we have that $x_1 R\nu \wedge x_2 \not\leq_{Tur} \nu$. So $\nu \in Y^*$ and $x_1 R\nu$.

(2) Let $x_1, x_2$ be given. We take $N \prec (H(\beth_3), \in)$ such that $x_1, x_2 \in N$. Let $T$ be Amoeba-generic over $N$. Then $T = x$ is as claimed in Definition 7.2(b): Let $\nu \in 2^{\omega}$ be such that $\nu \in A_T$. The closed set $T$ is a subset of $x_1$ by the Amoeba genericity of $T$. Hence $x_1 R_{random} \nu$. The set $\{\nu : x_2 \leq_{Tur} \nu\}$ is a tail set and hence has measure zero or one. Since every real recursive in a generic for random forcing is recursive (see the proof or equation (5.1) or [4, Proposition 14] or [5]) and since $x_2$ is not recursive, every generic real for the random forcing avoids the set. Hence it has measure zero and is disjoint from $A_T$, and therefore for $\nu \in A_T$, $\neg(x_2 \leq_{Tur} \nu)$. □

**Conclusion 7.4.** *Needed reals for $R_{random}$ and weakly needed reals for $R_{random}$ coincide and are just all the recursive reals.* □

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Current address: Heike Mildenberger, Institut für formale Logik, Universität Wien, Währinger Str. 25, 1090 Wien, Austria

Saharon Shelah, Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, 91904 Jerusalem, Israel

*Email address*: heike@logic.univie.ac.at

*Email address*: shelah@math.huji.ac.il