A SPACE WITH ONLY BOREL SUBSETS

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Miklós Laczkovich (Budapest) asked if there exists a Haussdorff (or even normal) space in which every subset is Borel yet it is not meager. The motivation of the last condition is that under MA_κ every subspace of the reals of cardinality κ has the property that all subsets are F_σ however Martin's axiom also implies that these subsets are meager. Here we answer Laczkovich' question. I thank Peter Komjath – the existence of this paper owes much to him.

Theorem. *The following are equiconsistent.*

1. *There exists a measurable cardinal.*
2. *There is a non-meager T_1 space with no isolated points in which every subset is Borel.*
3. *There is a non-meager T_4 space with no isolated points in which every subset is the union of an open and a closed set.*

Proof. Assume first that κ is measurable in the model V. Add κ Cohen reals, that is, force with the partial ordering $\mathrm{Add}(\omega,\kappa)$. Our model will be $V[G]$ where $G \subseteq \mathrm{Add}(\omega,\kappa)$ is generic. We first observe that in $V[G]$ there is a κ-complete ideal on κ such that the complete Boolean algebra $P(\kappa)/I$ is isomorphic to the Boolean algebra of the complete closure of $\mathrm{Add}(\omega, j(\kappa))$ where $j : V \to M$ is the corresponding elementary embedding. Indeed we let $X \in I$ if and only if $1 \Vdash \kappa \notin j(\tau)$ for some τ satisfying $X = \tau^G$, that is, τ is a name for $X \subseteq \kappa$. Moreover, the mapping $X \mapsto [\![\kappa \in j(\tau)]\!]$ is an isomorphism between $P(\kappa)/I$ and the regular Boolean algebra of $\mathrm{Add}(\omega, j(\kappa) \setminus \kappa)$ (where τ is a name for X). Notice that $|j(\kappa)| = 2^\kappa$.

We observe that this Boolean algebra has the following properties. There are 2^κ subsets $\{A_\alpha : \alpha < 2^\kappa\}$ which are independent mod I, that is, if s is a function from a finite subset of κ into $\{0,1\}$ then the intersection

$$B_s \stackrel{\text{def}}{=} \bigcap_{\alpha \in \mathrm{Dom}(s)} A_\alpha^{s(\alpha)}$$

is not in I (here $A^1 = A$ and $A^0 = \kappa \setminus A$). Moreover, if $A \subseteq \kappa$ then there are countably many pairwise contradictory functions $s_0, s_1, \dots$ as above, such that

$$A/I = B_{s_0}/I \vee B_{s_1}/I \vee \dots,$$

that is, A can be written as $B_{s_0} \cup B_{s_1} \cup \dots$ add-and-take-away a set in I.

By cardinality assumptions we can assume that for every pair (X, Y) of disjoint members of I there is some $\alpha < 2^\kappa$ with $X \subseteq A_\alpha$, $Y \subseteq \kappa \setminus A_\alpha$.

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We define a topology on κ by declaring the system

$$\{A_\alpha \setminus Z, A^1 \setminus Z : \alpha < 2^\kappa, Z \in I\}$$

a subbasis, or, what is the same, the collection of all sets of the form $B_s \setminus Z$ (where $Z \in I$) a basis.

We prove the following statements on the space.

Claim. *The space has the following properties.*

(1) *Every set of the form B_s is clopen, every set in I is closed.*
(2) *Every meager set is in I.*
(3) *Every set is the union of an open and a closed set.*
(4) *The closure of $B_s \setminus Z$ is B_s.*
(5) *The space is T_4.*

Proof. 1. Straightforward.

2. Every set not in I contains a subset of the form $B_s \setminus Z$ (by one of the properties of the Boolean algebra mentioned above), which is open, so every nowhere dense, therefore every meager set is in I.

3. If $A \subseteq \kappa$ then A/I can be written as $A/I = B_{s_0}/I \vee B_{s_1}/I \vee \dots$ and then clearly

$$A = \Big((B_{s_0} \setminus Z_0) \cup (B_{s_1} \setminus Z_1) \cup \dots\Big) \cup Z$$

for some sets $Z_0, Z_1, \dots, Z$ in I. But this is a decomposition into the union of an open and a closed set.

4. Clear.

5. Assume we are given the disjoint closed sets F and F'. They can be written as

$$F = (B_{s_0} \setminus Z_0) \cup (B_{s_1} \setminus Z_1) \cup \dots \cup Z$$

and

$$F' = (B_{s'_0} \setminus Z'_0) \cup (B_{s'_1} \setminus Z'_1) \cup \dots \cup Z'.$$

As F and F' are closed, using 4., we can assume that

$$Z_0 = Z_1 = \dots = Z'_0 = Z'_1 = \dots = \emptyset.$$

Set $G = B_{s_0} \cup B_{s_1} \cup \dots$, $G' = B_{s'_0} \cup B_{s'_1} \cup \dots$, then $F = G \cup Z$, $F' = G' \cup Z'$ and these four sets are pairwise disjoint. It suffices to separate each of the pairs (G, G'), (G, Z'), (G', Z), and (Z, Z'). There is no problem with the first case, as G, G' are open. For the last case we use our assumption that some A_α separates Z and Z'. For the second, we can assume that G is non empty hence B_{s_0} is well defined and disjoint to Z', now choose $\alpha < \kappa$ such that Z' is a subset of A_α, and so G, $A_\alpha \setminus B_{s_0}$ is a pair of disjoint open sets as required. Lastly the third case is similar to the second.

We have proved (1) $\longrightarrow$ (3), and (3) $\longrightarrow$ (2) is trivial; lastly for (2) $\longrightarrow$ (1) assume that $(X, \mathcal{T})$ is a non-meager T_1 space with no isolated points in which every subset is Borel. Let $\{G_\alpha : \alpha < \tau\}$ be a maximal system of disjoint, nonempty, meager open sets. Such a system exists by Zorn's lemma. Set $Y = \bigcup\{G_\alpha : \alpha < \tau\}$. Clearly, Y is meager. As the boundary of the open Y is nowhere dense, we get that even the closure of Y is meager. Then the nonempty subspace $Z = X - \overline{Y}$ has the property that no nonempty open set is meager and every subset is Borel. If I is the meager ideal on Z then every subset is equal to some open set mod I. We claim that I is precipitous on Z which implies that in some inner model there is a measurable cardinal (see [1], [2]).

For this, assume that $\mathcal{W}^0, \mathcal{W}^1, \ldots$ is a refining sequence of mod I partitions. That is, every $\mathcal{W}^n$ is a maximal system of I-almost disjoint open sets, and if A is a member of some $\mathcal{W}^{n+1}$ then there is some member of $\mathcal{W}^n$ which includes A mod I. We try to find a member $A_n \in \mathcal{W}^n$ such that $\bigcap\{A_n : n < \omega\}$ is nonempty. To this, observe that the intersection of two members in $\mathcal{W}^n$ is a meagre open set, hence is the empty set. Therefore, $\mathcal{W}^n$ is actually a decomposition of $Z \setminus Z_n$ into the union of disjoint open sets where Z_n is a meager set. Pick an element in $Z \setminus \bigcup\{Z_n : n < \omega\}$ then it is in some member of $\mathcal{W}^n$ for every n and we are done.

References

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