HISTORIC FORCING FOR Depth

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ABSTRACT. We show that, consistently, for some regular cardinals $\theta < \lambda$, there exists a Boolean algebra $\mathbb{B}$ such that $|\mathbb{B}| = \lambda^+$ and for every subalgebra $\mathbb{B}' \subseteq \mathbb{B}$ of size λ^+ we have $\mathrm{Depth}(\mathbb{B}') = \theta$.

0. Introduction

The present paper is concerned with forcing a Boolean algebra which has some prescribed properties of Depth. Let us recall that, for a Boolean algebra $\mathbb{B}$, its depth is defined as follows:

$$\begin{aligned}\mathrm{Depth}(\mathbb{B}) &= \sup\{|X| : X \subseteq \mathbb{B} \text{ is well-ordered by the Boolean ordering }\},\\ \mathrm{Depth}^+(\mathbb{B}) &= \sup\{|X|^+ : X \subseteq \mathbb{B} \text{ is well-ordered by the Boolean ordering }\}.\end{aligned}$$

($\mathrm{Depth}^+(\mathbb{B})$ is used to deal with attainment properties in the definition of $\mathrm{Depth}(\mathbb{B})$, see e.g. [RS98, §1].) The depth (of Boolean algebras) is among cardinal functions that have more algebraic origins, and their relations to "topological fellows" is often indirect, though sometimes very surprising. For example, if we define

$$\mathrm{Depth}_{\mathrm{H+}}(\mathbb{B}) = \sup\{\mathrm{Depth}(\mathbb{B}/I) : I \text{ is an ideal in } \mathbb{B}\ \},$$

then for any (infinite) Boolean algebra $\mathbb{B}$ we will have that $\mathrm{Depth}_{\mathrm{H+}}(\mathbb{B})$ is the tightness $t(\mathbb{B})$ of the algebra $\mathbb{B}$ (or the tightness of the topological space $\mathrm{Ult}(\mathbb{B})$ of ultrafilters on $\mathbb{B}$), see [Mon96, Theorem 4.21]. A somewhat similar function to $\mathrm{Depth}_{\mathrm{H+}}$ is obtained by taking $\sup\{\mathrm{Depth}(\mathbb{B}') : \mathbb{B}' \text{ is a subalgebra of } \mathbb{B}\ \}$, but clearly this brings nothing new: it is the old Depth. But if one wants to understand the behaviour of the depth for subalgebras of the considered Boolean algebra, then looking at the following *subalgebra* Depth *relation* may be very appropriate:

$$\mathrm{Depth}_{\mathrm{Sr}}(\mathbb{B}) = \{(\kappa,\mu) : \text{ there is an infinite subalgebra } \mathbb{B}' \text{ of } \mathbb{B} \text{ such that } |\mathbb{B}'| = \mu \text{ and } \mathrm{Depth}(\mathbb{B}') = \kappa\ \}.$$

A number of results related to this relation is presented by Monk in [Mon96, Chapter 4]. There he asks if there are a Boolean algebra $\mathbb{B}$ and an infinite cardinal θ such that $(\theta,(2^\theta)^+) \in \mathrm{Depth}_{\mathrm{Sr}}(\mathbb{B})$, while $(\omega,(2^\theta)^+) \notin \mathrm{Depth}_{\mathrm{Sr}}(\mathbb{B})$ (see Monk [Mon96, Problem 14]; we refer the reader to Chapter 4 of Monk's book [Mon96] for the motivation and background of this problem). Here we will partially answer this question, showing that it is consistent that there is such $\mathbb{B}$ and θ. The question if that can be done in ZFC remains open.

1991 *Mathematics Subject Classification.* Primary 03E35, 03G05; Secondary 03E05, 06Exx.
Key words and phrases. Boolean algebras, depth, historic forcing.
The first author thanks the KBN (Polish Committee of Scientific Research) for partial support through grant 2 P03 A 01109.
The research of the second author was partially supported by the Israel Science Foundation. Publication 733.

Our consistency result is obtained by forcing, and the construction of the required forcing notion is interesting *per se*. We use the method of *historic forcing* which was first applied in Shelah and Stanley [SS87]. The reader familiar with [SS87] will notice several correspondences between the construction here and the method used there. However, we do not relay on that paper and our presentation here is self-contained.

Let us describe how our historic forcing notion is built. So, we fix two (regular) cardinals θ, λ and our aim is to force a Boolean algebra $\dot{\mathbb{B}}^\theta_\lambda$ such that $|\dot{\mathbb{B}}^\theta_\lambda| = \lambda^+$ and for every subalgebra $\mathbb{B} \subseteq \dot{\mathbb{B}}^\theta_\lambda$ of size λ^+ we have $\mathrm{Depth}(\mathbb{B}) = \theta$. The algebra $\dot{\mathbb{B}}^\theta_\lambda$ will be generated by $\langle x_i : i \in \dot{U}\rangle$ for some set $\dot{U} \subseteq \lambda^+$. A condition p will be an approximation to the algebra $\dot{\mathbb{B}}^\theta_\lambda$, it will carry the information on what is the subalgebra $\mathbb{B}_p = \langle x_i : i \in u^p\rangle_{\dot{\mathbb{B}}^\theta_\lambda}$ for some $u^p \subseteq \lambda^+$. A natural way to describe algebras in this context is by listing ultrafilters (or: homomorphisms into $\{0,1\}$):

Definition 1. For a set w and a family $F \subseteq 2^w$ we define
$\mathrm{cl}(F) = \{g \in 2^w : (\forall u \in [w]^{<\omega})(\exists f \in F)(f \restriction u = g \restriction u)\}$,
$\mathbb{B}_{(w,F)}$ is the Boolean algebra generated freely by $\{x_\alpha : \alpha \in w\}$ except that
if $u_0, u_1 \in [w]^{<\omega}$ and there is no $f \in F$ such that $f \restriction u_0 \equiv 0$, $f \restriction u_1 \equiv 1$
then $\bigwedge\limits_{\alpha\in u_1} x_\alpha \wedge \bigwedge\limits_{\alpha\in u_0} (-x_\alpha) = 0$.

This description of algebras is easy to handle, for example:

Proposition 2 (see [She96, 2.6]). *Let $F \subseteq 2^w$. Then:*

(1) *Each $f \in F$ extends (uniquely) to a homomorphism from $\mathbb{B}_{(w,F)}$ to $\{0,1\}$ (i.e. it preserves the equalities from the definition of $\mathbb{B}_{(w,F)}$). If F is closed, then every homomorphism from $\mathbb{B}_{(w,F)}$ to $\{0,1\}$ extends exactly one element of F.*

(2) *If $\tau(y_0,\ldots,y_\ell)$ is a Boolean term and $\alpha_0,\ldots,\alpha_\ell \in w$ are distinct then*

$$\mathbb{B}_{(w,F)} \models \tau(x_{\alpha_0},\ldots,x_{\alpha_\ell}) \neq 0 \quad \text{if and only if} \quad (\exists f \in F)(\{0,1\} \models \tau(f(\alpha_0),\ldots,f(\alpha_k)) = 1).$$

(3) *If $w \subseteq w^*$, $F^* \subseteq 2^{w^*}$ and*

$$(\forall f \in F)(\exists g \in F^*)(f \subseteq g) \quad \text{and} \quad (\forall g \in F^*)(g \restriction w \in \mathrm{cl}(F))$$

then $\mathbb{B}_{(w,F)}$ is a subalgebra of $\mathbb{B}_{(w^*,F^*)}$.

So each condition p in our forcing notion $\mathbb{P}^\theta_\lambda$ will have a set $u^p \in [\lambda^+]^{<\lambda}$ and a closed set $F^p \subseteq 2^{u^p}$ (and the respective algebra will be $\mathbb{B}_p = \mathbb{B}_{(u^p,F^p)}$). But to make the forcing notion work, we will have to put more restrictions on our conditions, and we will be taking only those conditions that have to be taken to make the arguments work. For example, we want that cardinals are not collapsed by our forcing, and demanding that $\mathbb{P}^\theta_\lambda$ is λ^+-cc (and somewhat $({<}\lambda)$–closed) is natural in this context. How do we argue that a forcing notion is λ^+–cc? Typically we start with a sequence of λ^+ distinct conditions, we carry out some "cleaning procedure" (usually involving the Δ–lemma etc), and we end up with (at least two) conditions that "can be put together". Putting together two (or more) conditions that are approximations to a Boolean algebra means amalgamating them. There are various ways to amalgamate conditions - we will pick one that will work for several purposes. Then, once we declare that some conditions forming a "clean" Δ–sequence of length θ are in $\mathbb{P}^\theta_\lambda$, we will be bound to declare that the amalgamation is in our forcing notion. The amalgamation (and natural limits) will be the only way to build new conditions from the old ones, but the description above still misses an important factor. So far, a condition does not have to know what are the reasons for it to be called to $\mathbb{P}^\theta_\lambda$. This information is *the history of the condition* and it will be encoded by two functions h^p, g^p. (Actually, these functions will give histories of all elements of u^p describing why and how those points were incorporated to u^p. Thus both functions will be defined on $u^p \times \mathrm{ht}(p)$, were $\mathrm{ht}(p)$ is the height of the condition p, that is the step in our construction at which the condition p is created.) We will also want that our forcing is suitably closed, and getting "$(<\lambda)$–strategically closed" would be fine. To make that happen we will have to deal with two relations on $\mathbb{P}^\theta_\lambda$: $\leq_{\rm pr}$ and $\leq$. The first ("pure") is $(<\lambda)$–closed and it will help in getting the strategic closure of the second (main) one. In some sense, the relation $\leq_{\rm pr}$ represents "the official line in history", and sometimes we will have to rewrite that official history, see Definition 6 and Lemma 7 (on changing history see also Orwell [Orw77]).

The forcing notion $\mathbb{P}^\theta_\lambda$ has some other interesting features. (For example, conditions are very much like fractals, they contain many self-similar pieces (see Definition 10 and Lemma 11).) The method of historic forcing notions could be applicable to more problems, and this is why in our presentation we separated several observations of general character (presented in the first section) from the problem specific arguments (section 2)

Notation: Our notation is standard and compatible with that of classical textbooks on set theory (like Jech [Jec03]) and Boolean algebras (like Monk [Mon90], [Mon96]). However in forcing considerations we keep the older tradition that

the stronger condition is the greater one.

Let us list some of our notation and conventions.

1. Throughout the paper, θ, λ are fixed regular infinite cardinals, $\theta < \lambda$.
2. A name for an object in a forcing extension is denoted with a dot above (like $\dot{X}$) with one exception: the canonical name for a generic filter in a forcing notion $\mathbb{P}$ will be called $\Gamma_{\mathbb{P}}$. For a $\mathbb{P}$–name $\dot{X}$ and a $\mathbb{P}$–generic filter G over $\mathbf{V}$, the interpretation of the name $\dot{X}$ by G is denoted by $\dot{X}^G$.
3. $i, j, \alpha, \beta, \gamma, \delta, \ldots$ will denote ordinals.
4. For a set X and a cardinal λ, $[X]^{<\lambda}$ stands for the family of all subsets of X of size less than λ. The family of all functions from Y to X is called X^Y. If X is a set of ordinals then its order type is denoted by $\mathrm{otp}(X)$.
5. In Boolean algebras we use $\vee$ (and $\bigvee$), $\wedge$ (and $\bigwedge$) and $-$ for the Boolean operations. If $\mathbb{B}$ is a Boolean algebra, $x \in \mathbb{B}$ then $x^0 = x$, $x^1 = -x$.
6. For a subset Y of an algebra $\mathbb{B}$, the subalgebra of $\mathbb{B}$ generated by Y is denoted by $\langle Y \rangle_{\mathbb{B}}$.

Acknowledgements: We would like to thank the referee for valuable comments and suggestions.

1. The forcing and its basic properties

Let us start with the definition of the forcing notion $\mathbb{P}^\theta_\lambda$. By induction on $\alpha < \lambda$ we will define sets of conditions $P^{\theta,\lambda}_\alpha$, and for each $p \in P^{\theta,\lambda}_\alpha$ we will define $u^p, F^p, \mathrm{ht}(p), h^p$ and g^p. Also we will define relations $\leq^\alpha$ and $\leq^\alpha_{\rm pr}$ on $P^{\theta,\lambda}_\alpha$. Our inductive requirements are:

$(\mathrm{i})_\alpha$ for each $p \in P^{\theta,\lambda}_\alpha$:
$u^p \in [\lambda^+]^{<\lambda}$, $\mathrm{ht}(p) \le \alpha$, $F^p \subseteq 2^{u^p}$ is a non-empty closed set, g^p is a function with domain $\mathrm{dom}(g^p) = u^p \times \mathrm{ht}(p)$ and values of the form (ℓ,τ), where $\ell < 2$ and τ is a Boolean term, and $h^p : u^p \times \mathrm{ht}(p) \longrightarrow \theta + 2$ is a function,

$(\mathrm{ii})_\alpha$ $\le^\alpha, \le^\alpha_{\mathrm{pr}}$ are transitive and reflexive relations on $P^{\theta,\lambda}_\alpha$, and $\le^\alpha$ extends $\le^\alpha_{\mathrm{pr}}$,

$(\mathrm{iii})_\alpha$ if $p, q \in P^{\theta,\lambda}_\alpha$, $p \le^\alpha q$, then $u^p \subseteq u^q$, $\mathrm{ht}(p) \le \mathrm{ht}(q)$, and $F^p = \{f \restriction u^p : f \in F^q\}$, and if $p \le^\alpha_{\mathrm{pr}} q$, then for every $i \in u^p$ and $\xi < \mathrm{ht}(p)$ we have $h^p(i,\xi) = h^q(i,\xi)$ and $g^p(i,\xi) = g^q(i,\xi)$,

$(\mathrm{iv})_\alpha$ if $\beta < \alpha$ then $P^{\theta,\lambda}_\beta \subseteq P^{\theta,\lambda}_\alpha$, and $\le^\alpha_{\mathrm{pr}}$ extends $\le^\beta_{\mathrm{pr}}$, and $\le^\alpha$ extends $\le^\beta$.

For a condition $p \in P^{\theta,\lambda}_\alpha$, we will also declare that $\mathbb{B}^p = \mathbb{B}_{(u^p,F^p)}$ (the Boolean algebra defined in Definition 1).

We define $P^{\theta,\lambda}_0 = \{\langle\xi\rangle : \xi < \lambda^+\}$ and for $p = \langle\xi\rangle$ we let $F^p = 2^{\{\xi\}}$, $\mathrm{ht}(p) = 0$ and $h^p = \emptyset = g^p$. The relations $\le^0_{\mathrm{pr}}$ and $\le^0$ both are the equality. [Clearly these objects are as declared, i.e, clauses $(\mathrm{i})_0$–$(\mathrm{iv})_0$ hold true.]

If $\gamma < \lambda$ is a limit ordinal, then we put

$$P^*_\gamma = \big\{\langle p_\xi : \xi < \gamma\rangle : (\forall \xi < \zeta < \gamma)(p_\xi \in P^{\theta,\lambda}_\xi \ \&\ \mathrm{ht}(p_\xi) = \xi \ \&\ p_\xi \le^\zeta_{\mathrm{pr}} p_\zeta)\big\},$$
$$P^{\theta,\lambda}_\gamma = \bigcup_{\alpha<\gamma} P^{\theta,\lambda}_\alpha \cup P^*_\gamma,$$

and for $p = \langle p_\xi : \xi < \gamma\rangle \in P^*_\gamma$ we let

$$u^p = \bigcup_{\xi<\gamma} u^{p_\xi}, \quad F^p = \{f \in 2^{u^p} : (\forall \xi < \gamma)(f \restriction u^{p_\xi} \in F^{p_\xi})\}, \quad \mathrm{ht}(p) = \gamma$$

and $h^p = \bigcup\limits_{\xi<\gamma} h^{p_\xi}$ and $g^p = \bigcup\limits_{\xi<\gamma} g^{p_\xi}$. We define $\le^\gamma$ and $\le^\gamma_{\mathrm{pr}}$ by:

$p \le^\gamma_{\mathrm{pr}} q$ if and only if
either $p, q \in P^{\theta,\lambda}_\alpha$, $\alpha < \gamma$ and $p \le^\alpha_{\mathrm{pr}} q$,
or $q = \langle q_\xi : \xi < \gamma\rangle \in P^*_\gamma$, $p \in P^{\theta,\lambda}_\alpha$ and $p \le^\alpha_{\mathrm{pr}} q_\alpha$ for some $\alpha < \gamma$,
or $p = q$;

$p \le^\gamma q$ if and only if
either $p, q \in P^{\theta,\lambda}_\alpha$, $\alpha < \gamma$ and $p \le^\alpha q$,
or $q = \langle q_\xi : \xi < \gamma\rangle \in P^*_\gamma$, $p \in P^{\theta,\lambda}_\alpha$ and $p \le^\alpha q_\alpha$ for some $\alpha < \gamma$,
or $p = \langle p_\xi : \xi < \gamma\rangle \in P^*_\gamma$, $q = \langle q_\xi : \xi < \gamma\rangle \in P^*_\gamma$ and

$$(\exists \delta < \gamma)(\forall \xi < \gamma)(\delta \le \xi \ \Rightarrow\ p_\xi \le^\xi q_\xi).$$

[It is straightforward to show that clauses $(\mathrm{i})_\gamma$–$(\mathrm{iv})_\gamma$ hold true.]

Suppose now that $\alpha < \lambda$. Let $P^*_{\alpha+1}$ consist of all tuples

$$\langle \zeta^*, \tau^*, n^*, u^*, \langle p_\xi, v_\xi : \xi < \theta\rangle\rangle$$

such that for each $\xi_0 < \xi_1 < \theta$:

(α) $\zeta^* < \theta$, $n^* < \omega$, $\tau^* = \tau^*(y_1, \ldots, y_{n^*})$ is a Boolean term, $u^* \in [\lambda^+]^{<\lambda}$,

(β) $p_{\xi_0} \in P^{\theta,\lambda}_\alpha$, $\mathrm{ht}(p) = \alpha$, $v_{\xi_0} \in [u^{p_{\xi_0}}]^{n^*}$,

(γ) the family $\{u^{p_\xi} : \xi < \theta\}$ forms a Δ–system with heart u^* and $u^{p_{\xi_0}} \setminus u^* \neq \emptyset$ and

$$\sup(u^*) < \min(u^{p_{\xi_0}} \setminus u^*) \le \sup(u^{p_{\xi_0}} \setminus u^*) < \min(u^{p_{\xi_1}} \setminus u^*),$$

(δ) $\mathrm{otp}(u^{p_{\xi_0}}) = \mathrm{otp}(u^{p_{\xi_1}})$ and if $H : u^{p_{\xi_0}} \longrightarrow u^{p_{\xi_1}}$ is the order isomorphism then $H \restriction u^*$ is the identity on u^*, $F^{p_{\xi_0}} = \{f \circ H : f \in F^{p_{\xi_1}}\}$, $H[v_{\xi_0}] = v_{\xi_1}$ and

$$(\forall j \in u^{p_{\xi_0}})(\forall \beta < \alpha)(h^{p_{\xi_0}}(j,\beta) = h^{p_{\xi_1}}(H(j),\beta) \ \& \ g^{p_{\xi_0}}(j,\beta) = g^{p_{\xi_1}}(H(j),\beta)).$$

We put $P^{\theta,\lambda}_{\alpha+1} = P^{\theta,\lambda}_{\alpha} \cup P^*_{\alpha+1}$ and for $p = \langle \zeta^*, \tau^*, n^*, u^*, \langle p_\xi, v_\xi : \xi < \theta \rangle \rangle \in P^*_{\alpha+1}$ we let $u^p = \bigcup\limits_{\xi<\theta} u^{p_\xi}$ and

$$F^p = \{f \in 2^{u^p} : \quad (\forall \xi < \theta)(f \restriction u^{p_\xi} \in F^{p_\xi}) \text{ and for all } \xi < \zeta < \theta \\ f(\sigma_{\mathrm{maj}}(\tau_{3\cdot\xi}, \tau_{3\cdot\xi+1}, \tau_{3\cdot\xi+2})) \leq f(\sigma_{\mathrm{maj}}(\tau_{3\cdot\zeta}, \tau_{3\cdot\zeta+1}, \tau_{3\cdot\zeta+2}))\},$$

where $\tau_\xi = \tau^*(x_i : i \in v_\xi)$ for $\xi < \theta$ (so τ_ξ is an element of the algebra $\mathbb{B}^{p_\xi} = \mathbb{B}_{(u^{p_\xi}, F^{p_\xi})}$), and $\sigma_{\mathrm{maj}}(y_0, y_1, y_2) = (y_0 \wedge y_1) \vee (y_0 \wedge y_2) \vee (y_1 \wedge y_2)$. Next we let $\mathrm{ht}(p) = \alpha + 1$ and we define functions h^p, g^p on $u^p \times (\alpha+1)$ by

$$h^p(j,\beta) = \begin{cases} h^{p_\xi}(j,\beta) & \text{if } \ j \in u^{p_\xi},\ \xi < \theta,\ \beta < \alpha, \\ \theta & \text{if } \ j \in u^*,\ \beta = \alpha, \\ \theta + 1 & \text{if } \ j \in u^{p_{\zeta^*}} \setminus u^*,\ \beta = \alpha, \\ \xi & \text{if } \ j \in u^{p_\xi} \setminus u^*,\ \xi < \theta,\ \xi \neq \zeta^*,\ \beta = \alpha, \end{cases}$$

$$g^p(j,\beta) = \begin{cases} g^{p_\xi}(j,\beta) & \text{if } \ j \in u^{p_\xi},\ \xi < \theta,\ \beta < \alpha, \\ (1,\tau^*) & \text{if } \ j \in v_\xi,\ \xi < \theta,\ \beta = \alpha, \\ (0,\tau^*) & \text{if } \ j \in u^{p_\xi} \setminus v_\xi,\ \xi < \theta,\ \beta = \alpha. \end{cases}$$

Next we define the relations $\leq^{\alpha+1}_{\mathrm{pr}}$ and $\leq^{\alpha+1}$ by:

$p \leq^{\alpha+1}_{\mathrm{pr}} q$ if and only if
either $p, q \in P^{\theta,\lambda}_\alpha$ and $p \leq^\alpha_{\mathrm{pr}} q$,
or $q = \langle \zeta^*, \tau^*, n^*, u^*, \langle q_\xi, v_\xi : \xi < \theta \rangle \rangle \in P^*_{\alpha+1}$, $p \in P^{\theta,\lambda}_\alpha$, and $p \leq^\alpha_{\mathrm{pr}} q_{\zeta^*}$,
or $p = q$;

$p \leq^{\alpha+1} q$ if and only if
either $p, q \in P^{\theta,\lambda}_\alpha$ and $p \leq^\alpha q$,
or $q = \langle \zeta^*, \tau^*, n^*, u^*, \langle q_\xi, v_\xi : \xi < \theta \rangle \rangle \in P^*_{\alpha+1}$, $p \in P^{\theta,\lambda}_\alpha$, and $p \leq^\alpha q_\xi$ for some $\xi < \theta$,
or $p = \langle \zeta^{**}, \tau^*, n^*, u^*, \langle p_\xi, v_\xi : \xi < \theta \rangle \rangle$, $q = \langle \zeta^*, \tau^*, n^*, u^*, \langle q_\xi, v_\xi : \xi < \theta \rangle \rangle$ are from $P^*_{\alpha+1}$ and

$$(\forall \xi < \theta)(p_\xi \leq^\alpha q_\xi \ \& \ u^{p_\xi} = u^{q_\xi}).$$

[Again, it is easy to show that clauses $(\mathrm{i})_{\alpha+1}$–$(\mathrm{iv})_{\alpha+1}$ are satisfied.]

After the construction is carried out we let

$$\mathbb{P}^\theta_\lambda = \bigcup_{\alpha<\lambda} P^{\theta,\lambda}_\alpha \quad \text{and} \quad \leq_{\mathrm{pr}} \ = \bigcup_{\alpha<\lambda} \leq^\alpha_{\mathrm{pr}} \quad \text{and} \quad \leq \ = \bigcup_{\alpha<\lambda} \leq^\alpha.$$

One easily checks that $\leq_{\mathrm{pr}}$ is a partial order on $\mathbb{P}^\theta_\lambda$ and that the relation $\leq$ is transitive and reflexive, and that $\leq_{\mathrm{pr}} \subseteq \leq$.

Lemma 3. *Let $p, q \in \mathbb{P}^\theta_\lambda$.*

1. *If $p \leq q$ then $\mathrm{ht}(p) \leq \mathrm{ht}(q)$, $u^p \subseteq u^q$ and $F^p = \{f \restriction u^p : f \in F^q\}$ (so $\mathbb{B}^p$ is a subalgebra of $\mathbb{B}^q$). If $p \leq q$ and $\mathrm{ht}(p) = \mathrm{ht}(q)$, then $q \leq p$.*
2. *For each $j \in u^p$, the set $\{\beta < \mathrm{ht}(p) : h^p(j,\beta) < \theta\}$ is finite.*
3. *If $p \leq_{\mathrm{pr}} q$ and $i \in u^p$, then $h^q(i,\beta) \geq \theta$ for all β such that $\mathrm{ht}(p) \leq \beta < \mathrm{ht}(q)$.*

(4) *If* $i,j\in u^p$ *are distinct, then there is* $\beta<{\rm ht}(p)$ *such that* $\theta\neq h^p(i,\beta)\neq h^p(j,\beta)\neq\theta$.

(5) *For each finite set* $X\subseteq{\rm ht}(p)$ *there is* $i\in u^p$ *such that*

$$\{\beta<{\rm ht}(p):h^p(i,\beta)<\theta\}=X.$$

(6) *If* $p\leq_{\rm pr} q$ *then there is a* $\leq_{\rm pr}$*–increasing sequence* $\langle p_\xi:\xi\leq{\rm ht}(p)\rangle\subseteq\mathbb{P}^\theta_\lambda$ *such that* $p_{{\rm ht}(p)}=p$, $p_{{\rm ht}(q)}=q$ *and* ${\rm ht}(p_\xi)=\xi$ *(for* $\xi\leq{\rm ht}(p)$*). (In particular, if* $p\leq_{\rm pr} q$ *and* ${\rm ht}(p)={\rm ht}(q)$ *then* $p=q$*.)*

(7) *If* ${\rm ht}(p)=\gamma$ *is a limit ordinal,* $p=\langle p_\xi:\xi<\gamma\rangle$*, then for each* $i\in u^p$ *and* $\xi<\gamma$*:*

$$i\in u^{p_\xi}\quad\text{if and only if}\quad(\forall\zeta<\gamma)(\xi\leq\zeta\ \Rightarrow\ h^p(i,\zeta)\geq\theta).$$

Proof. 1) Should be clear (an easy induction).

2) Suppose that $p\in\mathbb{P}^\theta_\lambda$ and $j\in u^p$ are a counterexample with the minimal possible value of ${\rm ht}(p)$. Necessarily ${\rm ht}(p)$ is a limit ordinal, $p=\langle p_\xi:\xi<{\rm ht}(p)\rangle$, ${\rm ht}(p_\xi)=\xi$ and $\zeta<\xi<{\rm ht}(p)\ \Rightarrow\ p_\zeta\leq_{\rm pr} p_\xi$. Let $\xi<{\rm ht}(p)$ be the first ordinal such that $j\in u^{p_\xi}$. By the choice of p, the set $\{\beta\leq\xi:h^p(j,\beta)<\theta\}$ is finite, but clearly $h^p(j,\beta)\geq\theta$ for all $\beta\in(\xi,{\rm ht}(p))$.

3) An easy induction on ${\rm ht}(q)$ (with fixed p).

4) We show this by induction on ${\rm ht}(p)$. Suppose that ${\rm ht}(p)=\alpha+1$, so $p=\langle\zeta^*,\tau^*,n^*,u^*,\langle p_\xi,v_\xi:\xi<\theta\rangle\rangle$, and $i,j\in u^p$ are distinct. If $i,j\in u^{p_\xi}$ for some $\xi<\theta$, then by the inductive hypothesis we find $\beta<\alpha$ such that

$$\theta\neq h^p(i,\beta)=h^{p_\xi}(i,\beta)\neq h^{p_\xi}(j,\beta)=h^p(j,\beta)\neq\theta.$$

If $i\in u^{p_\xi}\setminus u^*$, $j\in u^{p_\zeta}\setminus u^*$ and $\xi,\zeta<\theta$ are distinct, then look at the definition of $h^p(i,\alpha)$, $h^p(j,\alpha)$ – these two values cannot be equal (and both are distinct from θ). Finally suppose that ${\rm ht}(p)$ is limit, so $p=\langle p_\xi:\xi<{\rm ht}(p)\rangle$. Take $\xi<{\rm ht}(p)$ such that $i,j\in u^{p_\xi}$ and apply the inductive hypothesis to p_ξ getting $\beta<\xi$ such that $h^p(i,\beta)\neq h^p(j,\beta)$ (and both are not θ).

5) Again, it goes by induction on ${\rm ht}(p)$. First consider a limit stage, and suppose that ${\rm ht}(p)=\gamma$ is a limit ordinal, $X\in[\gamma]^{<\omega}$ and $p=\langle p_\xi:\xi<\gamma\rangle$. Let $\xi<\gamma$ be such that $X\subseteq\xi$. By the inductive hypothesis we find $i\in u^{p_\xi}$ such that $\{\beta<\xi: h^p(i,\beta)<\theta\}=X$. Applying clause (3) we may conclude that this i is as required. Now consider a successor case ${\rm ht}(p)=\alpha+1$. Let $p=\langle\zeta^*,\tau^*,n^*,u^*,\langle p_\xi,v_\xi:\xi<\theta\rangle\rangle$, and let $\xi<\theta$ be ζ^* if $\alpha\in X$, and be ζ^*+1 otherwise. Apply the inductive hypothesis to p_ξ and $X\cap\alpha$ to get suitable $i\in u^{p_\xi}$, and note that this i works for p and X too.

6), 7) Straightforward. □

Definition 4. We say that conditions $p,q\in\mathbb{P}^\theta_\lambda$ are *isomorphic* if ${\rm ht}(p)={\rm ht}(q)$, ${\rm otp}(u^p)={\rm otp}(u^q)$, and if $H:u^p\longrightarrow u^q$ is the order isomorphism, then for every $\beta<{\rm ht}(p)$

$$(\forall j\in u^p)(h^p(j,\beta)=h^q(H(j),\beta)\ \&\ g^p(j,\beta)=g^p(H(j),\beta)).$$

[In this situation we may say that H is the isomorphism from p to q.]

Lemma 5. *Suppose that* $q_0,q_1\in\mathbb{P}^\theta_\lambda$ *are isomorphic conditions and* H *is the isomorphism from* q_0 *to* q_1*.*

(1) *If* $\mathrm{ht}(q_0) = \mathrm{ht}(q_1) = \gamma$ *is a limit ordinal,* $q_\ell = \langle q^\ell_\xi : \xi < \gamma\rangle$ *(for* $\ell < 2$*), then* $H \restriction u^{q^0_\xi}$ *is an isomorphism from* q^0_ξ *to* q^1_ξ.
(2) *If* $\mathrm{ht}(q_0) = \mathrm{ht}(q_1) = \alpha + 1$, $\alpha < \lambda$, *and* $q_\ell = \langle \zeta^*_\ell, \tau^*_\ell, n^*_\ell, u^*_\ell, \langle q^\ell_\xi, v^\ell_\xi : \xi < \theta\rangle\rangle$ *(for* $\ell < 2$*), then* $\zeta^*_0 = \zeta^*_1$, $\tau^*_0 = \tau^*_1$, $n^*_0 = n^*_1$, $H \restriction u^{q^0_\xi}$ *is an isomorphism from* q^0_ξ *to* q^1_ξ *and* $H[v^0_\xi] = v^1_\xi$ *(for* $\xi < \theta$*).*
(3) $F^{q_0} = \{f \circ H : f \in F^{q_1}\}$.
(4) *Assume* $p_0 \le q_0$. *Then there is a unique condition* $p_1 \le q_1$ *such that* $H \restriction u^{p_0}$ *is the isomorphism from* p_0 *to* p_1.
[The condition p_1 *will be called* $H(p_0)$*.]*

Proof. 1), 2) Straightforward (for (1) use Lemma 3(7)).
3), 4) Easy inductions on $\mathrm{ht}(q_0)$ using (1), (2) above. □

Definition 6. By induction on $\alpha < \lambda$, for conditions $p, q \in P^{\theta,\lambda}_\alpha$ such that $p \le^\alpha q$, we define *the* p*–transformation* $T_p(q)$ *of* q.

- If $\alpha = 0$ (so necessarily $p = q$) then $T_p(q) = p$.
- Assume that $\mathrm{ht}(q) = \alpha + 1$, $q = \langle \zeta^*, \tau^*, n^*, u^*, \langle q_\xi, v_\xi : \xi < \theta\rangle\rangle$.
  If $p \le q_\xi$ for some $\xi < \theta$, then let ξ^* be such that $p \le q_{\xi^*}$. Next for $\xi < \theta$ let $q'_\xi = T_{H_{\xi^*,\xi}(p)}(q_\xi)$, where $H_{\xi^*,\xi}$ is the isomorphism from q_{ξ^*} to q_ξ. Define $T_p(q) = \langle \xi^*, \tau^*, n^*, u^*, \langle q'_\xi, v_\xi : \xi < \theta\rangle\rangle$.
  Suppose now that $p = \langle \zeta^{**}, \tau^*, n^*, u^*, \langle p_\xi, v_\xi : \xi < \theta\rangle\rangle$ and $u^{p_\xi} = u^{q_\xi}$, $p_\xi \le q_\xi$ (for $\xi < \theta$). Let $q'_\xi = T_{p_\xi}(q_\xi)$ and put $T_p(q) = \langle \zeta^{**}, \tau^*, n^*, u^*, \langle q'_\xi, v_\xi : \xi < \theta\rangle\rangle$.
- Assume now that $\mathrm{ht}(q)$ is a limit ordinal and $q = \langle q_\xi : \xi < \mathrm{ht}(q)\rangle$.
  If $\mathrm{ht}(p) < \mathrm{ht}(q)$ then $p \le q_\varepsilon$ for some $\varepsilon < \mathrm{ht}(q)$, and we may choose q'_ξ (for $\xi < \mathrm{ht}(q)$) such that $\mathrm{ht}(q'_\xi) = \xi$, $\xi < \xi' < \mathrm{ht}(q) \Rightarrow q'_\xi \le_{\mathrm{pr}} q'_{\xi'}$, and $q'_\zeta = T_p(q_\zeta)$ for $\zeta \in [\varepsilon, \mathrm{ht}(q))$. Next we let $T_p(q) = \langle q'_\zeta : \zeta < \theta\rangle$.
  If $\mathrm{ht}(p) = \mathrm{ht}(q)$, $p = \langle p_\xi : \xi < \mathrm{ht}(p)\rangle$ and $p_\xi \le q_\xi$ for $\xi > \delta$ (for some $\delta < \mathrm{ht}(p)$) then we define $T_p(q) = p$.

To show that the definition of $T_p(q)$ is correct one proves inductively (parallely to the definition of the p–transformation of q) the following facts.

Lemma 7. *Assume* $p, q \in \mathbb{P}^\theta_\lambda$, $p \le q$. *Then:*

(1) $T_p(q) \in \mathbb{P}^\theta_\lambda$, $u^{T_p(q)} = u^q$, $\mathrm{ht}(T_p(q)) = \mathrm{ht}(q)$,
(2) $p \le_{\mathrm{pr}} T_p(q) \le q \le T_p(q)$,
(3) $\mathrm{ht}(p) = \mathrm{ht}(q) \Rightarrow T_p(q) = p$,
(4) *if* $q' \in \mathbb{P}^\theta_\lambda$ *is isomorphic to* q *and* $H : u^q \longrightarrow u^{q'}$ *is the isomorphism from* q *to* q', *then* H *is the isomorphism from* $T_p(q)$ *to* $T_{H(p)}(q')$,
(5) *if* $q \le_{\mathrm{pr}} q'$ *then* $T_p(q) \le_{\mathrm{pr}} T_p(q')$.

Proposition 8. *Every* $\le_{\mathrm{pr}}$*–increasing chain in* $\mathbb{P}^\theta_\lambda$ *of length* $< \lambda$ *has a* $\le_{\mathrm{pr}}$*–upper bound, that is the partial order* $(\mathbb{P}^\theta_\lambda, \le_{\mathrm{pr}})$ *is* $(< \lambda)$*–closed.*

Let us recall that a forcing notion $(\mathbb{Q}, \le)$ is $(<\lambda)$*–strategically closed* if the second player has a winning strategy in the following game $\Game_\lambda(\mathbb{Q})$.

The game $\Game_\lambda(\mathbb{Q})$ lasts λ moves. The first player starts with choosing a condition $p^* \in \mathbb{Q}$. Later, in her i^{th} move, the first player chooses an open dense subset D_i of $\mathbb{Q}$. The second player (in his i^{th} move) picks a condition $p_i \in \mathbb{Q}$ so that $p_0 \ge p^*$,

$p_i \in D_i$ and $p_i \geq p_j$ for all $j < i$. The second player looses the play if for some $i < \lambda$ he has no legal move.

It should be clear that $(<\lambda)$–strategically closed forcing notions do not add sequences of ordinals of length less than λ. The reader interested in this kind of properties of forcing notions and iterating them is referred to [She03a], [She03b].

Proposition 9. *Assume that $\theta < \lambda$ are regular cardinals, $\lambda^{<\lambda} = \lambda$. Then $(\mathbb{P}^\theta_\lambda, \leq)$ is a $(<\lambda)$–strategically closed λ^+–cc forcing notion.*

Proof. It follows from Lemma 7(2) that if $D \subseteq \mathbb{P}^\theta_\lambda$ is an open dense set, $p \in \mathbb{P}^\theta_\lambda$, then there is a condition $q \in D$ such that $p \leq_{\rm pr} q$. Therefore, to win the game $\Game_\lambda(\mathbb{P}^\theta_\lambda)$, the second player can play so that the conditions p_i that he chooses are $\leq_{\rm pr}$–increasing, and thus there are no problems with finding $\leq_{\rm pr}$–bounds (remember Proposition 8).

Now, to show that $\mathbb{P}^\theta_\lambda$ is λ^+–cc, suppose that $\langle p_\delta : \delta < \lambda^+\rangle$ is a sequence of distinct conditions from $\mathbb{P}^\theta_\lambda$. We may find a set $A \in [\lambda^+]^{\lambda^+}$ such that

- conditions $\{p_\delta : \delta \in A\}$ are pairwise isomorphic,
- the family $\{u^{p_\delta} : \delta \in A\}$ forms a Δ–system with heart u^*,
- if $\delta_0 < \delta_1$ are from A then

$$\sup(u^*) < \min(u^{p_{\delta_0}} \setminus u^*) \leq \sup(u^{p_{\delta_0}} \setminus u^*) < \min(u^{p_{\delta_0}} \setminus u^*).$$

Take an increasing sequence $\langle \delta_\xi : \xi < \theta\rangle$ of elements of A, let $\tau^* = \mathbf{1}$, $v_\xi = \emptyset$ (for $\xi < \theta$), and look at $p = \langle 0, \tau^*, 0, u^*, \langle p_{\delta_\xi}, v_\xi : \xi < \theta\rangle\rangle$. It is a condition in $\mathbb{P}^\theta_\lambda$ stronger than all p_{δ_ξ}'s. □

Definition 10. By induction on ${\rm ht}(p)$ we define *α–components of p* (for $p \in \mathbb{P}^\theta_\lambda$, $\alpha \leq {\rm ht}(p)$).

- First we declare that the only ${\rm ht}(p)$–component of p is the p itself.
- If ${\rm ht}(p) = \beta + 1$, $p = \langle \zeta^*, \tau^*, n^*, u^*, \langle p_\xi, v_\xi : \xi < \theta\rangle\rangle$ and $\alpha = \beta$, then α–components of p are p_ξ (for $\xi < \theta$); if $\alpha < \beta$, then α–components of p are those q which are α–components of p_ξ for some $\xi < \theta$.
- If ${\rm ht}(p)$ is a limit ordinal, $p = \langle p_\xi : \xi < {\rm ht}(p)\rangle$ and $\alpha < {\rm ht}(p)$, then α–components of p are α–components of p_ξ for $\xi \in [\alpha, {\rm ht}(p))$.

Lemma 11. *Assume $p \in \mathbb{P}^\theta_\lambda$ and $\alpha < {\rm ht}(p)$.*

(1) *If q is an α–component of p then $q \leq p$, ${\rm ht}(q) = \alpha$, and for all $j_0, j_1 \in u^q$ and every $\beta \in [\alpha, {\rm ht}(p))$:*

$$h^p(j_0,\beta) \neq \theta \ \&\ h^p(j_1,\beta) \neq \theta \quad \Rightarrow \quad h^p(j_0,\beta) = h^p(j_1,\beta).$$

Moreover, for each $i \in u^p$ there is a unique α–component q of p such that $i \in u^q$ and

$$(\forall j \in u^q)(\forall \beta \in [\alpha, {\rm ht}(p)))(h^p(i,\beta) \geq \theta \ \Rightarrow\ h^p(j,\beta) \geq \theta).$$

(2) *If H is an isomorphism from p onto $p' \in \mathbb{P}^\theta_\lambda$, and q is an α–component of p, then $H(q)$ is an α–component of p'. If q_0, q_1 are α–components of p then q_0, q_1 are isomorphic.*

(3) *There is a unique α–component q of p such that $q \leq_{\rm pr} p$.*

Proof. Easy inductions on ${\rm ht}(p)$. □

Definition 12. By induction on ${\rm ht}(p)$ we define when a set $Z \subseteq \lambda$ is p–closed for a condition $p \in \mathbb{P}^\theta_\lambda$.

- If $\mathrm{ht}(p)=0$ then every $Z\subseteq\lambda$ is p–closed;
- if $\mathrm{ht}(p)$ is limit, $p=\langle p_\xi:\xi<\mathrm{ht}(p)\rangle$, then Z is p–closed provided it is p_ξ–closed for each $\xi<\mathrm{ht}(p)$;
- if $\mathrm{ht}(p)=\alpha+1$, $p=\langle\zeta^*,\tau^*,n^*,u^*,\langle p_\xi,v_\xi:\xi<\theta\rangle\rangle$ and $\alpha\notin Z$, then Z is p–closed whenever it is p_{ζ^*}–closed;
- if $\mathrm{ht}(p)=\alpha+1$, $p=\langle\zeta^*,\tau^*,n^*,u^*,\langle p_\xi,v_\xi:\xi<\theta\rangle\rangle$ and $\alpha\in Z$, then Z is p–closed provided it is p_{ζ^*}–closed and

$$\{\beta<\alpha:(\exists j\in v_{\zeta^*}\cup\{\min(u^{p_{\zeta^*}}\setminus u^*)\})(h^{p_{\zeta^*}}(j,\beta)<\theta)\}\subseteq Z.$$

Lemma 13. (1) *If $p\in\mathbb{P}^\theta_\lambda$ and $w\in[\mathrm{ht}(p)]^{<\omega}$, then there is a finite p–closed set $Z\subseteq\mathrm{ht}(p)$ such that $w\subseteq Z$.*

(2) *If $p,q\in\mathbb{P}^\theta_\lambda$ are isomorphic and Z is p–closed, then Z is q–closed. If Z is p–closed, $\alpha<\mathrm{ht}(p)$ and p^* is an α–component of p, then $Z\cap\alpha$ is p^*–closed.*

Proof. Easy inductions on $\mathrm{ht}(p)$ (remember Lemma 3(2)). □

Definition 14. Suppose that $p\in\mathbb{P}^\theta_\lambda$ and $Z\subseteq\mathrm{ht}(p)$ is a finite p–closed set. Let $Z=\{\alpha_0,\dots,\alpha_{k-1}\}$ be the increasing enumeration.

(1) We define

$$U[p,Z]\stackrel{\rm def}{=}\{j\in u^p:(\forall\beta<\mathrm{ht}(p))(h^p(j,\beta)<\theta\ \Rightarrow\ \beta\in Z)\}.$$

(2) We let

$$\Upsilon_p(Z)=\langle\zeta_\ell,\tau_\ell,n_\ell,\langle g_\ell,h^\ell_0,\dots,h^\ell_{n_\ell-1}\rangle:\ell<k\rangle,$$

where, for $\ell<k$, ζ_ℓ is an ordinal below θ, τ_ℓ is a Boolean term, $n_\ell<\omega$ and $g_\ell,h^\ell_0,\dots,h^\ell_{n_\ell-1}:\ell\longrightarrow 2$, and they all are such that for every (equivalently: some) $\alpha_\ell+1$–component $q=\langle\zeta^*,\tau^*,n^*,u^*,\langle q_\xi,v_\xi:\xi<\theta\rangle\rangle$ of p we have: $\zeta_\ell=\zeta^*$, $\tau_\ell=\tau^*$, $n_\ell=n^*$ and if $v_\xi=\{j_0,\dots,j_{n_\ell-1}\}$ (the increasing enumeration) then

$$(\forall m<n_\ell)(\forall\ell'<\ell)(h^\ell_m(\ell')=h^q(j_m,\alpha_{\ell'})),$$

and if $i_0=\min(u^{q_{\zeta^*}}\setminus u^*)$ then $(\forall\ell'<\ell)(g_\ell(\ell')=h^q(i_0,\alpha_{\ell'}))$. (Note that $\zeta_\ell,\tau_\ell,n_\ell,\ g_\ell,h^\ell_0,\dots,h^\ell_{n_\ell-1}$ are well-defined by Lemma 11. Necessarily, for all $m<n_\ell$ and $\beta\in\alpha_\ell\setminus Z$ we have $h^q(i_0,\beta),h^q(j_m,\beta)\geq\theta$; remember that Z is p–closed.)

Note that if $Z\subseteq\mathrm{ht}(p)$ is a finite p–closed set, $\alpha=\max(Z)$ and p^* is the $\alpha+1$–component of p satisfying $p^*\leq_{\rm pr}p$ (see 11(3)), then $U[p,Z]\subseteq u^{p^*}$.

Lemma 15. *Suppose that $p\in\mathbb{P}^\theta_\lambda$ and $Z_0,Z_1\subseteq\mathrm{ht}(p)$ are finite p–closed sets such that $\Upsilon_p(Z_0)=\Upsilon_p(Z_1)$. Then $\mathrm{otp}(U[p,Z_0])=\mathrm{otp}(U[p,Z_1])$, and the order preserving isomorphism $\pi:U[p,Z_0]\longrightarrow U[p,Z_1]$ satisfies*

($\otimes$) $(\forall\ell<k)(h^p(i,\alpha^0_\ell)=h^p(\pi(i),\alpha^1_\ell))$,
where $\{\alpha^x_0,\dots,\alpha^x_{k-1}\}$ is the increasing enumeration of Z_x (for $x=0,1$).

Proof. We prove this by induction on $|Z_0|=|Z_1|$ (for all p,Z_0,Z_1 satisfying the assumptions).

Step $|Z_0|=|Z_1|=1$; $Z_0=\{\alpha^0_0\}$, $Z_1=\{\alpha^1_0\}$.
Take the α^x_0+1–component q_x of p such that $q_x\leq_{\rm pr}p$. Then, for $x=0,1$, $q_x=\langle\zeta,\tau,n,u^x,\langle q^x_\xi,v^x_\xi:\xi<\theta\rangle\rangle$, and for each $i\in v^x_\xi$, $\beta<\alpha^x_0$ we have $h^{q^x_\xi}(i,\beta)\geq\theta$.

Also, if $i^x_0 = \min(u^{q^x_\zeta}\setminus u^x)$ and $\beta<\alpha^x_0$, then $h^{q^x_\zeta}(i^x_0,\beta)\geq\theta$. Consequently, $n=|v^x_\xi|\leq 1$, and if $n=1$ then $\{i^x_0\}=v^x_\zeta$ (remember Lemma 3(4)). Moreover,

$$U[p,Z_x]=U[q_x,Z_x]=\{H^x_{\xi,\zeta}(i^x_0):\xi<\theta\},$$

where $H^x_{\xi,\zeta}$ is the isomorphism from q^x_ζ to q^x_ξ. Now it should be clear that the mapping $\pi: H^0_{\xi,\zeta}(i^0_0)\mapsto H^1_{\xi,\zeta}(i^1_0): U[p,Z_0]\longrightarrow U[p,Z_1]$ is the order preserving isomorphism (remember clause (γ) of the definition of $P^*_{\alpha+1}$), and it has the property described in $(\otimes)$.

Step $|Z_0|=|Z_1|=k+1$; $Z_0=\{\alpha^0_0,\ldots,\alpha^0_k\}$, $Z_1=\{\alpha^1_0,\ldots,\alpha^1_k\}$.
Let

$$\Upsilon_p(Z_0)=\Upsilon_p(Z_1)=\langle\zeta_\ell,\tau_\ell,n_\ell,\langle g_\ell,h^\ell_0,\ldots,h^\ell_{n_\ell-1}\rangle:\ell\leq k\rangle.$$

For $x=0,1$, let $q_x=\langle\zeta,\tau,n,u^x,\langle q^x_\xi,v^x_\xi:\xi<\theta\rangle\rangle$ be the α^x_k+1–component of p such that $q_x\leq_{\rm pr}p$. The sets $Z_x\cap\alpha^x_k$ (for $x=0,1$) are q^x_ξ–closed for every $\xi<\theta$, and clearly $\Upsilon_p(Z_0\cap\alpha^0_k)=\Upsilon_p(Z_1\cap\alpha^1_k)$. Hence, by the inductive hypothesis, ${\rm otp}(U[q^0_\xi,Z_0\setminus\{\alpha^0_k\}])={\rm otp}(U[q^1_\xi,Z_1\setminus\{\alpha^1_k\}])$ (for each $\xi<\theta$), and the order preserving mappings $\pi_\xi:U[q^0_\xi,Z_0\setminus\{\alpha^0_k\}]\longrightarrow U[q^1_\xi,Z_1\setminus\{\alpha^1_k\}]$ satisfy the demand in $(\otimes)$. Let $i^x_\xi=\min(u^{q^x_\xi}\setminus u^x)$. Then, as q^x_ξ and q^x_ζ are isomorphic and the isomorphism is the identity on u^x, we have $(\forall\ell<k)(h^p(i^x_\xi,\alpha^x_\ell)=g_k(\ell))$. Hence $\pi_\xi(i^0_\xi)=i^1_\xi$, and therefore $\pi_\xi[u^0\cap U[q^0_\xi,Z_0\setminus\{\alpha^0_k\}]]=u^1\cap U[q^1_\xi,Z_1\setminus\{\alpha^1_k\}]$. But since the mappings π_ξ are order preserving, the last equality implies that $\pi_\xi\restriction(u^0\cap U[q^0_\xi,Z_0\setminus\{\alpha^0_k\}])=\pi_\zeta\restriction(u^0\cap U[q^0_\zeta,Z_0\setminus\{\alpha^0_k\}])$, and hence $\pi=\bigcup\limits_{\xi<\theta}\pi_\xi$ is a function, and it is an order isomorphism from $U[q_0,Z_0]=U[p,Z_0]$ onto $U[q_1,Z_1]=U[p,Z_1]$ satisfying $(\otimes)$. $\square$

2. The algebra and why it is OK (in $\mathbf{V}^{\mathbb{P}^\theta_\lambda}$)

Let $\dot{\mathbb{B}}^\theta_\lambda$ and $\dot{U}$ be $\mathbb{P}^\theta_\lambda$–names such that

$$\Vdash_{\mathbb{P}^\theta_\lambda}\text{“ }\dot{\mathbb{B}}^\theta_\lambda=\bigcup\{\mathbb{B}^p:p\in\Gamma_{\mathbb{P}^\theta_\lambda}\}\text{ ”}\qquad\text{and}\qquad\Vdash_{\mathbb{P}^\theta_\lambda}\text{“ }\dot{U}=\bigcup\{u^p:p\in\Gamma_{\mathbb{P}^\theta_\lambda}\}\text{ ”}.$$

Note that $\dot{U}$ is (a name for) a subset of λ^+. Let $\dot{F}$ be a $\mathbb{P}^\theta_\lambda$–name such that

$$\Vdash_{\mathbb{P}^\theta_\lambda}\text{“ }\dot{F}=\{f\in 2^{\dot{U}}:(\forall p\in\Gamma_{\mathbb{P}^\theta_\lambda})(f\restriction u^p\in\dot{F}^p)\}\text{ ”}.$$

Proposition 16. *Assume $\theta<\lambda$ are regular, $\lambda^{<\lambda}=\lambda$. Then in $\mathbf{V}^{\mathbb{P}^\theta_\lambda}$:*

1. *$\dot{F}$ is a non-empty closed subset of $2^{\dot{U}}$, and $\dot{\mathbb{B}}^\theta_\lambda$ is the Boolean algebra generated $\mathbb{B}_{(\dot{U},\dot{F})}$ (see Definition 1);*
2. *$|\dot{U}|=|\dot{\mathbb{B}}^\theta_\lambda|=\lambda^+$;*
3. *For every subalgebra $\mathbb{B}\subseteq\dot{\mathbb{B}}^\theta_\lambda$ of size λ^+ we have* ${\rm Depth}^+(\mathbb{B})>\theta$.

Proof. 2) Note that if $p\in\mathbb{P}^\theta_\lambda$, $\sup(u^p)<j<\lambda^+$ then there is a condition $q\geq p$ such that $j\in u^q$. Hence $\Vdash|\dot{U}|=\lambda^+$. To show that, in $\mathbf{V}^{\mathbb{P}^\theta_\lambda}$, the algebra $\dot{\mathbb{B}}^\theta_\lambda$ is of size λ^+ it is enough to prove the following claim.

Claim 16.1. *Let $p\in\mathbb{P}^\theta_\lambda$, $j\in u^p$. Then $x_j\notin\langle x_i:i\in j\cap u^p\rangle_{\mathbb{B}^p}$.*

Proof of the claim. Suppose not, and let p,j be a counterexample with the smallest possible ${\rm ht}(p)$. Necessarily, ${\rm ht}(p)$ is a successor ordinal, say ${\rm ht}(p)=\alpha+1$. So let $p=\langle\zeta^*,\tau^*,n^*,u^*,\langle p_\xi,v_\xi:\xi<\theta\rangle\rangle$ and suppose that $v\in[u^p\cap j]^{<\omega}$ is such that $x_j\in\langle x_i:i\in v\rangle_{\mathbb{B}^p}$. If $j\in u^*$ then $v\subseteq u^*$ and we immediately get a contradiction (applying the inductive hypothesis to p_{ζ^*}). So let $\xi < \theta$ be such that $j \in u^{p_\xi} \setminus u^*$. We know that $x_j \notin \langle x_i : i \in u^* \cup (v \cap u^{p_\xi}) \rangle_{\mathbb{B}^{p_\xi}}$ (remember clause (γ) of the definition of $P^*_{\alpha+1}$), so we may take functions $f_0, f_1 \in F^{p_\xi}$ such that $f_0 \restriction (u^* \cup (v \cap u^{p_\xi})) = f_1 \restriction (u^* \cup (v \cap u^{p_\xi}))$, $f_0(j) = 0$, $f_1(j) = 1$. Let $g_0, g_1 : u^p \longrightarrow 2$ be such that $g_\ell \restriction u^{p_\xi} = f_\ell$, $g_\ell \restriction u^{p_\zeta} = f_0 \circ H_{\zeta,\xi}$ for $\zeta \neq \xi$ (where $H_{\zeta,\xi}$ is the order isomorphism from u^{p_ζ} to u^{p_ξ}). Now one easily checks that $g_0, g_1 \in F^p$ (remember the definition of the term $\sigma_{\rm maj}$). By our choices, $g_0(i) = g_1(i)$ for all $i \in v$, and $g_0(j) \neq g_1(j)$, and this is a clear contradiction with the choice of i and v. $\square$

3) Suppose that $\langle \dot{a}_\xi : \xi < \lambda^+ \rangle$ is a $\mathbb{P}^\theta_\lambda$–name for a λ^+–sequence of distinct members of $\dot{\mathbb{B}}^\theta_\lambda$ and let $p \in \mathbb{P}^\theta_\lambda$. Applying standard cleaning procedures we find a set $A \subseteq \lambda^+$ of the order type θ, an ordinal $\alpha < \lambda$ and τ^*, n^*, u^* and $\langle p_\xi, v_\xi : \xi \in A \rangle$ such that $p \leq p_\xi$, $\mathrm{ht}(p_\xi) = \alpha$, $p_\xi \Vdash \dot{a}_\xi = \tau^*(x_i : i \in v_\xi)$ and

$$q \stackrel{\rm def}{=} \langle 0, \tau^*, n^*, u^*, \langle p_\xi, v_\xi : \xi \in A \rangle \rangle \in P^*_{\alpha+1},$$

where A is identified with θ by the increasing enumeration (so we will think $A = \theta$). For $\xi < \theta$ let $\tau_\xi = \tau^*(x_i : i \in v_\xi) \in \mathbb{B}^{p_\xi}$. Since $\dot{a}_\xi$ were (forced to be) distinct we know that $\mathbb{B}^q \models \tau_\xi \neq \tau_\zeta$ for distinct ξ, ζ. Hence $\tau_\xi \notin \langle x_i : i \in u^* \rangle_{\mathbb{B}^{p_\xi}}$ (for each ξ) and therefore we may find functions $f^0_\xi, f^1_\xi \in F^{p_\xi}$ such that $f^0_\xi \restriction u^* = f^1_\xi \restriction u^*$, and $f^0_\xi(\tau_\xi) = 0$, $f^1_\xi(\tau_\xi) = 1$, and if $\xi < \zeta < \theta$, and $H_{\xi,\zeta}$ is the isomorphism from p_ξ to p_ζ, then $f^\ell_\xi = f^\ell_\zeta \circ H_{\xi,\zeta}$. Now fix $\xi < \zeta < \theta$ and let

$$g \stackrel{\rm def}{=} \bigcup_{\alpha \leq 3\cdot\xi+2} f^0_\alpha \cup \bigcup_{3\cdot\xi+2<\alpha<\theta} f^1_\alpha.$$

It should be clear that g is a function from u^q to 2, and moreover $g \in F^q$. Also easily

$$g(\sigma_{\rm maj}(\tau_{3\cdot\xi}, \tau_{3\cdot\xi+1}, \tau_{3\cdot\xi+2})) = 0 \quad \text{and} \quad g(\sigma_{\rm maj}(\tau_{3\cdot\zeta}, \tau_{3\cdot\zeta+1}, \tau_{3\cdot\zeta+2}))\} = 1.$$

Hence we may conclude that

$$\mathbb{B}^q \models \sigma_{\rm maj}(\tau_{3\cdot\xi}, \tau_{3\cdot\xi+1}, \tau_{3\cdot\xi+2}) < \sigma_{\rm maj}(\tau_{3\cdot\zeta}, \tau_{3\cdot\zeta+1}, \tau_{3\cdot\zeta+2})$$

for $\xi < \zeta < \theta$ (remember the definition of F^q and Proposition 2). Consequently we get $q \Vdash \mathrm{Depth}^+(\langle \dot{a}_\xi : \xi < \lambda^+ \rangle_{\dot{\mathbb{B}}^\theta_\lambda}) > \theta$, finishing the proof. $\square$

Theorem 17. *Assume $\theta < \lambda$ are regular, $\lambda = \lambda^{<\lambda}$. Then $\Vdash_{\mathbb{P}^\theta_\lambda} \mathrm{Depth}(\dot{\mathbb{B}}^\theta_\lambda) = \theta$.*

Proof. By Proposition 16 we know that $\Vdash \mathrm{Depth}^+(\dot{\mathbb{B}}^\theta_\lambda) > \theta$, so what we have to show is that there are no increasing sequences of length θ^+ of elements of $\dot{\mathbb{B}}^\theta_\lambda$. We will show this under an additional assumption that $\theta^+ < \lambda$ (after the proof is carried out, it will be clear how one modifies it to deal with the case $\lambda = \theta^+$). Due to this additional assumption, and since the forcing notion $\mathbb{P}^\theta_\lambda$ is $(<\lambda)$–strategically closed (by Proposition 9), it is enough to show that $\mathrm{Depth}(\mathbb{B}^p) \leq \theta$ for each $p \in \mathbb{P}^\theta_\lambda$.

So suppose that $p \in \mathbb{P}^\theta_\lambda$ is such that $\mathrm{Depth}(\mathbb{B}^p) \geq \theta^+$. Then we find a Boolean term τ, an integer n and sets $w_\rho \in [u^p]^n$ (for $\rho < \theta^+$) such that

$$\rho_0 < \rho_1 < \theta^+ \quad \Rightarrow \quad \mathbb{B}^p \models \tau(x_i : i \in w_{\rho_0}) < \tau(x_i : i \in w_{\rho_1}).$$

For each $\rho < \theta^+$ use Lemma 13 to choose a finite p–closed set $Z_\rho \subseteq \mathrm{ht}(p)$ containing the set

$$\{\beta < \mathrm{ht}(p) : (\exists j \in w_\rho)(h^p(j, \beta) < \theta)\}.$$

Look at $\Upsilon_p(Z_\rho)$ (see Definition 14). There are only θ possibilities for the values of $\Upsilon_p(Z_\rho)$, so we find $\rho_0<\rho_1<\theta^+$ such that

(i) $|Z_{\rho_0}|=|Z_{\rho_1}|$, $\Upsilon_p(Z_{\rho_0})=\Upsilon_p(Z_{\rho_1})=\langle\zeta_\ell,\tau_\ell,n_\ell,\langle g_\ell,h^\ell_0,\ldots,h^\ell_{n_\ell-1}\rangle:\ell<k\rangle$,

(ii) if $\pi^*:Z_{\rho_0}\longrightarrow Z_{\rho_1}$ is the order isomorphism then $\pi^*\restriction Z_{\rho_0}\cap Z_{\rho_1}$ is the identity on $Z_{\rho_0}\cap Z_{\rho_1}$,

(iii) if $\pi:U[p,Z_{\rho_0}]\longrightarrow U[p,Z_{\rho_1}]$ is the order isomorphism, then $\pi[w_{\rho_0}]=w_{\rho_1}$.

Note that, by Lemma 15, $\mathrm{otp}(U[p,Z_{\rho_0}])=\mathrm{otp}(U[p,Z_{\rho_1}])$ and the order isomorphism π satisfies

$$(\forall j\in U[p,Z_{\rho_0}])(\forall\beta\in Z_{\rho_0})(h^p(j,\beta)=h^p(\pi(j),\pi^*(\beta))),$$

and hence π is the identity on $U[p,Z_{\rho_0}]\cap U[p,Z_{\rho_1}]$ (remember Lemma 3).

For a function $f\in F^p$ let $G^{\rho_0}_{\rho_1}(f):u^p\longrightarrow 2$ be defined by

$$G^{\rho_0}_{\rho_1}(f)(j)=\begin{cases} f(\pi(j)) & \text{if} \quad j\in U[p,Z_{\rho_0}],\\ f(\pi^{-1}(j)) & \text{if} \quad j\in U[p,Z_{\rho_1}]\setminus U[p,\rho_0],\\ 0 & \qquad\ \text{otherwise.}\end{cases}$$

Claim 17.1. *For each $f\in F^p$, $G^{\rho_0}_{\rho_1}(f)\in F^p$.*

Proof of the claim. By induction on $\alpha\le\mathrm{ht}(p)$ we show that for each α–component q of p, the restriction $G^{\rho_0}_{\rho_1}(f)\restriction u^q$ is in F^q.

If α is limit, we may easily use the inductive hypothesis to show that, for any α–component q of p, $G^{\rho_0}_{\rho_1}(f)\restriction u^q\in F^q$.

Assume $\alpha=\beta+1$ and let $q=\langle\zeta^*,\tau^*,n^*,u^*,\langle q_\xi,v_\xi:\xi<\theta\rangle\rangle$ be an α–component of p. We will consider four cases.

Case 1: $\beta\notin Z_{\rho_0}\cup Z_{\rho_1}$.
Then $(U[p,Z_{\rho_0}]\cup U[p,Z_{\rho_1}])\cap u^q\subseteq u^{q_{\zeta^*}}$ and $G^{\rho_0}_{\rho_1}(f)\restriction(u^{q_\xi}\setminus u^*)\equiv 0$ for each $\xi\neq\zeta^*$. Since, by the inductive hypothesis, $G^{\rho_0}_{\rho_1}(f)\restriction u^{q_\xi}\in F^{q_\xi}$ for each $\xi<\theta$, we may use the definition of $P^*_{\beta+1}$ and conclude that $G^{\rho_0}_{\rho_1}(f)\restriction u^q\in F^q$ (remember the definition of the term $\sigma_{\rm maj}$).

Case 2: $\beta\in Z_{\rho_0}\setminus Z_{\rho_1}$.
Let $Z_{\rho_0}=\{\alpha_0,\ldots,\alpha_{k-1}\}$ be the increasing enumeration. Then $\beta=\alpha_\ell$ for some $\ell<k$ and $\zeta^*=\zeta_\ell$, $\tau^*=\tau_\ell$, $n^*=n_\ell$. Moreover, if $v_\xi=\{j^\xi_0,\ldots,j^\xi_{n_\ell-1}\}$ (the increasing enumeration), $\xi<\theta$, then for $m<n_\ell$:

$$(\forall\ell'<\ell)(h^\ell_m(\alpha_{\ell'})=h^q(j^\xi_m,\alpha_{\ell'}))\quad\text{and}\quad(\forall\gamma\in\beta\setminus Z_{\rho_0})(h^q(j^\xi_m,\gamma)\ge\theta).$$

Note that $U[p,Z_{\rho_1}]\cap u^q\subseteq u^{q_{\zeta^*}}$, so if $U[p,Z_{\rho_0}]\cap u^q=\emptyset$, then we may proceed as in the previous case. Therefore we may assume that $U[p,Z_{\rho_0}]\cap u^q\neq\emptyset$. So, for each $\gamma\in Z_{\rho_0}\setminus\alpha$ we may choose $i_\gamma\in U[p,Z_{\rho_0}]\cap u^q$ such that

$$(\forall i\in U[p,Z_{\rho_0}]\cap u^q)(h^p(i,\gamma)\neq\theta\ \Rightarrow\ h^p(i,\gamma)=h^p(i_\gamma,\gamma))$$

(remember Lemma 11(1)). Let $i^*=\max\{i_\gamma:\gamma\in Z_{\rho_0}\setminus\alpha\}$ (if $\beta=\max(Z_{\rho_0})$, then let i^* be any element of $U[p,Z_{\rho_0}]\cap u^q$). Note that then

$$(\forall i\in U[p,Z_{\rho_0}]\cap u^q)(\forall\gamma\in Z_{\rho_0}\setminus\alpha)(h^p(i,\gamma)\neq\theta\ \Rightarrow\ h^p(i,\gamma)=h^p(i^*,\gamma))$$

[Why? Remember Lemma 11(1) and the clause (γ) of the definition of $P^*_{\beta+1}$.] By Lemma 11, we find a $(\pi^*(\beta)+1)$–component $q'=\langle\zeta',\tau',n',u',\langle q'_\varepsilon,v'_\varepsilon:\varepsilon<\theta\rangle\rangle$ of p such that $\pi(i^*)\in u^{q'}$ and

$$(\forall j\in u^{q'})(\forall\gamma\in(\pi^*(\beta),\mathrm{ht}(p)))(h^p(\pi(i^*),\gamma)\ge\theta\ \Rightarrow\ h^p(j,\gamma)\ge\theta).$$

We claim that then

($\boxtimes$) $\quad (\forall j \in U[p, Z_{\rho_0}] \cap u^q)(\pi(j) \in u^{q'} \cap U[p, Z_{\rho_1}])$.

Why? Fix $j \in U[p, Z_{\rho_0}] \cap u^q$. Let r, r' be components of p such that $r \leq_{\rm pr} p$, $r' \leq_{\rm pr} p$, ${\rm ht}(r) = \beta + 1$, ${\rm ht}(r') = \pi^*(\beta) + 1$ (so r and q, and r', q', are isomorphic). The sets $Z_{\rho_0} \cap (\beta + 1)$ and $Z_{\rho_1} \cap (\pi^*(\beta) + 1)$ are p–closed, and they have the same values of Υ, and therefore $U[p, Z_{\rho_0} \cap (\beta + 1)]$ and $U[p, Z_{\rho_1} \cap (\pi^*(\beta) + 1)]$ are (order) isomorphic. Also, these two sets are included in u^r and $u^{r'}$, respectively. So looking back at our j, we may successively choose $j_0 \in u^r \cap U[p, Z_{\rho_0} \cap (\beta + 1)]$, $j_1 \in u^{r'} \cap U[p, Z_{\rho_1} \cap (\pi^*(\beta) + 1)]$, and $j^* \in u^q$ such that

- $(\forall \gamma \leq \beta)(h^q(j, \gamma) = h^r(j_0, \gamma))$,
- $(\forall \ell' \leq \ell)(h^r(j_0, \alpha_{\ell'}) = h^{r'}(j_1, \pi^*(\alpha_{\ell'})))$, and
- $(\forall \gamma \leq \pi^*(\beta))(h^{r'}(j, \gamma) = h^{q'}(j^*, \gamma))$.

Then we have

$$(\forall \ell' \leq \ell)(h^q(j, \alpha_{\ell'}) = h^{q'}(j^*, \pi^*(\alpha_{\ell'})) \quad \text{and} \quad (\forall \gamma \in \pi^*(\beta) \setminus Z_{\rho_1})(h^{q'}(j^*, \gamma) \geq \theta).$$

To conclude ($\boxtimes$) it is enough to show that $\pi(j) = j^*$. If this equality fails, then there is $\gamma < {\rm ht}(p)$ such that $\theta \neq h^p(\pi(j), \gamma) \neq h^p(j^*, \gamma) \neq \theta$. If $\gamma \leq \pi^*(\beta)$, then necessarily $\gamma \in Z_{\rho_1}$, and this is impossible (remember $h^p(j, \alpha_{\ell'}) = h^p(\pi(j), \pi^*(\alpha_{\ell'}))$ for $\ell' \leq \ell$). So $\gamma > \pi^*(\beta)$. If $h^p(\pi(j), \gamma) = \theta + 1$, then $h^p(j^*, \gamma) < \theta$ and (by the choice of q') $h^p(\pi(i^*), \gamma) < \theta$. Then $\gamma \in Z_{\rho_1}$ and $h^p(i^*, (\pi^*)^{-1}(\gamma)) < \theta$, and also $h^p(i^*, (\pi^*)^{-1}(\gamma)) = h^p(j, (\pi^*)^{-1}(\gamma)) = \theta + 1$ (by the choice of i^*), a contradiction. Thus necessarily $h^p(\pi(j), \gamma) < \theta$ (so $\gamma \in Z_{\rho_1}$) and therefore

$$\theta > h^p(j, (\pi^*)^{-1}(\gamma)) = h^p(i^*, (\pi^*)^{-1}(\gamma)) = h^p(\pi(i^*), \gamma) = h^p(j^*, \gamma)$$

(as the last is not θ), again a contradiction. Thus the statement in ($\boxtimes$) is proven.

Now we may finish considering the current case. By the definition of the function Υ (and by the choice of ρ_0, ρ_1) we have

$$\zeta' = \zeta_\ell, \quad \tau' = \tau_\ell, \quad n' = n_\ell, \quad \text{and } \pi[v_\xi] = v'_\xi \text{ for } \xi < \theta$$

(and $\pi \restriction v_\xi$ is order–preserving). Therefore

$$G^{\rho_0}_{\rho_1}(f)(\tau^*(x_i : i \in v_\xi)) = f(\tau'(x_i : i \in v'_\xi)) \qquad \text{(for every } \xi < \theta).$$

By the inductive hypothesis, $G^{\rho_0}_{\rho_1}(f) \restriction u^{q_\xi} \in F^{q_\xi}$ (for $\xi < \theta$), so as $f \in F^p$ (and hence $f \restriction u^{q'} \in F^{q'}$) we may conclude now that $G^{\rho_0}_{\rho_1}(f) \restriction u^q \in F^q$.

Case 3: $\beta \in Z_{\rho_1} \setminus Z_{\rho_0}$
Similar.

Case 3: $\beta \in Z_{\rho_0} \cap Z_{\rho_1}$
If $U[p, Z_{\rho_0}] \cap u^q = \emptyset = U[p, Z_{\rho_1}] \cap u^q$, then $G^{\rho_0}_{\rho_1}(f) \restriction u^q \equiv 0$ and we are easily done. If one of the intersections is non-empty, then we may follow exactly as in the respective case (2 or 3). $\square$

Now we may conclude the proof of the theorem. Since

$$\mathbb{B}^p \models \tau(x_i : i \in w_{\rho_0}) < \tau(x_i : i \in w_{\rho_1}),$$

we find $f \in F^p$ such that $f(\tau(x_i : i \in w_{\rho_0})) = 0$ and $f(\tau(x_i : i \in w_{\rho_1})) = 1$. It should be clear from the definition of the function $G^{\rho_0}_{\rho_1}(f)$ (and the choice of ρ_0, ρ_1) that

$$G^{\rho_0}_{\rho_1}(f)(\tau(x_i : i \in w_{\rho_0})) = 1 \quad \text{and} \quad G^{\rho_0}_{\rho_1}(f)(\tau(x_i : i \in w_{\rho_1})) = 0.$$

But it follows from Claim 17.1 that $G^{\rho_0}_{\rho_1}(f) \in F^p$, a contradiction. $\square$

Conclusion 18. It is consistent that for some uncountable cardinal θ there is a Boolean algebra $\mathbb{B}$ of size $(2^\theta)^+$ such that

$$\mathrm{Depth}(\mathbb{B}) = \theta \quad \text{but} \quad (\omega, (2^\theta)^+) \notin \mathrm{Depth}_{\mathrm{Sr}}(\mathbb{B}).$$

Problem 19. Assume $\theta < \lambda = \lambda^{<\lambda}$ are regular cardinals. Does there exist a Boolean algebra $\mathbb{B}$ such that $|\mathbb{B}| = \lambda^+$ and for every subalgebra $\mathbb{B}' \subseteq \mathbb{B}$ of size λ^+ we have $\mathrm{Depth}(\mathbb{B}') = \theta$?

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