

GROUPS ISOMORPHIC TO ALL THEIR NON-TRIVIAL NORMAL SUBGROUPS

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Abstract

In answer to a question of P. Hall, we supply another construction of a group which is isomorphic to each of its non-trivial normal subgroups.

1 Introduction

In the early seventies, the following question was asked by Philip Hall of J.C. Lennox (see [4]) and appeared later in [3] in the form :

Must a non-trivial group, which is isomorphic to each of its non-trivial normal subgroups, be either free of infinite rank, simple or infinite cyclic?

This was answered in the negative by Obraztsov in [6], where he used the technique of graded diagrams developed by Ol'shanskii in [7].

In this note we provide a different construction of a non-trivial group which is isomorphic to each of its non-trivial normal subgroups. This construction makes special use of the notion of homogeneous and universal structures in model theory and well-known facts about HNN extensions and free products with amalgamation. In our view, the following construction yields a transparent proof of the existence of such groups and shows how basic ideas from model theory can be used to resolve questions in other areas of mathematics. While Obraztsov's proof is based on the deep theory of

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the geometry of defining relations in groups, our approach is accessible to a graduate student with basic knowledge of model theory and group theory. Our main result is the following:

Main Theorem. Let κ be a cardinal such that $\kappa = \kappa^{<\kappa}$. Then there exists a group G of cardinal κ with a descending principal series

$$G \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

such that $\bigcap_{n < \omega} G_n = 1$ and any normal subgroup of G is some G_i , which is isomorphic to G .

2 Homogeneous and universal structures

In this section we consider a class of models \mathcal{M} and give a brief summary of its relevant properties. Notation and terminology observed here follow that in [1].

Let \mathcal{M} be a collection of relational structures in the language L of groups with constant symbol 1 (whose value in any model will be the group identity), the axioms of groups and an additional sequence P_n ($n \in \omega$) of unary predicates having the following properties: If A is a group and $P_n A$ denotes P_n applied to A , then $P_n A$ is a normal subgroup of A (denoted $P_n A \triangleleft A$) and $P_{n+1} A \subseteq P_n A$. In terms of first order language, this is to say that

- (1) There exists $1 \in P_n A$.
- (2) If $x, y \in P_n A$, then $xy^{-1} \in P_n A$.
- (3) If $x \in P_n A$ and $z \in A$, then $z^{-1}xz \in P_n A$.
- (4) If $x \in P_{n+1} A$, then $x \in P_n A$.

Thus the class \mathcal{M} of all such L -models is

$$\mathcal{M} = \{(P_n A)_{n < \omega} : P_n A \text{ is a group, } P_n A \triangleleft P_0 A \text{ and } P_{n+1} A \subseteq P_n A \text{ (} n < \omega)\}$$

To simplify notation, we let $P_0 A = A$ and abbreviate the sequence $(P_n A)_{n < \omega}$ of groups as $(P_n A)$.

Let $(P_n A), (P_n B) \in \mathcal{M}$. The relational structure $(P_n A)$ is said to be a *substructure* of $(P_n B)$ (equivalently, $(P_n B)$ is an *extension* of $(P_n A)$) if $P_n A = P_n B \cap P_0 A$ for all $n < \omega$. We denote this by $(P_n A) \leq (P_n B)$. Let $\varphi : P_0 A \rightarrow P_0 B$ be a map. We say that φ is a *homomorphism* of $(P_n A)$ into $(P_n B)$ if φ is a group homomorphism and $\varphi(P_n A) = P_n B \cap \varphi(P_0 A)$ ($n < \omega$). A homomorphism φ is called an *embedding*

of (P_nA) into (P_nB) if φ is one-to-one. Hence if $(P_nA) \leq (P_nB)$, then the identity map $id : (P_nA) \rightarrow (P_nB)$ is an embedding. If φ is one-to-one and onto, then φ is called an *isomorphism* and (P_nA) is said to be *isomorphic* to (P_nB) . This we denote by $(P_nA) \cong (P_nB)$. We refer to the cardinal of P_0A as the *cardinal of* (P_nA) .

Proposition 2.1 *The following properties hold in \mathcal{M} for an infinite cardinal κ :*

(I) \mathcal{M} contains structures of arbitrarily large cardinality.

(II) If $(P_nA) \in \mathcal{M}$ and $(P_nA) \cong (P_nB)$, then $(P_nB) \in \mathcal{M}$.

(III) If $(P_nA_i), (P_nB) \in \mathcal{M}$, there exist $(P_nC) \in \mathcal{M}$ and embeddings

$$\varphi_i : (P_nA_i) \hookrightarrow (P_nC) \quad (i = 1, 2).$$

(IV) (*Amalgamation Property*) Let $(P_nA), (P_nB_1), (P_nB_2) \in \mathcal{M}$ such that

$$\varphi_i : (P_nA) \hookrightarrow (P_nB_i) \quad (i = 1, 2)$$

are embeddings. Then there is some $(P_nC) \in \mathcal{M}$ and embeddings

$$\gamma_i : (P_nB_i) \hookrightarrow (P_nC) \quad (i = 1, 2) \text{ such that } \gamma_1 \circ \varphi_1 = \gamma_2 \circ \varphi_2.$$

(V) The union of any chain of structures $(P_nA) \in \mathcal{M}$ is still in \mathcal{M} .

(VI $_{\kappa}$) If $(P_nA) \in \mathcal{M}$ and $(P_nC) \leq (P_nA)$ such that (P_nC) is of cardinal $< \kappa$, then there exists $(P_nB) \in \mathcal{M}$ of cardinal $< \kappa$ such that $(P_nC) \leq (P_nB) \leq (P_nA)$.

(VII $_{\kappa}$) Each $(P_nA) \in \mathcal{M}$ has $< \kappa$ relations.

Proof. (II) This is clear by the definition of a homomorphism between elements of \mathcal{M} . (V) is obvious. (VII $_{\kappa}$) is clear, since, by definition, each element of \mathcal{M} has \aleph_0 relations.

(I) Let $P_0A = F_{\alpha}$ be a free group with rank α and $P_nA = F_{\alpha}^{(n)}$ be the n th derived group of F_{α} . Then $(P_nA) \in \mathcal{M}$ with arbitrarily large cardinality α .

(III) Take, for instance, $P_nC = P_nA_1 \oplus P_nA_2$ ($n < \omega$) and the natural injections $\varphi_i : P_0A_i \rightarrow P_0C$ ($i = 1, 2$).

(IV) Consider the free product with amalgamation $P_0C = P_0B_1 *_{P_0A} P_0B_2$ and the identity map $id_i : P_0B_i \rightarrow P_0C$. Then $id_1 \circ \varphi_1 = id_2 \circ \varphi_2$, since $\varphi_1(a) = \varphi_2(a)$ ($a \in P_0A$) when viewed as elements of P_0C . Define $P_nC = \langle P_nB_1, P_nB_2 \rangle^{P_0C}$, the normal subgroup generated by $\langle P_nB_1, P_nB_2 \rangle$ in P_0C . Clearly $(P_nC) \in \mathcal{M}$. It suffices to verify that $P_nC \cap P_0B_i = P_nB_i$ ($n < \omega$). Let $G_n^* = P_0B_1/P_nB_1 *_{P_0A/P_nA} P_0B_2/P_nB_2$ be a free product with amalgamation and $\varphi_{ni} : P_0A/P_nA \hookrightarrow P_0B_i/P_nB_i$ ($i = 1, 2$)

be the homomorphisms defined by $\bar{\varphi}_{ni}(aP_nA) = \varphi_i(a)P_nB_i$. Consider the canonical epimorphisms $\pi_{ni} : P_0B_i \rightarrow P_0B_i/P_nB_i$ ($n < \omega$, $i = 1, 2$). Then the map $\pi_{n1} \cup \pi_{n2} : P_0B_1 \cup P_0B_2 \rightarrow G_n^*$ extends to a homomorphism $\pi : P_0C \rightarrow G_n^*$ such that $\pi \upharpoonright_{P_0B_i} = \pi_{ni}$. If $x \in P_0B_i \setminus P_nB_i$, then $\pi_{ni}(x) \neq 1$ and so $\pi(x) \neq 1$. If $x \in P_nB_1 \cup P_nB_2$, then $\pi(x) = 1$. Thus $\ker \pi = \langle P_nB_1, P_nB_2 \rangle^{P_0C} = P_nC$ and so $P_nC \cap P_0B_i = P_nB_i$ ($i = 1, 2$). Thus $(P_nB_i) \leq (P_nC)$.

(VI $_{\kappa}$) Note that (P_nC) is itself in \mathcal{M} , hence take $(P_nB) = (P_nC)$. \square

The preceding properties (I)–(VII) $_{\kappa}$ show that \mathcal{M} is a κ -class, where κ is an infinite cardinal. Note that if \mathcal{M}^+ is defined to consist of all elements in \mathcal{M} such that the first n terms of the sequence are all equal (for a fixed n), then \mathcal{M}^+ is still a κ -class.

A relational structure $(P_nA) \in \mathcal{M}$ is said to be (\mathcal{M}, κ) -homogeneous if given any two $(P_nB_1), (P_nB_2) \in \mathcal{M}$ of cardinal $< \kappa$ which are substructures of (P_nA) and an isomorphism $\varphi : (P_nB_1) \rightarrow (P_nB_2)$, then there is an isomorphism $\gamma : (P_nA) \rightarrow (P_nA)$ such that $\gamma \upharpoonright_{(P_nB_1)} = \varphi$. A structure (P_nA) is said to be \mathcal{M} -homogeneous if (P_nA) is of cardinal κ and (\mathcal{M}, κ) -homogeneous. (P_nA) is said to be \mathcal{M} -universal, if (P_nA) is of cardinal κ and given any $(P_nB) \in \mathcal{M}$ of cardinal $< \kappa^+$, there is an embedding of (P_nB) into (P_nA) . By Jónsson's Theorem (see [1], p.213 or [2]), if κ is chosen such that $\kappa = \kappa^{<\kappa}$ (e.g., κ is regular and either κ is a limit beth number or the GCH holds), then \mathcal{M} contains an \mathcal{M} -homogeneous, \mathcal{M} -universal structure of cardinal κ which is unique up to isomorphism. We denote this structure by (P_nG) .

3 Normal subgroups of P_0G

We state and verify here special properties of P_0G and use them to obtain the required group.

Proposition 3.1 *Let (P_nG) be \mathcal{M} -homogeneous, \mathcal{M} -universal of cardinal κ . Then $P_nG \cong P_0G$ ($n < \omega$).*

Proof. Let $M = (P_nG)$. Define $M^n = (P_nG, P_nG, \dots, P_nG, P_{n+1}G, P_{n+2}G, \dots)$, where the first $n + 1$ terms in the sequence are all equal to P_nG and the succeeding terms agree with the corresponding terms in M , i.e., if $k > 1$, the $(n + k)$ th term of M^n is the $(n + k)$ th term of M . Define

$$\mathcal{M}^n = \{(P_nA) \in \mathcal{M} : \text{the first } n + 1 \text{ terms of } (P_nA) \text{ are equal}\}$$

which is a κ -class containing M^n . Since M is \mathcal{M} -homogeneous, \mathcal{M} -universal and $M^n \leq M$, it follows that M^n is \mathcal{M}^n -homogeneous and \mathcal{M}^n -universal.

Consider $M^- = (P_n G, P_{n+1} G, \dots)$. We show that M^- is \mathcal{M} -homogeneous and \mathcal{M} -universal. For then, by Jónsson's Theorem, M must be isomorphic to M^- and so we have shown that $P_0 G \cong P_n G$.

Suppose $X = (P_n A) \in \mathcal{M}$ of cardinal $< \kappa^+$. Define

$$X^0 = (P_0 A, P_0 A, \dots, P_0 A, P_0, P_1 A, P_2 A, \dots)$$

to be an element of \mathcal{M}^n , where the first n terms are all equal to $P_0 A$ and the succeeding terms of the chain X^0 are $P_n A$ ($n < \omega$). Since M^n is \mathcal{M}^n -universal, there exists an embedding $\varphi : X^0 \hookrightarrow M^n$. Hence $\varphi \upharpoonright_X : X \hookrightarrow M^-$ is also an embedding, and so M^- is \mathcal{M} -universal.

Let $X = (P_n A)$ and $Y = (P_n B)$ be substructures of M^- and $\varphi : (P_n A) \rightarrow (P_n B)$ be an isomorphism. We show that there exists an automorphism γ of M^- such that $\gamma \upharpoonright_X = \varphi$. As in the preceding paragraph, define the elements X^0 and Y^0 , which are substructures of M^n . Then $\varphi : X^0 \rightarrow Y^0$ is also an isomorphism. Since M^n is \mathcal{M}^n -homogeneous, there exists an automorphism ρ of M^n such that $\rho \upharpoonright_{X^0} = \varphi$. It is clear that $\gamma = \rho \upharpoonright_{M^-}$ is the required automorphism of M^- which extends φ . Thus we have shown that M^- is \mathcal{M} -homogeneous. \square

The following results will establish that if $x \in P_n G \setminus P_{n+1} G$, then $\langle x \rangle^{P_0 G} = P_n G$. Thus if N is a normal subgroup of $P_0 G$ which is not contained in $\bigcap_{n < \omega} P_n G$, then $N = P_k G$ for some k .

Lemma 3.2 *Let H be a normal subgroup of a group G and $x, y \in G$ with $|x| = |y|$. Assume that for all natural numbers k , $x^k \in H$ if and only if $y^k \in H$. If $G^* = \langle G, t \rangle$ is the HNN extension of G such that $x^t = y$, then $H^{G^*} \cap G = H$.*

Proof. Note that $H^{G^*} = H^{(t)}$, since H is normal in G . Clearly, $H \subseteq H^{(t)}$. Let $g \in H^{(t)} \cap G$. Then $g = h_1^{t^{n_1}} \cdots h_k^{t^{n_k}}$, for some $h_i \in H$ and integers n_i . Without loss of generality, assume that $h_i \neq 1$ and g cannot be written as a product of fewer than k such conjugates $h_i^{t^{n_i}}$. If $h_i = x^k$, for some k and $n_i \geq 0$, then replace $h_i^{t^{n_i}}$ by y^k . Similarly, if $h_i = y^k$ and $n_i \leq 0$, replace $h_i^{t^{n_i}}$ by x^k . Thus we can further assume that, if $h_i^{t^{n_i}} \in H$, then $n_i = 0$. If all the $n_i = 0$, then it is clear that $g \in H$. Now suppose, without loss of generality, that $n_1 \neq 0$. So when g is written in normal form, it will have length ≥ 1 (length of an element g refers to the number of occurrences of $t^{\pm 1}$ in its normal form). But the elements of G have length 0 (see [5], p. 178). Thus, all the n_i must be 0. \square

Theorem 3.3 *Suppose $x, y \in P_n G$ such that $|x| = |y|$ and $(\langle x \rangle \cup \langle y \rangle) \cap P_{n+1} G = 1$. Then there exists $t \in P_0 G$ such that $x^t = y$.*

Proof. Consider $P_0S = \langle x, y \rangle$ and $P_nS = P_nG \cap P_0S$ ($n < \omega$). Let $P_0T = \langle x, y, t \rangle$ be the HNN extension of P_0S and $P_nT = P_nS^{P_0T}$ be the normal subgroup generated by P_nS in P_0T . Then $(P_nS), (P_nT) \in \mathcal{M}$ and $(P_nS) \leq (P_nG)$. By Lemma 3.2, $(P_nS) \leq (P_nT)$. Since (P_nG) is \mathcal{M} -universal, there exists an embedding φ from (P_nT) to (P_nG) . Let $P_nU = \varphi(P_nT)$ and $P_nV = \varphi(P_nS)$. Then $(P_nV) \leq (P_nU) \leq (P_nG)$ and $(P_nS) \cong (P_nV)$ via φ . Since (P_nG) is \mathcal{M} -homogeneous, there exists an isomorphism $\gamma : (P_nG) \rightarrow (P_nG)$ such that $\gamma \upharpoonright_{(P_nS)} = \varphi$. Now $\varphi(t) \in P_0U$ and $\varphi(x^t) = \varphi(x)^{\varphi(t)} = \varphi(y)$ imply that there exists $t'' \in P_0G$ such that $\gamma(t'') = \varphi(t)$. Since $x, y \in P_0S$, $\gamma(x^{t''}) = \gamma(x)^{\gamma(t'')} = \varphi(x)^{\varphi(t)} = \varphi(y) = \gamma(y)$. Since γ is an isomorphism, $x^{t''} = y$ for some $t'' \in P_0G$. Hence x and y are conjugate in P_0G via the element t'' . \square

Recall that in Proposition 2.1 (I), we defined (P_nA) to be $P_nA = F_\alpha^{(n)}$, where F_α is a free group of infinite rank α . Since $P_nA/P_{n+1}A$ is free abelian of rank α , it is clear that there exist infinitely many elements $x \in P_nA$, which are independent modulo $P_{n+1}A$. In particular, $\langle x \rangle \cap P_{n+1}A = 1$. By the definition of (P_nG) as an \mathcal{M} -universal structure, it now follows that P_nG contains infinitely many elements x such that $\langle x \rangle \cap P_{n+1}G = 1$.

Theorem 3.4 *For all $x, y \in P_nG \setminus P_{n+1}G$, $x \in \langle y \rangle^{P_0G}$.*

Proof. If x and y satisfy the assumptions in Theorem 3.3, then we are done. Suppose we have an arbitrary element $x \in P_nG \setminus P_{n+1}G$. We show that there exists an element $a_x \in \langle x \rangle^{P_0G}$ such that the elements $a_x, xa_x \in P_nG \setminus P_{n+1}G$ have infinite order, and $\langle a_x \rangle \cup \langle xa_x \rangle \cap P_{n+1}G = 1$. For then, by Theorem 3.3,

$$\langle xa_x \rangle^{P_0G} = \langle a_x \rangle^{P_0G} = \langle a_y \rangle^{P_0G} = \langle ya_y \rangle^{P_0G},$$

and so $x \in \langle a_x \rangle^{P_0G} = \langle ya_y \rangle^{P_0G} \subseteq \langle y \rangle^{P_0G}$.

Suppose for now that the order of x is infinite. There exists a free group F of infinite rank and contained in P_0G such that $\langle F, x \rangle = F * \langle x \rangle$. Let $P_0H = F * \langle x \rangle$ and $P_nH = (F^{(n)})^{\langle x \rangle}$, where $F^{(n)}$ denotes the n th derived group of F . By Lemma 3.2, $(F^{(n)}) \leq (P_nH)$. Since (P_nG) is \mathcal{M} -universal, (P_nH) embeds in (P_nG) . Without loss of generality, assume $(P_nH) \leq (P_nG)$. Let $a \in F^{(n)} \setminus F^{(n+1)}$. Then $x^a x \in P_nG$ of infinite order. If $(x^a x)^k \in P_{n+1}G$, for some $k > 0$, then $x^a x = x^2(a^{-x^2}a^x)$ and

$$(x^a x)^k = (a^{-x^2}a^x)^{x^{-2}} \cdots (a^{-x^2}a^x)^{x^{-2k}} x^{2k} \in P_{n+1}G \cap P_0H = P_{n+1}H.$$

If $n \geq 1$, then $x^{2k} \in P_nH$, since $P_{n+1}H \subseteq P_nH$ and $a \in P_nH$. By the definition of P_nH , $x^{2k} = 1$. This gives a contradiction, since x was assumed to have infinite order. If $n = 0$, $(x^a x)^k \in P_1H \leq (P_0H)^{(1)}$. Since $P_0H/(P_0H)^{(1)}$ is free abelian (see [5], p. 24),

$$(x^a x)^k \equiv x^{2k} \equiv 1 \pmod{(P_0H)^{(1)}}.$$

This gives a contradiction, since $x(P_0H)^{(1)}$ has infinite order as an element of the quotient group $P_0H/(P_0H)^{(1)}$. Similarly, the element $x \cdot x^a x \in P_nG$ has infinite order and $\langle x \cdot x^a x \rangle \cap P_{n+1}G = 1$. Thus we can take $a_x = x^a x$.

If the order of x is finite, choose an element $g \in P_0G$ such that $\langle x, g \rangle = \langle x \rangle * \langle g \rangle$. Then $x^g x \in P_nG$ has infinite order. If $x^g x \notin P_{n+1}G$, apply the preceding to $x^g x$. If $x^g x \in P_{n+1}G$, then $xx^g x \in P_nG \setminus P_{n+1}G$ has infinite order and apply the preceding to $xx^g x$. \square

Theorem 3.5 *Let $I = \bigcap_{n < \omega} P_nG$. If H is a normal subgroup of P_0G such that H is not contained in I , then $H = P_nG$ for some $n < \omega$. Thus every non-trivial normal subgroup of P_0G/I is isomorphic to P_kG/I for some $k < \omega$.*

Proof. Let H be a normal subgroup of P_0G such that H is not contained in I . Then there exists a least n such that $H \not\subseteq P_nG$. So $H \subseteq P_{n-1}G$ and there exists $x \in H \setminus P_nG$. By Theorem 3.4, it follows that $H = P_{n-1}G$. \square

The proof of the main theorem will now follow, when we take $G = P_0G/I$ and $G_n = P_nG/I$ ($0 < n < \omega$).

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