

LARGE INTERVALS IN THE CLONE LATTICE

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ABSTRACT. We give three examples of cofinal intervals in the lattice of (local) clones on an infinite set X , whose structure is on the one hand non-trivial but on the other hand reasonably well understood. Specifically, we will exhibit clones $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ such that:

- (1) the interval $[\mathcal{C}_1, \mathcal{O}]$ in the lattice of local clones is (as a lattice) isomorphic to $\{0, 1, 2, \dots\}$ under the divisibility relation,
- (2) the interval $[\mathcal{C}_2, \mathcal{O}]$ in the lattice of local clones is isomorphic to the congruence lattice of an arbitrary semilattice,
- (3) the interval $[\mathcal{C}_3, \mathcal{O}]$ in the lattice of all clones is isomorphic to the lattice of all filters on X .

1. INTRODUCTION

Definition 1.1. Let X be a nonempty set. The *full clone* on X , called \mathcal{O}_X is the set of all finitary functions or (“operations”) on X : $\mathcal{O}_X = \bigcup_{n=1}^{\infty} \mathcal{O}^{(n)}$, where $\mathcal{O}^{(n)}$ is the set of all functions from X^n into X . A *clone* (on X) is a set $\mathcal{C} \subseteq \mathcal{O}$ which contains all projections and is closed under composition. Alternatively, \mathcal{C} is a clone if \mathcal{C} is the set of term functions of some universal algebra over X . Identifying a clone \mathcal{C} with the algebra (X, \mathcal{C}) (whose term functions are the elements of \mathcal{C}) allows us to talk about subalgebras and automorphisms of \mathcal{C} .

The set of clones over X forms a complete algebraic lattice with largest element \mathcal{O} . The coatoms of this lattice are called “precomplete clones” or “maximal clones”. (See also [Sze86], [PK79]).

Definition 1.2. A clone \mathcal{C} is called a *local clone*, iff each set $\mathcal{C} \cap \mathcal{O}^{(k)}$ is closed in the product topology (Tychonoff topology) on X^{X^k} , where X is taken to be discrete. In other words, \mathcal{C} is local iff:

Whenever $f \notin \mathcal{O}^{(k)}$, then there is a finite “witness” for it, i.e., there is a finite $A \subseteq X^k$ such that for all $g \in \mathcal{C}$: $g \upharpoonright A \neq f \upharpoonright A$.

The set of local clones over X forms again a complete lattice with largest element \mathcal{O} .

For any k -ary relation $\rho \subseteq X^k$ the set $\text{Pol}(\rho)$ is the set of all functions preserving ρ . We will only need two special cases of this construction:

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Definition 1.3. For any $A \subseteq X$ we let

$$\text{Pol}(A) := \bigcup_{n=1}^{\infty} \{f \in \mathcal{O}^{(n)} : f[A^n] \subseteq A\},$$

and for any unary (partial) function h we let

$$\text{Pol}(h) := \bigcup_{n=1}^{\infty} \{f \in \mathcal{O}^{(n)} : \forall \bar{x} = (x_1, \dots, x_n) : f(h(x_1), \dots, h(x_n)) = h(f(\bar{x}))\}.$$

Definition 1.4.

- The ternary discriminator t (on the base set X) is defined to be the function $t : X^3 \rightarrow X$ satisfying $t(x, x, z) = z$ and $t(x, y, z) = x$ whenever $x \neq y$.
- An *internal isomorphism* of an algebra (X, \mathcal{C}) is a bijection $h : U \rightarrow V$ between two subalgebras of (X, \mathcal{C}) which is compatible with all operations of \mathcal{C} , i.e.: $\mathcal{C} \subseteq \text{Pol}(h)$.
- A local clone \mathcal{C} is called *locally quasiprimal* iff the elements of \mathcal{C} are exactly the operations which are compatible with all internal isomorphism of \mathcal{C} , i.e., if

$$\mathcal{C} = \bigcap_{h \in \text{Iso}(\mathcal{C})} \text{Pol}(h).$$

(Note that the inclusion \subseteq holds by definition of $\text{Iso}(\mathcal{C})$.)

Theorem 1.5 (Pixley's theorem, see [Pix71] and [Pix82]). A local clone \mathcal{C} is locally quasiprimal if $t \in \mathcal{C}$, where t is the ternary discriminator.

In the following sections we will use Pixley's theorem to describe intervals in the lattice of local clones. As a warmup, consider the following example:

Example 1.6. Fix an infinite set X , and let $s : X \rightarrow X$ be a 1-1 onto map without cycles. For $n > 0$, s^n is the n -th iterate of s , s^{-n} is the inverse of s^n . s^0 is the identity function.

Then $\mathcal{C}_1 := \text{Pol}(s)$ is a local clone, and the local clones containing $\text{Pol}(s)$ are exactly the clones $\text{Pol}(s^n)$, $n \in \mathbb{Z}$; we have $\text{Pol}(s^n) \subseteq \text{Pol}(s^k)$ iff n divides k .

(These clones, and also the unbounded chain $\text{Pol}(s^{2^n})$ were already considered in [RS84].)

Proof sketch. First we note that $\text{Pol}(s)$ contains the discriminator, hence $\text{Pol}(s)$, as well as any local clone containing it, must be locally quasiprimal.

Next, note that:

(*) For all $a, b \in X$ there is a map $f \in \text{Pol}(s)$ with $f(a) = b$.

[Why? Define $f(s^n(a)) := s^n(b)$ for all $n \in \mathbb{Z}$, and $f(x) = x$ for all x not of the form $s^n(a)$.]

Hence $(X, \text{Pol}(s))$ has no proper subalgebras, so the internal isomorphisms of $(X, \text{Pol}(s))$ are exactly the automorphisms of $(X, \text{Pol}(s))$. Clearly s is an automorphism of this structure, and using (*) it is easy to see that every automorphism must be of the form s^n for some $n \in \mathbb{Z}$.

Now let \mathcal{D} be a local clone above $\text{Pol}(s)$. Let I be the set of internal isomorphisms (=automorphisms) of (X, \mathcal{D}) . Then I is a subset and even a subgroup of $\{s^n : n \in \mathbb{Z}\}$, say $I = \{s^{nk_0} : n \in \mathbb{Z}\}$ for some $k_0 \in \mathbb{Z}$.

Hence \mathcal{D} and $\text{Pol}(s^{k_0})$ have the same set of internal isomorphisms; as both clones are locally quasiprimal, they must be equal. \square

2. A LARGE INTERVAL OF LOCAL CLONES

Theorem 2.1. Let (X, \vee) be a semilattice and let $Con(X, \vee)$ be the lattice of congruences on (X, \vee) . Let \mathcal{C}_2 be the clone of all operations that are bounded by the sup function of the appropriate arity:

$$\mathcal{C}_2 := \bigcup_{k=1}^{\infty} \{f \in \mathcal{O}^{(k)} : \forall x_1 \cdots x_k \ f(x_1, \dots, x_k) \leq x_1 \vee \cdots \vee x_k\}.$$

(Here $x \leq y \Leftrightarrow x \vee y = y$ is the usual semilattice order.)

Then $[\mathcal{C}_2, \mathcal{O}_X] \simeq Con(X, \vee)$. That is, there is a lattice isomorphism between the set of local clones above \mathcal{C}_2 and the set of congruences of (X, \vee) .

Remark 2.2. If $\emptyset \subsetneq I \subsetneq X$ is an ideal, then the partition $\{I, X \setminus I\}$ corresponds to a congruence relation which is a coatom in $Con(X, \vee)$. In fact, all coatoms are obtained in this form. It is clear that $Con(X, \vee)$ is dually atomic.

It will be notationally more convenient to deal with congruence orders rather than congruence relations:

Definition 2.3. Let (X, \vee) be a semilattice. We call $\preceq \subseteq X \times X$ a *congruence order* on (X, \vee) if \preceq is transitive, extends the semilattice order \leq and satisfies

$$(**) \quad \forall x, y, z \in X : x \preceq z \ \& \ y \preceq z \Rightarrow (x \vee y) \preceq z.$$

The following fact is easy to check:

Fact 2.4. The maps

$$\begin{aligned} \preceq &\mapsto \{(x, y) : x \preceq y \ \& \ y \preceq x\} \\ \theta &\mapsto \{(x, y) : (x \vee y, y) \in \theta\} \end{aligned}$$

are monotone bijections between the congruence relations θ and congruence orders \preceq on (X, \vee) , and they are inverses of each other.

Definition 2.5. For any clone \mathcal{C} on the set X , and any subset $E \subseteq X$ we write $\langle E \rangle_{\mathcal{C}}$ for the subalgebra of (X, \mathcal{C}) that is generated by E . In other words: $\langle E \rangle_{\mathcal{C}} = \bigcup_{k=1}^{\infty} \{f(\bar{a}) : \bar{a} \in X^k, f \in \mathcal{C}^{(k)}\}$.

Definition 2.6. We define a correspondence between clones on X and preorders (quasiorders) on X through two maps $\mathcal{C} \mapsto R_{\mathcal{C}}$ and $\preceq \mapsto \mathcal{E}(\preceq)$.

- For any clone \mathcal{C} on X , let $R_{\mathcal{C}}$ be the preorder on X defined by

$$x R_{\mathcal{C}} y \Leftrightarrow x \in \langle y \rangle_{\mathcal{C}}.$$

The associated equivalence relation $\sim_{\mathcal{C}}$ is then given by $\langle x \rangle_{\mathcal{C}} = \langle y \rangle_{\mathcal{C}}$, and the algebra $\langle x \rangle_{\mathcal{C}}$ generated by x is just the half-open interval

$$(x]_{R_{\mathcal{C}}} := \{y \in X : y R_{\mathcal{C}} x\}.$$

- For any preorder \preceq on X let the clone $\mathcal{E}(\preceq)$ be defined by

$$\mathcal{E}(\preceq) = \bigcap_{a \in X} \text{Pol}((a]_{\preceq}).$$

Lemma 2.7. Let \preceq be a preorder on X . Then the following are equivalent for all $a, b \in X$:

- $a \preceq b$.
- $\chi_{a,b} \in \mathcal{E}(\preceq)$, where $\chi_{a,b}$ maps b to a , and is the identity otherwise.

- (iii) There is a unary function $f \in \mathcal{E}(\preceq)$ with $f(b) = a$.
- (iv) $a \in \langle b \rangle_{\mathcal{E}(\preceq)}$, i.e., $a R_{\mathcal{E}(\preceq)} b$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) and (iii) \Leftrightarrow (iv) are all easy. \square

Lemma 2.8. Let \mathcal{C} be a clone on X , $\mathcal{D} := \mathcal{E}(R_{\mathcal{C}})$. Then \mathcal{C} and \mathcal{D} have the same 1-generated subalgebras: $\forall b \in X : \langle b \rangle_{\mathcal{C}} = \langle b \rangle_{\mathcal{D}}$.

Proof. By the equivalence (i) \Leftrightarrow (iv) in Lemma 2.7, the relations $R_{\mathcal{C}}$ and $R_{\mathcal{D}}$ coincide. Now $\langle b \rangle_{\mathcal{C}} = (b]_{R_{\mathcal{C}}} = (b]_{R_{\mathcal{D}}} = \langle b \rangle_{\mathcal{D}}$. \square

The relation $\langle x \rangle_{\mathcal{C}} \subseteq \langle y \rangle_{\mathcal{C}}$ carries information only about the unary functions of \mathcal{C} ; in our context, however, this is sufficient, because our clones \mathcal{C} are generated by $\mathcal{C}^{(1)} \cup \{\vee\}$. The ‘‘encoding’’ property defined below will help us to reduce questions about subalgebras to questions about 1-generated subalgebras.

Definition 2.9. Let \mathcal{C} be a clone on X , $*$ a binary function in \mathcal{C} . We say that $(\mathcal{C}, *)$ *encodes pairs* iff $\langle x, y \rangle_{\mathcal{C}} = \langle x * y \rangle_{\mathcal{C}}$ for all $x, y \in X$.

Fact 2.10. Assume that both $(\mathcal{C}, *)$ and $(\mathcal{D}, *)$ encode pairs. Then the following are equivalent:

- (1) \mathcal{C} and \mathcal{D} have the same subalgebras.
- (2) \mathcal{C} and \mathcal{D} have the same finitely generated subalgebras.
- (3) $\langle x \rangle_{\mathcal{C}} = \langle x \rangle_{\mathcal{D}}$ for all $x \in X$.

Lemma 2.11. Let (X, \vee) be a semilattice and let $x \leq y$ iff $x \vee y = y$. Let $\mathcal{C} := \mathcal{E}_{\leq}$. Then

- (1) \mathcal{C} is a local clone containing the binary function \vee as well as the ternary discriminator. In fact:

$$\mathcal{C} = \bigcup_{k=1}^{\infty} \{ f : \forall x_1, \dots, x_n f(x_1, \dots, x_n) \leq x_1 \vee \dots \vee x_n \}.$$
- (2) $\langle x, y \rangle_{\mathcal{C}} = \langle x \vee y \rangle_{\mathcal{C}}$, and similarly $\langle x, y \rangle_{\mathcal{D}} = \langle x \vee y \rangle_{\mathcal{D}}$ for all clones $\mathcal{D} \supseteq \mathcal{C}$.
- (3) If $U, V \leq (X, \mathcal{C})$, and $h : U \rightarrow V$ is an isomorphism with respect to the operations in \mathcal{C} , then one of the following holds:
 - $U = V$, and h is the identity on U .
 - U and V are singleton subalgebras.

Proof. (1) and (2) are obvious.

(3) For any a, b define $\psi_{a,b}(a, b) = a$, and $\psi_{a,b}(x, y) = y$ otherwise. Clearly $\psi_{a,b} \in \mathcal{C}$.

Assume that U contains at least 2 elements, and let $u \in U$ with $h(u) \neq u$. If there is some element $a < u$ in U , then the inequality

$$h(\psi_{a,u}(a, u)) = h(a) \neq h(u) = \psi_{a,u}(h(a), h(u))$$

shows that h is not an internal isomorphism. Otherwise let $u' \in U \setminus \{u\}$ and $b := u \vee u' \in U$, then $u < b$, and we get

$$h(\psi_{u,b}(u, b)) = h(u) \neq h(b) = \psi_{u,b}(h(u), h(b)).$$

\square

Proof of Theorem 2.1. We just need to collect a few implications:

- (1) The maps $\preceq \rightarrow \mathcal{E}(\preceq)$ and $\mathcal{D} \mapsto R_{\mathcal{D}}$ are monotone.
- (2) For any local clone $\mathcal{D} \supseteq \mathcal{C}_2$, the relation $R_{\mathcal{D}}$ is a congruence order. [The main property to check is 2.3(**): If $\langle x \rangle_{\mathcal{D}} \subseteq \langle z \rangle_{\mathcal{D}}$ and $\langle y \rangle_{\mathcal{D}} \subseteq \langle z \rangle_{\mathcal{D}}$, then $\langle x \vee y \rangle_{\mathcal{D}} = \langle x, y \rangle_{\mathcal{D}} \subseteq \langle z \rangle_{\mathcal{D}}$.]
- (3) For any congruence order \preceq , the clone $\mathcal{E}(\preceq)$ is a local clone extending \mathcal{C}_2 . [Obvious.]
- (4) Any congruence order \preceq coincides with $R_{\mathcal{E}(\preceq)}$. [By 2.7.]
- (5) Finally we claim that each local clone $\mathcal{D} \subseteq \mathcal{C}$ coincides with $\mathcal{D}' := \mathcal{E}(R_{\mathcal{D}})$: From 2.8 we know that \mathcal{D} and \mathcal{D}' have the same 1-generated subalgebras, so from 2.10 we conclude that they have the same subalgebras. By 2.11, they have the same internal isomorphisms, so by Pixley's theorem they must be equal.

□

Example 2.12. Let $(X, <)$ be a linearly ordered set. Then the congruence relations on (X, \max) are exactly the equivalence relations with convex classes. As a special case, consider the semilattice (\mathbb{N}, \max) . A congruence relation is just a partition of \mathbb{N} into disjoint intervals.

The map $\theta \mapsto A_{\theta} := \{\max E : E \text{ is a finite congruence class}\}$ is an antitone 1-1 map from the congruence relations onto $\mathcal{P}(\mathbb{N})$, the power set of \mathbb{N} . The empty set corresponds to \mathcal{O} , or to the equivalence relation with a single class; the set \mathbb{N} itself corresponds to \mathcal{C}_2 , or to the equivalence with singleton classes.

3. A LARGE INTERVAL OF CLONES

On any infinite set X we will define a clone \mathcal{C}_3 such that the interval $[\mathcal{C}_3, \mathcal{O}]$ in the full clone lattice is very large (with $2^{2^{|X|}}$ precomplete elements), but still reasonably well understood.

Definition 3.1.

- (1) For $A \subseteq X$, $n \geq 1$ let $\Delta_n(A) := \{(a, \dots, a) \in X^n : a \in A\}$.
- (2) For any function $f \in \mathcal{O}^{(n)}$, let $f^{(1)} \in \mathcal{O}^{(1)}$ be defined by $f^{(1)}(x) = f(x, \dots, x)$.
- (3) For any function $f \in \mathcal{O}^{(n)}$, we let

$$\text{fix}(f) = \{x : f^{(1)}(x) = x\}, \quad \text{nix}(f) = \{x : f^{(1)}(x) \neq x\}$$
- (4) For any clone \mathcal{C} we define $\text{fix}(\mathcal{C}) := \{\text{fix}(f) : f \in \mathcal{C}\}$, $\text{nix}(\mathcal{C}) := \{\text{nix}(f) : f \in \mathcal{C}\}$.
- (5) For any family $T \subseteq \mathcal{P}(X)$ we define

$$\mathcal{E}_T := \{f \in \mathcal{O} : \text{fix}(f) \in T\} = \bigcup_{A \in T} \bigcap_{a \in A} \text{Pol}(\{a\})$$

Definition 3.2. Let $\mathcal{C}_3 := \mathcal{E}_{\{X\}}$ be the clone of “idempotent” functions, i.e., of all functions satisfying $f(x, \dots, x) = x$ for all x .

Theorem 3.3. The map $T \rightarrow \mathcal{E}_T$ is an order isomorphism between the set of all filters on X (including the improper filter $\mathcal{P}(X)$) and the set of all clones above \mathcal{C}_3 .

In particular, the precomplete clones above \mathcal{C}_3 are exactly the clones of the form \mathcal{E}_U , where U is an ultrafilter on X .

Remark 3.4. The subalgebras of \mathcal{C}_3 are exactly all singleton sets, and \mathcal{C}_3 contains the ternary discriminator. Hence every local clone above \mathcal{C}_3 is determined by its subalgebras. For every $A \subseteq X$ the clone

$$\mathcal{E}_{\{A\}} := \bigcap_{a \in A} \text{Pol}(\{a\}) = \{f : f(a, a, \dots, a) = a \text{ for all } a \in A\}$$

is a local clone whose subalgebras are exactly the singleton sets $\{a\}$ with $a \in A$. Hence the local clones above \mathcal{C}_3 are exactly the clones of the form $\mathcal{E}_{\{A\}}$, $A \subseteq X$. In the language of theorem 3.3: the local clones above \mathcal{C}_3 are exactly the clones corresponding to principal filters.

Lemma 3.5. Assume that $\mathcal{D} \supseteq \mathcal{C}_3$ is a clone, $f \in \mathcal{D}$, $\text{fix}(f) \subseteq \text{fix}(g)$. Then $g \in \mathcal{D}$. Hence, every clone $\mathcal{D} \supseteq \mathcal{C}_3$ is determined by $\text{nix}(\mathcal{D})$: $f \in \mathcal{D} \Leftrightarrow \text{nix}(f) \in \text{nix}(\mathcal{D})$.

Proof. For $\bar{x} = (x_1, \dots, x_n)$ define

$$G(\bar{x}, y) = \begin{cases} x_1 & \text{if } x_1 = \dots = x_n = y \\ g(\bar{x}) & \text{otherwise} \end{cases}$$

Clearly $G \in \mathcal{C}_3$. For $\bar{x} \in \Delta_n(\text{fix}(f)) \subseteq \Delta_n(\text{fix}(g))$, both $g(\bar{x})$ and $G(\bar{x}, f^{(1)}(x_1))$ have the value x_1 . If $\bar{x} \in \Delta_n(X) \setminus \Delta_n(\text{fix}(f))$, then $f^{(1)}(x_1) \neq x_1$, so $G(\bar{x}, f^{(1)}(x_1)) = g(\bar{x})$ by definition; the same holds for $\bar{x} \notin \Delta_n(X)$. Hence $g(\bar{x}) = G(\bar{x}, f^{(1)}(x_1))$ for all $\bar{x} \in X^n$. \square

Lemma 3.6. Let \mathcal{D} be a clone with $\mathcal{C}_3 \subseteq \mathcal{D}$, $\mathcal{D} \neq \mathcal{C}$. Then $\text{nix}(\mathcal{D})$ is an ideal and $\text{fix}(\mathcal{D})$ is a filter.

Proof. Lemma 3.5 shows that $\text{nix}(\mathcal{D})$ is downward closed.

Now let $A_\ell = \text{nix}(f_\ell)$, $f_\ell \in \mathcal{D}$ for $\ell = 1, 2$, and assume that $A_1 \cap A_2 = \emptyset$.

Let $B = X \setminus (A_1 \cup A_2)$. We may assume that either $|A_1| \geq 2$, or $B \neq \emptyset$ (or both).

In either case there is a unary function f'_1 with $\text{nix}(f'_1) = A_1$, and f'_1 maps A_1 into $A_1 \cup B$. By lemma 3.5, $f'_1 \in \mathcal{D}$. So $f_2 \circ f'_1 \in \mathcal{D}$. Since $\text{nix}(f_2 \circ f'_1) = A_1 \cup A_2$ we see that $\text{nix}(\mathcal{D})$ is closed under \cup . \square

Fact 3.7. For all filters T and all $A \in T$ there is a unary function f with $\text{fix}(f) = A$.

Proof of Theorem 3.3. Again we just collect some implications.

- (1) The maps $\mathcal{D} \mapsto \text{fix}(\mathcal{D})$ and $T \mapsto \mathcal{E}_T$ are clearly monotone with respect to set inclusion.
- (2) For every clone \mathcal{C} the set $\text{fix}(\mathcal{C})$ is a filter (by Lemma 3.6).
- (3) For every filter T the set \mathcal{E}_T is a clone. (Obvious.)
- (4) For every filter T we have $T = \text{fix}(\mathcal{E}_T)$. (By Fact 3.7.)
- (5) For every clone $\mathcal{D} \supseteq \mathcal{C}_3$ we have $\mathcal{D} = \mathcal{E}_{\text{fix}(\mathcal{D})}$:
Let $\mathcal{D}' := \mathcal{E}_{\text{fix}(\mathcal{D})}$. From the previous item we conclude $\text{fix}(\mathcal{D}') = \text{fix}(\mathcal{D})$.
Now by Lemma 3.5 we see $\mathcal{D} = \mathcal{D}'$.

\square

Remark 3.8. If we regard the set X as a discrete topological space, then the Stone-Cech compactification of X is

$$\beta X = \{U : U \text{ is an ultrafilter on } X\}$$

There is a canonical 1-1 order-preserving correspondence between the filters on X (ordered by \subseteq) and the closed subsets of βX (ordered by \supseteq).

So the interval $[\mathcal{C}_3, \mathcal{O}]$ in the full clone lattice is isomorphic (as a complete lattice) to the family of closed subsets of βX , ordered by reverse inclusion: \mathcal{O} corresponds to the empty set, each precomplete clone in $[\mathcal{C}_3, \mathcal{O}]$ corresponds to a singleton set.

Note that for any closed subset $F \subseteq \beta X$ and any $p \in \beta X \setminus F$, also $F \cup \{p\}$ is closed, and moreover:

$$\begin{aligned} F \text{ covers } G \text{ (i.e., } F \supset G, \text{ and the interval } (G, F) \text{ is empty) iff} \\ G = F \cup \{p\} \text{ for some } p \in \beta X \setminus F \end{aligned}$$

In particular, let $\mathcal{C}_{\text{bd}} \supseteq \mathcal{C}_3$ be the clone corresponding to the ideal of small sets, i.e.,

$$\mathcal{C}_{\text{bd}} := \{f \in \mathcal{O}^{(\cdot)} \mid \exists B \subseteq X, |B| < |X|, \forall x \in X \setminus B : f(x, \dots, x) = x\}$$

Then every clone $\mathcal{C} \supsetneq \mathcal{C}_{\text{bd}}$ has exactly $2^{2^{|\mathcal{X}|}}$ lower neighbors in the clone lattice; the clone corresponds to a closed set F , and the lower neighbors correspond to closed sets $F \cup \{p\}$. This is a special case of a theorem of [Mar81].

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