ON THE EXISTENCE OF PRECOVERS

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Abstract. It is proved consistent with ZFC + GCH that for every Whitehead group A of infinite rank, there is a Whitehead group H_A such that $\mathrm{Ext}(H_A, A) \neq 0$. This is a strong generalization of the consistency of the existence of non-free Whitehead groups. A consequence is that it is undecidable in ZFC + GCH whether every $\mathbb{Z}$-module has a ${}^{\perp}\{\mathbb{Z}\}$-precover. Moreover, for a large class of $\mathbb{Z}$-modules N, it is proved consistent that a known sufficient condition for the existence of ${}^{\perp}\{N\}$-precovers is not satisfied.

Dedicated to the memory of Reinhold Baer, who was a pioneer in the study of Ext

Let $\mathcal{F}$ be a class of R-modules of the form

$$^{\perp}\mathcal{C} = \{A : \mathrm{Ext}(A, C) = 0 \text{ for all } C \in \mathcal{C}\}$$

for some class $\mathcal{C}$ of R-modules. Note that if $\mathcal{C}$ is a set (not a proper class) then $\mathcal{F} = {}^{\perp}\{N\}$ where N is the direct product of the elements of $\mathcal{C}$.

A homomorphism $\phi \in \mathrm{Hom}(A, M)$ with $A \in \mathcal{F}$ is called an $\mathcal{F}$-precover of M if the induced map $\mathrm{Hom}(A', A) \to \mathrm{Hom}(A', M)$ is surjective for all $A' \in \mathcal{F}$. (See [10] or [23].)

The first author and Jan Trlifaj proved [8] that a sufficient condition for every module M to have an $\mathcal{F}$-precover is the following:

(†) there is a module B such that $\mathcal{F}^{\perp} = \{B\}^{\perp}$.

(Here $\mathcal{F}^{\perp} = \{A : \mathrm{Ext}(M, A) = 0 \text{ for all } M \in \mathcal{F}\}$.)

In [9], generalizing a method used by Enochs [1] to prove the Flat Cover Conjecture, it is proved that (†) holds whenever $\mathcal{C}$ is a class of pure-injective modules; moreover, for R a Dedekind domain, the sufficient condition holds whenever $\mathcal{C}$ is a class of cotorsion modules. The following is also proved in [9]:

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Theorem 1. *Assuming $V = L$, for any hereditary ring R and any R-module N, there is an R-module B such that $({}^{\perp}\{N\})^{\perp} = \{B\}^{\perp}$ and hence every R-module has a ${}^{\perp}\{N\}$-precover.*

For the case $R = N = \mathbb{Z}$ this is an easy consequence of the second author's proof that V = L implies that all Whitehead groups are free (cf. [15]). Indeed, ${}^{\perp}\{\mathbb{Z}\}$ is, by definition, the class $\mathcal{W}$ of all Whitehead groups, so assuming V = L, $({}^{\perp}\{\mathbb{Z}\})^{\perp} = \{B\}^{\perp} =$ the class of all abelian groups, for any free abelian group B.

Our main results here are that the conclusions of Theorem 1 are not provable in ZFC + GCH for $R = \mathbb{Z}$. First, we will prove in the next section the following result:

Theorem 2. *It is consistent with ZFC + GCH that for every* set $\mathcal{C}$ *of abelian groups which contains a non-zero cotorsion-free group, there is no B such that $B^{\perp} = \{{}^{\perp}\mathcal{C}\}^{\perp}$.*

For countable torsion-free groups this settles the question of when it is provable in ZFC that ${}^{\perp}\{N\}$ satisfies (†):

Corollary 3. *Let N be a countable torsion-free abelian group. It is provable in ZFC that there is a group B such that $B^{\perp} = \{{}^{\perp}N\}^{\perp}$ if and only if N is divisible.*

For the case of $N = \mathbb{Z}$ we can prove more. The rest of the paper is devoted to the proof of the following result:

Theorem 4. *It is consistent with ZFC + GCH that there is an abelian group, namely $\mathbb{Q}$, which does not have a $\mathcal{W}$-precover.*

Theorem 2 for $\mathcal{C} = \{\mathbb{Z}\}$ is easily seen to be equivalent to the statement that it is consistent with ZFC + GCH that for every Whitehead group B we can find a Whitehead group $A \in \{B\}^{\perp}$ such that there is a Whitehead group H_A with $\mathrm{Ext}(H_A, A) \neq 0$. For the proof of Theorem 4 we will need to prove the stronger fact that it is consistent with ZFC + GCH that for every Whitehead group A of infinite rank, there is a Whitehead group H_A with $\mathrm{Ext}(H_A, A) \neq 0$.

The consistency results 2 and 4 will each be proved by citing the consistency of a known combinatorial property (involving so-called uniformization properties introduced by the second author) and then using the combinatorial property to prove the algebraic facts needed.

From now on, we will deal exclusively with $\mathbb{Z}$-modules, that is, abelian groups (though the results generalize trivially to modules over a countable p.i.d.). We will use the word "group" to mean "abelian group." We will sometimes write W-group instead of Whitehead group. It is well-known that W-groups are $\aleph_1$-free (that is, every countable subgroup is free). Moreover, CH implies that a W-group A is strongly $\aleph_1$-free, that is, every countable subset of A is contained in a countable subgroup C such that A/C is $\aleph_1$-free. (For facts about W-groups see, for example, [5, Chap. XII] or [6, Chaps. XII & XIII].)

1. Proof of Theorem 2

The proof will make use of the following consequence of Theorem 2 of [8]. The last assertion follows from Lemma 1 of [8].

Theorem 5. *Let μ be a cardinal $> \kappa$ such that $\mu^\kappa = \kappa$ and let B be a group of cardinality $\leq \kappa$. Then there is a group $A \in \{B\}^\perp$ such that $A = \bigcup_{\nu<\mu} A_\nu$ (continuous), $A_0 = 0$, and such that for all $\nu < \mu$, $A_{\nu+1}/A_\nu$ is isomorphic to B.*

Moreover, if $B \in {}^\perp G$, then so does A/A_ν for all $\nu < \mu$. ∎

We will have occasion to use the $\mathbb{Z}$-adic topology on a reduced torsion-free group M, that is, the metrizable linear topology whose base of neighborhoods of 0 consists of the subgroups $(n+1)!M$ ($n \in \omega$). We use $\sum_{n\in\omega} n!t_n$ to denote the limit of the sequence $\left\langle \sum_{j\leq n} j!t_j : n \in \omega \right\rangle$. We denote by $\widehat{M}$ the completion of M in the $\mathbb{Z}$-adic topology.

The following sums up some well-known facts (cf. [11, §7] or [6, §I.3]:

Lemma 6. *Let M be a reduced torsion-free abelian group. Then M is not cotorsion if and only if M is not pure-injective if and only if M is not complete in the $\mathbb{Z}$-adic topology if and only if there are elements $\{t_n : n \in \omega\}$ such that the system of equations*

$$(n+1)y_{n+1} = y_n - t_n$$

in the unknowns y_n ($n \in \omega$) does not have a solution in M. ∎

We will also need the following result. Recall that a group is *cotorsion-free* if it does not contain any non-zero subgroups which are cotorsion, or, equivalently, is reduced and torsion-free and does not contain a subgroup isomorphic to J_p for any prime p (cf. [12] or [6, §V.2]).

Lemma 7. *If G is a non-zero cotorsion-free group and A is a torsion-free group in ${}^\perp G$, then A is not pure-injective.*

Proof. By hypothesis, G is reduced and torsion-free and not pure-injective, so G is not equal to $\widehat{G}$. There is an exact sequence

$$0 \to G \to \widehat{G} \to \widehat{G}/G \to 0$$

where G is pure in $\widehat{G}$ and $\widehat{G}/G$ is (torsion-free) divisible and non-zero. This exact sequence induces the exact sequence

$$0 \to \text{Hom}(A, G) \to \text{Hom}(A, \widehat{G}) \to \text{Hom}(A, \widehat{G}/G) \to \text{Ext}(A, G) = 0.$$

We claim that $\text{Hom}(A, G) \neq 0$. Indeed, otherwise, $\text{Hom}(A, \widehat{G}) \cong \text{Hom}(A, \widehat{G}/G)$, but $\text{Hom}(A, \widehat{G})$ is reduced, because $\widehat{G}$ is torsion-free and reduced; and $\text{Hom}(A, \widehat{G}/G)$ is a non-zero divisible group. Now if A were pure-injective, G would contain a non-zero homomorphic image of A, that is, a non-zero cotorsion group; but that is impossible by the assumption on G. ∎

If S is a subset of an uncountable cardinal μ which consists of ordinals of cofinality σ, a *ladder system* on S is a family $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$ of functions $\zeta_\delta : \sigma \to \delta$ which are strictly increasing and cofinal in δ. For a cardinal λ, we say that $\bar{\eta}$ has the λ-*uniformization property* if for any functions $c_\delta : \sigma \to \lambda$ for $\delta \in S$, there is a pair (f, f^*) where $f : \mu \to \omega$ and $f^* : S \to \sigma$ such that for all $\delta \in S$, $f(\zeta_\delta(\nu)) = c_\delta(\nu)$ whenever $f^*(\delta) \leq \nu < \sigma$.

Proof of Theorem 2. We will use the fact that the following principle is consistent with ZFC + GCH (cf. [7]):

> **(UP$^+$)** For every cardinal μ of the form τ^+ where τ is singular of cofinality ω there is a stationary subset S of μ consisting of limit ordinals of cofinality ω and a ladder system $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$ which has the λ-uniformization property for every $\lambda < \tau$.

We work in a model of GCH plus (UP$^+$). Given $B \in {}^\perp\mathcal{C}$, let $\kappa \geq \max(|B|, \sup\{|G| : G \in \mathcal{C}\})$ and let $\mu = \tau^+ = 2^\tau$ where $\tau > \kappa$ is a singular cardinal of cofinality ω. Then $\mu^\kappa = \mu$. Let $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$ be as in (UP$^+$) for this μ. Let $A = \cup_{\nu<\mu} A_\nu$ be as in Theorem 5 for this B and μ.

Let $H_A = F/K$ where F is the free group on symbols $\{y_{\delta,n} : \delta \in S, n \in \omega\} \cup \{x_j : j < \mu\}$ and K is the subgroup with basis $\{w_{\delta,n} : \delta \in S, n \in \omega\}$ where

$$w_{\delta,n} = y_{\delta,n} - (n+1)y_{\delta,n+1} + x_{\zeta_\delta(n)}. \tag{1}$$

Then H_A is a group of cardinality μ and the uniformization property of $\bar{\zeta}$ implies that $H_A \in {}^\perp\mathcal{C}$. (See [6, §XIII.0] or [22].) It suffices to show that $\text{Ext}(H_A, A) \neq 0$ for then A belongs to $B^\perp$ but not to $\{{}^\perp\mathcal{C}\}^\perp$.

We will show that $\text{Ext}(H_A, A) \neq 0$ by defining $\psi : K \to A$ which does not extend to a homomorphism from F to A.

For all $\delta < \mu$, A/A_δ belongs to ${}^\perp G$ for every $G \in \mathcal{C}$, so A/A_δ is not pure-injective by hypothesis on $\mathcal{C}$ and Lemma 7. Thus by Lemma 6, there is an element $a_\delta = \Sigma_{n\in\omega} n!(t_{\delta,n} + A_\delta)$ in the $\mathbb{Z}$-adic completion of A/A_δ which is not in A/A_δ. Define $\psi : K \to A$ such that $\psi(w_{\delta,n}) = t_{\delta,n}$ for all $\delta \in S$, $n \in \omega$. Suppose, to obtain a contradiction, that ψ extends to a homomorphism $\varphi : F \to A$. The set of $\delta < \mu$ such that $\varphi(x_j) \in A_\delta$ for all $j < \delta$ is a club, C, in μ, so there exists $\delta \in S \cap C$. We will contradict the choice of a_δ for this δ.

We work in A/A_δ. Let $c_n = \varphi(y_{\delta,n}) + A_\delta$. Then by applying φ to the equations (1) and since $\varphi(x_j) \in A_\delta$ for all $j < \delta$ we have that for all $n \in \omega$,

$$t_{\delta,n} + A_\delta = c_n - (n+1)c_{n+1}.$$

It follows that $a_\delta = c_0$ is in A/A_δ, a contradiction. ∎

Proof of Corollary 3. It is easy to see that the condition that N is divisible is sufficient; for example, it follows from the main theorem of [9], since N is pure-injective. On the other hand, if N is not divisible, then $N = G \oplus D$ where G is reduced and non-zero and D is divisible. But then ${}^\perp N = {}^\perp\{G\}$ and G is cotorsion-free, so by the theorem it is consistent that there is no such B. ∎

2. Building Whitehead groups

We now begin the proof of Theorem 4. For this proof we will need the fact that the members of ${}^\perp\{\mathbb{Z}\}$ have a stronger property than not being pure-injective, namely, they are $\aleph_1$-free, even strongly $\aleph_1$-free. It will suffice to prove the following:

Theorem 8. *It is consistent with ZFC + GCH that for every Whitehead group B there is an uncountable Whitehead group $G = G_B$ such that every homomorphism from G to B has finitely-generated range.*

Proof of Theorem 4 from Theorem 8. Suppose that $f : B \to \mathbb{Q}$ is a $\mathcal{W}$-precover of $\mathbb{Q}$. Let G be as in Theorem 8 for this B. Since $\mathbb{Q}$ is injective and G has infinite rank, there is a surjective homomorphism $g : G \to \mathbb{Q}$. But then clearly there is no $h : G \to B$ such that $f \circ h = g$. ∎

Our method of proving 8 is based on the following lemma. In its proof, as well as in later results, we will use the result of Gregory and Shelah (cf. [13], [18]) that GCH implies $\diamondsuit_\lambda$ for every successor cardinal $\lambda > \aleph_1$.

Lemma 9. *Assume GCH. Suppose that for every Whitehead group A of infinite rank, there is a Whitehead group H_A of cardinality $\leq |A|^+$ such that* $\mathrm{Ext}(H_A, A) \neq 0$. *Then for every Whitehead group B there* *is an uncountable Whitehead group G such that every homomorphism from G to B has finitely-generated range.*

Proof. Let $\lambda = \mu^+$ where $\mu > |B| + \aleph_1$. Then $\diamondsuit_\lambda$ holds, and we will use it to construct the group structure on a set G of size λ. We can write $G = \bigcup_{\nu<\lambda} G_\nu$ as the union of a continuous chain of sets such that for all $\nu < \lambda$, $|G_{\nu+1} - G_\nu| = \mu$. Now $\diamondsuit_\lambda$ gives us a family $\{h_\nu : \nu \in \lambda\}$ of set functions $h_\nu : G_\nu \to B$ such that for every function $f : G \to B$, $\{\nu \in \lambda : f \restriction G_\nu = h_\nu\}$ is stationary.

Suppose that the group structure on G_ν has been defined and consider h_ν; if h_ν is not a homomorphism or the range of h_ν is of finite rank, define the group structure on $G_{\nu+1}$ in any way which extends that on G_ν. Otherwise, let A be the range of h_ν and let H_A be as in the hypothesis. Without loss of generality, $|H_A| = \mu$. (Just add a free summand to H_A if necessary.) Write $H_A = F/K$ where F is a free group of rank μ. Since $\mathrm{Ext}(H_A, A) \neq 0$, a standard homological argument implies that there is a homomorphism $\psi : K \to A$ which does not extend to a homomorphism $: F \to A$. Since K is free and $h_\nu : G_\nu \to B$ is onto A, there is a homomorphism $\theta : K \to G_\nu$ such that $h_\nu \circ \theta = \psi$. Now form the pushout

$$\begin{array}{ccc} F & \to & G_{\nu+1} \\ \uparrow & & \uparrow \\ K & \stackrel{\theta}{\to} & G_\nu \end{array}$$

to define the group structure on $G_{\nu+1}$ (cf. [8, proof of Theorem 2]). Then $G_{\nu+1}/G_\nu \cong F/K \cong H_A$ so it is Whitehead. Moreover, h_ν does not extend to a homomorphism from $G_{\nu+1}$ into A, else ψ extends to a homomorphism on F. This completes the definition of G. Notice that G is a Whitehead group since all quotients $G_{\nu+1}/G_\nu$ are isomorphic to F/K and hence Whitehead (cf. [8, Lemma 1]).

Now given any homomorphism $f : G \to B$, let $A \subseteq B$ be the range of f. Since $|A| < |G| = \lambda$, $\{\nu \in \lambda : f[G_\nu] = A\}$ is a club in λ; hence there exists $\nu \in \lambda$ such that $f \restriction G_\nu = h_\nu$ and the range of h_ν is A. If A is of infinite rank, we have constructed $G_{\nu+1}$ so that $f \restriction G_\nu$ does not extend to $G_{\nu+1}$, which is a contradiction. So we must conclude that the range of f is of finite rank. ∎

Thus our goal is to show that there is a model of ZFC + GCH such that for every W-group A of infinite rank, there is a W-group H_A of cardinality $\leq |A|^+$ such that $\mathrm{Ext}(H_A, A) \neq 0$. The W-groups H_A will be constructed in the following manner. The definition is in the spirit of the general constructions in, for example, [22] or [6, XIII.1.4] but is a little more complicated since it is "two step": involving a system of ladders of length $\mathrm{cf}(\mu)$ and another system of ladders of length ω (if $\mathrm{cf}(\mu) > \aleph_0$).

Definition 1. *Let μ be a cardinal of cofinality σ ($\leq \mu$). Let S be a subset of $\lambda = \mu^+$ consisting of ordinals of cofinality σ and $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ a ladder system on S. If $\sigma > \aleph_0$, let E be a stationary subset of σ consisting of limit ordinals of cofinality ω and let $\bar{\zeta} = \{\zeta_\nu : \nu \in E\}$ be a ladder system on E. We will say that H is* the group built on $\bar{\eta}$ and $\bar{\zeta}$ *if $H \cong F/K$ where F is the free group on symbols $\{y_{\delta,\nu,n} : \delta \in S, \nu \in E, n \in \omega\} \cup \{z_{\delta,j} : \delta \in S, j \in \sigma\} \cup \{x_\beta : \beta \in \lambda\}$ and K is the subgroup with basis $\{w_{\delta,\nu,n} : \delta \in S, \nu \in E, n \in \omega\}$ where*

$$w_{\delta,\nu,n} = y_{\delta,\nu,n} - 2y_{\delta,\nu,n+1} - z_{\delta,\zeta_\nu(n)} + x_{\eta_\delta(\nu+n)}. \tag{2}$$

(If $\sigma = \aleph_0$, let $E = \{0\}$ and omit $\bar{\zeta}$ and the $z_{\delta,j}$.) For future reference, for $\alpha \in \lambda$, let F_α be the subgroup of F generated by $\{y_{\delta,\nu,n} : \delta \in S \cap \alpha, \nu \in E, n \in \omega\} \cup \{z_{\delta,j} : \delta \in S \cap \alpha, j < \sigma\} \cup \{x_\beta : \beta < \alpha\}$ and for $\alpha \in S$ and $\tau < \sigma$ let $F_{\alpha,\tau}$ be the subgroup generated by $\{z_{\alpha,j} : j < \tau\}$.

Theorem 10. *Suppose that H is built from $\bar{\eta}$ and $\bar{\zeta}$ as in Definition 1 and that E is a non-reflecting subset of σ. If, in addition, $\bar{\eta}$ has the ω-uniformization property, then H is a Whitehead group.*

Proof. We assume $\sigma > \aleph_0$ since this is known otherwise (cf. [16], [22]). If F and K are as in Definition 1, it suffices to show that every homomorphism $\psi : K \to \mathbb{Z}$ extends to a homomorphism $\varphi : F \to \mathbb{Z}$. Given ψ, for all $n \in \omega$ define $c_\delta(\nu + n)$ to be $\psi(w_{\delta,\nu,n})$ if $\nu \in E$, and arbitrary otherwise. Let (f, f^*) be the uniformizing pair. Define $\varphi(x_\beta) = f(\beta)$. For each $\delta \in S$ we must still define $\varphi(y_{\delta,\nu,n})$ and $\varphi(z_{\delta,j})$ for $\nu, j \in \sigma$ and $n \in \omega$. Fix δ and let $\rho = f^*(\delta)$; without loss of generality $\rho \notin E$. Let F' (resp. F'_ρ) be the subgroup of F generated by $\{y_{\delta,\nu,n} : \nu \in E, n \in \omega\} \cup \{z_{\delta,j} : j < \sigma\} \cup \{x_\beta : \beta < \delta\}$ (resp. by $\{y_{\delta,\nu,n} : \nu \in E \cap \rho, n \in \omega\} \cup \{z_{\delta,j} : j < \rho\} \cup \{x_\beta : \beta < \delta\}$) and K' (resp., K'_ρ) the subgroup generated by $\{w_{\delta,\nu,n} : \nu \in E, n \in \omega\} \cup \{x_\beta : \beta < \delta\}$ (resp., by $\{w_{\delta,\nu,n} : \nu \in E \cap \rho, n \in \omega\} \cup \{x_\beta : \beta < \rho\}$). Now F'/K' is σ-free since E is non-reflecting (cf. [6, §VII.1]), so $F'_\rho + K/K \cong F'_\rho/K'_\rho$ is free and hence K'_ρ is a summand of F'_ρ; then it is easy to extend $\psi \restriction \{w_{\delta,\nu,n} : \nu \in E \cap \rho, n \in \omega\} + \varphi \restriction \{x_\beta : \beta < \rho\}$ to $\varphi : F'_\rho \to \mathbb{Z}$. For $\nu \in E$ with $\nu > \rho$ we have $\varphi(x_{\eta_\delta(\nu+n)}) = \psi(w_{\delta,\nu,n})$ for all $n \in \omega$. For some m_ν, $\zeta_\nu(n) \geq \rho$ when $n \geq m_\nu$. Then we can satisfy the equations

$$\psi(w_{\delta,\nu,n}) = 2\varphi(y_{\delta,\nu,n+1}) - \varphi(y_{\delta,\nu,n}) - \varphi(z_{\delta,\zeta_\nu(n)}) + \varphi(x_{\eta_\delta(\nu+n)})$$

by setting $\varphi(y_{\delta,\nu,n}) = 0 = \varphi(z_{\delta,\zeta_\nu(n)})$ for $n \geq m_\nu$. For $\zeta_\nu(n) < \rho$, $\varphi(z_{\delta,\zeta_\nu(n)})$ is already defined; we can define $\varphi(y_{\delta,\nu,n})$ by downward induction on $n < m_\nu$ (cf. [6, proof of XIII.1.4]). ∎

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3. How to make Ext not vanish

Next we need to show how groups H defined as in 1 can satisfy $\text{Ext}(H, A) \neq 0$ for a given W-group A. In the proof of 2, we used a description of A as the union of a chain of subgroups which came from the construction of A. Now we have only what we can learn from the fact that A is Whitehead, assuming GCH. We begin by proving some general properties of decompositions of Whitehead groups assuming GCH. Besides the result of Gregory and Shelah that GCH implies $\diamondsuit_\lambda$ for successor cardinals $\lambda > \aleph_1$, we will use the result of Devlin and Shelah [3] that CH implies weak diamond, $\Phi_{\aleph_1}$, at $\aleph_1$. We will also make repeated use of the following crucial fact (cf. [14], [3], [6, XII.1.10]):

Proposition 11. *Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ be a λ-filtration of a group of cardinality λ, that is $\{A_\alpha : \alpha < \lambda\}$ is a continuous chain of subgroups of A of cardinality $< \lambda$. Let Z be any group of cardinality $\leq \lambda$. Suppose that $\diamondsuit_\lambda(E)$ or the weak diamond principle $\Phi_\lambda(E)$ holds, where $E = \{\alpha \in \lambda : \exists \beta > \alpha \text{ s.t. } \text{Ext}(A_\beta/A_\alpha, Z) \neq 0\}$. Then $\text{Ext}(A, Z) \neq 0$.* ∎

Corollary 12. *Let A be a Whitehead group of cardinality $\lambda = \mu^+$ and let $A = \bigcup_{\alpha<\lambda} A_\alpha$ be a λ-filtration of A. Let $S(A) \stackrel{def}{=} \{\alpha \in \lambda : A_\tau/A_\alpha$ is Whitehead for all $\tau > \alpha\}$. If $\Phi_\lambda(Y)$ holds for some subset Y of λ, then $Y \cap S(A)$ is stationary. In particular, assuming GCH, $S(A)$ is stationary.*

Proof. Suppose $Y \cap S(A)$ is not stationary in λ, and let C be a club in its complement. Then $\Phi_\lambda(Y \cap C)$ holds and $\alpha \in Y \cap C$ implies that $\alpha \notin S(A)$, so by 11 (with $Z = \mathbb{Z}$), A is not Whitehead, a contradiction. ∎

We will say that A/A_α is *locally Whitehead* when $\alpha \in S(A)$, that is, every subgroup of A/A_α of cardinality $< \lambda$ is Whitehead.

Lemma 13. *Assume GCH. Let A be a Whitehead group of cardinality μ (possibly a singular cardinal). Then we can write $A = \bigcup_{\nu<\mu} A_\nu$ as the continuous union of a chain of subgroups of cardinality $< \mu$ such that for all $\nu < \mu$, $A/A_{\nu+1}$ is $\aleph_1$-free.*

Proof. If suffices to show that every subgroup X of A of cardinality $\kappa < \mu$ is contained in a subgroup N of cardinality κ such that N'/N is free whenever $N \subseteq N' \subseteq A$ and N'/N is countable. But if X is a counterexample, then we can build a chain $\{N_\alpha : \alpha < \kappa^+\}$ such that $N_0 = X$ and for all $\alpha < \kappa^+$, $N_{\alpha+1}/N_\alpha$ is countable and not free, and hence is not Whitehead. We obtain a contradiction since then Φ_{κ^+} implies that $\bigcup_{\alpha<\kappa^+} N_\alpha$ is not Whitehead. ∎

We now give sufficient conditions for $\mathrm{Ext}(H, A)$ to be non-zero, when H is defined as in 1. The analysis will be divided into cases, depending on whether the cardinality of A is singular, the successor of a regular cardinal, or the successor of a singular cardinal.

When the cardinality of A is singular, we will use a special case of a recent result of the second author (cf. [20]).

Lemma 14. *Assume GCH. Let μ be a singular cardinal and let $\sigma = \mathrm{cof}(\mu) < \mu$ and $\lambda = \mu^+$. Suppose that S is a stationary subset of λ consisting of ordinals of cofinality σ and $\{\eta_\delta : \delta \in S\}$ is a ladder system on S. Then for each $\delta \in S$ there is a sequence of sets $D^\delta = \langle D^\delta_\nu : \nu < \sigma\rangle$ such that*

(a) for all $\delta \in S$ and $\nu \in \sigma$, $D^\delta_\nu \subseteq \lambda$, $\sup(D^\delta_\nu) < \delta$ and $|D^\delta_\nu| < \mu$; and

(b) for every function $h : \lambda \to \lambda$, $\{\delta \in S : h(\eta_\delta(\nu)) \in D^\delta_\nu$ for all $\nu \in \sigma\}$ is stationary in λ.

Proof. Fix $\delta \in S$. Let $\langle b^\delta_\nu : \nu < \sigma\rangle$ be an increasing continuous union of subsets of δ whose union is δ and such that $\sup(b^\delta_\nu) < \delta$ and $|b^\delta_\nu| < \mu$. Let $\theta = \sigma^\sigma = 2^\sigma = \sigma^+(< \mu)$ and let $\langle g_i : i < \theta\rangle$ be a list of all functions from σ to σ. Also let $\langle f_\gamma : \gamma < \lambda\rangle$ list all functions from θ to λ $(= 2^\mu = \lambda^\theta)$; without loss of generality, $f_\gamma(i) < \gamma$ for all $i \in \theta$. For each $i \in \theta$ and $\nu \in \theta$, define $D^{i,\delta}_\nu = \{f_\gamma(i) : \gamma \in b^\delta_{g_i(\nu)}\}$.

We claim that for some $i \in \theta$, the sets $\{D^{i,\delta} = \langle D^{i,\delta}_\nu : \nu < \sigma\rangle : \delta \in S\}$ will work in (b). Assuming the contrary, for each $i \in \theta$, let $h_i : \lambda \to \lambda$ be a counterexample, i.e., there is a club C_i in λ such that for each $\delta \in C_i \cap S$, there is $\nu \in \sigma$ such that $h_i(\eta_\delta(\nu)) \notin D^{i,\delta}_\nu$.

For each $\alpha \in \lambda$, there is $h(\alpha) \in \lambda$ such that for all $i \in \theta$, $h_i(\alpha) = f_{h(\alpha)}(i)$. There exists $\delta_* \in \bigcap_{i\in\theta} C_i \cap S$ such that for all $\alpha < \delta_*$, $h(\alpha) \in \delta_*$. Denote $h(\eta_{\delta_*}(\nu))$ by γ_ν. There exists $i_* \in \theta$ such that for all $\nu < \sigma$,

$$g_{i_*}(\nu) = \min\{j < \sigma : \gamma_\nu \in b^{\delta_*}_j\}.$$

(Note that the right-hand side exists since $\delta_* = \cup_{j<\sigma} b^{\delta_*}_j$ and $\gamma_\nu \in \delta_*$.) Thus

$$\gamma_\nu \in b^{\delta_*}_{g_{i_*}(\nu)}.$$

But then, (letting $\alpha = \eta_{\delta_*}(\nu)$ in the definition of h),

$$h_{i_*}(\eta_{\delta_*}(\nu)) = f_{h(\eta_{\delta_*}(\nu))}(i_*) = f_{\gamma_\nu}(i_*) \in D^{i_*,\delta_*}_\nu.$$

Since this holds for all $\nu \in \sigma$, the fact that h_{i_*} is a counterexample implies that $\delta_* \notin C_{i_*} \cap S$. But this contradicts the choice of δ_*. ∎

Theorem 15. *Assume GCH. Let μ be a singular cardinal of cofinality σ. If H is a group of cardinality $\lambda = \mu^+$ built on $\bar{\eta}$ and $\bar{\zeta}$ as in Definition 1 and A is a Whitehead group of cardinality μ, then* $\mathrm{Ext}(H, A) \neq 0$.

Proof. Let the sets $\{D^\delta = \langle D^\delta_\nu : \nu \in \sigma \rangle : \delta \in S\}$ be as in Lemma 14 for this ladder system. Write $A = \bigcup_{\nu<\mu} A_\nu$ as in Lemma 13. Without loss of generality we can assume that the universe of A is μ and that for all ν, $A_{\nu+1}/A_\nu$ is non-zero.

We claim that for all $\beta < \mu$, the 2-adic completion of A/A_β has rank $\geq \mu$ over A/A_β. For notational convenience we will prove the case $\beta = 0$, but the argument is the same in general using the decomposition $A/A_\beta = \bigcup_{\beta\leq\alpha<\mu} A_\alpha/A_\beta$. For every successor ordinal α, since $A_{\alpha+1}/A_\alpha$ is $\aleph_1$-free and non-zero, there are $s^\alpha_n \in A_{\alpha+1}$ such that the element $\Sigma_{n\in\omega} 2^n(s^\alpha_n + A_\alpha)$ of the 2-adic completion of $A_{\alpha+1}/A_\alpha$ is not in $A_{\alpha+1}/A_\alpha$. We claim that the elements $\{\Sigma_{n\in\omega} 2^n s^\alpha_n : \alpha = \nu+1, \nu \in \mu\}$ of the 2-adic completion of A are linearly independent over A. Suppose not, and let

$$\Sigma^m_{i=1} k_i(\Sigma_{n\in\omega} 2^n s^{\alpha(i)}_n) = a$$

be a counterexample; so $a \in A$; $k_i \in \mathbb{Z} - \{0\}$; and $\alpha(1) < \alpha(2) < \ldots < \alpha(m) < \mu$. Let $\gamma = \alpha(m)$ and $k = k_\gamma$. We claim that the element $k\Sigma_{n\in\omega} 2^n(s^\gamma_n + A_\gamma)$ of the 2-adic completion of $A_{\gamma+1}/A_\gamma$ belongs to $A_{\gamma+1}/A_\gamma$ which is a contradiction of the choice of the s^γ_n. Since $A/A_{\gamma+1}$ is $\aleph_1$-free, we can write $\langle A_{\gamma+1}, a\rangle_* = A_{\gamma+1} \oplus C$ for some C, and let a' be the projection of a on the first factor. For every $r \in \omega$, 2^{r+1} divides $a - \Sigma^m_{i=1} k_i(\Sigma^r_{n=0} 2^n s^{\alpha(i)}_n)$ in A and hence 2^{r+1} divides $a' - \Sigma^m_{i=1} k_i(\Sigma^r_{n=0} 2^n s^{\alpha(i)}_n)$ in $A_{\gamma+1}$. But then 2^{r+1} divides $(a' + A_\gamma) - k\Sigma^r_{n=0} 2^n(s^\gamma_n + A_\gamma)$ in $A_{\gamma+1}/A_\gamma$; since this holds for all $r \in \omega$, $k\Sigma_{n\in\omega} 2^n(s^\gamma_n + A_\gamma) = a' + A_\gamma$, and we have a contradiction.

Choose a strictly increasing continuous function $\xi : \sigma \to \mu$ whose range is cofinal in μ. For each $\delta \in S$ and $\nu \in E$, there is an element $a_{\delta,\nu} = \Sigma_{n\in\omega} 2^n(a(\delta, \nu, n) + A_{\xi(\nu)+1})$ in the 2-adic completion of $A/A_{\xi(\nu)+1}$ which is not in the subgroup generated by $A/A_{\xi(\nu)+1}$ and the 2-adic completion of $\{d + A_{\xi(\nu)+1} : d \in D^\delta_\nu \cap A\}$. (Note that the latter has cardinality $< \mu$ since $|D^\delta_\nu|^{\aleph_0} < \mu$ by the GCH.)

Now define $\psi : K \to A$ such that $\psi(w_{\delta,\nu,n}) = a(\delta, \nu, n)$. We claim that ψ does not extend to a homomorphism $\varphi : F \to A$. Suppose, to the contrary, that it does. Then by Lemma 14, there is $\delta \in S$ such that $\varphi(x_{\eta_\delta(\nu)}) \in D^\delta_\nu$ for all $\nu \in \sigma$. Now there exists $\nu \in E$ such that $\varphi(z_{\delta,j}) \in A_{\xi(\nu)}$ for all $j < \nu$. We will contradict the choice of $a_{\delta,\nu}$ for this δ and ν.

We work in $A/A_{\xi(\nu)+1}$. Let $c_n = \varphi(y_{\delta,\nu,n}) + A_{\xi(\nu)+1}$, $d_n = \varphi(x_{\eta_\delta(\nu+n)}) + A_{\xi(\nu)+1}$. Then by applying φ to the equations (2) and since $\varphi(z_{\delta,j}) \in$

$A_{\xi(\nu)}$ for all $j < \nu$ we have that for all $n \in \omega$,

$$a(\delta, \nu, n) + A_{\xi(\nu)+1} = c_n - 2c_{n+1} + d_n.$$

It follows that $a_{\delta,\nu} = c_0 + \Sigma_{n\in\omega} 2^n d_n$ is in the subgroup generated by $A/A_{\xi(\nu)+1}$ and the 2-adic completion of $\{d + A_{\xi(\nu)+1} : d \in D^{\delta}_{\nu} \cap A\}$, which contradicts the choice of $a_{\delta,\nu}$. ∎

We now turn to the cases when the cardinality of A is a successor cardinal. Though the two arguments could be combined into one, following the argument in Theorem 17, we prefer to introduce the method with the somewhat simpler argument for the successor of regular case.

Theorem 16. *Assume GCH. Let $\lambda = \mu^+$ where μ is a regular cardinal (so $\sigma = \mathrm{cf}(\mu) = \mu$). Suppose H is built on $\bar\eta = \{\eta_\delta : \delta \in S\}$ and $\bar\zeta = \{\zeta_\nu : \nu \in E\}$ as in Definition 1. Suppose also, for $\mu > \aleph_0$, that $\diamondsuit_\mu(E')$ holds for all stationary subsets E' of E. If A is a Whitehead group of cardinality $\lambda = \mu^+$, then* $\mathrm{Ext}(H, A) \neq 0$.

Proof. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ and $S(A)$ be as in Lemma 12. Note that (here and in the next theorem) we make no assumption about the relation of S and $S(A)$; maybe $S \cap S(A) = \emptyset$. Without loss of generality, for all $\delta \in S(A)$, $A_{\delta+1}/A_\delta$ is Whitehead of rank μ and $A/A_{\delta+1}$ is locally Whitehead. Assume $\mu > \aleph_0$; the proof for $\aleph_0$ is simpler. For each $\alpha < \lambda$, write A_α as the union of a continuous chain of subgroups of cardinality $< \mu$: $A_\alpha = \bigcup_{\nu<\mu} B_{\alpha,\nu}$. Thus $A_{\delta+1}/A_\delta = \bigcup_{\nu<\mu}(A_\delta + B_{\delta+1,\nu})/A_\delta$; for $\delta \in S(A)$, since $\diamondsuit_\mu(E)$ holds, we can assume that the set of $\nu \in E$ such that $A_{\delta+1}/(A_\delta + B_{\delta+1,\nu})$ is locally Whitehead is stationary; for such ν, $A_{\delta+1}/A_\delta + B_{\delta+1,\nu}$ is then strongly $\aleph_1$-free since CH holds. Thus for ν in a stationary subset E_δ of E we can assume that $A_\delta + B_{\delta+1,\nu+1}/A_\delta + B_{\delta+1,\nu}$ is free of rank $\aleph_0$ and $A_{\delta+1}/A_\delta + B_{\delta+1,\nu+1}$ is $\aleph_1$-free. Say $\{t_{\delta,\nu,n} + A_\delta + B_{\delta+1,\nu} : n \in \omega\}$ is a basis of $A_\delta + B_{\delta+1,\nu+1}/A_\delta + B_{\delta+1,\nu}$.

For each $\delta_1 \in S$, let δ_1^+ be the least member of $S(A)$ which is $\geq \delta_1$. Define

$$\psi(w_{\delta_1,\nu,n}) = t_{\delta_1^+,\nu,n}$$

for all $n \in \omega$ if $\nu \in E_{\delta_1^+}$. Define ψ arbitrarily otherwise. We claim that ψ does not extend to $\varphi : F \to A$. Suppose to the contrary that it does. Let $M = \varphi[F]$, $M_\alpha = \varphi[F_\alpha]$, $M_{\alpha,\tau} = \varphi[F_{\alpha,\tau}]$. Then there is a club C in λ such that for $\alpha \in C$, $M_\alpha \subseteq A_\alpha$. Fix δ_1 in $C \cap S$. Let δ denote δ_1^+ and choose $\gamma \in C$ such that $\gamma > \delta$. There is a club C' in μ such that for $\nu \in C'$, $M_{\delta_1,\nu} \subseteq B_{\gamma,\nu}$ and $A_{\delta+1} \cap B_{\gamma,\nu} \subseteq B_{\delta+1,\nu}$. Since $\diamondsuit_\mu(E_\delta)$ holds and $A_\gamma/A_{\delta+1}$ is Whitehead, there is, by Lemma 12, $\nu \in E_\delta \cap C'$ such that $A_\gamma/(A_{\delta+1} + B_{\gamma,\nu})$ is locally Whitehead, and hence $\aleph_1$-free.

We will obtain a contradiction of Lemma **??** with $L = A_\gamma/(A_\delta + B_{\gamma,\nu})$ and $L' = (B_{\delta+1,\nu+1} + A_\delta + B_{\gamma,\nu})/(A_\delta + B_{\gamma,\nu})$ and $t_n = t_{\delta,\nu,n} + A_\delta + B_{\gamma,\nu}$. Notice that modulo $A_\delta + B_{\gamma,\nu}$ we have

$$2\varphi(y_{\delta_1,\nu,n+1}) = \varphi(y_{\delta_1,\nu,n}) - t_{\delta,\nu,n}$$

for all $n \in \omega$ since $\varphi(x_{\eta_{\delta_1}(\nu+n)}) \in M_{\delta_1} \subseteq A_{\delta_1} \subseteq A_\delta$ and $\varphi(z_{\delta_1,\zeta_\nu(n)}) \in M_{\delta_1,\nu} \subseteq B_{\gamma,\nu}$. Moreover, $\{t_n : n \in \omega\}$ is a basis of a summand of L' since L' is naturally isomorphic to $A_\delta + B_{\delta+1,\nu+1}/A_\delta + (B_{\gamma,\nu} \cap (A_\delta + B_{\delta+1,\nu+1}))$ and the latter has a natural epimorphism onto $A_\delta + B_{\delta+1,\nu+1}/A_\delta + B_{\delta+1,\nu}$ which is free on the basis $\{t_{\delta,\nu,n} + A_\delta + B_{\delta+1,\nu} : n \in \omega\}$. It remains to show that L/L' is $\aleph_1$-free. Now

$$0 \to (A_{\delta+1}+B_{\gamma,\nu})/(B_{\delta+1,\nu+1}+A_\delta+B_{\gamma,\nu}) \to L/L' \to A_\gamma/(A_{\delta+1}+B_{\gamma,\nu}) \to 0$$

is exact and $A_\gamma/(A_{\delta+1} + B_{\gamma,\nu})$ is $\aleph_1$-free by choice of ν, so it suffices to show that $(A_{\delta+1} + B_{\gamma,\nu})/(B_{\delta+1,\nu+1} + A_\delta + B_{\gamma,\nu})$ is $\aleph_1$-free. But this is isomorphic to $A_{\delta+1}/((A_\delta + B_{\delta+1,\nu+1}) + (A_{\delta+1} \cap B_{\gamma,\nu}))$, which (since $A_{\delta+1} \cap B_{\gamma,\nu} \subseteq B_{\delta+1,\nu} \subseteq B_{\delta+1,\nu+1}$) equals $A_{\delta+1}/(A_\delta + B_{\delta+1,\nu+1})$, which was chosen $\aleph_1$-free. ∎

The proof of the following is similar, but requires elementary submodels.

Theorem 17. *Assume GCH. Let $\lambda = \mu^+$ where μ is a (singular) cardinal of cofinality σ. Suppose H is built on $\bar\eta = \{\eta_\delta : \delta \in S\}$ and $\bar\zeta = \{\zeta_\nu : \nu \in E\}$ as in Definition 1. Suppose also that $\diamondsuit_\lambda(Y)$ holds for some subset Y of λ consisting of limit ordinals of cofinality σ and that, if $\sigma > \aleph_0$, $\diamondsuit_\sigma(E)$ holds. If A is a Whitehead group of cardinality $\lambda = \mu^+$, then* $\mathrm{Ext}(H, A) \neq 0$.

Proof. Without loss of generality, for all $\delta \in S(A)$, $A_{\delta+1}/A_\delta$ is Whitehead of rank μ. For each $\delta \in S$, choose a strictly increasing sequence $\langle \xi_{\delta,\nu} : \nu < \sigma \rangle$ of elements of $S(A)$ such that $\xi_{\delta,0} \geq \delta+1$ and whose limit, denoted $\xi_{\delta,\sigma}$, belongs to $S(A)$. This is possible because, by Lemma 12, $Y \cap S(A)$ is stationary so we can choose $\xi_{\delta,\sigma}$ to be an element of the intersection of $Y \cap S(A)$ with the closure of $\{\alpha \in S(A) : \alpha > \delta\}$. Let $B_{\nu+1,\nu} = A_{\xi_{\delta,\nu}}$. (Note the difference from the last proof.) We can then modify the sequence so that $B_{\delta+1,\nu+1}/B_{\delta+1,\nu}$ is free on a countable set $\{t_{\delta,\nu,n} + B_{\delta+1,\nu}\}$ and $A/B_{\delta+1,\nu+1}$ is $\aleph_1$-free when $\nu \in E$. (We no longer require $\xi_{\delta,\nu+1} \in S(A)$.)

For each $\delta_1 \in S$, let δ_1^+ be the least member of $S(A)$ which is $\geq \delta_1$. Define

$$\psi(w_{\delta_1,\nu,n}) = t_{\delta_1^+,\nu,n}$$

for all $n \in \omega$. We claim that ψ does not extend to $\varphi : F \to A$. Suppose to the contrary that it does. As before, let $M = \varphi[F]$, $M_\alpha = \varphi[F_\alpha]$, $M_{\alpha,\tau} = \varphi[F_{\alpha,\tau}]$ and let C be a club such that for $\alpha \in C$, $M_\alpha \subseteq A_\alpha$. Fix δ_1 in $C \cap S$. Let δ be δ_1^+ and choose $\gamma \in C$ such that $\gamma > \delta$.

Let $N = \bigcup_{\nu<\sigma} N_\nu$ be the continuous union of a chain of elementary submodels of $H(\chi)$ for large enough χ such that each N_ν has cardinality $< \sigma$, $N_\nu \in N_{\nu+1}$ and such that δ, σ, A, $\{A_\alpha : \alpha < \lambda\}$, $\{\varphi(z_{\delta_1,j}) : j < \sigma\}$, $\{\varphi(x_{\eta_{\delta_1}(j+n)}) : j < \sigma\}$ (for each $n \in \omega$), $\{t_{\delta,j,n} : j < \sigma, n \in \omega\}$ and $\{\xi_{\delta,j} : j \le \sigma\}$ all belong to N_0 and

$$\{\varphi(z_{\delta_1,j}) : j < \sigma\} \cup \{\varphi(x_{\eta_{\delta_1}(j)}) : j < \sigma\} \cup \{t_{\delta,j,n} : j < \sigma, n \in \omega\} \cup \sigma \subseteq N.$$

Moreover, by intersecting with a club, we can assume that for all ν, $N_\nu \cap \sigma = \nu$ and $N_\nu \cap B_{\delta+1,\sigma} \subseteq B_{\delta+1,\nu}$ and hence $\{\xi_{\delta,j} : j < \nu\}$, $\{\varphi(z_{\delta_1,j}) : j < \nu\}$, $\{t_{\delta,j,n} : j < \nu, n \in \omega\}$, and $\{\varphi(x_{\eta_{\delta_1}(j+n)}) : j < \nu\}$ (for all $n \in \omega$) are all subsets of N_ν. We claim that there is a $\nu \in E$ such that $A/(B_{\delta+1,\sigma} + (N_\nu \cap A))$ is $\aleph_1$-free. Assuming this for the moment, we show how to obtain a contradiction of Lemma **??** with

$$L = (N \cap A)/((N \cap A_\delta) + (N_\nu \cap A)),$$
$$L' = ((N \cap B_{\delta+1,\nu+1}) + (N_\nu \cap A))/((N \cap A_\delta) + (N_\nu \cap A))$$

and

$$t_n = t_{\delta,\nu,n} + ((N \cap A_\delta) + (N_\nu \cap A)).$$

Notice that for all $n \in \omega$, $\varphi(x_{\eta_{\delta_1}(\nu+n)}) \in (N \cap A_\delta)$ and $\varphi(z_{\delta_1,\zeta_\nu(n)}) \in N_\nu$. Moreover, $\{t_n : n \in \omega\}$ is a basis of a summand of L' because L' is naturally isomorphic to $(N \cap B_{\delta+1,\nu+1})/(N \cap A_\delta) + (N_\nu \cap B_{\delta+1,\nu})$ and the latter has epimorphic image $(N \cap B_{\delta+1,\nu+1})/(N \cap B_{\delta+1,\nu})$ which is free on the basis $\{t_{\delta,\nu,n} + (N \cap B_{\delta+1,\nu}) : n \in \omega\}$ by choice of N. To see that L/L' is $\aleph_1$-free, use the short exact sequence

$$\begin{aligned} 0 \;\to\; & ((N \cap B_{\delta+1,\sigma}) + (N_\nu \cap A))/((N \cap B_{\delta+1,\nu+1}) + (N_\nu \cap A)) \to L/L' \\ \to\; & (N \cap A)/((N \cap B_{\delta+1,\sigma}) + (N_\nu \cap A)) \to 0 \end{aligned}$$

The last term is $\aleph_1$-free by choice of ν and since N is an elementary submodel of $H(\chi)$. Moreover, the first term is isomorphic to $(N \cap B_{\delta+1,\sigma})/(N \cap B_{\delta+1,\nu+1})$ (since $N_\nu \cap B_{\delta+1,\sigma} \subseteq B_{\delta+1,\nu}$) and thus is $\aleph_1$-free since $A/B_{\delta+1,\nu+1}$ is $\aleph_1$-free.

It remains to show that there is a $\nu \in E$ such that $A/(B_{\delta+1,\sigma} + (N_\nu \cap A))$ is $\aleph_1$-free. If not, then for all $\nu \in E$, $(B_{\delta+1,\sigma} + (N_{\nu+1} \cap A))/(B_{\delta+1,\sigma} + (N_\nu \cap A))$ is not $\aleph_1$-free (and hence not Whitehead), since A, $B_{\delta+1,\sigma}$ and N_ν belong to the elementary submodel $N_{\nu+1}$. But then $\diamondsuit_\sigma(E)$ implies that $\bigcup_{\nu<\sigma}(B_{\delta+1,\sigma} + (N_\nu \cap A))/B_{\delta+1,\sigma}$ is a group of cardinality σ which is not a Whitehead group, contradicting the fact that $A/B_{\delta+1,\sigma} = A/A_{\xi_{\delta,\sigma}}$ is locally Whitehead. ∎

4. Finishing the proof of Theorem 8

Finally we can put the pieces together to prove the consistency of the hypothesis of Lemma 9:

Theorem 18. *There is a model of ZFC + GCH such that for every Whitehead group A of infinite rank, there is a Whitehead group H_A of cardinality $\leq |A|^+$ such that* $\mathrm{Ext}(H_A, A) \neq 0$.

Proof. By a forcing construction (cf. [21]) there is a model of ZFC + GCH such that the following holds (where S^λ_μ denotes the set of ordinals $< \lambda$ of cofinality μ):

(i) for every infinite successor cardinal $\lambda = \mu^+$ there is a stationary subset S of $S^\lambda_{\mathrm{cf}(\mu)}$ with a ladder system $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ which satisfies ω-uniformization (or even κ-uniformization for every $\kappa < \mu$);

(ii) for every infinite successor cardinal $\lambda = \mu^+$ there is a stationary subset Y of $S^\lambda_{\mathrm{cf}(\mu)}$ such that $\diamondsuit_\lambda(Y)$ holds;

(iii) for every regular uncountable cardinal σ, there is a non-reflecting stationary subset E of S^σ_ω such that $\diamondsuit_\sigma(E')$ holds for every stationary subset E' of E;

(iv) there is a tree-like ladder system on a stationary subset of ω_1 which satisfies 2-uniformization but not ω-uniformization.

We work in this model. Let A be a Whitehead group of infinite rank. If the rank of A is $\aleph_0$, then A is isomorphic to $\mathbb{Z}^{(\omega)}$ and it is well-known (cf. [16], [6, XIII.0.6]) that (iv) implies that there is a Whitehead group H which is not $\aleph_1$-coseparable, i.e., $\mathrm{Ext}(H, \mathbb{Z}^{(\omega)}) \neq 0$. If the cardinality of A is either singular or a successor cardinal, then for $\lambda = |A|$ if $|A|$ is regular, or $\lambda = |A|^+$ if $|A|$ is singular, the properties (i), (ii) and (iii) allow us to build a group H_A of cardinality λ as in Definition 1, which is Whitehead by Theorem 10 and such that by Theorem 15, 16 or 17, $\mathrm{Ext}(H_A, A) \neq 0$.

It is also consistent to assume that there are no regular limit (i.e. inaccessible) cardinals, in which case we have covered all possibilities for the cardinality of A and we are done. Another approach is to allow inaccessible cardinals but force the model to satisfy in addition:

(v) for every inaccessible cardinal λ there is a stationary subset S of S^λ_ω with a ladder system $\bar{\eta} = \{\eta_\delta : \delta \in S\}$ which satisfies ω-uniformization; moreover $\diamondsuit_\lambda$ holds.

As in Lemma 12, one can show that $S(A)$ is stationary and then the proof is similar to that in Theorem 16. ∎

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