ON λ STRONG HOMOGENEITY EXISTENCE FOR COFINALITY LOGIC
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ABSTRACT. Let $C \subset_{\neq}$ Reg be a non-empty class (of regular cardinals). Then the logic $\mathbb{L}(Q^{\rm cf}_C)$ has additional nice properties: it has the homogeneous model existence property.

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0. Introduction

We deal with logics gotten by strengthening of first order logic by generalized quantifiers, in particular compact ones. We continue [She85] (and [She75a])

A natural quantifier is the cofinality quantifier, $Q^{\rm cf}_{\le\lambda}$ (or $Q^{\rm cf}_C$), introduced in [She75a] as the first example of compact logic (stronger than first order logic, of course). Recall that the "uncountably many x's" quantifier $Q^{\rm card}_{\ge\aleph_1}$, is $\aleph_0$-compact but not compact. Also note that $\mathbb{L}(Q^{\rm cf}_{\le\lambda})$ is a very nice logic, e.g. with a nice axiomatization (in particular finitely many schemes) like the one of $\mathbb{L}(Q^{\rm card}_{\ge\aleph_1})$ of Keisler. By [She85], e.g. for $\lambda = 2^{\aleph_0}$, its Beth closure is compact, giving the first compact logic with the Beth property (i.e. implicit definition implies explicit definition).

Earlier there were indications that having the Beth property is rare for such logic, see e.g. in Makowsky [Mak85]. A weaker version of the Beth property is the weak Beth property dealing with implicit definition which always works; H. Friedman claim that historically this was the question. Mekler-Shelah [MS85] prove that at least consistently, $\mathbb{L}(Q^{\rm card}_{\ge\aleph_1})$ satisfies the weak Beth property. Väänänen in the mid nineties motivated by the result of Mekler-Shelah [MS85] asked whether we can find a parallel proof for $\mathbb{L}(Q^{\rm cf}_{\le\lambda})$ in ZFC.

A natural property for a logic $\mathcal{L}$ is

Definition 0.1. A logic $\mathcal{L}$ has the (strong) homogeneous model existence property <u>when</u> every theory $T \subseteq \mathcal{L}(\tau)$, (so has a model) has a strongly $(\mathcal{L},\aleph_0)$-homogeneous model M, so $\tau_M = \tau$ and M is a model of T and M satisfies: if $\bar a, \bar b \in {}^{\omega>}M$ realize the same $\mathcal{L}(\tau)$-type in M then there is an automorphism of M mapping $\bar a$ to $\bar b$.

This property was introduced in [She85] being natural and also as it helps to investigate the weak Beth property.

In §1 we prove that $\mathbb{L}(Q^{\rm cf}_C)$ has the strongly $\aleph_0$-saturated model existence property. The situation concerning the weak Beth property is not clear.

Question 0.2. 1) Does the logic $\mathbb{L}(Q^{\rm cf}_C)$ have the weak Beth property?
2) Does the logic $\mathbb{L}(Q^{\rm cf}_{\le\lambda_1}, Q^{\rm cf}_{\le\lambda_2})$ has the homogeneous model existence property?

The first version of this work was done in 1996.

Notation 0.3. 1) τ denotes a vocabulary, $\mathcal{L}$ a logic, $\mathcal{L}(\tau)$ the language for the logic $\mathcal{L}$ and the vocabulary τ.
2) Let $\mathbb{L}$ be first order logic, $\mathbb{L}(Q_*)$ be first order logic when we add the quantifier Q_*.
3) For a model M and ultrafilter D on a cardinal λ, let M^λ/D be the ultrapower and $\mathbf{j}_{M,D} = \mathbf{j}^\lambda_{M,D}$ be the canonical embedding of M into M^λ/D; of course, we can replace λ by any set.
4) Let LST (theorem/argument) stand for Löwenheim-Skolem-Tarski (on existence of elementary submodels).

Concerning 0.1, more generally

Definition 0.4. 1) M is strongly $(\mathcal{L},\theta)$-saturated (in $\mathcal{L} = \mathbb{L}$ we may write just θ) <u>when</u>

- (a) it is θ-saturated (i.e. every set of $\mathcal{L}(\tau_M)$-formulas with $< \theta$ parameters from M and $< \theta$ free variables which is finitely satisfiable in M is realized in M

(*b*) if $\zeta < \theta$ and $\bar{a}, \bar{b} \in {}^{\zeta}M$ realize the same $\mathcal{L}(\tau_M)$-type in M, then some automorphism of M maps $\bar{a}$ to $\bar{b}$.

2) M is a strongly sequence $(\mathcal{L}, \theta)$-homogeneous when clause (b) above holds.
3) M is sequence (Δ, κ)-homogeneous when: $\Delta \subseteq \mathcal{L}(\tau_M)$ and if $\zeta < \kappa, \bar{a} \in {}^{\zeta}M, \bar{b} \in {}^{\zeta}M$ and $\text{tp}_\Delta(\bar{a}, \emptyset, M) = \text{tp}_\Delta(\bar{b}, \emptyset, M)$ then for every $c \in M$ for some $d \in M$ we have $\text{tp}_\Delta(\bar{b}\hat{\ }\langle d \rangle, \emptyset, M) = \text{tp}_\Delta(\bar{a}\hat{\ }\langle c \rangle, M)$.
3A) $\Sigma_1(\tau)$ is the set of formulas of the form $\varphi(\bar{x}) = (\exists \bar{y})\vartheta(x, \bar{y})$ where $\vartheta(\bar{x}, \bar{y})$ is quantifier free first order formula in the vocabulary τ.
4) We may omit "sequence".

Definition 0.5. 1) The logic $\mathcal{L}$ has "the strong κ-homogeneous existence property" when every theory $T \subseteq \mathcal{L}(\tau_1)$ has a strongly $(\mathcal{L}, \kappa)$-homogeneous model.
2) Similarly "the strong κ-saturated existence property", etc.

1. On strongly saturated models

We prove that any theory in $\mathbb{L}(Q^{\rm cf}_C)$ has strongly $(\mathbb{L}(Q^{\rm cf}_C),\theta)$-saturated model when $C\backslash\theta \notin \{\emptyset, {\rm Reg}\backslash\theta\}$ of course.

Definition 1.1. Let $\iota \in \{1,2\}$ and C be a class of regular cardinals such that $C \neq \emptyset$, Reg.
1) The quantifier $Q^{{\rm cf}(\iota)}_C$ is defined as follows:

<u>syntactically</u>: it bounds two variables, i.e. we can form $(Q^{{\rm cf}(\iota)}_C x,y)\varphi$, with its set of free variables being defined as ${\rm FVar}(\varphi)\backslash\{x,y\}$.

<u>semantically</u>: $M \models (Q^{{\rm cf}(\iota)}_C x,y)\varphi(x,y,\bar a)$ <u>iff</u> (a) + (b) holds where

- (a) relevancy demand:
  <u>the case $\iota = 1$</u>: the formula $\varphi(-,-;\bar a)^M$ define in M a linear order with no last element called $\leq^\varphi_{M,\bar a}$ on the non-empty set ${\rm Dom}(\leq^\varphi_{M,\bar a}) = \{b \in M : M \models (\exists y)(\varphi(b,y;\bar a)\}$
  <u>The case $\iota = 2$</u>: similarly but $\leq^\varphi_{M,\bar a}$ is a quasi linear order on its domain
- (b) the actual demand: $\leq^\varphi_{M,\bar a}$ has cofinality ${\rm cf}(\leq^\varphi_{M,\bar a})$, (necessarily an infinite regular cardinal) which belongs to C.

Convention 1.2. 1) Writing $Q^{\rm cf}_C$ we mean that this holds for $Q^{{\rm cf}(\iota)}_C$ for $\iota = 1$ and for $\iota = 2$.
2) Let ι-order mean order when $\iota = 1$ and quasi order when $\iota = 2$; but when we are using $Q^{{\rm cf}(\iota)}_C$ then order means ι-order.

Definition 1.3. 1) As $\{\psi \in \mathbb{L}(Q^{\rm cf}_C) : \psi$ has a model$\}$ does not depend on C (and is compact, see [She75a]) we may use the formal quantifier $Q_{\rm cf}$, so the syntex is determined but not the semantics, i.e. the satisfaction relation $\models$. We shall write $M \models_C \psi$ or $M \models_C T$ for the interpretation of $Q_{\rm cf}$ as $Q^{\rm cf}_C$, but also can say "$T \subseteq \mathbb{L}(Q_{\rm cf})(\tau)$ has model/is consistent".
2) If C is clear from the context, <u>then</u> $Q^{\rm cf}_\ell$ stands for $Q^{\rm cf}_C$ if $\ell = 1$ and $Q^{\rm cf}_{{\rm Reg}\backslash C}$ if $\ell = 0$.

Convention 1.4. 1) T^* is a complete (consistent $\equiv$ has models) theory in $\mathbb{L}(Q_{\rm cf})$ which is closed under definitions i.e. every formula $\varphi = \varphi(\bar x)$ is equivalent to a predicate $P_\varphi(\bar x)$ so $P_\varphi \in \tau(T^*)$, as in [**?**], i.e. $T^* \vdash (\forall \bar z)[\varphi(\bar z) \equiv P_\varphi(\bar z)]$.
2) Let $T = T^* \cap$ (first order logic), i.e. $T = T^* \cap \mathbb{L}(\tau_{T^*})$, it is a complete first order theory.
3) $C \subsetneq$ Reg, we let $C_1 = C$ and $C_0 =$ Reg$\backslash C$, both non-empty.

Theorem 1.5. *Assume* $\chi = {\rm cf}(\chi), \mu = \mu^{<\theta} \geq 2^{|T|} + \chi + \kappa, \theta = {\rm cf}(\theta) \leq \lambda, \theta \leq \min\{\chi,\kappa\}, \chi \neq \kappa = {\rm cf}(\kappa)$ *and*

$$\mu_\ell = \begin{cases} \chi & \ell = 0 \\ \kappa & \ell = 1 \end{cases}$$

<u>*Then*</u> *there is a* $\tau(T)$*-model* M *such that*

- (a) $M \models T, \|M\| = \mu$ *and* M *is* θ*-saturated*

(b) *if* $\varphi(\bar z) = (Q^{\rm cf}_\ell)\psi(x,y;\bar z)$ *then:* $M \models P_{\varphi(\bar x)}[\bar a]$ *iff* $\varphi(y,z;\bar a)$ *define in* M *a linear order with no last element and cofinality* μ_ℓ

(c) M *is strongly*[1] θ*-saturated model of* T^*.

Remark 1.6. 1) We can now change χ,κ,μ and $\|M\|$ by LST. Almost till the end instead $\mu \ge 2^{|T|}+\chi+\kappa$ just $\mu \ge |T|+\chi+\kappa$ suffice. The proof is broken to a series of definitions and claims. The "$\ge 2^{|T|}$" is necessary for $\aleph_0$-saturativity.
3) We can assume **V** satisfies GCH high enough and then use LST. So $\mu^+ = 2^\mu$ below is not a real burden.

Definition 1.7. 0) Mod_T is the class of models of T.
1)

(a) $K = \{(M,N) : M \prec N$ are from $\mathrm{Mod}_T\}$

(b) $K_\alpha = \{\bar M : \bar M = \langle M_i : i<\alpha\rangle$ satisfies $M_i \in \mathrm{Mod}_T$ and $i<j \Rightarrow M_i \prec M_j\}$ (so $K = K_2$)

(c) $K^\alpha_\mu = \{\bar M \in K_\alpha : \|M_i\| \le \mu$ for $i<\alpha\}$, but then we (naturally) assume $\alpha < \mu^+$

(d) let $\tau_\alpha = \tau_T \cup \{P_\beta : \beta<\alpha\} \cup \{R_{\varphi(x,y,\bar z),\beta} : \varphi(x,y,\bar z) \in \mathbb{L}(\tau_T), \beta<\alpha\}$, each P_β a unary predicate and each $R_{\varphi(x,y,\bar z),\beta}$ is an $(\ell g(\bar z)+1)$-place predicate and no incidental identification (so $P_\alpha \notin \tau$, etc.)

(e) for $\bar M \in K_\alpha$ let $\mathbf{m}(\bar M)$ be the τ_α-model M with
- universe $\cup\{M_\beta : \beta<\alpha\}$
- $M{\restriction}\tau_T = \cup\{M_\beta : \beta<\alpha\}$
- $P^M_\beta = M_\beta$
- $R^M_{\varphi(x,y,\bar z),\beta} = \{\langle c\rangle\hat{\ }\bar a : \varphi(x,y,\bar a)$ a linear order, $\bar a \in {}^{\ell g(\bar z)}(P^M_\beta)$ such that $M \models P_{(Q^{\rm cf}_0 x,y)\varphi(x,y,\bar z)}[\bar a]$ and $c \in \mathrm{Dom}(\le^\varphi_{M,\bar a})$ and $[b \in \mathrm{Dom}(\le^\varphi_{M,\bar a}) \cap P^M_\beta \Rightarrow b \le^\varphi_{M,\bar a} c]\}$

(f) let $\mathbf{m}_0(\bar M)$ be the τ-model $\cup\{M_\beta : \beta<\alpha\}$ so $\mathbf{m}_0(\bar M) = \mathbf{m}(\bar M){\restriction}\tau$.

2) Assume $(M^\ell,N^\ell) \in K$ for $\ell=1,2$ let $(M^1,N^1) \le (M^2,N^2)$ mean that clauses (a),(b),(c) below hold and let $(M^1,N^1) \le_K (M^2,N^2)$ mean that in addition clause (d) below holds, where:

(a) $M^1 \prec M^2$

(b) $M^2 \cap N^1 = M^1$

(c) $N^1 \prec N^2$

(d) if $M^1 \models P_{(Q^{\rm cf}_0 x,y)\varphi(x,y,\bar z)}[\bar a], c \in N^1, c \in \mathrm{Dom}(\le^\varphi_{N_1,\bar a})$ and in N^1 the element c is $\le^\varphi_{N^1,\bar a}$-above all $d \in \mathrm{Dom}(\le^\varphi_{M^1,\bar a})$, <u>then</u> in N^2 the element c is $\le^\varphi_{N^2,\bar a}$-above all $d \in \mathrm{Dom}(\le^\varphi_{M^2,\bar a})$.

3) For $\bar M^1, \bar M^2 \in K_\alpha$ let $\bar M^1 \le \bar M^2$ <u>means</u> $\gamma<\beta<\alpha \Rightarrow (M^1_\gamma, M^1_\beta) \le (M^2_\gamma, M^2_\beta)$; similarly $\bar M^1 \le_{K_\alpha} \bar M^2$ means $\bar M^1, \bar M^2 \in K_\alpha$ and $\gamma<\beta<\alpha \Rightarrow (M^1_\delta, M^1_\beta) \le_K (M^2_\gamma, M^2_\beta)$.
4) For $\bar M \in K_\alpha$, D an ultrafilter on λ we define $\bar N = \bar M^\lambda/D, \mathbf{j}_{M,D} = \mathbf{j}^\lambda_{\bar M,D}$ naturally: $N_\beta = M^\lambda_\beta/D$ for $\beta<\alpha$ and $\mathbf{j}_{\bar M,D} = \cup\{\mathbf{j}_{M_\beta,D} : \beta<\alpha\}$, recalling 0.3.

[1]as T^* has elimination of quantifiers, doing it for $\mathbb{L}(Q^{\rm cf}_C)$ or for $\mathbb{L}$ is the same

Fact 1.8. 0) For $\bar M^1, \bar M^2 \in K_\alpha$ we have

(a) $\bar M^1 \le_{K_\alpha} \bar M^2$ iff $\mathbf{m}(M^1) \subseteq \mathbf{m}(\bar M^2)$
(b) $(\mathbf{m}(\bar M^\ell) \restriction P^{M_\beta}) \restriction \tau_T = M^\ell_\beta$
(c) $\bar M^1 \le_{K_\alpha} \bar M^1$ implies $\bar M^1 \le \bar M^2$.

1) $(K_\alpha, \le)$ and $(K_\alpha, \le_{K_\alpha})$ are partial orders.
2a) If $\bar M^1 \le_{K_\alpha} \bar M^2$ in K_α and $0 < \gamma < \beta \le \alpha$ <u>then</u> $(\bigcup_{\varepsilon<\gamma} M^1_\varepsilon, \bigcup_{\varepsilon<\beta} M^1_\varepsilon) \le (\bigcup_{\varepsilon<\gamma} M^2_\varepsilon, \bigcup_{\varepsilon<\beta} M^2_\varepsilon)$ moreover $\langle \bigcup_{i<1+\varepsilon} M^1_i : 1+\varepsilon \le \alpha\rangle \le_{K_\xi} \langle \bigcup_{i<1+\varepsilon} M^2_i : 1+\varepsilon \le \alpha\rangle$ where ξ is α if $\alpha < \omega$ and is $\alpha+1$ if $\alpha \ge \omega$.
2b) If $\langle \bar M^i : i < \delta\rangle$ is a $\le_{K_\alpha}$-increasing sequence (of members of K_α) and we define $\bar M^\delta = \langle M^\delta_\varepsilon : \varepsilon < \alpha\rangle$ by $M^\delta_\varepsilon = \cup\{M^i_\varepsilon : i < \delta\}$ then $i < \delta \Rightarrow \bar M^i \le_{K_\alpha} \bar M^\delta$ and the sequence $\langle \bar M^i : i \le \delta\rangle$ is continuous in δ.
3) In part (2b), if in addition $i < \delta \Rightarrow \bar M^i \le_{K_\alpha} \bar N$ so $\bar N \in K_\alpha$ <u>then</u> $\bar M^\delta \le_{K_\alpha} \bar N$.
4) In part (2b), if $\delta < \mu^+$ and $i < \delta \Rightarrow \bar M^i \in K^\alpha_\mu$ then $\bar M^\delta \in K^\alpha_\mu$.
5) If $\bar M \le_{K_\alpha} \bar N$ and $Y_\varepsilon \subseteq N_\varepsilon$ for $\varepsilon < \alpha$ and $\Sigma\{\|M_\varepsilon\| + |Y_\varepsilon| : \varepsilon < \alpha\} + |\tau| + |\alpha| \le \lambda$ <u>then</u> there is $\bar N' \in K^\alpha_\lambda$ such that $\bar M \le_{K_\alpha} \bar N' \le_{K_\alpha} \bar N$ and $\varepsilon < \alpha \Rightarrow Y_\varepsilon \subseteq N'_\varepsilon$.
6) Assume $\bar M^i \in K^{\alpha(i)}_\mu$ for $i < \delta < \mu^+, \langle \alpha(i) : i < \delta\rangle$ is a non-decreasing sequence of ordinals and $i < j < \delta \Rightarrow \bar M^i \le_{K_{\alpha(i)}}, \bar M^j \restriction \alpha(i)$ and we define $\alpha(\delta) = \cup\{\alpha(i) : i < \delta\}, \bar M^\delta = \langle M^\delta_\beta : \beta < \alpha(\delta)\rangle$ where $M^\delta_\beta = \cup\{M^i_\beta : \beta < \delta$ satisfies $\beta < \alpha(i)\}$ <u>then</u> $\bar M^\delta \in K^{\alpha(\delta)}_\mu$ and $i < \delta \Rightarrow \bar M^i \le_{K_{\alpha(i)}} \bar M^\delta \restriction \alpha(i)$.
7) If $\bar M^\ell \le_{K_\alpha} \bar N$ for $\ell = 1,2$ and $[a \in \mathbf{m}(\bar M^1) \Rightarrow a \in \mathbf{m}(\bar M^2)]$ then $\bar M^1 \le_{K_\alpha} \bar M^2$.
8) Parts (2)-(7) holds also when we replace $\le_{K_\alpha}$ by $\le$.

Proof. Check. $\square_{1.8}$

Fact 1.9. 1) If $(M_0, M_1) \in K^2_\mu$ and $(M_0, M'_1) \in K^2_\mu$ <u>then</u> there are M_2, f such that

(a) $M'_1 \prec M_2 \in K_\mu$
(b) f is an elementary embedding of M_1 into M_2
(c) $f \restriction M_0 = \mathrm{id}_{M_0}$
(d) $(M_0, M'_1) \le_{K_2} (f(M_1), M_2)$.

2) If $\bar M \in K_\alpha, \bar x = \langle x_\varepsilon : \varepsilon < \zeta\rangle$ and Γ is a set of first order formulas from $\mathbb{L}(\tau^+_\alpha)$ in the variables $\bar x$ with parameters from the model $\mathbf{m}(\bar M)$ finitely satisfiable in $\mathbf{m}(M)$ such that $\varepsilon < \zeta \Rightarrow \bigvee_{\beta<\alpha} P_\beta(x_\varepsilon) \in \Gamma$, <u>then</u> there is $\bar N \in K_\alpha$ such that $\bar M \le_{K_\alpha} \bar N$ and Γ is realized in $\mathbf{m}(\bar N)$.
3) If Γ is a type over $\mathbf{m}_0(\bar M)$ of cardinality[2] $<$ cf(α) <u>then</u> it is included in some Γ' as in part (2).
4) If $\bar M \in K^\alpha_\mu, D$ an ultrafilter on θ and $M'_\beta = (M_\beta)^\theta/D$ for $\beta < \alpha$ then

(a) $\bar M' = \langle M'_\beta : \beta < \alpha\rangle \in K_\alpha$
(b) $\mathbf{j}^\theta_{\bar M,D} := \cup\{\mathbf{j}^\theta_{M_\beta,D} : \beta < \alpha\}$ is a $\le_{K_\alpha}$-embedding of $\bar M$ into $\bar M'$, i.e.
(b)′ $\langle \mathbf{j}^\theta_{M_\beta,D}(M_\beta) : \beta < \alpha\rangle := \bar M' \le_{K_\alpha} \langle M'_\beta : \beta < \alpha\rangle$, so

[2] also if cf$(\alpha) = 1$, i.e. α is a successor ordinal

(c) for many $Y \in [\cup\{M'_\beta : \beta < \alpha\}]^\mu$ we have $\mathbf{j}^\theta_{\bar M,D}(\bar M) \le_{K_\alpha} \langle M'_\beta \restriction Y : \beta < \alpha\rangle \in K^\alpha_\mu$; see 1.8(5), 1.12(3).

Proof. 1) See [She85, §4]; just let D be a regular ultrafilter on $\lambda \ge \|M_1\| + |\tau|$, let g an elementary embedding of M_1 into $(M_0)^\lambda/D$ extending $\mathbf{j} = \mathbf{j}^\lambda_{M_0,D}$, necessarily exists.

Lastly, let $M_2 \prec (M'_1)^\lambda/D$ include $\mathbf{j}^\lambda_{M_1,D}(M'_1) \cup g(M_1)$ be of cardinality μ. Identifying M'_1 with $\mathbf{j}^\lambda_{M'_1,D}(M'_1) \prec (M'_1)^\lambda/D$ we are done.
2) Similarly.
3) Trivial.
4) Should be clear. $\square_{1.9}$

Definition 1.10. $K^{\rm ec}_\alpha$ is the class of $\bar M \in K_\alpha$ such that: if $\bar M \le_{K_\alpha} \bar N \in K_\alpha$, <u>then</u> $\mathbf{m}(\bar M) \le_{\Sigma_1} \mathbf{m}(\bar N)$, i.e. $(*)$ below and $K^{{\rm ec},\alpha}_\lambda = K^{\rm ec}_\alpha \cap K_\lambda$ where

$(*)$ if $a_1, \dots, a_n \in \mathbf{m}(\bar M), b_1, \dots, b_k \in \mathbf{m}(\bar N), \varphi \in \mathbb{L}(\tau^+_\alpha)$ is quantifier free and $\mathbf{m}(\bar N) \models \varphi[a_1, \dots, a_n, b_1, \dots, b_k]$ <u>then</u> for some $b'_1, \dots, b'_k \in \bigcup\limits_{\beta<\alpha} M_\beta$ we have $\mathbf{m}(\bar M) \models \varphi[a_1, \dots, a_n, b'_1, \dots, b'_k]$.

Claim 1.11. *1) $K^{{\rm ec},\alpha}_\mu$ is dense in K^α_μ when $\mu \ge |\tau_T| + |\alpha|$ of course.*
2) $K^{{\rm ec},\alpha}_\mu$ is closed under union of increasing chains of length $< \mu^+$.
3) In Definition 1.10, if $|\alpha| + |\tau_T| \le \mu$ and $\bar M \in K^\alpha_\mu$ <u>then</u> without loss of generality $\bar N \in K^\alpha_\mu$.

Proof. 1) Given $\bar M_0 \in K^\alpha_\mu$ we try to choose $\bar M_\varepsilon \in K^\alpha_\mu$ by induction on $\varepsilon < \mu^+$ such that $\langle \bar M_\zeta : \zeta \le \varepsilon\rangle$ is $\le_{K_\alpha}$-increasing continuous and $\varepsilon = \zeta + 1 \Rightarrow \mathbf{m}(\bar M_\zeta) \nleq_{\Sigma_1} \mathbf{m}(M_\varepsilon)$. For $\varepsilon = 0$ the sequence is given, for ε limit use 1.8(2), for $\varepsilon = \zeta + 1$ if we cannot choose then by 1.8(5) we get $\bar M_\zeta \in K^{{\rm ec},\alpha}_\mu$ is as required. But if we succeed to choose $\langle \bar M_\varepsilon : \varepsilon < \mu^+\rangle$ we get contradiction by Fodor lemma.
2) Think on the definitions.
3) By LST. $\square_{1.11}$

Claim 1.12. *1) If $\bar M, \bar N \in K^\alpha_\mu$ and $\bar M \le_{\Sigma_1} \bar N$ and $\bar N \in K^{\rm ec}_\alpha$ <u>then</u> $\bar M \in K^{\rm ec}_\alpha$.*
2) If $\bar N \in K^{{\rm ec},\alpha}_\mu, Y \subseteq \mathbf{m}_0(\bar N)$ and $\lambda = |\tau_T| + |\alpha| + |Y|$ <u>then</u> there is $\bar M \in K^{{\rm ec},\alpha}_\lambda$ such that $\bar M \le_{K_\alpha} \bar N$ and $Y \subseteq \mathbf{m}_0(\bar M)$.
3) Assume $\bar M^\ell \in K^\alpha_\mu$ and $\bar M^0 \le_{K_\alpha} \bar M^1$ and $\bar M^0 \le \bar M^2$. If $\bar M^0 \in K^{{\rm ec},\alpha}_\mu, \bar M^0 \le_{K_\alpha} \bar M^2$ <u>or</u> $\mathbf{m}(\bar M^0) \le_{\Sigma_1} \mathbf{m}(\bar M^2)$, <u>then</u> we can find $(\bar N, f_2)$ such that:
$\bar M^1 \le_{K_\alpha} \bar N \in K^\alpha_\mu$, moreover $\bar N \in K^{{\rm ec},\alpha}_\mu$ and f_2 is a $\le_{K_\alpha}$-embedding of $\bar M^2$ into $\bar N$ over $\bar M^0$.

Proof. 1) By part (3).
2) By part (1) and the LST argument.
3) By the definition of $\bar M^0 \in K^{{\rm ec},\alpha}_\mu$ in both cases we can assume $\bar M^0 \le_{\Sigma_1} \bar M^2$. Let $\bar{\mathbf{a}} = \langle a_\varepsilon : \varepsilon < \zeta\rangle$ list the elements of $\mathbf{m}(\bar M^2)$ and let $\Gamma = \ {\rm tp}_{\rm qf}(\bar{\mathbf{a}}, \emptyset, \mathbf{m}(\bar M^2)) = \{\varphi(x_{\varepsilon_0}, \dots, x_{\varepsilon_{n-1}}, \bar b) : \varphi \in \mathbb{L}(\tau^+_\alpha)$ is quantifier free, $\bar b \subseteq \mathbf{m}(\bar M^0)$ and $\mathbf{m}(\bar M^2) \models \varphi[a_{\varepsilon_0}, \dots, a_{\varepsilon_{n-1}}, \bar b]\}$; note that $P_\beta(x_\varepsilon))^{\mathbf{t}(\varepsilon,\beta)} \in \Gamma$ when $\beta < \alpha, \varepsilon < \zeta$ and $\mathbf{t}(\varepsilon, \beta)$ is the truth value of $a_\varepsilon \in M^2_\beta$.

Now let D be a regular ultrafilter on $\lambda = \|\mathbf{m}(\bar M^2)\|$ and use 1.9(2),(3). This is fine to get $(f_2, \bar N)$ with $\bar N \in K_\alpha$ and by 1.8(5) without loss of generality $\bar N \in K^\alpha_\mu$ and by 1.11(1) without loss of generality $\bar N \in K^{{\rm ec},\alpha}_\mu$. $\square_{1.12}$

Claim 1.13. *1)* $(K^{\text{ec},\alpha}_\mu, \leq_{K^\alpha_\mu})$ *has the JEP.*
2) Suppose $\bar M^1, \bar M^2 \in K^\alpha_\mu, \beta \leq \alpha, f$ *is an elementary embedding of* $\bigcup\limits_{\gamma<\beta} M^1_\gamma$ *into* $\bigcup\limits_{\gamma<\beta} M^2_\gamma$ *such that* $\langle f(M_\gamma) : \gamma < \beta\rangle \leq_{K_\mu} \langle M^2_\gamma : \gamma < \beta\rangle$*, equivalently* f *is an embedding of* $\mathbf{m}(\bar M^\beta)$ *into* $\mathbf{m}(\bar M^2 \restriction \beta)$ *(so if* $\beta = 0$ *then* $f = \emptyset$ *and there is no demand).*

<u>*Then*</u> *we can find* $\bar M^3, f^+$ *such that:*

- (a) $\bar M^2 \leq_{K_\mu} \bar M^3 \in K^\alpha_\mu$
- (b) $f \subseteq f^+$
- (c) f^+ *is an elementary embedding of* $\bigcup\limits_{\gamma<\alpha} M^1_\gamma$ *into* $\bigcup\limits_{\gamma<\alpha} M^3_\gamma$
- (d) $\langle f^+(M^1_\gamma)) : \gamma < \alpha\rangle \leq_{K_\alpha} \langle M^3_\gamma : \gamma < \alpha\rangle$.

Proof. 1) A special case of part (2) recalling 1.11(1).
2) By induction on α.

<u>$\alpha = 0$</u>: nothing to do.

<u>$\beta = \alpha$</u>: nothing to do.

<u>$\alpha = 1$</u>: so $\beta = 0$ which is trivial or $\beta = \alpha$, a case done above.

<u>α successor</u>: by the induction hypothesis and transitive nature of conclusion replacing $\bar M^2$ without loss of generality $\beta = \alpha - 1$, then use 1.9(1).

<u>α limit</u>: By $\alpha - \beta$ successive uses of induction hypothesis using 1.8(2b). $\square_{1.13}$

Conclusion 1.14. $(K^{\text{ec}}_\alpha, \leq_{K_\alpha})$, or formally $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ defined by $K_{\mathfrak{k}} := \{\mathbf{m}(\bar M) : \bar M \in K^{\text{ec}}_\alpha\}, \mathbf{m}(\bar M^1) \leq_{\mathfrak{k}} \mathbf{m}(\bar M^2) \Leftrightarrow \mathbf{m}(M^1) \subseteq \mathbf{m}(\bar M^2)$, is an a.e.c. with amalgamation, the JEP and $\text{LST}(\mathfrak{k}) \leq |\tau_T| + |\alpha| + \aleph_0$.

Proof. By the above, on a.e.c. see [She09, Ch.I], i.e. [She] and history there. $\square_{1.14}$

Fact 1.15. Assume $\lambda = \lambda^{<\lambda} > |\tau_T| + \aleph_0 + |\alpha|$. <u>Then</u> there is $\bar M$ such that

- (a) $\bar M \in K^{\text{ec}}_\alpha$ is universal for $(K^{\text{ec}}_\alpha, \leq_{K_\alpha})$ in cardinality λ
- (b) $\mathbf{m}(\bar M)$ is model homogeneous for $(K^{\text{ec}}_\alpha, \leq_{K_\alpha})$ of cardinality λ
- (c) $\mathbf{m}(\bar M)$ is sequence $(\Sigma_1(\tau^+_\alpha), \lambda)$-homogeneous, see 0.4(3).

Proof. Clause (a) + (b) are straight by 1.12 + 1.13(1), or use 1.14 and see [She09, Ch.I,§2] = [She, §2]. Now clause (c) follows: just think. $\square_{1.15}$

Fact 1.16. Assume $\bar M \in K^\alpha_\mu, \beta + 1 < \alpha, \ell \in \{0, 1\}$ and $M_\beta \models P_{(Q^{\text{cf}}_\ell x,y)\varphi(x,y,\bar z)}[\bar a]$ <u>then</u> there are $\bar N, c$ such that $\bar M \leq_{K_\alpha} \bar N \in K^\alpha_\mu$ and:

- $(*)_1$ if $\ell = 1$ then $c \in \text{Dom}(\leq^\varphi_{N_\beta,\bar a})$ and c is $\leq^\varphi_{N_\gamma,\bar a}$-above $d \in \text{Dom}(\leq^\varphi_{M_\gamma,\bar a})$ for any $\gamma \in [\beta, \alpha)$
- $(*)_2$ if $\ell = 0$ then $c \in \text{Dom}(\leq^\varphi_{N_{\beta+1},\bar a})$ and is $\leq^\varphi_{N_{\beta+1},\bar a}$-above any $d \in \text{Dom}(\leq^\varphi_{N_\beta,\bar a})$.

Proof. First assume $\ell = 1$, without loss of generality $\beta = 0$ as we can let $\bar N{\restriction}\beta = \bar M{\restriction}\beta$.

By 1.8(2a) without loss of generality $\bar M$ is increasing continuous; we prove by induction on α that without loss of generality $\alpha = 2$. Now this is obvious by [She75a], [She85]; in details by [She75a] there is a μ^+-saturated model M_* of T such that $M_1 \prec M_*$ and $M_* \models_{C_*} T^*$ whenever, e.g. $\mu^{++} \in C_* \wedge \mu^+ \notin C_*$. Let $\{\varphi_i(x,y,\bar a^*_i) : i < \mu\}$ list $\{\varphi(x,y,\bar a') : \varphi \in \mathbb{L}(\tau_T), M_0 \models P_{(Q^{\rm cf}_0 x,y)\varphi(x,y;\bar z)}[\bar a']\}$, and for each $i < \mu$ let $\langle c_{i,\varepsilon} : \varepsilon < \mu^+\rangle$ be $\le^{\varphi_i}_{M_*,\bar a^*_i}$-increasing and cofinal. For $\varepsilon < \mu^+$ let f_ε be an elementary embedding of M_1 into M_* over M_0 such that:

- $(*)$ if $c \in \mathrm{Dom}(\le^{\varphi_i}_{M_*,\bar a^*_i})$ is a $\le^{\varphi_i}_{M_*,\bar a^*_i}$-upper bound of $\mathrm{Dom}(\le^{\varphi_i}_{M_0,\bar a^*_i})$ then $c_{i,\varepsilon} \le^{\varphi_i}_{M_*,\bar a^*_i} c$.

Let $c_* \in M_*$ be a $\le^{\varphi}_{M_*,\bar a}$-upper bound of $\mathrm{Dom}(\le^{\varphi}_{M_0,\bar a})$. Choose $N_0 \prec M_*$ of cardinality μ be such that $M_0 \cup \{c_*\} \subseteq N_0$ and choose $\varepsilon < \mu^+$ large enough such that:

- $(*)$ if $i < \mu$ and $d \in N_0$ is a $\le^{\varphi_i}_{M_*,\bar a_i}$-upper bound of $\mathrm{Dom}(\le^{\varphi_i}_{M_i,\bar a_i})$ then $d \le^{\varphi_i}_{M_*,\bar a_i} c_{i,\varepsilon}$.

Let $N_1 \prec M_*$ be of cardinality μ be such that $N_0 \cup f_\varepsilon(M_1) \subseteq N_1$. Renaming, f_ε is the identity and (N_0, N_1) is as required.

Second, assume $\ell = 0$ is even easier (again without loss of generality first, $\alpha = \beta + 2$ and second $\beta = 0, \alpha = 2$ and use $N_0 = M_0, N_1$ satisfies $M_1 \prec N_1$ and $\|N_1\| = \mu$ and N_1 realizes the relevant upper). $\square_{1.16}$

Conclusion 1.17. In 1.15 the model $M^* = \mathbf{m}(\bar M^*) = \bigcup\limits_{\beta<\alpha} M^*_\beta$ satisfies

- (a) if $M^* \models P_{(Q^{\rm cf}_1 x,y)\varphi}[\bar a]$ then the order $\le^{\varphi}_{M^*,\bar a}$ has cofinality λ
- (b) if α is a limit ordinal and $M^* \models P_{(Q^{\rm cf}_0 x,y)\varphi}[\bar a]$ <u>then</u> the linear order $\le^{\varphi}_{M^*,\bar a}$ has cofinality $\mathrm{cf}(\alpha)$
- (c) M^* is $\mathrm{cf}(\alpha)$-saturated
- (d) if $\lambda \in C$ and $\mathrm{cf}(\alpha) \in \mathrm{Reg}\backslash C$ then M^* is a model of T^*.

Claim 1.18. *Assume $\bar M \in K^{\rm ec}_\alpha$. If $\zeta \le \mu$ and $\bar a, \bar b \in {}^\zeta(M^*_0)$ realize the same type (equivalently q.f. type) in M_0 <u>then</u> they realize the same Σ_1-type in $\mathbf{m}(\bar M)$.*

Proof. We choose $(N_\beta, f_\beta, g_\beta, h_\beta)$ by induction on $\beta < \alpha$ such that:

- (a) N_β is a model of T
- (b) N_β is $\prec$-increasing continuous with β
- (c) f_β, g_β are $\le_{K_{1+\beta}}$-embedding of $\bar M{\restriction}(1+\beta)$ into $\langle N_\gamma : \gamma < 1+\beta\rangle \in K_{1+\beta}$
- (d) $f_0(\bar a) = g_0(\bar b)$
- (e) if $\gamma < \beta$ then $f_\gamma \subseteq f_\beta, g_\gamma \subseteq g_\beta$.

For $\beta = 0$ this speaks just on Mod_T.

For β successor use 1.9.

For β limit as in the successor case, recalling we translated it to the successor case (by 1.8(2a)).

Having carried the induction $f = \cup\{f_\beta : \beta < \alpha\}$ and $g = \cup\{g_\beta : \beta < \alpha\}$ are $\le_{K_\alpha}$-embedding of $\bar M$ into $\bar N = \langle N_\beta : \beta < \alpha\rangle$. By 1.11(1) there is $\bar N' \in K^{\rm ec}_\alpha$ which is $\le_{K_\alpha}$-above $\bar N$. Now as $\bar M \in K^{\rm ec}_\alpha$, the Σ_1-type of $\bar a$ in $\mathbf{m}(\bar M)$ is equal to the Σ_1-type of $f(\bar a)$ in $\mathbf{m}(\bar N')$, and the Σ_1-type of $\bar b$ in $\mathbf{m}(\bar M)$ is equal to the Σ_1-type of $f(\bar a)$ in $\mathbf{m}(\bar N')$. But $f(\bar a) = f_0(\bar a) = g_0(\bar b) = g(\bar b)$, so we have gotten the promised equality of Σ_1-types. $\square_{1.18}$

Observation 1.19. *1) If $\bar M \in K^{\rm ec}_\alpha$ and $\beta < \alpha$ then $\bar M' := \bar M \restriction [\beta,\alpha) = \langle M_{\beta+\gamma} : \gamma < \alpha - \beta\rangle$ belongs to $K^{\rm ec}_{\alpha-\beta}$.*
2) If $\bar M \in K_\alpha, \beta < \alpha$ and $\bar M \restriction [\beta,\alpha) \le_{K_{\alpha,\beta}} \bar N'$ then for some $\bar N \in K_\alpha$ we have $\bar M \le_{K_\alpha} \bar N$ and $\bar N \restriction [\beta,\alpha) = \bar N'$.

Proof. 1) If not, then there is $\bar N' \in K_{\alpha-\beta}$ such that $\bar M' \le_{K_{\alpha-\beta}} \bar N'$ but $\mathbf{m}(\bar M') \not\le_{\Sigma_1} \mathbf{m}(\bar N')$. Define $\bar N = \langle N_\gamma : \gamma < \alpha\rangle$ by: N_γ is M_γ if $\gamma < \beta$ and is $N'_{\gamma-\beta}$ if $\gamma \in [\beta,\alpha)$. Easily $\bar M \le_{K_\alpha} \bar N \in K_\alpha$ but $\mathbf{m}(\bar M) \not\le_{\Sigma_1} \mathbf{m}(\bar N)$, contradiction to the assumption $\bar M \in K^{\rm ec}_\alpha$.
2) The proof is included in the proof of part (1). $\square_{1.19}$

Claim 1.20. *In 1.15 for each $\beta < \alpha$ we have*

- *(a) $\langle M^*_{\beta+\gamma} : \gamma < \alpha - \beta\rangle$ is homogeneous universal for $K^{\alpha-\beta}_\mu$*
- *(b) if $\alpha = \alpha - \beta$, i.e. $\beta + \alpha = \alpha$ then there is an isomorphism from $\bar M^*$ onto $\langle M^*_{\beta+\gamma} : \gamma < \alpha - \beta\rangle$, in fact, we can determine $f(\bar a) = \bar b$ if $\bar a \in {}^\zeta(M^*_0), \bar b \in {}^\zeta(M^*_\beta)$ and $\mathrm{tp}(\bar a, \emptyset, M^*_\beta) = \mathrm{tp}(\bar b, \emptyset, M^*_\beta)$.*

Proof. Chase arrows as usual recalling 1.19. $\square_{1.20}$

Proof. Proof of Theorem 1.5:
Without loss of generality there is $\sigma = \sigma^{<\theta} \ge \mu$ such that $2^\sigma = \sigma^+$ (why? let $\sigma > \mu$ be regular, work in $\mathbf{V}^{\mathrm{Levy}(\sigma^+,2^\sigma)}$ and use absoluteness argument, or choose set A of ordinals such that $\mathcal{P}(\mu) \in \mathbf{L}[A]$ hence $T, T^* \in \mathbf{L}[A]$ and regular σ large enough such that $\mathbf{L}[A] \models$ "$2^\sigma = \sigma^+$", work in $\mathbf{L}[A]$ a little more; and for the desired conclusion (there is a model of cardinality μ such that ...) it makes no difference). Let $\alpha = \kappa$ and let $\bar M^* \in K^{\mathrm{ec},\alpha}_\lambda$ be as in 1.15 for $\lambda := \sigma^+$ and let $M_* = \cup\{M^*_\beta : \beta < \alpha\}$.
Now

$(*)_1$ M_* is a model of T^* by the $\{\mu^+\}$-interpretation.

[Why? By 1.17.]

$(*)_2$ M_* is θ-saturated.

[Why? Clearly M^*_β is θ-saturated for each $\beta < \theta$. As κ is regular and $\langle M^*_\beta : \beta < \theta\rangle$ is increasing with union M_*, also M_* is θ-saturated.]

$(*)_3$ M_* is strongly $\aleph_0$-saturated and even strongly θ-saturated, see Definition 0.4(1).

[Why? Let $\zeta < \theta$ and $\bar a, \bar b \in {}^\zeta(M_*)$ realize the same q.f.-type (equivalently the first order type) in M_*. As $\zeta < \theta$ for some $\beta < \theta$ we have $\bar a, \bar b \in {}^\zeta(M_\beta)$. Now by 1.20 we know that $\langle M^*_{\beta+\gamma} : \gamma < \theta\rangle \cong \langle M^*_\gamma : \gamma < \theta\rangle$, and by 1.18 the sequences $\bar a, \bar b$ realize the same Σ_1-type in $\mathbf{m}(\langle M^*_{\beta+\gamma} : \gamma < \theta\rangle)$ hence by clause (c) of 1.15 there is an automorphism π of it mapping $\bar a$ to $\bar b$. So π is also an automorphism of M_* mapping $\bar a$ to $\bar b$ as required.]

Lastly, we have to go back to models of cardinality $\mu = \mu^{<\theta} \geq \lambda + \kappa + 2^{|T|}$, this is done by the LST argument recalling 1.17.

More fully, first let $\langle \bar{M}^\varepsilon : \varepsilon < \lambda \rangle$ be $\leq_{K^\alpha_\sigma}$-increasing continuous sequence with union $\bar{M}^*$. For $\zeta < \theta$ and $\bar{a}, \bar{b} \in {}^\zeta(M_*)$ let $f_{\bar{a},\bar{b}}$ be an automorphism of M_* mapping $\bar{a}$ to $\bar{b}$. Now the set of $\delta < \lambda$ satisfying $\circledast_\delta$ below is a club of λ hence if $\mathrm{cf}(\delta) = \chi$ then $M = \cup\{M^\varepsilon_\beta : \beta < \lambda\}$ is as required except of being of cardinality μ, where

$\circledast_\delta$ (a) if $\varepsilon < \delta, \zeta < \theta$ and $\bar{a}, \bar{b} \in {}^\zeta(\cup\{M^\varepsilon_\beta : \beta < \alpha\})$ realize the same Σ_1-type in $\bar{M}^\zeta$ then $\cup\{M^\delta_\beta : \beta < \alpha\}$ is closed under $f_{\bar{a},\bar{b}}$ and under $f^{-1}_{\bar{a},\bar{b}}$

(b) the witnesses for the cofinality work, i.e.

$\bullet_1$ if $\beta < \alpha, \bar{a} \in {}^{\omega>}(M^\delta_\beta), M^\delta_\beta \models P_{(Q^{\mathrm{cf}}_0 y,z)\varphi(y,z,\bar{x})}[\bar{a}]$ then for some $\varepsilon < \delta$ we have $\bar{a} \subseteq M^\varepsilon_\beta$ and for every $\gamma \in (\beta, \alpha)$ there is $c = c_{\varphi,\bar{a},\gamma} \in M^\varepsilon_{\gamma+1}$ which is a $\leq^\varphi_{M^\varepsilon_{\gamma+1},\bar{a}}$-upper bound of $\mathrm{Dom}(\leq^\varphi_{M^\varepsilon_\gamma,\bar{a}})$, hence this holds for any $\varepsilon' \in [\varepsilon, \lambda)$

$\bullet_2$ if $\beta < \alpha, \bar{a} \in {}^{\omega>}(M^\gamma_\beta)$ and $M^\delta_\beta \models P_{(Q^{\mathrm{cf}}_1 y,z)\varphi(y,z,\bar{x})}[\bar{a}]$ <u>then</u> for arbitrarily large $\varepsilon < \delta$ we have $\bar{a} \subseteq M^\varepsilon_\beta$ and there is $c = c_{\varphi,\bar{a}} \in M^{\varepsilon+1}_\beta$ which is a $(\leq^\varphi_{M^{\varepsilon+1}_\gamma,\bar{a}})$-upper bound of $\mathrm{Dom}(\leq^\varphi_{M^\varepsilon_\gamma,\bar{a}})$ for every $\gamma \in [\beta, \alpha)$.

By a similar use of the LST argument we get a model of T^* of cardinal μ. $\square_{1.5}$

Remark 1.21. If you do not like the use of (set theoretic absoluteness) you may do the following. Use 1.22 below, which is legitimate as

(a) the class $(K^{\mathrm{ec}}_\alpha, \leq_{K_\alpha})$ is an a.e.c. with LST number $\leq |T| + \aleph_0$ and amalgamation, so 1.22(1) apply

(b) using Σ_1-types, it falls under [She71] more exactly [She75b], so 1.22(3) apply

(c) we can define $K^{\mathrm{ec}(\varepsilon)}_\alpha$ by induction on $\varepsilon \leq \omega$
<u>$\varepsilon = 0$</u>: K_α
<u>$\varepsilon = 1$</u>: K^{ec}_α
<u>$\varepsilon = n + 1$</u>: $K^{\mathrm{ec}(n+1)}_\alpha = \{\bar{M} \in K^{\mathrm{ec}(n)}_\alpha$: if $\bar{M} \subseteq N \in K^{\mathrm{ec}(n)}_\alpha$ then $\mathbf{m}(M) \leq_{\Sigma_{n+1}} \mathbf{m}(\bar{N})\}$
<u>$\varepsilon = \omega$</u>: $K^{\mathrm{ec}(\varepsilon)}_\alpha = \cap\{K^{\mathrm{ec}(n)}_\alpha : n < \omega\}$.

On $K^{\mathrm{ec}(\omega)}_\alpha$ apply 1.22(2).

Remark 1.22. 1) Assume $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$ is a a.e.c. satisfying amalgamation and the JEP with $\lambda > \mathrm{LST}(\mathfrak{k})$ and $\mu = \mu^{<\lambda}$. For any $M \in K_\mu$ there is a strongly model λ-homogeneous $N \in K_\mu$ which $\leq_\mathfrak{k}$-extend M, which means: if $M \in K_\mathfrak{k}$ has cardinality $< \lambda$ and f_1, f_2 are $\leq_\mathfrak{k}$-embedding of M into N then for some automorphism g of N we have $f_2 = g \circ f_1$.
2) Let D be a good finite diagram as in [She71] and let K_D be as below in part (3) for $\Delta = \mathbb{L}(\tau)$. If $\lambda = \lambda^{<\theta} \geq |D|$ and $M \in K_D$ has cardinality λ then there is $N \in K_D$ of cardinality λ which $\prec$-extend M and is strongly (D, θ)-homogenous, i.e.

(a) if $\zeta < \theta, \bar{a}, \bar{b} \in {}^\zeta N$ realizes the same type <u>then</u> some automorphism f of N maps $\bar{a}$ to $\bar{b}$

(b) $D = \{\mathrm{tp}(\bar{a}, \emptyset, N) : \bar{a} \in {}^{\omega>}N\}$.

3) Assume $\Delta \subseteq \mathbb{L}(\tau)$, not necessarily closed under negation, D is a set of Δ-types, K_D is the class of τ-models such that $\bar{a} \in {}^{\omega >}M \Rightarrow \mathrm{tp}_\Delta(\bar{a}, \emptyset, M) \in D$ and $M \le_D N$ iff $M \subseteq N$ are from K_D and $\bar{a} \in {}^{\omega >}M \Rightarrow \mathrm{tp}_\Delta(\bar{a}, \emptyset, M) = \mathrm{tp}_\Delta(\bar{a}, \emptyset, N)$. Assume further D is good, i.e. for every $M \in K_D$ and λ there is a sequence (D, λ)-homogeneous model $N \in K_D$ which $\le_D$-extends M. Then for every $\lambda = \lambda^{<\theta} > |T| + \aleph_0$ and $M \in K_D$ of cardinality λ there is a strongly sequence (Δ, λ)-homogeneous.

Conclusion 1.23. 1) The logic $\mathbb{L}(Q^{\mathrm{cf}}_C)$ has the strong $\aleph_0$-saturated model existence property (hence the strong $\aleph_0$-homogeneous model existence property).
2) If $\kappa = \mathrm{cf}(\kappa) \le \mathrm{Min}(C)$ and $\kappa \le \mathrm{Min}(\mathrm{Reg} \setminus C)$ then in part (1) we can replace $\aleph_0$ by κ.

Proof. Choose $\chi \in C, \kappa \in \mathrm{Reg} \setminus C$ and apply 1.5. $\square_{1.23}$

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