WHITEHEAD MODULES OVER LARGE PRINCIPAL IDEAL DOMAINS

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Abstract. We consider the Whitehead problem for principal ideal domains of large size. It is proved, in ZFC, that some p.i.d.'s of size $\geq \aleph_2$ have non-free Whitehead modules even though they are not complete discrete valuation rings.

A module M is a *Whitehead module* if $\mathrm{Ext}^1_R(M, R) = 0$. The second author proved that the problem of whether every Whitehead $\mathbb{Z}$-module is free is independent of ZFC + GCH (cf. [5], [6], [7]). This was extended in [1] to modules over principal ideal domains of cardinality at most $\aleph_1$. Here we consider the Whitehead problem for modules over principal ideal domains (p.i.d.'s) of cardinality $> \aleph_1$.

If R is any p.i.d. which is not a complete discrete valuation ring, then an R-module of countable rank is Whitehead if and only if it is free (cf. [3]). On the other hand, if R is a complete discrete valuation ring, then it is cotorsion and hence every torsion-free R-module is a Whitehead module (cf. [2, XII.1.17]).

It will be convenient to decree that a field is not a p.i.d. and to use the term "slender" to designate a p.i.d. which is not a complete discrete valuation ring, or equivalently, is not cotorsion (cf. [2, III.2.9]). We will say that a module is κ-generated if it is generated by a subset of size $\leq \kappa$ and that it is κ-free if every submodule generated by $< \kappa$ elements is free. (Note that, by Pontryagin's Criterion and induction on κ, every $\aleph_1$-free module which has rank $\leq \kappa$ is κ-generated.)

An argument due to the second author (cf. [7] or [8]) shows that it is consistent with ZFC + GCH that for any p.i.d. R (of arbitrary size), there are Whitehead R-modules of rank $\geq |R|$ which are not free.

If the p.i.d. R is slender and has cardinality at most $\aleph_1$, the Axiom of Constructibility (V = L) implies that every Whitehead R-module is free (cf. [1]). Our main result is that the story is different for p.i.d.'s of larger size. We will prove the following theorems in ZFC.

Theorem 1. *There is a slender p.i.d. R of cardinality $2^{\aleph_1}$ such that every $\aleph_1$-free $\aleph_1$-generated R-module is a Whitehead module. Hence there are non-free Whitehead R-modules which are $\aleph_1$-generated.*

Theorem 2. *There is a p.i.d. R of cardinality $\aleph_2$ such that an $\aleph_1$-generated R-module is Whitehead only if it is free.*

Assuming V = L and using the existing theory (cf. [1]) one easily obtains the following:

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Corollary 3. *($V = L$) There are principal ideal domains R_1 and R_2 each of cardinality $\aleph_2$ and non-slender such that:*

(1) an R_1-module M (of arbitrary cardinality) is Whitehead if and only if M is the union of a continuous chain, $M = \bigcup_{\alpha<\lambda} M_\alpha$ for some λ, such that for all $\alpha < \lambda$, $M_{\alpha+1}/M_\alpha$ is $\aleph_1$-free and $\aleph_1$-generated;

(2) an R_2-module M (of arbitrary cardinality) is Whitehead if and only if M is free. ∎

The theorems can be generalized to other cardinals: see Theorems 6 and 7 at the end of the sections.

1. Proof of Theorem 1

The ring R in Theorem 1 will be constructed by a transfinite induction so that for every module F/K (F free) which is $\aleph_1$-free and $\aleph_1$-generated, $\mathrm{Ext}(F/K, R) = 0$, i.e., every homomorphism from K to R extends to a homomorphism from F to R. The following proposition provides the inductive step.

Proposition 4. *Let R be a local slender p.i.d. with maximal ideal pR, and let $K \subseteq F$ be free R-modules of rank $\aleph_1$ such that F/K is $\aleph_1$-free. Let $\psi : K \to R$ be an R-homomorphism. Then there is a local slender p.i.d. R^+ containing R as subring, with maximal ideal pR^+ and of cardinality $= |R| + \aleph_1$ such that the R^+-homomorphism $1_{R^+} \otimes_R \psi : R^+ \otimes_R K \to R^+ \otimes_R R$ extends to an R^+-homomorphism $\varphi : R^+ \otimes_R F \to R^+ \otimes_R R$.*

Proof. Write $F = \bigcup_{\alpha<\omega_1} F_\alpha$ as a continuous union of submodules of countable rank with $F_0 = 0$. For each $\alpha < \omega_1$, $F_\alpha + K/K$ is free; let $\{b_i^\alpha : i \in I_\alpha\}$ be a linearly independent subset of F_α such that $\{b_i^\alpha + K : i \in I_\alpha\}$ is a basis of $F_\alpha + K/K$. ($I_0 = \emptyset$.) Then for all $\alpha < \beta < \omega_1$ and all $i \in I_\alpha$, $b_i^\alpha = \sum_{j \in I_\beta} r_{i,j}^{\alpha,\beta} b_j^\beta + k_i^{\alpha,\beta}$ for some unique $r_{i,j}^{\alpha,\beta} \in R$ (which are equal to 0 for almost all j) and $k_i^{\alpha,\beta} \in K$. Let $s_i^{\alpha,\beta} = \psi(k_i^{\alpha,\beta})$.

We claim that there is a local slender p.i.d. R^+ of cardinality $= |R| + \aleph_1$ containing R as subring and with maximal ideal pR^+ and elements $x_i^\alpha \in R^+$ ($\alpha < \omega_1$, $i \in I_\alpha$) such that $x_i^\alpha = \sum_{j\in I_\beta} r_{i,j}^{\alpha,\beta} x_j^\beta + s_i^{\alpha,\beta}$ for all $\alpha < \beta < \omega_1$ and $i \in I_\alpha$. Supposing this for the moment, let us finish the proof. Clearly $\{b_i^\alpha : \alpha < \omega_1, i \in I_\alpha\} \cup K$ generates $R^+ \otimes_R F$ as R^+-module. Define φ extending $1_{R^+} \otimes_R \psi$ by $\varphi(1 \otimes b_i^\alpha) = x_i^\alpha \otimes 1$. We must check that this is well-defined. For this it suffices to prove that $\varphi(1 \otimes b_i^\alpha) = \sum_{j\in I_\beta} r_{i,j}^{\alpha,\beta} \varphi(1 \otimes b_j^\beta) + (1 \otimes \psi)(1 \otimes k_i^{\alpha,\beta})$ for all $\alpha < \beta < \omega_1$ and $i \in I_\alpha$. But this is implied by the assumption that $x_i^\alpha = \sum_{j\in I_\beta} r_{i,j}^{\alpha,\beta} x_j^\beta + s_i^{\alpha,\beta}$.

So it remains to define R^+. Let $R^0 = R$ and for $0 < \alpha < \omega_1$, let $R^\alpha = R[\{x_i^\alpha : i \in I_\alpha\}]$, the polynomial ring over R in the commuting indeterminates x_i^α, $i \in I_\alpha$. For $\alpha < \beta < \omega_1$, let $\pi_\beta^\alpha : R^\alpha \to R^\beta$ be the ring homomorphism which is the identity on R and takes x_i^α to $\sum_{j\in I_\beta} r_{i,j}^{\alpha,\beta} x_j^\beta + s_i^{\alpha,\beta}$. It is easy to check, using the fact that the $\{b_i^\gamma : i \in I_\gamma\}$ are linearly independent, that $\pi_\gamma^\beta \circ \pi_\beta^\alpha = \pi_\gamma^\alpha$ whenever $\alpha < \beta < \gamma < \omega_1$. Let R' with maps $\pi^\alpha : R^\alpha \to R'$ be the direct limit of this $\aleph_1$-directed system of homomorphisms. Clearly each R^α is a unique factorization domain such that p is prime in R^α. Since the system is directed, R' is an integral domain and p is prime in R'. Moreover, since the system is $\aleph_1$-directed, $\bigcap_{n\in\omega} p^n R' = 0$ since the same is true in each R^α. If $\{a_n : n \in \omega\}$ is a Cauchy sequence in R which does not have a limit (in the p-adic topology), then $\{t\pi^0_\alpha(a_n) : n \in \omega\}$ does not have a limit in the p-adic topology on R^α for all $t \in R^\alpha - pR^\alpha$. Hence, by the $\aleph_1$-directedness, the same holds for $\{\pi^0(a_n) : n \in \omega\}$ in R'.

Finally, let R^+ be the localization of R' at the prime p. We appeal to the following elementary Lemma to finish. ∎

Lemma 5. *Suppose R' is an integral domain with a prime p such that $\bigcap_{n\in\omega} p^n R' = 0$. Then the localization $R'_{(p)}$ of R' at p is a p.i.d.*

Proof. Given a non-zero proper ideal I of $R'_{(p)}$, let $I' = I \cap R'$ $(= \{r \in R' : \frac{r}{1} \in I\})$. Let m be minimal such that $I' \cap (p^m R' - p^{m+1}R') \neq \emptyset$. Clearly m exists, by hypothesis and since I' is non-zero. We claim that $I = p^m R'_{(p)}$. Let $a \in I' \cap (p^m R' - p^{m+1}R')$; then $a = p^m r$ for some $r \in R'$ and $r \notin pR'$; so r is a unit in $R'_{(p)}$ and thus $p^m \in I$. Now for any non-zero $\frac{b}{t} \in I$, $b \in I' - \{0\}$ so $b \in I' \cap (p^n R' - p^{n+1}R')$ for some $n \geq m$. Thus $b = p^n c$ for some $c \in R'$ and $n \geq m$. But then $\frac{b}{t} = p^n \frac{c}{t} \in p^m R'_{(p)}$. Therefore $I = p^m R'_{(p)}$. ∎

Proof of Theorem 1. Let $\lambda = 2^{\aleph_1}$. We define a ring R on the set λ which is the union of a continuous chain of rings R_ν $(\nu < \lambda)$ such that for each $\nu < \lambda$, $R_{\nu+1}$ is of the form $(R_\nu)^+$ for some quadruple $(R_\nu, K_\nu, F_\nu, \psi_\nu)$ satisfying the hypotheses of the Proposition. We begin, for example, with $R_0 = \mathbb{Z}_{(p)}$. It is easy to see that R is a local p.i.d. with prime p. Moreover, the proof of the Proposition shows that a witnessing Cauchy sequence to the incompleteness of R_0 is preserved at each stage and therefore also in R since ω_1 has cofinality $> \omega$. Because $\lambda^{\aleph_1} = \lambda$, we can choose the enumeration of quadruples $(R_\nu, K_\nu, F_\nu, \psi_\nu)$ such that for every $\aleph_1$-generated $\aleph_1$-free R-module F/K (where $K \subseteq F$ are free R-modules) and every R-homomorphism $\psi : K \to R$, there is a $\nu < \lambda$ such that $R \otimes_{R_\nu} F_\nu$ is isomorphic to F under an isomorphism which takes $R \otimes_{R_\nu} K_\nu$ to K and identifies $1_R \otimes_{R_\nu} \psi_\nu$ with ψ under the natural isomorphism of $R \otimes_{R_\nu} R_\nu$ with R. (Note that $K \subseteq F$ and ψ can each be completely described by a sequence of $\aleph_1$ elements of $R = \lambda$.) ∎

By using a direct system indexed by the countable rank submodules of F/K in the proof of the Proposition, we can prove the following more general version of the theorem. Part (1) of Corollary 3 can be correspondingly generalized.

Theorem 6. *For any cardinal $\kappa \geq \aleph_1$, there is a local slender p.i.d. R of cardinality 2^κ such that every $\aleph_1$-free κ-generated R-module is a Whitehead module.* ∎

2. Proof of Theorem 2

Let R be the polynomial ring $\mathcal{F}[X]$ where $\mathcal{F} = \mathbb{Q}(\{t_\nu : \nu < \omega_2\})$ and $\{t_\nu : \nu < \omega_2\}$ is an algebraically independent set.

Let A be an $\aleph_1$-generated $\aleph_1$-free R-module which is not free and let $A = \bigcup_{\alpha<\omega_1} A_\alpha$ be an $\aleph_1$-filtration of A. Then there is a stationary set S of limit ordinals such that for $\gamma \in S$, $A_{\gamma+1}/A_\gamma$ is not free. Without loss of generality we can assume that there is a $d \in \omega$ such that for all $\gamma \in S$, $A_{\gamma+1}/A_\gamma$ is of rank $d+1$ and not free but every submodule of rank $\leq d$ is free. (Note that we allow $A_{\alpha+1}/A_\alpha$ to be non-free for $\alpha \notin S$.) Thus $A_{\gamma+1}/A_\gamma$ is isomorphic to F'_γ/K'_γ where F'_γ is free on $\{y_{\gamma,n} : n \in \omega\} \cup \{x_{\gamma,\ell} : \ell < d\}$ and K'_γ has a basis $\{w'_{\gamma,n} : n \in \omega\}$ where

$$w'_{\gamma,n} = p_{\gamma,n} y_{\gamma,n+1} - y_{\gamma,n} - \sum_{\ell<d} s_{\gamma,n,\ell} x_{\gamma,\ell}$$

for some $p_{\gamma,n}$, $s_{\gamma,n,\ell} \in R$ where the $p_{\gamma,n}$ are non-units of R (not necessarily prime). (Compare, for example, [4, Observation 3.1].)

Let $F = \oplus_{\beta<\omega_1} F_\beta$ and $K = \oplus_{\beta<\omega_1} K_\beta$ be as in [2, Lemma XII.1.4]; that is, for all $\alpha < \omega_1$, $\oplus_{\beta<\alpha} F_\beta / \oplus_{\beta<\alpha} K_\alpha \cong A_\alpha$ and $\oplus_{\beta\leq\alpha} F_\beta/(\oplus_{\beta<\alpha} F_\beta + K_\alpha) \cong A_{\alpha+1}/A_\alpha$. Moreover, by the proof of [2, Lemma XII.1.4], we can assume that for $\gamma \in S$, F'_γ is a summand of F_γ and K_γ has a basis which includes $\{w_{\gamma,n} : n \in \omega\}$ where

$$w_{\gamma,n} = w'_{\gamma,n} - a_{\gamma,n}$$

for some $a_{\gamma,n} \in \oplus_{\beta<\gamma} F_\beta$ (and $\psi_\gamma(w'_{\gamma,n}) = \varphi_\gamma(a_{\gamma,n}) \in A_\gamma$). Fix a basis $\mathcal{B}$ of F which is the union of a basis $\mathcal{B}_\beta$ for each F_β and which includes $\bigcup_{\gamma\in S}\{y_{\gamma,n} : n \in \omega\} \cup \{x_{\gamma,\ell} : \ell < d\}$. Also fix a basis of K which includes $\bigcup_{\gamma\in S}\{w_{\gamma,n} : n \in \omega\}$. Given an element r of R, we will say $\mu \in \omega_2$ *occurs in* r if r does not belong to $\mathbb{Q}(\{t_\nu : \nu \in \omega_2 - \{\mu\}\})[X]$. Given an element z of F we will say that μ *occurs in* z if it occurs in some coefficient of the unique linear combination of elements of $\mathcal{B}$ which equals z. There is a subset I of ω_2 of cardinality $\aleph_1$ such that all of the $p_{\gamma,n}$ and $s_{\gamma,n,\ell}$ ($\gamma \in S$, $n \in \omega$, $\ell < d$) belong to $\mathbb{Q}(\{t_i : i \in I\})[X]$. Moreover, we can choose I such that it contains every μ which occurs in some coefficient of a linear combination of elements of $\mathcal{B}$ which equals some $a_{\gamma,n}$ ($\gamma \in S$, $n \in \omega$). Without loss of generality (by renumbering the t_ν), $I = \omega_1$.

Now we define $\psi : K \to R$ by defining

$$\psi(w_{\gamma,n}) = t_{\omega_1+\omega\gamma+n}$$

and letting ψ be arbitrary on the other basis elements of K. We will show that $\text{Ext}(A, R) \neq 0$ by showing that ψ cannot be extended to a homomorphism from F into R. Suppose to the contrary that there is a homomorphism $\varphi : F \to R$ extending ψ. For each $\alpha < \omega_1$, let T_α be the set of all $\mu \in \omega_2$ which occur in $\varphi(b)$ for some $b \in \bigcup\{\mathcal{B}_\beta : \beta < \alpha\}$. Then the T_α ($\alpha \in \omega_1$) form a continuous chain of countable subsets of ω_2 and there is $\delta \in S$ such that $T_\delta \cap \{\omega_1+\beta : \beta < \omega_1\} \subseteq \{\omega_1+\beta : \beta < \delta\}$. There is a finite subset Z of ω_2 such that every μ which occurs in $\varphi(y_{\delta,0})$ or in $\varphi(x_{\delta,\ell})$ for some $\ell < d$ belongs to Z. Let $R^* = \mathbb{Q}(\{t_\nu : \nu \in \omega_1 \cup T_\delta \cup Z\})[X]$, a subring of $R = \mathcal{F}[X]$. Now for all $n \in \omega$ we have $\varphi(w_{\delta,n}) = \psi(w_{\delta,n}) =$

$$t_{\omega_1+\omega\delta+n} = p_{\delta,n}\varphi(y_{\delta,n+1}) - \varphi(y_{\delta,n}) - \sum_{\ell<d} s_{\delta,n,\ell}\varphi(x_{\delta,\ell}) - \varphi(a_{\delta,n}).$$

If we can show that this implies that $t_{\omega_1+\omega\delta+n}$ belongs to R^* for all $n \in \omega$, we will have a contradiction of the choice of T_δ and the fact that Z is finite. We will show this by induction on n along with simultaneously proving that $\varphi(y_{\delta,n+1}) \in R^*$. We begin with $n = -1$: $\varphi(y_{\delta,0})$ belongs to R^* by definition of Z. Now suppose the inductive hypothesis is true for $n-1$ and we prove it for n. By the last displayed formula, the inductive hypothesis and the choice of R^*, there is an element $r_n \in R^*$ such that $p_{\delta,n}\varphi(y_{\delta,n+1}) = r_n - t_{\omega_1+\omega\delta+n}$. If $t_{\omega_1+\omega\delta+n} \notin R^*$, there is an automorphism Θ of R which fixes R^* and takes $t_{\omega_1+\omega\delta+n}$ to t_τ for some $\tau \notin T_\delta$. Then $p_{\delta,n}\Theta(\varphi(y_{\delta,n+1})) = r_n - t_\tau$. (Remember that $p_{\delta,n} \in R^*$.) Therefore, subtracting, $p_{\delta,n}$ divides $t_{\omega_1+\omega\delta+n} - t_\tau$,which is impossible since $p_{\delta,n}$ is a non-unit. Thus $t_{\omega_1+\omega\delta+n}$ and hence $p_{\delta,n}\varphi(y_{\delta,n+1})$ belong to R^*. But then since $p_{\delta,n} \in R^*$ we can prove by induction on m that the coefficient of X^m in $\varphi(y_{\delta,n+1}) \in \mathcal{F}[X]$, belongs to $\mathbb{Q}(\{t_\nu : \nu \in \omega_1 \cup T_\delta \cup Z\})$, and hence that $\varphi(y_{\delta,n+1})$ belongs to R^*. ∎

We can even find a principal ideal domain of cardinality $\aleph_1$ which satisfies the conclusion of Theorem 2. Namely, let $R = \mathcal{F}_1[X]$ where $\mathcal{F}_1 = \mathbb{Q}(\{t_\nu : \nu < \omega_1\})$.

Define $\psi(w_{\delta,n})$ to be $t_{\omega\delta+\sigma_\delta+n}$ where $\omega\delta+\sigma_\delta$ is larger than any μ which occurs in any $p_{\delta,k}$ or $s_{\delta,k,\ell}$ for $k \in \omega$, $\ell < d$. Define T_α as before and choose $\delta \in S$ such that $T_\delta \cap \omega_1 \subseteq \omega\delta$. Let $R^* = \mathbb{Q}(\{t_\nu : \nu \in \omega\delta + \sigma_\delta \cup T_\delta \cup Z\})[X]$.

We can also localize without affecting the property of the ring that we desire. More generally, we have:

Theorem 7. *For any $\kappa \geq \aleph_1$ there is a local p.i.d. R of cardinality κ such that an R-module of cardinality $\leq \kappa$ is Whitehead only if it is free.* ∎

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