THE SPLITTING NUMBER CAN BE SMALLER THAN THE MATRIX CHAOS NUMBER

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Abstract. Let χ be the minimum cardinal of a subset of ${}^\omega 2$ that cannot be made convergent by multiplication with a single Toeplitz matrix. By an application of creature forcing we show that $\mathfrak{s} < \chi$ is consistent. We thus answer a question by Vojtáš. We give two kinds of models for the strict inequality. The first is the combination of an $\aleph_2$-iteration of some proper forcing with adding $\aleph_1$ random reals. The second kind of models is got by adding δ random reals to a model of $\mathrm{MA}_{<\kappa}$ for some $\delta \in [\aleph_1, \kappa)$. It was a conjecture of Blass that $\mathfrak{s} = \aleph_1 < \chi = \kappa$ holds in such a model. For the analysis of the second model we again use the creature forcing from the first model.

0. Introduction

We consider products of $\omega \times \omega$ matrices of reals $A = (a_{i,j})_{i,j<\omega}$ and functions from ω to 2 or to some bounded interval of the reals. We define

$$A \lim f := \lim_{i\to\infty} \sum_{j=0}^{\infty} (a_{i,j} \cdot f(j)).$$

Toeplitz (cf. [2]) showed: $A \lim$ is an extension of the ordinary limit iff A is a regular matrix, i.e. iff $\exists m \; \forall i \; \sum_{j=0}^{\infty} |a_{i,j}| < m$ and $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{i,j} = 1$ and $\forall j \; \lim_{i\to\infty} a_{i,j} = 0$. Regular matrices are also called Toeplitz matrices.

We are interested whether for many f's simultaneously there is one A such that all $A \lim f$ exist, and formulate our question in terms of cardinal characteristics.

Let ℓ^∞ denote the set of bounded real sequences, and let $\mathbb{M}$ denote the set of all Toeplitz matrices. Vojtáš [10] defined for $\mathbb{A} \subseteq \mathbb{M}$ the chaos relations $\chi_{\mathbb{A},\infty}$ and their norms $\|\chi_{\mathbb{A},\infty}\|$ as follows:

$$\begin{aligned} \chi_{\mathbb{A},\infty} &= \{(A,f) : A \in \mathbb{A} \;\wedge\; f \in \ell^\infty \;\wedge\; A \lim f \text{ does not exist}\}, \\ \|\chi_{\mathbb{A},\infty}\| &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \ell^\infty \;\wedge \\ &\qquad (\forall A \in \mathbb{A})\,(\exists f \in \mathcal{F})\; A \lim f \text{ does not exist}\}. \end{aligned}$$

By replacing ℓ^∞ by ${}^\omega 2$, the set of ω-sequences with values in 2, we get the variations $\chi_{\mathbb{A},2}$. In [6] we showed that for the cardinals we are interested in, ${}^\omega 2$ and ℓ^∞ give the same result. From now on we shall work with ${}^\omega 2$.

Vojtáš (cf. [11]) also gave some bounds valid for any $\mathbb{A}$ that contains at least all matrices which have exactly one non-zero entry in each line:

$$\mathfrak{s} \leq \|\chi_{\mathbb{A},2}\| \leq \mathfrak{b} \cdot \mathfrak{s}.$$

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We write χ for $\|\chi_{\mathbb{M},2}\|$.

In [6] we showed that $\chi < \mathfrak{b} \cdot \mathfrak{s}$ is consistent relative to ZFC. Here, we show the complementary consistency result, that $\mathfrak{s} < \chi$ is consistent. We get the convergence with positive matrices.

Now we recall here the definitions of the cardinal characteristics $\mathfrak{b}$ and $\mathfrak{s}$ involved: The order of eventual dominance $\leq^*$ is defined as follows: For $f, g \in {}^\omega\omega$ we say $f \leq^* g$ if there is $k \in \omega$ such that for all $n \geq k$ we have $f(n) \leq g(n)$.

The unbounding number $\mathfrak{b}$ is the smallest size of a subset $\mathcal{B} \subseteq {}^\omega\omega$ such that for each $f \in {}^\omega\omega$ there is some $b \in \mathcal{B}$ such that $b \not\leq^* f$. The splitting number $\mathfrak{s}$ is the smallest size of a subset $\mathcal{S} \subseteq [\omega]^\omega$ such that for each $X \in [\omega]^\omega$ there is some $S \in \mathcal{S}$ such that $X \cap S$ and $X \setminus S$ are both infinite. The latter is expressed as "S splits X", and $\mathcal{S}$ is called a splitting family. For more information on these cardinal characteristics, we refer the reader to the survey articles [1, 3, 9].

If $A \lim f$ exists, then also $A' \lim f$ exists for any A' that is gotten from A be erasing rows and moving the remaining (infinitely many) rows together. We may further change A' by keeping only finitely many non-zero entries in each row, such that the neglected ones have a negligible absolute sum, and then possibly multiplying the remaining ones such that they again sum up to 1. Hence, after possibly further deleting of lines we may restrict the set of Toeplitz matrices to linear Toeplitz matrices. A matrix is linear iff each column j has at most one entry $a_{i,j} \neq 0$ and for $j < j'$ the i with $a_{i,j} \neq 0$ is smaller or equal to the i with $a_{i,j'} \neq 0$ if both exist, in picture

$$\begin{pmatrix} c_0(0) & \dots c_0(m_{\mathrm{up}}(c_0)-1) & 0 & \dots & 0 & 0 & \dots \\ 0 & \dots 0 & c_1(m_{\mathrm{dn}}(c_1)) & \dots & c_1(m_{\mathrm{up}}(c_1)-1) & 0 & \dots \\ 0 & \dots 0 & 0 & \dots & 0 & c_2(m_{\mathrm{dn}}(c_2)) & \dots \\ \vdots & & & & & & \end{pmatrix}$$

Linear matrices can be naturally (as in the picture) read as $\langle c_i \rangle_{i<\omega}$ where $c_i \colon [m_{\mathrm{dn}}(c_i), m_{\mathrm{up}}(c_i)) \to [0,1]$, $c_i(j) = a_{i,j}$, give the finitely many non-zero entries in row i, and $m_{\mathrm{up}}(c_{i-1}) = m_{\mathrm{dn}}(c_i)$. The c_i's are special instances of the weak creatures in the sense of [7]. In the next two sections we shall show: The c_i's coming from the trunks of the conditions in the generic filter of our forcing Q give matrices that make, after multiplication, members of ${}^\omega 2$ from the ground model and members of ${}^\omega 2$ of any random extension convergent.

Now, that we have used the word "creature" several times, we should explain it. Roughly speaking creatures are certain partial functions. Countably many of them are put into an arrangement which serves as one forcing condition. Stronger conditions are gotten by composing (unsually finitely many) of the partial functions and changing the arrangment in a certain way. We need the exact definitions only at one point in our work, when we want to cite some result on properness from [7]. For this purpose, we have to verify that our forcing is an instance of a creature forcing built from a finitary creating pair, such that the forcing conditions fulfil some conditions on the growth of the norms of their building blocks. We shall explain these notions in the next section.

1. A creature forcing

In this section, we give a self-contained description of the creature forcing Q which is the main tool for building the two kinds of models in the next section.

Moreover, we explain the connections and give the references to [7], so that the reader can identify it as a special case of an extensive framework.

Definition 1.1. *a) We define a notion of forcing Q. We say that $p \in Q$ iff $p = (n, \langle c_i \rangle_{i<\omega})$ (or $p = (n^p, \langle c^p_i \rangle_{i<\omega})$ when we want to avoid confusion comparing two elements of Q) and*

(1) $n < \omega$,

(2) *for each i there are natural numbers $m_{\rm dn}(c_i) < m_{\rm up}(c_i) < \omega$ such that $c_i \colon [m_{\rm dn}(c_i), m_{\rm up}(c_i)) \to [0, 1]$,*

(3) $(\forall i < \omega)(\forall k \in {\rm dom}(c_i))(c_i(k) \cdot k! \in \mathbb{Z})$,

(4) $(\forall i < \omega) \sum_{k \in {\rm dom}(c_i)} c_i(k) = 1$,

(5) $m_{\rm up}(c_i) = m_{\rm dn}(c_{i+1})$.

We let $p \leq q$ ("q is stronger than p") if

(6) $n^p \leq n^q$,

(7) $(\forall 0 \leq k < n^p) c^p_k = c^q_k$,

(8) *there exists an increasing sequence of natural numbers $\langle k_t \rangle_{t \geq n^p}$ such that for each $t \geq n^p$ there exist a non-empty set $u_t \subseteq [k_t, k_{t+1})$ and positive rationals $\{d_\ell : \ell \in u_t\}$ such that $\sum_{\ell \in u_t} d_\ell = 1$ and $c^q_t = \sum_{\ell \in u_t} d_\ell \cdot c^p_\ell$.*

Observe that if $p \leq q$ then for each $t \geq n^p$ we have that $m_{\rm dn}(c^q_t) = m_{\rm dn}(c^p_{k_t})$ and $m_{\rm up}(c^q_t) = m_{\rm up}(c^p_{k_{t+1}-1})$.

We write $p \leq_i q$ iff $n^p = n^q$ and $c^q_j = c^p_j$ for $j < n^p + i$.

Remark 1.2. *The notation we used in 1.1 is natural to describe our forcing in a compact manner. However, it does not coincide with the notation given for the general framework in [7]. Here is a translation: We write $(\langle c^p_i \rangle_{i<n^p}, \langle c^p_i \rangle_{i \geq n^p})$ instead of $(n^p, \langle c^p_i \rangle_{i<\omega})$, which contains the same information. Then we write*

$$(*) \qquad (\langle c^p_i \rangle_{i<n^p}, \langle c^p_i \rangle_{i \geq n^p}) = (w^p, \langle t^p_i \rangle_{i<\omega}),$$

i.e. we shift the indices.

Now we want to show that Q is proper. This follows from the work in [7] once we have verified that Q, in the form $(*)$, fulfils all the conditions made on the notions of forcing in [7, 2.1.6].

We claim that there is are a set of creatures K with respect to a finitary $\mathbf{H}$ and a subcompositions function Σ such that (K, Σ) is finitary [7, 1.1.3 (2)] and such that our forcing is $Q^*_{s,\infty}(K, \Sigma)$ in the notation of [7].

We take $\mathbf{H} \colon \omega \to V$ such that for $i \in \omega$, $\mathbf{H}(i) = \left\{0, \frac{1}{i!}, \frac{2}{i!}, \dots, \frac{i!-1}{i!}, 1\right\}$. As usual we write $\lhd$ for the proper initial segment relation. The set of all weak creatures with respect to $\mathbf{H}$ is the set of all $t = (\mathbf{nor}(t), \mathbf{val}(t), \mathbf{dis}(t))$ such that $\mathbf{nor}(t) \in \mathbb{R}$, $\mathbf{nor}(t) > 0$, $\mathbf{val}(t)$ is a non-empty subset of

$$\left\{ (x, y) \in \bigcup_{m_0 < m_1 < \omega} \left[\prod_{i<m_0} \mathbf{H}(i) \times \prod_{i<m_1} \mathbf{H}(i) \right] \; : \; x \lhd y \right\}.$$

K will be a subset of the set of al weak creatures. The function $\mathbf{dis}$ is the empty function in our case. Now, in our case there are the following requirements on $t \in K$: For $i \in \omega$, t_i from $(*)$ is a part of such a t in the following sense:

$\mathbf{nor}(t) = m_{\mathrm{dn}}(t_i)$, range$(\mathbf{val}(t)) = \{t_i\}$. For dom$(\mathbf{val}(t))$ one can take the maximal set fitting to $\mathbf{val}(t)$, since the range does not depend on the domain in our case. We have that $\mathbf{H}$ and K are finitary in the sense of [7, 1.1.3(2)]. Now we take $\Sigma\colon [K]^{<\omega} \to \mathcal{P}(K)$, $\Sigma(\{t_0, \dots t_{n-1}\}) = \emptyset$ unless $t_i\colon [m_{\mathrm{dn}}(t_i), m_{\mathrm{up}}(t_i)) \to \mathbb{R}$, $t_i \in K$, and $m_{\mathrm{up}}(t_i) = m_{\mathrm{dn}}(t_{i+1})$, in which case $\Sigma(\{t_0, \dots t_{n-1}\})$ is the set of all $t\colon [m_{\mathrm{dn}}(t_0), m_{\mathrm{up}}(t_{n-1})) \to \mathbb{R}$, $t \in K$ such $\exists d_\ell, \ell \in n$, with $\sum_{\ell \in n} d_\ell = 1$ and such that for each ℓ, for $m \in [m_{\mathrm{dn}}(t_\ell), m_{\mathrm{up}}(t_\ell))$, $t(m) = d_\ell \cdot t_\ell(m)$. Our Σ is a sub-composition operation in the sense of [7, 1.1.4]. Now some further quite long definitions (the interested reader may look at 1.1.6 to 1.1.10, 1.2.1 to 1.2.6 in [7]) give that our instance (K, Σ) is a finitary creating pair and that our Q is $Q^*_{s\infty}(K, \Sigma)$ in Rosłanowski's and Shelah's framework. Now their work shows:

Lemma 1.3. *([7, Corollary 2.1.6]) The forcing notion Q is proper.*

2. The effect of Q on random reals

Let G be Q-generic over V. We set $c_i^G = c_i^q$ for $q \in G$ and $n^q > i$. This is well-defined. Let $\underset{\sim}{c_i}$ be a name for it. Our aim is to show that multiplication by the matrix whose i-th row is c_i makes any real from the ground model and even any real from a random extension of the ground model convergent. For background information about random reals we refer the reader to [5, §42]. The Lebesgue measure ist denoted by λ. With "adding κ random reals" we mean forcing with the measure algebra R_κ on $2^{\omega \times \kappa}$, that is adding κ random reals at once or "side-by-side" and not successively.

Definition 2.1. *(1) Let* $\mathrm{may}_k(p) = \{c_i^r \,:\, p \leq_k r, i \geq n^p + k\}$.

(2) For a function $c\colon \mathrm{dom}(c) \to \mathbb{R}$ with finite domain and $\eta \in {}^\omega 2$ let $\mathrm{av}(\eta, c) = \sum_{k \in \mathrm{dom}(c)} c(k)\eta(k)$.

Main Lemma 2.2. *Assume that*

(A) $\underset{\sim}{\eta}$ is a random name of a member of ${}^\omega 2$, $\underset{\sim}{\eta} = f(\underset{\sim}{r})$ where f is Borel and $\underset{\sim}{r}$ is as name of the random generic real,

(B) $p \in Q$,

(C) $k^* < \omega$.

Then for every $k \geq k^*$ there is some $q(k) \in Q$ such that

(α) $p \leq_{k^*} q(k)$,

(β) for all ℓ, if $k^* \leq k < \ell < \omega$ and $c_1, c_2 \in \mathrm{may}_\ell(q(k))$ then

$$\frac{1}{\ell!} > \lambda\left\{r \,:\, \frac{3}{2^k} \leq |\mathrm{av}(f(r), c_1) - \mathrm{av}(f(r), c_0)|\right\}.$$

Proof. For $q \in Q$ and $k, \ell \in \omega$, $i \in \{0, 1, \dots, 2^k\}$ we set

$$\begin{aligned} \mathrm{err}_{k,i}(\underset{\sim}{\eta}, c) &= \int_0^1 \left|\mathrm{av}(f(r), c) - \frac{i}{2^k}\right| \, dr, \\ \mathrm{e}^\ell_{k,i}(\underset{\sim}{\eta}, q) &= \inf\{\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c) \,:\, c \in \mathrm{may}_\ell(q)\}. \end{aligned}$$

Note that $\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c)$ is a real and no longer a random name. So the infimum is well-defined.

Now, if $\ell_1 < \ell_2$ then $\mathrm{may}_{\ell_1}(q) \supseteq \mathrm{may}_{\ell_2}(q)$ and hence

$$\mathrm{e}^{\ell_1}_{k,i}(\underset{\sim}{\eta}, q) \leq \mathrm{e}^{\ell_2}_{k,i}(\underset{\sim}{\eta}, q).$$

So $\langle \mathrm{e}^\ell_{k,i}(\underset{\sim}{\eta}, q)\rangle_{\ell<\omega}$ increasing bounded sequence and

$$\mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, q) = \lim_{\ell\to\infty} \mathrm{e}^\ell_{k,i}(\underset{\sim}{\eta}, q)$$

is well-defined.

We fix $i \le 2^k$, until Subclaim 4, when we start looking at all i together.

Subclaim 1: There is some $q_1^{k,i} = q_1 \ge_{k^*} p$ such that for $\ell \ge k^*$

$$\mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, p) - \frac{1}{\ell} \le \mathrm{err}_{k,i}(\underset{\sim}{\eta}, c^{q_1}_\ell) \le \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, p) + \frac{1}{\ell}.$$

Moreover, if $m_{\mathrm{dn}}(c^{q_1}_{\ell'}) = m_{\mathrm{dn}}(c^p_\ell)$ then $\mathrm{e}^{\ell'}_{k,i}(\underset{\sim}{\eta}, q_1) \ge \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, p) - \frac{1}{\ell}$.

Why? We choose $c^{q_1}_\ell$ by induction on ℓ: For $\ell \le n^p + k^*$, we take $c^{q_1}_\ell = c^p_\ell$. Suppose that we have chosen $c^{q_1}_m$ for $m < \ell$ and that we are to choose $c^{q_1}_\ell$, $\ell > n^p + k^*$. We set $\varepsilon = \frac{1}{\ell}$. By possibly end-extending $c^{q_1}_{\ell-1}$ by zeroes we may assume that $m_{\mathrm{up}}(c^{q_1}_{\ell-1}) = m_{\mathrm{up}}(c^p_{\ell'})$ for such a large $\ell' \ge \ell$ such that for all $\ell'' \ge \ell'$, $\mathrm{e}^{\ell''}_{k,i}(\underset{\sim}{\eta}, p) \ge \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, p) - \varepsilon$. Then we take $c_\ell = c^{q_1}_\ell \in \mathrm{may}_{\ell''}(p)$ such that $\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c^{q_1}_\ell) \le \mathrm{e}^{\ell''}_{k,i}(\underset{\sim}{\eta}, p) + \varepsilon \le \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, p) + \varepsilon$. On the other side we have that $\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c^{q_1}_\ell) \ge \mathrm{e}^{\ell''}_{k,i}(\underset{\sim}{\eta}, p) \ge \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, p) - \varepsilon$. The fact that this holds also for $\ell' \le \ell$ if $m_{\mathrm{dn}}(c^{\tilde{q}_1}_{\ell'}) = m_{\mathrm{dn}}(c^p_\ell)$ yields the "moreover" part.

Subclaim 2: In Claim 1, if $\ell \ge k^*$ and $q_1^{k,i} \le_\ell q_2$ then

$$\mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, q) - \frac{1}{\ell} \le \mathrm{err}_{k,i}(\underset{\sim}{\eta}, c^{q_2}_\ell) \le \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, q) + \frac{1}{\ell}.$$

.

Why? By the definition if suffices to show:

$$(\otimes)\qquad \begin{aligned} &\text{if } \ell_1 < \cdots < \ell_t < \omega \text{ and } d_1, \ldots d_t \ge 0 \text{ and } d_1 + \cdots + d_t = 1,\\ &\text{and } c^{q_2}_\ell = d_1 c^{q_1}_1 + \cdots + d_t c^{q_1}_t,\\ &\text{then } \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, q_1) - \frac{1}{\ell} \le \mathrm{err}_{k,i}(\underset{\sim}{\eta}, c^{q_2}_\ell) \le \mathrm{e}^*_{k,i}(\underset{\sim}{\eta}, q_1) + \frac{1}{\ell}.\end{aligned}$$

The first inequality holds by the "moreover" after the first inequality in the previous claim. For the second inequality it suffices to show that

$$\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c) \le \sum_{s=1}^t d_s \mathrm{err}_{k,i}(\underset{\sim}{\eta}, c^{q_1}_s).$$

For this is suffices to see that

$$\int_0^1 \left(\left|\mathrm{av}(f(r), c) - \frac{i}{2^k}\right|\right) dr \le \sum_{s=1}^t d_s \int_0^1 \left(\left|\mathrm{av}(f(r), c^{q_1}_s) - \frac{i}{2^k}\right|\right) dr,$$

and since $d_s \ge 0$ and $\sum_s d_s = 1$ we finish by the the triangular inequality.

Subclaim 3: Let $q^{k,i}$ be as in Subclaim 2. For all ℓ, if $c_0, c_1 \in \mathrm{may}_\ell(q_1^{k,i})$, then

$$\frac{2^{k+1}}{\ell} \ge \lambda\left\{r \,:\, \mathrm{av}(f(r), c_0) \ge \frac{i+1}{2^k} \wedge \mathrm{av}(f(r), c_1) \le \frac{i-1}{2^k}\right\}.$$

Why? Consider $c = \frac{1}{2}c_0 + \frac{1}{2}c_1 \in \mathrm{may}_\ell(q_1)$. Write $A = \left\{r \,:\, \mathrm{av}(f(r), c_0) \ge \frac{i+1}{2^k} \wedge \mathrm{av}(f(r), c_1) \le \frac{i-1}{2^k}\right\}$.

$$\begin{aligned}
\frac{2}{\ell} &\geq \frac{1}{2}\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c_0) + \frac{1}{2}\mathrm{err}_{k,i}(\underset{\sim}{\eta}, c_1) - \mathrm{err}_{k,i}(\underset{\sim}{\eta}, c) \\
&= \int_0^1 \Bigg(\frac{1}{2}\left|\mathrm{av}(f(r), c_0) - \frac{i}{2^k}\right| + \frac{1}{2}\left|\mathrm{av}(f(r), c_1) - \frac{i}{2^k}\right| - \\
&\quad \left|\mathrm{av}(f(r)), c) - \frac{i}{2^k}\right|\Bigg)\, dr \\
&\geq \int_A \Bigg(\frac{1}{2}\left|\mathrm{av}(f(r), c_0) - \frac{i}{2^k}\right| + \frac{1}{2}\left|\mathrm{av}(f(r), c_1) - \frac{i}{2^k}\right| - \\
&\quad \left|\mathrm{av}(f(r)), c) - \frac{i}{2^k}\right|\Bigg)\, dr \\
&\geq \frac{1}{2^k}\,\lambda(A).
\end{aligned}$$

Subclaim 4: For every $q \in Q$ and k^* we can find q^k such that

α) $q \leq_{k^*} q^k$,

β) if $\ell \in [k, \omega)$ and $c_0, c_1 \in \mathrm{may}_\ell(q^k)$ and $i \in \{1, 2, \ldots, 2^k - 1\}$ then $\frac{2^{k+1}}{\ell} > \lambda\left\{r : \mathrm{av}(f(r), c_0) \geq \frac{i+1}{2^k} \wedge \mathrm{av}(f(r), c_1) \leq \frac{i-1}{2^k}\right\}$.

γ) This holds also for every $q^* \geq q^k$.

Why? Repeat Subclaims 1 and 2 and 3 choosing $q^{k,i}$, $i = 0, 1, \ldots, 2^k$. We let $q_0 = q$ and choose $q^{k,i+1}$ such that it relates to $q^{k,i}$ like q_1 to q.

Now $q^k = q^{k,2^k}$ is o.k. Note that according to ($\otimes$) thinning and averaging can only help.

Subclaim 5: Let q^k be as in Subclaim 4. For $\ell \geq k$ there is $q(k, \ell) \geq_{\ell-1} q^k$ such that for $c_0, c_1 \in \mathrm{may}_\ell(q(k, \ell))$,

$$\frac{1}{\ell!} > \lambda\left\{r : \frac{3}{2^k} \leq |\mathrm{av}(f(r), c_1) - \mathrm{av}(f(r), c_0)|\right\}.$$

Why? The event $\frac{3}{2^k} \leq |\mathrm{av}(f(r), c_1) - \mathrm{av}(f(r), c_0)|$ implies that for some $i \in \{1, 2, \ldots, 2^k - 1\}$ we have $\mathrm{av}(f(r), c_1) \geq \frac{i+1}{2^k} \wedge \mathrm{av}(f(r), c_2) \leq \frac{i-1}{2^k}$ or vice versa. So it is incuded in the union of $2 \times (2^k - 1)$ events, each of measure $\leq \frac{2^{k+1}}{\ell}$. Hence it itself has measure $\leq \frac{2^{2k+2}}{\ell}$. By thinning out q^k (by moving the former ℓ far out by putting in a lot of zeroes and thus having as new c_ℓ's partial functions that were formerly labelled with a much larger ℓ and thus giving a much smaller quotient according to Subclaim 4) we replace $\frac{2^{2k+2}}{\ell}$ by $\frac{1}{\ell!}$.

Subclaim 6: Finally we come to the $q(k)$ from part (β) of the lemma: For any k there is $q(k)$ such that $q \leq_{k^*} q(k)$ and for any $\ell \geq k$ and any $c_0, c_1 \in \mathrm{may}_\ell(q^*)$ then

$$\frac{1}{\ell!} > \lambda\left\{r : \frac{3}{2^k} \leq |\mathrm{av}(f(r), c_1) - \mathrm{av}(f(r), c_0)|\right\}.$$

Why? Like in the previous claim we choose inductively $q(k, \ell)$ such that $q_0 = p$ and $q(k, \ell + 1) \geq_\ell q(k, \ell)$ and $(q(k, \ell + 1), q(k, \ell), \ell)$ are like $(q(k, \ell), q, \ell)$ from

Subclaim 5, but for larger and larger ℓ. Now

$$q(k) = (n^p + k, c_0^p, \dots, c_{n^p+k}^p, c_{n^p+k+1}^{q(k,n^p+k+1)}, c_{n^p+k+1}^{q(k,n^p+k+2)}, \dots)$$

is as required in (α) and (β) of the conclusion; we have even $q(k) \geq_k p$. $\square$

Now we use the Main Lemma in an iteration. Any notion of forcing Q' that preserves $\aleph_1$ and $\aleph_2$ is suitable. In the application, Q' will be an end segment of length $\aleph_2$ of a countable support iteration of Q.

Conclusion 2.3. *Let Q' be any notion of forcing. Then we have: $\Vdash_Q$ "if $\underset{\sim}{\eta} \in V$ is a random name for a real in $V^{Q * \underset{\sim}{Q'} * \underset{\sim}{R_\omega}}$ then "$\Vdash_{\underset{\sim}{Q'} * \underset{\sim}{R_\omega}} \langle \mathrm{av}(\underset{\sim}{\eta}, \underset{\sim}{c}_n) : n \in \omega\rangle$ converges" "*

Proof. Let $q \in Q$ and $\varepsilon > 0$ be given. Let $\underset{\sim}{\eta} = f(\underset{\sim}{r})$, $f \in V$, be a random name for a real. We take k_0 such that $\frac{3}{2^k} < \varepsilon$. Then we take for $q(k) \geq q$ as in the Main Lemma. We set

$$A_{k,c_0,c_1} = \left\{ r : \frac{3}{2^k} > |\mathrm{av}(f(r), c_1) - \mathrm{av}(f(r), c_0)| \right\}.$$

Since $\sum_{\ell \geq 1} \frac{1}{\ell!} < \infty$, we can apply the Borel-Cantelli lemma and get:

For any sequence $\langle c_\ell \rangle_{\ell < \omega}$ such that $c_\ell \in \mathrm{may}_\ell(q^*(k))$ we have that

$$\lambda \left(\bigcup_{K \in [k,\omega)} \bigcap_{\ell \geq K} A_{k_0, c_\ell, c_{\ell+1}} \right) = 1.$$

So $r \in \bigcap_{\ell \geq K} A_{k, c_\ell, c_{\ell+1}}$ for some $K \geq k$. So $q(k)$ forces (also in $V[G]$ where G is $Q * \underset{\sim}{Q'} * \underset{\sim}{R_\omega}$-generic over V) that $\langle c_\ell \rangle_{\ell<\omega}$ describes a matrix whose product with η lies eventually within an ε interval. Now we take smaller and smaller ε's and a density argument. $\square$

Conclusion 2.4. *Let $P_{\omega_2} = \langle P_i, \underset{\sim}{Q}_j : i \leq \omega_2, j < \omega_2\rangle$ be a countable support iteration of $\underset{\sim}{Q}_i$, where Q_i is Q defined in V^{P_i}, and let $\underset{\sim}{R}_{\omega_1}$ be a P_{ω_2} name of the $\aleph_1$-random algebra. Then in $V^{P_{\omega_2} * \underset{\sim}{R}_{\omega_1}}$ we have $\mathfrak{s} = \aleph_1$ and $\chi > \aleph_1$.*

Proof. Dow proves in [4, Lemma 2.3] that $s = \aleph_1$ after adding $\aleph_1$ or more random reals, over any ground model. In order to show $\chi > \aleph_1$, let η_i , $i < \omega_1$, be reals in $V^{P_{\omega_2} * \underset{\sim}{R}_{\omega_1}}$. Over $V^{P_{\omega_2}}$, each η_i has a R_{ω_1}-name $\underset{\sim}{\eta_i}$. Since the random algebra is c.c.c, there are w.l.o.g. only countably many of the $\aleph_1$ random reals mentioned in $\underset{\sim}{\eta_i}$. Let $\underset{\sim}{\eta}'_i$ be got from $\underset{\sim}{\eta_i}$ by replacing these countably many by the first ω ones and then doing as if it were just one random real. This is possible because R_1 and R_ω are equivalent forcings. Since the random algebra is c.c.c., the name $\underset{\sim}{\eta}'_i$ can be coded as a single real r_i in $V^{P_{\omega_2}}$. Now, by [8, V.4.4.] and by the properness of the $\underset{\sim}{Q}_j$, this name r_i appears at some stage $\alpha(\eta_i) < \aleph_2$ in the iteration P_{ω_2}. We take the supremum α of all the $\alpha(\eta_i)$, $i < \omega_1$. We apply 2.3 with $Q = Q_\alpha$, $Q' = \langle P_i, \underset{\sim}{Q}_j : \alpha < i \leq \omega_2, \alpha < j < \omega_2\rangle$ and R_ω to the $\underset{\sim}{\eta}'_i$. Thus Q_α adds a Toeplitz matrix, that makes after multiplication all the η'_i convergent. Since Conclusion 2.3 applies to all random algebras simultaneously, this matrix makes also the η_i convergent. $\square$

Definition 2.5. *(1)* $Q_{pr} = \{p \in Q : n^p = 0\}$ *is called the pure part of* Q.

(2) *We write* $p \leq^* q$ *if there are some* w, n *such that* $p \leq (w, t^q_n, t^q_{n+1} \dots)$. *So, it is up to a finite "mistake"* $p \leq q$.

Fact 2.6. *If* $\langle p_i : i < \gamma \rangle$ *is* $\leq^*$*-increasing in* Q *and* $\mathrm{MA}_{|\gamma|}$ *holds, then there is* $p \in Q_{pr}$ *such that for all* $i < \delta$, $p_i \leq^* p$.

Proof. We apply $\mathrm{MA}_{|\gamma|}$ to the following partial order P: Conditions are (s, F) where $s = (t^p_0, \dots, t^p_n)$ is an initial segment of a condition in Q_{pr} and $F \subset \gamma$ is a finite set. We let $(s, F) \leq_P (t, G)$ iff $s \trianglelefteq t$ and $F \subseteq G$ and $(\forall n \in \lg(t) - \lg(s))(\forall \alpha \in F)(n >$ (all mistakes between the $p_\alpha) \rightarrow t_n \in \Sigma(c^{p_\alpha}_i : i \in \mathcal{S}(\alpha, n)$ for suitable $\mathcal{S}(\alpha, n)))$. This forcing is c.c.c., because conditions with the same first component are compatible and because there are only countably many possibilities for the first component. It is easy to see that for $\alpha < \delta$ the sets $D_\alpha = \{(s, F) : \alpha \in F\}$ is dense and that for $n \in \omega$ the sets $D^n = \{(s, F) : \lg(s) \geq n\}$ are dense. Hence if G is generic, then $p = \bigcup\{s : \exists F(s, F) \in G\} \geq^* p_\alpha$ for all α. □

Conclusion 2.7. *If* $V \models \mathrm{MA}_\kappa$ *and* $\kappa > \delta > \aleph_0$, *then in* V^{R_δ} *then matrix number is at least* κ *and the splitting number is* $\aleph_1$.

Proof. As mentioned, [4] shows the the result on the splitting number. For the matrix number, let random names $\underset{\sim}{\eta}_i$, $i < \gamma$, be given in V, $\gamma < \kappa$. We fix $\varepsilon > 0$ and K as in the proof of 2.3. We choose for $i < \gamma$, $p^i = \langle c^i_k \rangle_{k<\omega}$ as in the end of the proof of 2.3 for $\underset{\sim}{\eta}_i$ and use and Fact 2.6. $\gamma + 1$ times iteratively and find a pure condition $p = \langle c_k \rangle_{k<\omega} \geq^* p^i$ for all $i < \gamma$, that gives the lines of a matrix which brings everything into an ε-range. We denote these c_k by $c_k = c_k(\varepsilon)$. Now by induction we choose c_k: $c_0 = c_0(1)$, and $c_k = c_{k'}(\frac{1}{k'+1})$ if $k' > k$ is the first k'' such that $m_{\mathrm{dn}}(c_{k''}(\frac{1}{k''+1})) > m_{\mathrm{dn}}(c_{k-1})$. The matrix with c_k in the kth line acts as desired. (Now $m_{\mathrm{up}}(c_k) > m_{\mathrm{dn}}(c_{k+1})$ is possible but this does not harm.) □

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References

[1] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In Matthew Foreman, Akihiro Kanamori, and Menachem Magidor, editors, *Handbook of Set Theory*. Kluwer, To appear.
[2] Richard Cooke. *Infinite Matrices and Sequence Spaces*. MacMillan Co., 1950.
[3] Eric van Douwen. The integers and topology. In Kenneth Kunen and Jerry Vaughan, editors, *Handbook of Set Theoretic Topology*, pages 111–167. North-Holland, 1984.
[4] Alan Dow. Tree π-bases for $\beta\mathbb{N} - \mathbb{N}$. *Topology Appl.*, 33:3–19, 1989.
[5] Thomas Jech. *Set Theory*. Addison Wesley, 1978.
[6] Heike Mildenberger. Convergence and Toeplitz Matrices. *Fundamenta Mathematicae*, pages 187–189, 2000.
[7] Andrzej Rosłanowski and Saharon Shelah. *Norms on Possibilities I: Forcing with Trees and Creatures*, volume 141 (no. 671) of *Memoirs of the American Mathematical Society*. 1999, [RoSh:470].
[8] Saharon Shelah. *Proper Forcing*, volume 940 of *Springer Lecture Notes in Mathematics*. Springer, 1982.
[9] Jerry E. Vaughan. Small uncountable cardinals and topology. In Jan van Mill and G. Reed, editors, *Open Problems in Topology*. Elsevier, 1990.

[10] Peter Vojtáš. More on set theoretic characteristics of summability by regular (Toeplitz) matrices. *Commentationes Mathematicae Universitatis Carolinae*, 29:97–102, 1988.

[11] Peter Vojtáš. Series and Toeplitz Matrices (A Global Implicit Approach). *Tatra Mountain Mathematical Journal*, 14:269–281, 1998.

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