WEAK DIAMOND

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ABSTRACT. Under some cardinal arithmetic assumptions, we prove that every stationary subset of λ of a right cofinality has weak diamond. This is a strong negation of uniformization. We then deal with a weaker version of the weak diamond that involves restricting the domain of the colourings. We then deal with semi- saturated (normal) filters.

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1. Weak Diamond: sufficient condition

On the weak diamond see [DS78], [She98, Appendix §1], [She85], [She]; there will be subsequent work on the middle diamond.

Definition 1.1. For regular uncountable λ,

(1) We say $S \subseteq \lambda$ is small if it is $\mathbf{F}$-small for some function $\mathbf{F}$ from ${}^{\lambda>}\lambda$ to $\{0,1\}$, which means
$(*)_{\mathbf{F},S}$ for every $\bar{c} \in {}^S2$ there is $\eta \in {}^\lambda\lambda$ such that $\{\lambda \in S : \mathbf{F}(\eta \restriction \delta) = c_\delta\}$ is not stationary.

(2) Let $D^{\rm wd}_\lambda = \{A \subseteq \lambda : \lambda \setminus A$ is small $\}$, it is a normal ideal (the weak diamond ideal).

Claim 1.2. *Assume*

(a) $\lambda = \lambda^{<\lambda} = 2^\mu$

(b) $\Theta = \{\theta : \theta = {\rm cf}(\theta)$ *and for every* $\alpha < \lambda$, *we have* $|\alpha|^{<\theta>} < \lambda$ *or just* $|\alpha|^{<{\rm tr},\theta>} < \lambda\}$ *(see below; so if* $\lambda > \beth_\omega$ *every large enough regular* $\theta < \beth_\omega$ *is in* Θ*)*

(c) $S \subseteq \{\delta < \lambda : {\rm cf}(\delta) \in \Theta$, *and* μ^ω *divides* $\delta\}$ *is stationary.*

Then S *is not in the ideal* $D^{\rm wd}_\lambda$ *of small subsets of* λ.

Definition 1.3. (1) Let $\chi^{\langle\theta\rangle} = {\rm Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\theta$ and every $A \in [\chi]^\theta$ is included in the union of $< \theta$ members of $\mathcal{P}\}$.

(2) $\chi^{\langle\theta\rangle_{\rm tr}} = \sup\{|\lim_\theta(t)| : t$ is a tree with $\leq \chi$ nodes and θ levels $\}$

Remark 1.4. (1) On $\chi^{\langle\theta\rangle_{\rm tr}}$ see [She00a], on $\chi^{\langle\theta\rangle}$ see there and in [She00b] but no real knowledge is assumed here.

(2) The interesting case of 1.2 is λ (weakly) inaccessible; for λ successor we know more; but in later results even if 2^μ is successor we say on it new things.

(3) Actually only $\mathbf{F} \restriction (\bigcup\limits_{\delta \in S} {}^\delta\delta$ mark. ??

Proof. Let $\mathbf{F}$ be a function from $\bigcup\limits_{\delta\in S} {}^\delta\lambda$ to $\{0,1\}$, i.e., $\mathbf{F}$ is a colouring, and we shall find $f \in {}^S\lambda$ as required for it.

Let $\{\nu_i : i < \lambda\}$ list $\bigcup\limits_{\alpha<\lambda} {}^\alpha\lambda$ such that

$$\alpha < \lg(\nu_i) \Rightarrow \nu_i \restriction \alpha \in \{\nu_j : j < i\}.$$

For $\delta \in S$ let $\mathcal{P}_\delta = \{\eta \in {}^\delta\delta : \quad (\forall \alpha < \delta)(\eta \restriction \alpha \in \{\nu_i : i < \delta\})\}$.

Clearly $\delta \in S \Rightarrow |\mathcal{P}_\delta| \leq |\delta|^{<{\rm tr},\theta>} < \lambda$ by assumption (c). For each $\eta \in \mathcal{P}_\delta$ we define $h_\eta \in {}^\mu2$ by: $h_\eta(\varepsilon) = \mathbf{F}(g_{\eta,\varepsilon})$ where for $\varepsilon < \mu$, we let $g_{\eta,\varepsilon} \in {}^\delta2$ be defined by $g_{\eta,\varepsilon}(\alpha) = \eta(\mu\alpha + \varepsilon)$ for $\alpha < \delta$, recalling that μ^ω divides δ as $\delta \in S$. So $\{h_\eta : \eta \in \mathcal{P}_\delta\}$ is a subset of ${}^\mu2$ of cardinality $\leq |\mathcal{P}_\delta| < \lambda = 2^\mu$ hence we can choose $g^*_\delta \in {}^\mu2 \setminus \{g_\eta : \eta \in \mathcal{P}_\delta\}$. For $\varepsilon < \mu$ let $f_\varepsilon \in {}^S2$ be $f_\varepsilon(\delta) = 1 - g^*_\delta(\varepsilon)$. If for some $\varepsilon < \mu$ the function f_ε serve as a weak diamond sequence for $\mathbf{F}$, we are done so assume that (for each $\varepsilon < \mu$) there are η_ε and E_ε such that:

(a) E_ε is a club of λ.
(b) $\eta_\varepsilon \in {}^\lambda\lambda$.
(c) if $\delta \in E_\varepsilon \cap S$ then $\mathbf{F}(\eta_\varepsilon \restriction \delta) = 1 - f_\varepsilon(\delta)$ and $\eta_\varepsilon \restriction \delta \in {}^\delta\delta$.

Now define $\eta \in {}^\delta 2$ by $\eta(\mu\alpha + \varepsilon) = \eta_\varepsilon(\alpha)$ for $\alpha < \lambda, \varepsilon < \mu$.

Let $E = \{\delta < \lambda : \delta$ is divisible by μ^ω and $\varepsilon < \mu \Rightarrow \delta \in E_\varepsilon$ and $(\forall \alpha < \delta)[\eta \restriction \alpha \in \{\eta_i : i < \delta\}]\}$. Clearly E is a club of λ hence we can find $\delta \in E \cap S$. So by the definition of $\mathcal{P}_\delta$ we have $\eta \restriction \delta \in \mathcal{P}_\delta$ and for $\varepsilon < \mu$ we have $g_{\eta\restriction\delta,\varepsilon} \in {}^\delta\delta$ is equal to $\eta_\varepsilon \restriction \delta$ (Why? note that $\mu\delta = \mu$ as $\delta \in E$ and see the definition of $g_{\eta\restriction\delta,\varepsilon}$ and of η, so : $\alpha < \delta \Rightarrow g_{\eta\restriction\delta,\varepsilon}(\alpha) = \eta(\mu\alpha+\varepsilon) = \eta_\varepsilon(\alpha)$). Hence $h_{\eta\restriction\delta} \in {}^\mu 2$ is well defined and by the choice of η we have $\varepsilon < \mu \Rightarrow g_{\eta\restriction\delta,\varepsilon} = \eta_\varepsilon \restriction \delta$ so by its definition, $h_{\eta\restriction\delta}$ for each $\varepsilon < \mu$ satisfies $h_{\eta\restriction\delta}(\varepsilon) = \mathbf{F}(g_{\eta\restriction\delta,\varepsilon}) = \mathbf{F} = (\eta_\varepsilon \restriction \delta)$. Now by clause (c) and the choice of f_ε we have $\mathbf{F}(\eta_\varepsilon \restriction \delta) = 1 - f_\varepsilon(\delta) = g^*_\delta(\varepsilon)$ so $h_{\eta\restriction\delta} = g^*_\delta$, but $h_{\eta\restriction\delta} \in \mathcal{P}_\delta$ whereas we have chosen g^*_δ such that $g^*_\delta \notin \mathcal{P}_\delta$, a contradiction. $\square$

We may consider a generalization.

Definition 1.5. (1) We say $\bar{C}$ is a $\lambda-$Wd-parameter if:

(a) λ is a regular uncountable,
(b) S a stationary subsets of λ,
(c) $\bar{C} = \langle C_\delta : \delta \in S\rangle, C_\delta \subseteq \delta$

(1A) We say $\bar{C}$ is a (λ, κ, χ) -Wd-parameter <u>if</u> in addition $(\forall \delta \in S)[\mathrm{cf}(\delta) = \kappa \wedge |C_\delta| < \chi]$. We may also say that $\bar{C}$ is (S, κ, χ)-parameter.

(2) We say that $\mathbf{F}$ is a $\bar{C}$ -colouring if: $\bar{C}$ is a λ-Wd-parameter and $\mathbf{F}$ is a function from ${}^{\lambda>}\lambda$ to 2 such that :

if $\delta \in S$, $\eta_0, \eta_1 \in {}^\delta\delta$ and $\eta_0 \restriction C_\delta = \eta_1 \restriction C_\delta$ then $\mathbf{F}(\eta_0) = \mathbf{F}(\eta_1)$.

(2A) If $\bar{C} = \langle \delta : \delta \in S\rangle$ we may omit it writing S - colouring

(2B) In part (2) we can replace $\mathbf{F}$ by $\langle F_\delta : \delta \in S\rangle$ where $F_\delta : {}^{(C_\delta)}\delta \to 2$ such that $\eta \in {}^\delta\delta \wedge \delta \in S \to \mathbf{F}(\eta) = F_\delta(\eta \restriction C_\delta)$. So abusing notation we may write $\mathbf{F}(\eta \restriction C_\delta)$

(3) Assume $\mathbf{F}$ is a $\bar{C}$-clouring, $\bar{C}$ a λ-Wd-parameter.

We say $\bar{c} \in {}^S 2$ (or $\bar{c} \in {}^\lambda 2$) is an $\mathbf{F}$-wd-sequence if :

(*) for every $\eta \in {}^\lambda\lambda$, the set $\{\delta \in S : \mathbf{F}(\eta \restriction \delta) = c_\delta\}$ is a stationary subset of λ.

We also may say $\bar{c}$ is an $(\mathbf{F}, S)$-Wd-sequence.

(3A) We say $\bar{c} \in {}^S 2$ is a $D - \mathbf{F}$-Wd-sequence <u>if</u> D is a filter on λ to which S belongs and

(*)for every $\eta \in {}^\lambda\lambda$ we have

$$\{\delta \in S : \mathbf{F}(\eta \restriction \delta) = c_\delta\} \neq \emptyset \bmod D$$

(4) We say $\bar{C}$ is a good λ-Wd-parameter, <u>if</u> for every $\alpha < \lambda$ we have $\lambda > |\{C_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in C_\delta\}|$.

Similarly to 1.2 we have

Claim 1.6. *Assume*

(a) $\bar{C}$ *is a good* (λ, κ, χ)*-Wd-parameter.*
(b) $|\alpha|^{\langle \mathrm{tr}, \kappa \rangle} < \lambda$ *for every* $\alpha < \lambda$.
(c) $\lambda = 2^\mu$ *and* $\lambda = \lambda^{<\chi}$
(d) $\mathbf{F}$ *is a* $\bar{C}$*- colouring.*

Then there is a $\mathbf{F}$*-Wd-sequence.*

Proof. Let cd be a 1-to-1 function from ${}^\mu\lambda$ onto λ, for simplicity, and without loss of generality

$$\alpha = \mathrm{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu \rangle) \Rightarrow \alpha \geq \sup\{\alpha_\varepsilon : \varepsilon < \mu\}$$

and let the function $\mathrm{cd}_i : \lambda \to \lambda$ for $i < \mu$ be such that $\mathrm{cd}_i(\langle \mathrm{cd}(\alpha_\varepsilon : \varepsilon < \mu) \rangle) = \alpha_i$.

Let $T = \{\eta :$ for some $C \subseteq \lambda$ of cardinality $< \chi$, we have $\eta \in {}^C\lambda\}$, so by assumption (c) clearly $|T| = \lambda$, so let us list T as $\{\eta_\alpha : \alpha < \lambda\}$ with no repetitions, and let $T_{<\alpha} = \{\eta_\beta : \beta < \alpha\}$. For $\delta \in S$ let $\mathcal{P}_\delta = \{\eta : \eta$ a function from C_δ to δ such that for every $\alpha \in C_\delta$ we have $\eta \restriction (C_\delta \cap \alpha) \in T_{<\delta}$.

By $\bar{C}$ being good and clause (b) of the assumption necessarily $\mathcal{P}_\delta$ has cardinality $< \lambda$. For each $\eta \in \mathcal{P}_\delta$ and $\varepsilon < \mu$ we define $\nu_{\eta,\varepsilon} \in {}^{C_\delta}\delta$ by $\nu_{\eta,\varepsilon}(\alpha) = \mathrm{cd}_\varepsilon(\eta(\alpha))$ for $\alpha \in C_\delta$. Now for $\eta \in \mathcal{P}_\delta$, clearly $\rho_\eta =: \langle \mathbf{F}(\nu_{\eta,\varepsilon}) : \varepsilon < \mu \rangle$ belongs to ${}^\mu 2$. Clearly $\{\rho_\eta : \eta \in \mathcal{P}_\delta\}$ is a subset of ${}^\mu 2$ of cardinality $\leq |\mathcal{P}_\delta|$ which as said above is $< \lambda$. But $|{}^\mu 2| = 2^\mu = \lambda$ by clause (c) of the assumption, so we can find $\rho^*_\delta \in {}^\mu 2 \setminus \{\rho_\eta : \eta \in \mathcal{P}_\delta\}$.

For each $\varepsilon < \mu$ we can consider the sequence $\bar{c}^\varepsilon = \langle 1 - \rho^*_\delta(\varepsilon) : \delta \in S \rangle$ as a candidate for being an $\mathbf{F}$ -Wd-sequence. If one of then is, we are done. So assume toward contradiction that for each $\varepsilon < \mu$ there is $\eta_\varepsilon \in {}^\lambda\lambda$ which exemplify its failure, so there is a club E_ε of λ such that

$\boxtimes_1$ $\delta \in S \cap E_\varepsilon \Rightarrow \mathbf{F}(\eta_\varepsilon \restriction C_\delta) \neq c^\varepsilon_\delta$
and without loss of generality

$\boxtimes_2$ $\alpha < \delta \in E_\varepsilon \Rightarrow \eta_\varepsilon(\alpha) < \delta$.
But $c^\varepsilon_\delta = 1 - \rho^*_\delta(\varepsilon)$ and so $z \in \{0,1\} \,\&\, z \neq c^\varepsilon_\delta \Rightarrow z = \rho^*_\delta(\varepsilon)$ hence we have got

$\boxtimes_3$ $\delta \in S \cap E \Rightarrow \mathbf{F}(\eta_\varepsilon \restriction C_\delta) = \rho^*_\delta(\varepsilon)$

Define $\eta^* \in {}^\lambda\lambda$ by $\eta^*(\alpha) = \mathrm{cd}(\langle \eta_\varepsilon(\alpha) : \varepsilon < \mu \rangle)$, now as λ is regular uncountable clearly $E =: \{\delta < \lambda :$ for every $\alpha < \delta$ we have $\eta^*(\alpha) < \delta$ and if $\delta' \in S, C' = C_{\delta'} \cap \alpha \& \alpha \in C_{\delta'}$ then $\eta^* \restriction C' \in T_{<\delta}\}$ is a club of λ (see the choice of $T, T_{<\delta}$, recall that by assumption (a) the sequence $\bar{C}$ is good, see Definition 1.5(4)).

Clearly $E^* = \cap\{E_\varepsilon : \varepsilon < \mu\} \cap E$ is a club of λ. Now for each $\delta \in E^* \cap S$, clearly $\eta^* \restriction C_\delta \in \mathcal{P}_\delta$; just check the definitions of $\mathcal{P}_\delta$ and E, E^*. Now recall $\nu_{\eta^* \restriction C_\delta, \varepsilon}$ is the function from C_δ to δ defined by

$$\nu_{\eta^* \restriction C_\delta, \varepsilon}(\alpha) = \mathrm{cd}_\varepsilon(\eta^*(\alpha)).$$

But by our choice of η^* clearly $\mathrm{cd}_\varepsilon(\alpha)) = \eta_\varepsilon(\alpha)$, so

$$\alpha \in C_\delta \Rightarrow \nu_{\eta^* \restriction C_\delta, \varepsilon}(\alpha) = \eta_\varepsilon(\alpha) \quad \text{so} \quad \nu_{\eta^* \restriction C_\delta, \varepsilon} = \eta_\varepsilon \restriction C_\delta,$$

Hence $\mathbf{F}(\nu_{\eta^* \restriction C_{\delta,\varepsilon}}) = \mathbf{F}(\eta_\varepsilon \restriction C_\delta)$, however as $\delta \in E^* \subseteq E_\varepsilon$ clearly $\mathbf{F}(\eta_\varepsilon \restriction C_\delta) = \rho^*_\delta(\varepsilon)$, together $\mathbf{F}(\nu_{\eta^* \restriction C_\delta, \varepsilon}) = \rho^*_\delta(\varepsilon)$.

As $\eta^* \restriction C_\delta \in \mathcal{P}_\delta$ clearly $\rho_{\eta^* \restriction C_\delta} \in {}^\mu 2$, moreover for each $\varepsilon < \mu$ we know that $\rho_{\eta^* \restriction C_\delta}(\varepsilon)$, see its definition above, is equal to $\mathbf{F}(\nu_{\eta^* \restriction C_\delta, \varepsilon})$ which by the previous sentence is equal to $\rho^*_\delta(\varepsilon)$. As this holds for every $\varepsilon < \mu$ and $\rho_{\eta^* \restriction C_\delta}, \rho^*_\delta$ are members of ${}^\mu 2$, clearly they are equal. But $\eta^* \restriction C_\delta \in \mathcal{P}_\delta$ so $\rho_{\eta^* \restriction C_\delta} \in \{\rho_\eta : \eta \in \mathcal{P}_\delta\}$ whereas ρ^*_δ has been chosen outside this set, contradiction. □

Well, are there good $(\lambda, \kappa, \kappa)$-parameters? (on $I[\lambda]$ see [She93, §1]).

Claim 1.7. (1) *If S is a stationary subset of the regular cardinal λ and $S \in I[\lambda]$ and $(\forall \delta \in S)\mathrm{cf}(\delta) = \kappa$ then for some club E of λ, there is a good $(S \cap E, \kappa, \kappa)$-parameter.*

(2) *If $\kappa = \mathrm{cf}(\kappa), \kappa^+ < \lambda = \mathrm{cf}(\lambda)$ then there is a stationary $S \in I[\lambda]$ with $(\forall \delta \in S)[\mathrm{cf}(\delta) = \kappa]$.*

Proof. (1) By the definition of $I[\lambda]$

(2) By [She93, §1].

□

We can note

Claim 1.8. (1) *Assume the assumption of 1.6 or 1.2 with $C_\delta = \delta$ and D is a μ^+- complete filter on $\lambda, S \in D$, and D include the club filter. Then we can get that there is a $D - \mathbf{F}$-Wd-sequence.*

(2) *In 1.6, we can weaken the demand $\lambda = 2^\mu$ to $\lambda = \mathrm{cf}(2^\mu)$ that is, assume*

(a) *$\bar{C}$ is a good (λ, κ, χ)-Wd-parameter.*

(b) *$|\alpha|^{\langle \mathrm{tr}, \kappa \rangle} < 2^\mu$ for every $\alpha < \lambda$.*

(c) *$\lambda = \mathrm{cf}(2^\mu)$ and $2^\mu = (2^\mu)^{<\chi}$*

(d) *$\mathbf{F}$ is a $\bar{C}$-colouring*

(e) *D is a μ^+-complete filter on λ extending the club filter to which $\mathrm{Dom}(\bar{C})$ belongs.*

Then[1] *there is a $D - \mathbf{F}$-Wd-sequence.*

(3) *In 1.6+1.8(2) we can omit "λ regular".*

Proof. (1) The same proof.

(2) Let $H^* : \lambda \to 2^\mu$ be increasing continuous with unbounded range and let $S \in I[\lambda]$ be stationary, such that $(\forall \delta \in S)\mathrm{cf}(\delta) = \kappa$, and $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a good $(\mathrm{cf}(\lambda), \kappa, \kappa)$-Wd- parameter, let

$$S' = \{h^*(\alpha) : \alpha \in S\}, \; C'_{h^*(\delta)} = \{h^*(\alpha) : \alpha \in C_\delta\}, \bar{C}' = \langle C_\beta : \beta \in S' \rangle$$

[1] in fact if $\lambda = \mathrm{cf}(2^\mu) < 2^\mu$ then the demand "$\bar{C}$ is good" is not necessary; see more in [She05]

and repeat the proof using $\lambda' = 2^\mu, \bar{C}' = \langle C'_\delta : \delta \in S' \rangle$ instead $\lambda, \bar{C}$. Except that in the choice of the club E we should use $E' = \{\delta < \lambda$: for every $\alpha \in \delta \cap$ Rang (h^*) we have $\eta^*(\alpha) < \delta$ and δ is a limit ordinal and $\delta' \in S' \wedge C' = C'_\delta \cap \alpha \Rightarrow \eta^* \restriction C' \in T_{<\delta}\}$.

(3) Similarly.

□

This lead to considering the natural related ideal.

Definition 1.9. Let $\bar{C}$ be a (λ, κ, χ)- parameter.

(1) For a family $\mathcal{F}$ of $\bar{C}$-colouring and $\mathcal{P} \subseteq {}^\lambda 2$, let $\mathrm{id}_{\bar{C},\mathcal{F},\mathcal{P}}$ be
$\{W \subseteq \lambda :$ for some $\mathbf{F} \in \mathcal{F}$ for every $\bar{c} \in \mathcal{P}$ for some $\eta \in {}^\lambda\lambda$ the set
$$\{\delta \in W \cap S : \mathbf{F}(\eta \restriction C_\delta) = c_\delta\} \quad \text{is not stationary}\}.$$

(2) If $\mathcal{P}$ is the family of all $\bar{C}$- colouring we may omit it. If we write Def instead $\mathcal{F}$ this mean as in [She01, §1].

We can strengthen 1.6 as follows.

Definition 1.10. We say the λ-colouring $\mathbf{F}$ is (S, χ)- good if:

(a) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) < \chi\}$ is stationary
(b) we can find E and $\langle C_\delta : \delta \in S \cap E \rangle$ such that
  (α) E a club of λ.
  (β) C_δ is an unbounded subset of $\delta, |C_\delta| < \chi$.
  (γ) if $\rho, \rho' \in {}^\delta\delta$, $\delta \in S \cap E$, and $\rho' \restriction C_\delta = f \restriction C_\delta$ then $\mathbf{F}(\rho') = \mathbf{F}(\rho)$
  (δ) for every $\alpha < \lambda$ we have
  $$\lambda > |\{C_\delta \cap \alpha : \delta \in S \cap E \quad \text{and} \quad \alpha \in C_\alpha\}|$$
  (ϵ) $\delta \in S \Rightarrow |\delta_{\mathrm{tr}}^{\langle \mathrm{cf}(\delta) \rangle}]$:

Claim 1.11. *Assume*

(a) $\lambda = \mathrm{cf}(2^\mu)$
(b) $\mathbf{F}$ *is an* (S, κ)*- good* λ*-colouring.*

Then there is a $(\mathbf{F}, S)$*-Wd-sequence, see Definition 1.5(3).*

Remark 1.12. So if $\lambda = \mathrm{cf}(2^\mu)$ and we let $\Theta_\lambda =: \{\theta = \mathrm{cf}(\theta)$ and $(\forall \alpha < \lambda)(|\alpha|^{\langle \mathrm{tr}, \theta \rangle} < \lambda)\}$ then

(a) Θ_λ "large" (e.g. contains every large enough $\theta \in \mathrm{Reg} \cap \beth_\omega$ if $\beth_\omega < \lambda$) and
(b) if $\theta = \mathrm{cf}(\theta) \wedge \theta^+ < \lambda$ then there is a stationary $S \in I[\lambda]$ such that $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \theta$.
(c) if $\theta \in \Theta, S$ are as above then there is a good $\langle C_\delta : \delta \in S \rangle$
(d) for $\theta, S, \bar{C}$ as above, if $\mathbf{F} = \langle F_\delta : \delta \in S \rangle$ and $F_\delta(\eta)$ depend just on $\eta \restriction C_\delta$ and D is a normal ultrafilter on λ (or less), and lastly $S \in D$ then there is an $D - \mathbf{F}$-Wd-sequence; see Definition 1.5(3A).

2. On versions of precipitousness

Definition 2.1. (1) We say the D is $(\mathbb{P}, \underset{\sim}{D})$ -precipituous if
(a) D is a normal filter on λ, a regular uncountable cardinal.
(b) $\mathbb{P}$ is forcing notion with $\emptyset_{\mathbb{P}}$ minimal.
(c) $\underset{\sim}{D}$ a $\mathbb{P}$-name of an ultrafilter of the Boolean Algebra $\mathcal{P}(\lambda)$
(d) letting for $p \in \mathbb{P}$

$$D_{p,\underset{\sim}{D}} =: \{A \subseteq \lambda : p \Vdash A \in \underset{\sim}{D}\}$$

we have:
(α) $D_{\emptyset_{\mathbb{P}},\underset{\sim}{D}} = D$ and
(β) $D_{p,\underset{\sim}{D}}$ is normal filter on λ
(e) $\Vdash_{\mathbb{P}}$ "$\mathbf{V}^{\lambda}/\underset{\sim}{D}$ is well founded".

(2) For λ regular uncountable and D a normal filter on λ let $\mathrm{NOR}_D = \{D' : D'$ a normal filter on λ extending $D\}$ ordered by inclusion and $\underset{\sim}{D} = \cup\{D' : D' \in \underset{\sim}{G}_{\mathrm{NOR}_D}\}$

Woodin [W99] defined and was interested in semi-saturation for $\lambda = \aleph_2$, where!.

(1A) If $\underset{\sim}{D}$ is clear from the context (as in part (2)) we may omit $\underset{\sim}{D}$.

Definition 2.2. For λ regular uncountable cardinal, a normal filter D on λ is called semi-saturated when for every forcing notion $\mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{D}$ of a normal (for regressive $f \in \mathbf{V}$) ultrafilter on $\mathcal{P}(\lambda)^{\mathbf{V}}$, we have: D is $(\mathbb{P}, \underset{\sim}{D})$-precipitous.

Woodin proved Con($D_{\omega_2} \restriction S^2_0$ is semi saturated), he proved that the existence of such filter has large consistency strength by proving 2.3 below. This is related to [She94, V].

Claim 2.3. *If $\lambda = \mu^+$, D a semi-saturated filter or λ, <u>then</u> every $f \in {}^\lambda\lambda$ is $<_D$- than the α-th function for some $\alpha < \lambda^+$ (on the α-th function see e.g* [She94, XVII, §3]*)*

In fact

Claim 2.4. *If $\lambda = \mu^+$ and D is NOR_λ-precipitous <u>then</u> every $f \in {}^\lambda\lambda$ is $<_D$- smaller than the α-th function for some $\alpha < \lambda^+$*

Proof. The point is that
(a) if D is a normal filter on $\lambda, \langle f_\alpha : \alpha < \lambda^+\rangle$ is ${}^<D$ -increasing in λ and $f \in {}^\lambda\lambda, \alpha < \lambda^+ \Rightarrow \neg(f \leq_D f_\alpha)$ <u>then</u> there is a normal filter D_1 on λ extending D such that $\alpha < \lambda^+ \Rightarrow f_\alpha <_{D_1} f$
(b) if $\langle f_\alpha : \alpha \leq \lambda^+\rangle$ is $<_D$- increasing $f_\alpha \in {}^\lambda\lambda$, and $\lambda = \mu^+$ and $X = \{\delta < \lambda : \mathrm{cf}(f_{\lambda^+}(\delta)) = \theta\} \neq \emptyset \mod D$ then there are functions $g_i \in {}^\lambda\lambda$ for $i < \theta$ such that $g_i < f_{\lambda^+} \mod (D + X)$, and $(\forall\alpha < \lambda^+)(\exists i < \theta)(\neg g_i <_D f_\alpha)$.

[In detials let $\Gamma = \{(D_1, f, \alpha) : D_1 \in \mathrm{NOR}_\lambda, f \in, {}^\lambda\lambda, D_1 \Vdash_{\mathrm{NOR}_\lambda}$ "$f/\underset{\sim}{D}$ is the α-the ordinal in $\mathbf{V}^\lambda/\underset{\sim}{D}$ and $\neg f \leq f_\alpha \mod D_1$ for $\alpha < \lambda^+$, for some

$f_\alpha \in {}^\frown\lambda :< D_1$- increasing with $\alpha\}$. If the conclusion fails then $\Gamma \neq 0$, choose $(D_1, f, \alpha) \in \Gamma$ with α minimal and by clause (a) without loss of generality $\alpha < \lambda^+ \Rightarrow f_\alpha < f \bmod D_1$. By (b) there is $g < f \bmod D_1$ such that $\alpha < \lambda^+ \Rightarrow \neg(g < f_\alpha \bmod D_1)$, without loss of generality $\alpha < \lambda^+ \Rightarrow f_\alpha < g \bmod D_1$ and for some $\beta < \alpha$ and $D_2 \in \mathrm{NOR}_\lambda$ extending $D_1, D_2 \Vdash_{\mathrm{NOR}_\lambda}$" $g/\underset{\sim}{D}$ is the β the ordinal of $\mathbf{V}^\lambda/\underset{\sim}{D}$, contradiction to the minimality of λ] $\square$

Claim 2.5. *(1) If $\lambda = \mu^+ \geq \beth_\omega$ <u>then</u> the club filter on λ is not semi-saturated.*

(2) If $\lambda = \mu^+ \geq \beth_\omega$ <u>then</u> for every large enough regular $\kappa < \beth_\omega$, there is no semi-saturated normal filter D^* on λ to which $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ belongs.

(3) If $2^{2^\kappa} < \lambda\lambda = \mu^+ > \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and for every $f \in {}^\kappa\lambda$ we have $\mathrm{rk}_{J^{bd}_\kappa}(f) < \lambda$ <u>then</u> there is no semi-saturated normal filter D^* on λ to which $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ belongs.

(4) In 1), 2), 3), if "D is Nor_D-semi-saturated" <u>then</u> the conclusion holds for D.

Remark: We can replace $\beth_\omega$ by any strong limit uncountable cardinal.

Proof. (1) Follows by (2)

(2) By [She00b] for some $\kappa_0 < \beth_\omega$, for every regular $\kappa \in (\kappa_0, \beth_\omega)$ we have: $\mu^{\langle\kappa\rangle} = \mu$, see 1.3. Let $D = \{A \subseteq \kappa : \sup(\kappa \setminus A) < \kappa\}$.

By part (3) it is enough to prove

⊠ if $f \in {}^\kappa\lambda$ then $\mathrm{rk}_D(f) < \lambda$

<u>proof of ⊠</u> If not then for every $\alpha < \lambda$ there is

$$f_\alpha \in {}^\kappa\lambda \quad \text{such that} \quad f_\alpha <_D f \quad \text{and} \quad \mathrm{rk}_D(f) = \alpha$$

and define

$$D_\alpha =: \{A \subseteq \kappa :\ A \in D \quad \underline{\text{or}} \quad \kappa \setminus A \notin D, \text{and} \quad \mathrm{rk}_{D+(\kappa\setminus A)}(f_\alpha) < \alpha\}.$$

This is a κ-complete filter on κ see [She00a]. So for some D^* the set $A = \{\alpha : D_\alpha = D^*\}$ is unbounded in λ. By [She00a, §4] (alternatively use [She94, V] on normal filters)

(*) for $\alpha < \beta$ from A, $f_\alpha <_{D^*} f_\beta$ and D^* is a κ-complete filter on κ.

But as $\mu = \mu^{\langle\kappa\rangle}$ letting $\alpha^* = \sup(\mathrm{Rang}(f)) + 1$ which is $< \lambda$, so $|\alpha^*| \leq \mu$, there is a family $\mathcal{P} \subseteq [\alpha^*]^\kappa$ such that for every $a \in [\alpha^*]^\kappa$, for some $i(*) < \kappa$ and $a_i \in \mathcal{P}$ for $i < i(*)$ we have $a \subseteq \bigcup_{i<i(*)} a_i$ hence for every $\alpha \in A$, for some $a_\alpha \in \mathcal{P}$ we have

$$\{i < \kappa : f_\alpha(i) \in a_\alpha\} \neq \emptyset \bmod D^*.$$

So for some a^* and unbounded $B \subseteq A$ we have $\alpha \in B \Rightarrow a_\alpha = a^*$ and moreover for some $b^* \subseteq \kappa$ we have $\alpha \in B \Rightarrow b^* = \{i < \kappa : f_\alpha(i) \in a^*\}$ and moreover $\alpha \in B \Rightarrow f_\alpha \restriction b^* = f^*$. But this contradict (*).

(3) We can find $\langle u_{\alpha,\varepsilon} : \varepsilon < \lambda, \alpha < \lambda^+ \rangle$ such that:
  (a) $\langle u_\alpha, \varepsilon : \varepsilon < \lambda \rangle$ is $\subseteq$-increasing continuous such that $|u_{\alpha,\varepsilon}| < \lambda$, and $\cup \{u_{\alpha,\varepsilon} : \varepsilon < \lambda\} = \alpha$.
  (b) if $\alpha < \beta < \lambda^+$ and $\alpha \in u_{\beta,\varepsilon}$ then $u_{\beta,\varepsilon} \cap \alpha = u_{\alpha,\varepsilon}$.

Let $f_\alpha \in {}^\lambda\lambda$ be $f_\alpha(\varepsilon) = \mathrm{otp}(u_{\alpha,\varepsilon})$, so it is well known that f_α / D_λ is the α-th function, in particular $\alpha < \beta \Rightarrow f_\alpha <_{D_\lambda} f_\beta$ where D_λ is the club filter on λ; in fact $\alpha < \beta < \lambda^+ \Rightarrow f_\alpha <_{J^{bd}_\lambda} f_\beta$. Choose[2] $\bar{C} = \langle C_\delta : \delta \in S^\lambda_\kappa \rangle$, C_δ a club of δ of order type κ, and let $g_\delta \in {}^\kappa\delta$ enumerate C_δ, i.e. $g_\delta(i)$ is the i-th member of C_δ

For $\zeta < \lambda$ let $g^*_\zeta \in {}^\kappa\lambda$ be constantly ζ, and let $g^* \in {}^\lambda\lambda$ be defined by $g^*(\zeta) = \mathrm{rk}_{J^{bd}_\kappa}(g^*_\zeta)$

$(*)_0$ $g^* \in {}^\lambda\lambda$ and $\zeta \le g^*(\zeta)$
  [why? by an assumption]
  For $\alpha < \lambda^+$ we define $f^*_\alpha \in {}^\lambda\lambda$ by:

$$f^*_\alpha(\varepsilon) = \begin{cases} \mathrm{rk}_{J^{bd}_\kappa}(f_\alpha \circ g_\epsilon) & \text{if} \quad \varepsilon \in S^\lambda_\kappa \\ 0 & \text{if} \quad \varepsilon \in \lambda \setminus S^\lambda_\kappa \end{cases}$$

  Note that $f_\alpha \circ g_\delta$ is a function from κ to λ.
  Now

$(*)_1$ $f^*_\alpha \in {}^\lambda\lambda$ for $\alpha < \lambda^+$
  [Why? as $f_\alpha \circ g_\delta \in {}^\kappa\lambda$, so by a hypothesis $\mathrm{rk}_{J^{bd}_\kappa}(f_\alpha \circ g_\delta) < \lambda$]

$(*)_2$ for $\alpha < \lambda^+$

$$(*)^2_\alpha \; E_\alpha = \{\delta < \lambda : \text{if} \quad \varepsilon < \delta \quad \text{then} \quad f^*_\alpha(\varepsilon) < \delta\}$$

  is a club of λ
  [Why? Obvious]

$(*)_3$ for $\alpha < \lambda^+$ we have

$$\delta \in E_\alpha \Rightarrow f^*_\alpha(\delta) < g^*(\delta), \text{ so} \quad f^*_\alpha <_{D_\lambda} g^* \in {}^\lambda\lambda$$

  [Why? the first statement by the definition of E_α, of f^*_α and of $g^*(\delta)$. The second by the first $(*)_0$.]

$(*)_4$ if $\alpha < \beta < \lambda^+$ <u>then</u> $f^*_\alpha <_{J^{bd}_\lambda} f^*_\beta$ hence $f^*_\alpha <_{D_\lambda} f^*_\beta$
  [Why? the first as $f_\alpha <_{J^{bd}_\lambda} f_\beta$ hence for some $\varepsilon < \lambda$, we have

$$\varepsilon < \zeta < \lambda \to f_\alpha(\zeta) < f_\beta(\zeta) \text{ hence} \quad \delta \in S^\lambda_\kappa \setminus (\varepsilon + 1) \Rightarrow$$

$$f_\alpha \restriction C_\delta <_{J^{bd}_{C_\delta}} f_\beta \restriction C_\delta \Rightarrow f_\alpha \circ g_\delta <_{J^{bd}_\kappa} f_\beta \circ g_\delta \Rightarrow rk_{J^{bd}_\kappa}(f_\delta \circ g_\delta) < rk_{J^{bd}_\kappa}(f_\beta \circ g_\delta) \Rightarrow$$

$$f^*_\alpha(\delta) < f^*_\beta(\delta)$$

Let $f^*_{\lambda^+} =: g^*$, so

$$(*) \, \alpha \le \lambda^+ \Rightarrow f^*_\alpha \in {}^\lambda\lambda \quad \text{and} \quad \alpha < \beta \le \lambda^+ \Rightarrow f_\alpha <_{D_\lambda} f_\beta$$

This of course suffices by **??**.

[2] recall $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$

(4) The same proof.

□

REMARK: In the proof of 2.5(2) it is enough that $\mathbf{U}_{J^{bd}_\kappa}(\mu) = \mu$ (see [She00a]).

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