Evasion and prediction IV
Strong forms of constant prediction

Jörg Brendle*
The Graduate School of Science and Technology
Kobe University
Rokko–dai 1–1, Nada–ku
Kobe 657–8501, Japan
email: brendle@kurt.cla.kobe-u.ac.jp

Saharon Shelah†
Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903, USA

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Abstract

Say that a function $\pi : n^{<\omega} \to n$ (henceforth called a predictor) k-constantly predicts a real $x \in n^\omega$ if for almost all intervals I of length k, there is $i \in I$ such that $x(i) = \pi(x{\restriction}i)$. We study the k-constant

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prediction number $\mathfrak{v}_n^{\mathrm{const}}(k)$, that is, the size of the least family of predictors needed to k-constantly predict all reals, for different values of n and k, and investigate their relationship.

Introduction

This work is about evasion and prediction, a combinatorial concept originally introduced by Blass when studying set–theoretic aspects of the Specker phenomenon in abelian group theory [Bl1]. The motivation for our investigation came from a (still open) question of Kamo, as well as from an argument in a proof by the first author. Let us explain this in some detail.

For our purposes, let $n \leq \omega$ and call a function $\pi : n^{<\omega} \to n$ a *predictor*. Say π *k-constantly predicts* a real $x \in n^\omega$ if for almost all intervals I of length k, there is $i \in I$ such that $x(i) = \pi(x{\restriction}i)$. In case π k-constantly predicts x for some k, say that π *constantly predicts* x. The *constant prediction number* $\mathfrak{v}_n^{\mathrm{const}}$, introduced by Kamo in [Ka1], is the smallest size of a set of predictors Π such that every $x \in n^\omega$ is constantly predicted by some $\pi \in \Pi$. Kamo [Ka1] showed that $\mathfrak{v}_\omega^{\mathrm{const}}$ may be larger than all the $\mathfrak{v}_n^{\mathrm{const}}$ where $n \in \omega$. He asked

Question. (Kamo [Ka2]) *Is $\mathfrak{v}_2^{\mathrm{const}} = \mathfrak{v}_n^{\mathrm{const}}$ for all $n \in \omega$.*

Some time ago, the first author answered another question of Kamo's by showing that $\mathfrak{b} \leq \mathfrak{v}_2^{\mathrm{const}}$ where $\mathfrak{b}$ is the unbounding number [Br]. Now, the standard approach to such a result would have been to show that, given a model M of ZFC such that there is a dominating real f over M, there must be a real which is not constantly predicted by any predictor from M. This, however, is far from being true. In fact, one needs a sequence of $2^k - 1$ models M_i and dominating reals f_i over M_i belonging to M_{i+1} to be able to construct a real which is not k-constantly predicted by any predictor from M_0, and this result is optimal (see [Br] for details). This means k-constant prediction gets easier in a strong sense the larger k gets, and one can expect interesting results when investigating the cardinal invariants which can be distilled out of this phenomenon.

Accordingly, let us define the *k-constant prediction number* $\mathfrak{v}_n^{\mathrm{const}}(k)$ to be the size of the smallest set of predictors Π such that every $x \in n^\omega$ is k-constantly predicted by some $\pi \in \Pi$. Interestingly enough, Kamo's question cited above has a positive answer when relativized to the new situation. Namely, we shall show in Section 1 that $\mathfrak{v}_2^{\mathrm{const}}(k) = \mathfrak{v}_n^{\mathrm{const}}(k)$ for all $k, n < \omega$

(see 1.4). Moreover, for $k < \ell$, one may well have $\mathfrak{v}_2^{\text{const}}(\ell) < \mathfrak{v}_2^{\text{const}}(k)$ (Theorem 2.1). Any hope to use Theorem 1.4 as an intermediate step to answer Kamo's question is dashed, however, by Theorem 2.2 which says that $\mathfrak{v}_2^{\text{const}}$ may be strictly smaller than the minimum of all $\mathfrak{v}_2^{\text{const}}(k)$'s.

In Section 3, we dualize Theorem 2.1 to a consistency result about evasion numbers and establish a connection between those and Martin's axiom for σ-k-linked partial orders (see Theorem 3.7).

We keep our notation fairly standard. For basics concerning the cardinal invariants considered here, as well as the forcing techniques, see [BJ] and [Bl2].

The results in this paper were obtained in September 2000 during and shortly after the second author's visit to Kobe. The results in Sections 1 and 2 are due to the second author. The remainder is the first author's work.

1 The ZFC–results

Temporarily say that $\pi : n^{<\omega} \to n$ *weakly k-constantly predicts* $x \in n^\omega$ if for almost all m there is $i < k$ such that $\pi(x \restriction mk + i) = x(mk + i)$. This notion is obviously weaker than k-constant prediction (and stronger than $(2k-1)$-constant prediction). It is often more convenient, however. We shall see soon that in terms of cardinal invariants the two notions are the same.

Put $G = \{\bar{g} = \langle g_i;\ i < k\rangle;\ g_i : n^k \to 2\}$.

Theorem 1.1 *There are functions $\bar{\pi} = \langle \pi^{\bar{g},j};\ (\bar{g}, j) \in G \times k\rangle \mapsto \psi_{\bar{\pi}}$ (where $\pi^{\bar{g},j} : 2^{<\omega} \to 2$ and $\psi_{\bar{\pi}} : n^{<\omega} \to n$) and $y \mapsto \langle y^{\bar{g},j};\ (\bar{g}, j) \in G \times k\rangle$ (where $y \in n^\omega$ and $y^{\bar{g},j} \in 2^\omega$) such that if $\pi^{\bar{g},j}$ weakly k-constantly predicts $y^{\bar{g},j}$ for all pairs $(\bar{g}, j)$, then $\psi_{\bar{\pi}}$ k-constantly predicts y.*

Proof. Given $y \in n^\omega$, define $y^{\bar{g},j}$ by

$$y^{\bar{g},j}(mk + i) = g_i(y \restriction [mk + j, (m + 1)k + j)).$$

Also, for $\sigma \in n^{<\omega}$, say $|\sigma| = m_0k + j$, define $\sigma^{\bar{g},j}$ by

$$\sigma^{\bar{g},j}(mk + i) = g_i(\sigma \restriction [mk + j, (m + 1)k + j))$$

for all $m < m_0$. So $|\sigma^{\bar{g},j}| = m_0k$.

Given $\bar{\pi} = \langle \pi^{\bar{g},j};\ (\bar{g},j) \in G \times k \rangle$, a sequence of predictors for the space 2^ω, and $\sigma \in n^{<\omega}$, say $|\sigma| = mk + j$, put

$$A^k_\sigma = \{\tau \supset \sigma;\ |\tau| = |\sigma| + k \text{ and } \forall \bar{g}\ \exists i\ (\tau^{\bar{g},j}(mk+i) = \pi^{\bar{g},j}(\tau^{\bar{g},j}\restriction mk+i))\}.$$

For $i < k$, define $A^i_\sigma = \{\tau \supset \sigma;\ \tau \in A^k_{\sigma\restriction|\sigma|-k+i}\}$. So, if $\tau \in A^i_\sigma$, $|\tau| = |\sigma| + i$.

Claim 1.2 *$|A^k_\sigma| < 2^k$ for all σ.*

Proof. Assume that, for some σ, we have $|A^k_\sigma| \geq 2^k$. List pairwise distinct $\{\tau_\ell;\ \ell < 2^k\} \subseteq A^k_\sigma$ and list $2^k = \{\sigma_\ell;\ \ell < 2^k\}$. Fix m and j such that $|\sigma| = mk + j$. Define $g_i(\tau_\ell\restriction[mk+j,(m+1)k+j)) = \sigma_\ell(i)$ and consider $\bar{g} = \langle g_i;\ i < k\rangle$. Then $\tau^{\bar{g},j}_\ell\restriction[mk,(m+1)k) = \sigma_\ell$. This is a contradiction to the definition of A^k_σ for it would mean $\pi^{\bar{g},j}$ cannot predict correctly all $\tau^{\bar{g},j}_\ell$ somewhere in the interval $[mk,(m+1)k)$. $\square$

For $\sigma \in n^{<\omega}$ define $\psi_{\bar{\pi}}(\sigma)$ as follows. First let $i \leq k$ be minimal such that $|A^i_\sigma| < 2^i$. Such i exists by the claim. Since $A^0_\sigma = \{\sigma\}$, we necessarily have $i \geq 1$. Then let $\psi_{\bar{\pi}}(\sigma)$ be any ℓ such that $A^{i-1}_{\sigma\hat{}\langle\ell\rangle}$ is of maximal size.

To see that this works, let $y \in n^\omega$. Let $\pi^{\bar{g},j}$ be predictors such that for all $\bar{g},j$ and almost all m, there is i such that $y^{\bar{g},j}(mk+i) = \pi^{\bar{g},j}(y^{\bar{g},j}\restriction mk+i)$. Fix m_0 such that for all $m \geq m_0$ and all $\bar{g},j$, there is i such that $y^{\bar{g},j}(mk+i) = \pi^{\bar{g},j}(y^{\bar{g},j}\restriction mk+i)$. Let $mk+j \in \omega$ with $m \geq m_0$. Thus $y\restriction mk+j+i \in A^i_{y\restriction mk+j}$ for all $i \leq k$. We need to find $i < k$ such that $\psi_{\bar{\pi}}(y\restriction mk+j+i) = y(mk+j+i)$. To this end simply note that if i is such that $\psi_{\bar{\pi}}(y\restriction mk+j+i) \neq y(mk+j+i)$, then, by definition of $\psi_{\bar{\pi}}$,

$$|A^{\ell_i-1}_{y\restriction mk+j+i+1}| \leq \frac{|A^{\ell_i}_{y\restriction mk+j+i}|}{2}$$

where ℓ_i is minimal with $|A^{\ell_i}_{y\restriction mk+j+i}| < 2^{\ell_i}$. This means in particular $|A^{\ell_i-1}_{y\restriction mk+j+i+1}| < 2^{\ell_i-1}$. A fortiori, $\ell_{i+1} \leq \ell_i - 1$. Since $\ell_0 \leq k$, this entails that if we had $\psi_{\bar{\pi}}(y\restriction mk+j+i) \neq y(mk+j+i)$ for all $i < k$, we would get $\ell_i = 0$ for some $i \leq k$. Thus $|A^0_{y\restriction mk+j+i}| < 2^0 = 1$. So $A^0_{y\restriction mk+j+i} = \emptyset$. However $y\restriction mk+j+i \in A^0_{y\restriction mk+j+i}$, a contradiction. This completes the proof of the theorem. $\square$

Define the *k-constant evasion number* $\mathfrak{e}^{\text{const}}_n(k)$ to be the dual of $\mathfrak{v}^{\text{const}}_n(k)$, namely the size of the smallest set of functions $F \subseteq n^\omega$ such that for every predictor π there is $x \in F$ which is not k-constantly predicted by π. Similarly, define the *constant evasion number* $\mathfrak{e}_n^{\rm const}$.

Let $\bar{\mathfrak{v}}_n^{\rm const}(k)$ denote the size of the least family Π of predictors $\pi : n^{<\omega} \to n$ such that every $y \in n^\omega$ is weakly k-constantly predicted by a member of Π. Dually, $\bar{\mathfrak{e}}_n^{\rm const}(k)$ is the size of the least family $F \subseteq n^\omega$ such that no predictor $\pi : n^{<\omega} \to n$ weakly k-constantly predicts all members of F. The above theorem entails

Corollary 1.3 $\mathfrak{v}_n^{\rm const}(k) \leq \bar{\mathfrak{v}}_2^{\rm const}(k)$. *Dually,* $\mathfrak{e}_n^{\rm const}(k) \geq \bar{\mathfrak{e}}_2^{\rm const}(k)$.

Proof. Let Π be a family of predictors in 2^ω weakly k-constantly predicting all functions. Put $\Psi = \{\psi_{\bar{\pi}};\ \bar{\pi} = \langle \pi^{\bar{g},j};\ (\bar{g},j) \in G \times k \rangle \in \Pi^{<\omega}\}$. By the theorem, every $y \in n^\omega$ is k-constantly predicted by a member of Ψ. This shows $\mathfrak{v}_n^{\rm const}(k) \leq \bar{\mathfrak{v}}_2^{\rm const}(k)$.

Next let $F \subseteq n^\omega$ be a family of functions such that no predictor k-constantly predicts all of F. Let $Y = \{y^{\bar{g},j};\ (\bar{g},j) \in G \times k \text{ and } y \in F\} \subseteq 2^\omega$. Assume $\pi : 2^{<\omega} \to 2$ weakly k-constantly predicts all members of Y. Then $\psi_{\bar{\pi}}$ k-constantly predicts all members of F, where we put $\bar{\pi} = \langle \pi^{\bar{g},j};\ (\bar{g},j) \in G \times k \rangle$ with $\pi^{\bar{g},j} = \pi$ for all $(\bar{g},j) \in G \times k$, a contradiction. $\square$

Since the other inequalities are trivial, we get

Theorem 1.4 $\bar{\mathfrak{v}}_n^{\rm const}(k) = \mathfrak{v}_n^{\rm const}(k) = \mathfrak{v}_2^{\rm const}(k)$ *for all* n. *Dually,* $\bar{\mathfrak{e}}_n^{\rm const}(k) = \mathfrak{e}_n^{\rm const}(k) = \mathfrak{e}_2^{\rm const}(k)$ *for all* n.

A fortiori, we also get $\min\{\mathfrak{v}_n^{\rm const}(k);\ k \in \omega\} = \min\{\mathfrak{v}_2^{\rm const}(k);\ k \in \omega\}$ and $\sup\{\mathfrak{e}_n^{\rm const}(k);\ k \in \omega\} = \sup\{\mathfrak{e}_2^{\rm const}(k);\ k \in \omega\}$ for all n.

2 Prediction and relatives of Sacks forcing

For $2 \leq k < \omega$, define k–ary Sacks forcing $\mathbb{S}^k$ to be the set of all subtrees $T \subseteq k^{<\omega}$ such that below each node $s \in T$, there is $t \supset s$ whose k immediate successor nodes $t\hat{\ }\langle i \rangle$ $(i < k)$ all belong to T. $\mathbb{S}^k$ is ordered by inclusion. Obviously $\mathbb{S}^2$ is nothing but standard Sacks forcing $\mathbb{S}$.

Iterating $\mathbb{S}^k$ ω_2 many times with countable support over a model for CH yields a model where $\mathfrak{v}_2^{\rm const}(\ell)$ is large if $2^\ell \leq k$ and small otherwise. This has been observed independently around the same time by Kada [Kd2]. However, one can get better consistency results by using large countable support products instead. The following is in the spirit of [GSh].

Theorem 2.1 *Assume CH. Let* $2 \le k_1 < ... < k_{n-1}$*. Also let* κ_i*,* $i \le n$*, be cardinals with* $\kappa_i^\omega = \kappa_i$ *and* $\kappa_n < ... < \kappa_0$*. Then there is a generic extension satisfying* $\mathfrak{v}_2^{\text{const}} = \min\{\mathfrak{v}_2^{\text{const}}(k);\ k \in \omega\} = \mathfrak{v}_2^{\text{const}}(k_{n-1}+1) = \kappa_n$*,* $\mathfrak{v}_2^{\text{const}}(k_i) = \mathfrak{v}_2^{\text{const}}(k_{i-1}+1) = \kappa_i$ *for* $0 < i < n$ *(where we put* $k_0 = 1$*) and* $\mathfrak{c} = \kappa_0$*.*

Proof. We force with the countable support product $\mathbb{P} = \prod_{\alpha<\kappa_0} \mathbb{Q}_\alpha$ where

- $\mathbb{Q}_\alpha$ is Sacks forcing $\mathbb{S}_\alpha$ for $\kappa_1 \le \alpha < \kappa_0$,
- $\mathbb{Q}_\alpha$ is 2^{k_i}–ary Sacks forcing $\mathbb{S}_\alpha^{2^{k_i}}$ for $0 < i < n$ and $\kappa_{i+1} \le \alpha < \kappa_i$, and
- $\mathbb{Q}_\alpha$ is $\mathbb{S}_\alpha^{\ell_\alpha}$ where $|\{\alpha;\ \ell = \ell_\alpha\}| = \kappa_n$ for all ℓ, for $\alpha < \kappa_n$.

By CH, $\mathbb{P}$ preserves cardinals and cofinalities. $\mathfrak{c} = \kappa_0$ is also immediate.

Note that if $X \subseteq 2^\omega$ and $|X| < \kappa_i$, then there is $A \subseteq \kappa_0$ of size $< \kappa_i$ such that $X \in V[G_A]$, the generic extension by conditions with support contained in A, i.e. via the ordering $\prod_{\alpha\in A} \mathbb{Q}_\alpha$. So there is $\alpha \in (\kappa_i \setminus \kappa_{i+1}) \setminus A$. Let $f_\alpha \in (2^{k_i})^\omega$ be the generic real added by $\mathbb{Q}_\alpha = \mathbb{S}_\alpha^{2^{k_i}}$. Using a standard bijection ϕ^{k_i} between 2^{k_i} as a set of numbers and 2^{k_i} as a set of binary sequences of length k_i, we define $x_\alpha \in 2^\omega$ by $x_\alpha(mk_i + j) = (\phi^{k_i}(f_\alpha(m)))(j)$ for $j < k_i$. Then x_α is not k_i-constantly predicted by any predictor from $V[G_A]$. This shows $\mathfrak{v}_2^{\text{const}}(k_i) \ge \kappa_i$. Similarly, given $A \subseteq \kappa_0$ of size $< \kappa_n$ such that $X \in V[G_A]$, choose $\alpha_\ell \in \kappa_n \setminus A$ such that $\ell_{\alpha_\ell} = 2^\ell$ for all ℓ, and let $f_{\alpha_\ell} \in (2^\ell)^\omega$ be the corresponding generic. Next choose a partition $\langle I_m^\ell;\ \ell, m \in \omega\rangle$ of ω into intervals with $|I_m^\ell| = \ell$, and define $x \in 2^\omega$ by $x \restriction I_m^\ell = \phi^\ell(f_{\alpha_\ell}(m))$. Then x is not constantly predicted by any predictor from $V[G_A]$, and $\mathfrak{v}_2^{\text{const}} \ge \kappa_n$ follows.

So it remains to see that $\mathfrak{v}_2^{\text{const}}(k_{i_0-1}+1) \le \kappa_{i_0}$ for $0 < i_0 \le n$. Put $\ell = k_{i_0-1}+1$. Let $\dot f$ be a $\mathbb{P}$–name for a function in 2^ω. By a standard fusion argument we can recursively construct

- a strictly increasing sequence m_j, $j \in \omega$,
- $A \subseteq \kappa_0$ countable,
- $\langle D_\alpha;\ \alpha \in A\rangle$, a partition of ω into countable sets,
- a condition $p = \langle p_\alpha;\ \alpha \in A\rangle \in \mathbb{P}$, and
- a tree $T \subseteq 2^{<\omega}$

such that

(a) if $\sigma \in T \cap 2^{m_j}$, $j \in D_\alpha$, and $\alpha \in \kappa_i \setminus \kappa_{i+1}$ $(i < n)$, then $|\{\tau \in T \cap 2^{m_{j+1}};\ \sigma \subseteq \tau\}| = 2^{k_i}$,

(b) $p \Vdash \dot{f} \in [T]$, and

(c) whenever $q \leq p$ where $q = \langle q_\beta;\ \beta \in B\rangle$ with $A \subseteq B$, $\sigma \in T \cap 2^{m_j}$, and $j \in D_\alpha$ are such that $q \Vdash \sigma \subseteq \dot{f}$, then there are $r_\alpha \leq q_\alpha$ and $\tau \in T \cap 2^{m_{j+1}}$ with $\tau \supseteq \sigma$, such that $r \Vdash \tau \subseteq \dot{f}$ where $r = \langle r_\beta;\ \beta \in B\rangle$ with $r_\beta = q_\beta$ for $\beta \neq \alpha$.

Now let $G_{\kappa_{i_0}}$ be $\prod_{\alpha < \kappa_{i_0}} \mathbb{Q}_\alpha$–generic with $p \restriction \kappa_{i_0} \in G_{\kappa_{i_0}}$. By (c) above, there is, in $V[G_{\kappa_{i_0}}]$, a tree $S \subseteq T$ such that for all $\alpha \in A \cap \kappa_{i_0}$, $j \in D_\alpha$ and $\sigma \in S \cap 2^{m_j}$, there is a unique $\tau \in S \cap 2^{m_{j+1}}$ extending σ, and such that $\dot{f}$ is forced to be a branch of S by the remainder of the forcing below p. By (a), we also have that for all $\alpha \in A \setminus \kappa_{i_0}$, $j \in D_\alpha$ and $\sigma \in S \cap 2^{m_j}$, there are at most $2^{k_{i_0 - 1}}$ many $\tau \in S \cap 2^{m_{j+1}}$ extending σ. This means we can recursively construct a predictor $\pi \in V[G_{\kappa_{i_0}}]$ which ℓ-constantly predicts all branches of S. A fortiori, $\dot{f}$ is forced to be predicted by π by the remainder of the forcing below p. On the other hand, $V[G_{\kappa_{i_0}}]$ satisfies $\mathfrak{c} = \kappa_{i_0}$ so that there are a total number of κ_{i_0} many predictors in $V[G_{\kappa_{i_0}}]$, and they ℓ-constantly predict all reals of the final extension. This completes the argument. $\square$

It is easy to see that in models obtained by such product constructions, $\mathfrak{v}_2^{\mathrm{const}} = \min\{\mathfrak{v}_2^{\mathrm{const}}(k);\ k \in \omega\}$ must always hold. To distinguish between these two cardinals, we must turn once again to a countable support iteration.

Theorem 2.2 *Assume CH. There is a generic extension satisfying* $\mathfrak{v}_2^{\mathrm{const}} = \aleph_1 < \min\{\mathfrak{v}_2^{\mathrm{const}}(k);\ k \in \omega\} = \mathfrak{c} = \aleph_2$.

Proof. Let $\langle k_\alpha;\ \alpha < \omega_2\rangle$ be a sequence of natural numbers ≥ 2 in which each k appears ω_2 often and such that in each limit ordinal, the set of α with $k_\alpha = 2$ is cofinal.

We perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \omega_2\rangle$ such that

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{S}}^{k_\alpha}, \text{ that is } k_\alpha\text{–ary Sacks forcing.”}$$

By CH, $\mathbb{P}_{\omega_2}$ preserves cardinals and cofinalities. As in the previous proof, we see $\mathfrak{v}_2^{\mathrm{const}}(k) = \mathfrak{c} = \aleph_2$ for all k. We are left with showing that $\mathfrak{v}_2^{\mathrm{const}} = \aleph_1$.

For $\ell \geq 2$, $p \in \mathbb{S}^\ell$ and $s \in p$, say s is a *splitting node* of p if all ℓ immediate successor nodes of s belong to p. Define recursively the *n–th splitting level* of p such that the 0–th splitting level consists of the least splitting node and the $(n+1)$–st splitting level consists of the least splitting nodes beyond the n–th splitting level. For $p, q \in \mathbb{S}^\ell$, say $q \leq_n p$ if $q \leq p$ and the n–th splitting levels of p and q are the same.

Let $\dot{f}$ be a $\mathbb{P}_{\omega_2}$–name for a function in 2^ω. Notice given any $p_0 \in \mathbb{P}_{\omega_2}$, we can find $p \leq p_0$ and $\alpha < \omega_2$ such that

$$p \Vdash \dot{f} \in V[\dot{G}_\alpha] \setminus \bigcup_{\beta<\alpha} V[\dot{G}_\beta].$$

First consider the case α is a successor ordinal, say $\alpha = \beta + 1$. Let ℓ be such that $2^\ell > k_\beta$. The following is the main point.

Main Claim 2.3 *There are $q \leq p$ and a predictor $\pi \in V$ such that*

$$q \Vdash \text{“}\pi \ \ \ell\text{-constantly predicts } \dot{f}.\text{”}$$

Proof. We construct recursively

- $A \subseteq \alpha$ countable (intended as the domain of the fusion q),
- $\langle D_\gamma;\ \gamma \in A\rangle$, a partition of ω into countable sets (its purpose being that at step j of the construction we preserve one more splitting level of the γ–th coordinate of the condition where $j \in D_\gamma$),
- finite partial functions $a_j : A \to \omega$, $j \in \omega$ (keeping track of how often the γ–th coordinate has been worked through),
- conditions $p_j \in \mathbb{P}_\alpha$, $j \in \omega$ (intended as a fusion sequence),
- a strictly increasing sequence m_j, $j \in \omega$,
- a tree $T \subseteq 2^{<\omega}$, and
- a predictor $\pi : 2^{<\omega} \to 2$

such that

(a) $\beta \in A$,

(b) $a_0 = \emptyset$,

(c) if $j \in D_\gamma$, then $\mathrm{dom}(a_{j+1}) = \mathrm{dom}(a_j) \cup \{\gamma\}$; in case $\gamma \notin \mathrm{dom}(a_j)$, we have $a_{j+1}(\gamma) = 0$, otherwise $a_{j+1}(\gamma) = a_j(\gamma) + 1$; $a_{j+1}(\delta) = a_j(\delta)$ for $\delta \neq \gamma$,

(d) $p_0 = p$,

(e) $p_{j+1} \leq p_j$; furthermore for all $\gamma \in \mathrm{dom}(a_{j+1})$, $p_{j+1}\restriction\gamma \Vdash_\gamma p_{j+1}(\gamma) \leq_{a_{j+1}(\gamma)} p_j(\gamma)$,

(f) $\bigcup_j \mathrm{dom}(p_j) = \bigcup_j \mathrm{dom}(a_j) = A$,

(g) if $\sigma \in T \cap 2^{m_j}$, $j \in D_\gamma$, then $|\{\tau \in T \cap 2^{m_{j+1}};\ \sigma \subseteq \tau\}| = k_\gamma$,

(h) for each $\sigma \in T \cap 2^{m_j}$, there is $p_j^\sigma \leq p_j$ which forces $\sigma \subseteq \dot{f}$; furthermore $p_j \Vdash \dot{f}\restriction m_j \in T \cap 2^{m_j}$, and

(i) π ℓ-constantly predicts all branches of T.

Most of this is standard. There is, however, one trick involved, and we describe the construction. For $j = 0$, there is nothing to do. So assume we arrived at stage j, and we are supposed to produce the required objects for $j + 1$. This proceeds by recursion on $\sigma \in T \cap 2^{m_j}$. Since the recursion is straightforward, we confine ourselves to describing a single step.

Fix $\sigma \in T \cap 2^{m_j}$. Let γ be such that $j \in D_\gamma$. Without loss $\gamma < \beta$ (the case $\gamma = \beta$ being easier). Consider p_j^σ. Step momentarily into $V[G_\beta]$ with $p_j^\sigma\restriction\beta \in G_\beta$. Then $p_j^\sigma(\beta) \Vdash_{\mathbb{Q}_\beta} \sigma \subseteq \dot{f}$. Since $\dot{f}$ is forced not to be in $V[G_\beta]$, we can find $m^\sigma \in \omega$, pairwise incompatible $r_i^\sigma \leq p_j^\sigma(\beta)$, and distinct $\tau_i^\sigma \in 2^{m^\sigma}$ extending σ where $i < k_\gamma$ such that $r_i^\sigma \Vdash_{\mathbb{Q}_\beta} \tau_i^\sigma \subseteq \dot{f}$. As $\mathbb{Q}_\beta$ is k_β–ary Sacks forcing, we may do this in such a way that the predictor π can be extended to ℓ-constantly predict all τ_i^σ.

Back in V, by extending the condition p_j^σ if necessary, we may without loss assume that it decides m^σ and the τ_i^σ. We therefore have the extension of π which ℓ-constantly predicts all τ_i^σ already in the ground model V. We may also suppose that $p_j^\sigma\restriction\gamma$ decides the stem of $p_j^\sigma(\gamma)$, say $p_j^\sigma\restriction\gamma \Vdash_\gamma \mathrm{stem}(p_j^\sigma(\gamma)) = t$. For $i < k_\gamma$ define $p_{j+1}^{\tau_i^\sigma}$ such that

- $p_{j+1}^{\tau_i^\sigma}\restriction\gamma = p_j^\sigma\restriction\gamma$, $p_{j+1}^{\tau_i^\sigma}\restriction[\gamma+1, \beta) = p_j^\sigma\restriction[\gamma+1, \beta)$,

- $p_{j+1}^{\tau_i^\sigma}\restriction\gamma \Vdash_\gamma p_{j+1}^{\tau_i^\sigma}(\gamma) = (p_j^\sigma(\gamma))_{t^\frown\langle i\rangle}$,
- $p_{j+1}^{\tau_i^\sigma}\restriction\beta \Vdash_\beta p_{j+1}^{\tau_i^\sigma}(\beta) = \dot{r}_i^\sigma$.

Doing this (in a recursive construction) for all $\sigma \in T \cap 2^{m_j}$ and increasing m^σ if necessary, we may assume there is m_{j+1} with $m_{j+1} = m^\sigma$ for all σ. Finally p_{j+1} is the least upper bound of all the $p_{j+1}^{\tau_i^\sigma}$.

This completes the construction. By (c), (e), and (f), the sequence of p_j's has a lower bound $q \in \mathbb{P}_\alpha$. By (d), $q \leq p$. By (h), $q \Vdash$ "$\dot{f} \in [T]$" which means that (i) entails $q \Vdash$ "$\dot{f}$ is ℓ-constantly predicted by π," as required. $\square$

Now let α be a limit ordinal. Using a similar argument and the fact that below α, $\dot{\mathbb{Q}}_\beta$ is cofinally often Sacks forcing, we see

Claim 2.4 *There are $q \leq p$ and a predictor $\pi \in V$ such that*

$$q \Vdash \text{“}\pi \ \ 2\text{-}\textit{constantly predicts } \dot{f}.\text{”}$$

This completes the proof of the theorem. $\square$

3 Evasion and fragments of MA(σ-linked)

Let $k \geq 2$. Recall that a partial order $\mathbb{P}$ is said to be *σ-k-linked* if it can be written as a countable union of sets P_n such that each P_n is k-linked, that is, any k many elements from P_n have a common extension. Clearly every σ-centered forcing is σ-k-linked for all k, and a σ-k-linked partial order is also σ-$(k-1)$-linked. Random forcing is an example of a partial order which is σ-k-linked for all k, yet not σ-centered. A partial order with the former property shall be called *σ-∞-linked* henceforth. We shall deal with partial orders which arise naturally in connection with constant prediction and which are σ-$(k-1)$-linked but not σ-k-linked for some k. Let $\mathfrak{m}$(σ-k-linked) denote the least cardinal κ such that for some σ-k-linked partial order $\mathbb{P}$, Martin's axiom MA_κ fails for $\mathbb{P}$.

Lemma 3.1 *Let $\mathbb{P}$ be σ-2^k-linked, and assume $\dot{\phi}$ is a $\mathbb{P}$–name for a function $\bigcup_i 2^{ik} \to 2^k$. Then there is a countable set Ψ of functions $\bigcup_i 2^{ik} \to 2^k$ such that whenever $g \in 2^\omega$ is such that for all $\psi \in \Psi$ there are infinitely many i with $\psi(g\restriction ik) = g\restriction[ik,(i+1)k)$, then*

$$\Vdash \text{“}\textit{there are infinitely many } i \textit{ with } \dot{\phi}(g\restriction ik) = g\restriction[ik,(i+1)k).\text{”}$$

Proof. Assume $\mathbb{P} = \bigcup_n P_n$ where each P_n is 2^k-linked. Define $\psi_n : \bigcup_i 2^{ik} \to 2^k$ such that, for each $\sigma \in 2^{ik}$, $\psi_n(\sigma)$ is a τ such that no $p \in P_n$ forces $\dot\phi(\sigma) \neq \tau$. (Such a τ clearly exists. For otherwise, for each $\tau \in 2^k$ we could find $p_\tau \in P_n$ forcing $\dot\phi(\sigma) \neq \tau$. Since P_n is 2^k-linked, the p_τ would have a common extension which would force $\dot\phi(\sigma) \notin 2^k$, a contradiction.) Let $\Psi = \{\psi_n;\ n \in \omega\}$.

Now choose $g \in 2^\omega$ such that for all $\psi \in \Psi$ there are infinitely many i with $\psi(g\restriction ik) = g\restriction[ik,(i+1)k)$. Fix i_0 and $p \in \mathbb{P}$. There is n such that $p \in P_n$. We can find $i \geq i_0$ such that $\psi_n(g\restriction ik) = g\restriction[ik,(i+1)k)$. By definition of ψ_n, there is $q \leq p$ such that $q \Vdash \dot\phi(g\restriction ik) = \psi_n(g\restriction ik)$. Thus $q \Vdash \dot\phi(g\restriction ik) = g\restriction[ik,(i+1)k)$, as required. $\square$

Lemma 3.2 *Let $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n;\ n \in \omega\rangle$ be a finite support iteration, and assume $\dot\phi$ is a $\mathbb{P}_\omega$–name for a function $\bigcup_i 2^{ik} \to 2^k$. Also assume for each n and each $\mathbb{P}_n$–name $\dot\phi_n$ for a function $\bigcup_i 2^{ik} \to 2^k$, there is a countable set Ψ_n of functions $\bigcup_i 2^{ik} \to 2^k$ such that $\forall g \in 2^\omega$, if $\forall \psi \in \Psi_n\ \exists^\infty i\ (\psi(g\restriction ik) = g\restriction[ik,(i+1)k))$, then*

$$\Vdash_n \text{“}\exists^\infty i\ (\dot\phi_n(g\restriction ik) = g\restriction[ik,(i+1)k)).\text{”}$$

Then there is a countable set Ψ of functions $\bigcup_i 2^{ik} \to 2^k$ such that $\forall g \in 2^\omega$, if $\forall \psi \in \Psi\ \exists^\infty i\ (\psi(g\restriction ik) = g\restriction[ik,(i+1)k))$, then

$$\Vdash_\omega \text{“}\exists^\infty i\ (\dot\phi(g\restriction ik) = g\restriction[ik,(i+1)k)).\text{”}$$

Proof. This is a standard argument which we leave to the reader. $\square$

Lemma 3.3 *Let $\mathbb{P}$ be a partial order of size κ, and assume $\dot\phi$ is a $\mathbb{P}$–name for a function $\bigcup_i 2^{ik} \to 2^k$. Then there is a set Ψ of size κ of functions $\bigcup_i 2^{ik} \to 2^k$ such that $\forall g \in 2^\omega$, if $\forall \psi \in \Psi\ \exists^\infty i\ (\psi(g\restriction ik) = g\restriction[ik,(i+1)k))$, then*

$$\Vdash \text{“}\exists^\infty i\ (\dot\phi(g\restriction ik) = g\restriction[ik,(i+1)k)).\text{”}$$

Proof. This is well–known and trivial. $\square$

Using the first two of these three lemmata we see that if we iterate σ-2^k-linked forcing over a model V containing a family $\mathcal{F} \subseteq 2^\omega$ such that

($\star$) for all countable sets Ψ of functions $\bigcup_i 2^{ik} \to 2^k$ there is $g \in \mathcal{F}$ such that for all $\psi \in \Psi$, $\exists^\infty i\ (\psi(g\restriction ik) = g\restriction[ik,(i+1)k))$, then $\mathcal{F}$ still satisfies $(\star)$ in the final extension. We also have

Lemma 3.4 *If $\mathcal{F}$ satisfies $(\star)$, then $\mathfrak{e}_2^{\text{const}}(k) \leq |\mathcal{F}|$.*

Proof. Simply note $\mathcal{F}$ is a witness for $\mathfrak{e}_2^{\text{const}}(k)$. For given a predictor $\pi : 2^{<\omega} \to 2$, define $\phi : \bigcup_i 2^{ik} \to 2^k$ by $\phi(\sigma) =$ the unique $\tau \in 2^k$ such that π predicts $\sigma\hat{\ }\tau$ incorrectly on the whole interval $[ik, (i+1)k)$ where $|\sigma| = ik$. If $g \in \mathcal{F}$ is such that $\exists^\infty i \; (\phi(g \restriction ik) = g \restriction [ik, (i+1)k))$, then π does not k-constantly predict g. $\square$

Let $2 \leq k$. The partial order $\mathbb{P}^k$ for adjoining a generic predictor k-constantly predicting all ground model reals is defined as follows. Conditions are triples (ℓ, σ, F) such that $\ell \in \omega$, $\sigma : 2^{<\omega} \to 2$ is a finite partial function, and $F \subseteq 2^\omega$ is finite, and such that the following requirements are met:

- $\text{dom}(\sigma) = 2^{\leq \ell}$,
- $f \restriction \ell \neq g \restriction \ell$ for all $f \neq g$ belonging to F,
- $\sigma(f \restriction \ell) = f(\ell)$ for all $f \in F$.

The order is given by: $(m, \tau, G) \leq (\ell, \sigma, F)$ if and only if $m \geq \ell$, $\tau \supseteq \sigma$, $G \supseteq F$, and for all $f \in F$ and all intervals $I \subseteq (\ell, m)$ of length k there is $i \in I$ with $\tau(f \restriction i) = f(i)$. This is a variation of a partial order originally introduced in [Br]. It has been considered as well by Kada [Kd1], who also obtained the following lemma.

Lemma 3.5 *$\mathbb{P}^k$ is σ-$(2^k - 1)$-linked.*

Proof. Simply adapt the argument from [Br, Lemma 3.2], or see [Kd1, Proposition 3.3]. $\square$

Corollary 3.6 (Kada [Kd1, Corollary 3.5]) $\mathfrak{m}(\sigma\text{-}(2^k - 1)\text{-linked}) \leq \mathfrak{e}_2^{\text{const}}(k)$.

We are ready to prove a result which is dual to Theorem 2.1.

Theorem 3.7 *Let $\langle \kappa_k ;\ 2 \leq k \in \omega \rangle$ be a sequence of uncountable regular cardinals with $\kappa_k \leq \kappa_{k+1}$. Also assume $\lambda = \lambda^{<\lambda}$ is above the κ_k. Then there is a generic extension satisfying $\mathfrak{e}_2^{\text{const}}(k) = \kappa_k$ for all k and $\mathfrak{c} = \lambda$. We may also get $\mathfrak{m}(\sigma\text{-}(2^k - 1)\text{-linked}) = \kappa_k$ for all k.*

Proof. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \lambda \rangle$ be a finite support iteration of ccc forcing such that each factor $\dot{\mathbb{Q}}_\alpha$ is forced to be a σ-$(2^k - 1)$-linked forcing notion of size less than κ_k for some $k \geq 2$. Also guarantee we take care of all such forcing notions by a book–keeping argument. Then $\mathfrak{m}(\sigma\text{-}(2^k - 1)\text{-linked}) \geq \kappa_k$ is straightforward. In view of Corollary 3.6 it suffices to prove $\mathfrak{e}_2^{\rm const}(k) \leq \kappa_k$ for all k. So fix k. Note that by stage κ_k of the iteration we have adjoined a family $\mathcal{F}$ of size κ_k satisfying $(\star)$ above with *countable* replaced by *less than* κ_k (for example, let $\mathcal{F}$ be the collection of Cohen reals added at limit stages of countable cofinality below κ_k). Show by induction on the remainder of the iteration that $\mathcal{F}$ continues to satisfy this version of $(\star)$. The limit step is taken care of by Lemma 3.2 because no new reals appear at limit steps of uncountable cofinality. For the successor step, in case $\dot{\mathbb{Q}}_\alpha$ is σ-2^ℓ-linked for some $\ell \geq k$, use Lemma 3.1, and in case it is not σ-2^k-linked (and thus of size less than κ_k), use Lemma 3.3. By Lemma 3.4, $\mathfrak{e}_2^{\rm const}(k) \leq \kappa_k$ follows. $\square$

By somewhat changing the above proof, we can dualize Kamo's $CON(\mathfrak{v}_2^{\rm const} > \mathsf{cof}(\mathcal{N}))$ (and thus answer a question of his, see [Ka2]), and reprove his result as well.

Theorem 3.8 (a) *$\mathfrak{e}_2^{\rm const} < \mathsf{add}(\mathcal{N})$ is consistent; in fact, given $\kappa < \lambda = \lambda^{<\kappa}$ regular uncountable, there is a partial order $\mathbb{P}$ forcing $\mathfrak{e}_2^{\rm const} = \kappa$ and $\mathsf{add}(\mathcal{N}) = \mathfrak{c} = \lambda$.*

(b) (Kamo, [Ka1]) *$\mathfrak{v}_2^{\rm const} > \mathsf{cof}(\mathcal{N})$ is consistent; in fact, given κ regular uncountable and $\lambda = \lambda^\omega > \kappa$, there is a partial order $\mathbb{P}$ forcing $\mathfrak{v}_2^{\rm const} = \mathfrak{c} = \lambda$ and $\mathsf{cof}(\mathcal{N}) = \kappa$.*

Proof. (a) Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \lambda \rangle$ be a finite support iteration of ccc forcing such that

- for even α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is amoeba forcing,
- for odd α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is a subforcing of some $\mathbb{P}^k$ of size less than κ.

Guarantee that we go through all such subforcings by a book–keeping argument. Then $\mathfrak{e}_2^{\rm const} \geq \kappa$ is straightforward, as is $\mathsf{add}(\mathcal{N}) = \mathfrak{c} = \lambda$. Now note that amoeba forcing is σ-∞-linked (like random forcing). Therefore we can apply Lemmata 3.1, 3.2, and 3.3 for all k simultaneously, and see that there is a family $\mathcal{F}$ of size κ which satisfies the appropriate modified version of $(\star)$ (such a family is adjoined after the first κ stages of the iteration).

(b) First add λ many Cohen reals. Then make a κ–stage finite support iteration of amoeba forcing. Again, $\mathsf{cof}(\mathcal{N}) = \kappa$ is clear. $\mathfrak{v}_2^{\mathrm{const}} = \mathfrak{c} = \lambda$ follows from Lemmata 3.1 and 3.2 using standard arguments. $\square$

One can even strengthen Theorem 3.7 in the following way. Say a partial order $\mathbb{P}$ *satisfies property* K_k if for all uncountable $X \subseteq \mathbb{P}$ there is $Y \subseteq X$ uncountable such that any k many elements from Y have a common extension. Property K_k is a weaker relative of σ-k-linkedness. Let $\mathfrak{m}(K_k)$ denote the least cardinal κ such that MA_κ fails for property K_k partial orders.

Lemma 3.9 *Assume CH. $\mathbb{P}^k$ does not have property K_{2^k}. In fact no property K_{2^k} partial order adds a predictor which k-constantly predicts all ground model reals.*

Proof. List all predictors as $\{\pi_\alpha;\ \alpha < \omega_1\}$. Choose reals $f_\alpha \in 2^\omega$ such that π_α does not k-constantly predict f_β for $\beta \geq \alpha$. Let $X = \{f_\alpha;\ \alpha < \omega_1\}$.

Let $\mathbb{P}$ have property K_{2^k}. Also let $\dot\pi$ be a $\mathbb{P}$–name for a predictor. Assume there are conditions $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash$ "$\dot\pi$ k-constantly predicts f_α from m_α onwards." Without loss $m_\alpha = m$ for all α, and any 2^k many p_α have a common extension. Let $T \subseteq 2^{<\omega}$ be the tree of initial segments of members of X. Given $\sigma \in T$ with $|\sigma| \geq m$, let $A_\sigma^k = \{\tau \in T;\ \sigma \subset \tau$ and $|\tau| = |\sigma| + k\}$. Note that if $|A_\sigma^k| < 2^k$ for all such σ, then we could construct a predictor π k-constantly predicting all of X past m as in the proof of Theorem 1.1. So there is $\sigma \in T$ with $|A_\sigma^k| = 2^k$. Find $\alpha_0, ..., \alpha_{2^k-1}$ such that $A_\sigma^k = \{f_{\alpha_i} \restriction |\sigma| + k;\ i < 2^k\}$ and notice that a common extension of the p_{α_i} forces a contradiction. $\square$

Note that some assumption is necessary for the above result for $MA_{\aleph_1}$ implies all ccc partial orders have property K_k for all k. We now get

Theorem 3.10 *Assume CH. Let $2 \leq k < \omega$. Then there is a generic extension satisfying $\mathfrak{e}_2^{\mathrm{const}}(k) = \aleph_1$ and $\mathfrak{m}(K_{2^k}) = \aleph_2$.*

Proof. Use the lemma and the folklore fact that the iteration of property K_ℓ partial orders has property K_ℓ. $\square$

Since we saw in Corollary 3.6 that $\mathfrak{e}_2^{\mathrm{const}}(k) \geq \mathfrak{m}(\sigma\text{-}(2^k-1)\text{-linked})$. one may ask, on the other hand, whether $\mathfrak{e}_2^{\mathrm{const}}(k) > \mathfrak{m}(\sigma\text{-}(2^k-1)\text{-linked})$ is consistent. This, however, is easy, for the forcing $\mathbb{P}^k$ is Suslin ccc [BJ] while it is well–known that iterating Suslin ccc forcing keeps numbers like $\mathfrak{m}(\sigma$-(2^k-1)-linked) small (it even keeps the splitting number $\mathfrak{s}$ small).

The results in this section are related to work of Blass [Bl2, Section 10]. We briefly sketch the connection. Fix $k \geq 2$. Momentarily call a function $\pi : \omega^{<\omega} \to [\omega]^k$ a *predictor*. Say that π *globally predicts* $f \in \omega^\omega$ if $f(n) \in \pi(f{\restriction}n)$ holds for almost all n. The *global evasion number* $\mathfrak{e}^{\rm gl}(k)$ is the size of the least $F \subseteq \omega^\omega$ such that for every predictor π there is $f \in F$ which is not globally predicted by π. (The concept is due to Blass [Bl2] while the notation is due to Kada [Kd1].) Then $\mathfrak{m}(\sigma\text{-}k\text{-linked}) \leq \mathfrak{e}^{\rm gl}(k) \leq \mathsf{add}(\mathcal{N})$ [Bl2]. Also, Corollary 3.6 can be improved to $\mathfrak{e}^{\rm gl}(2^k-1) \leq \mathfrak{e}^{\rm const}(k)$ [Kd2]. On the other hand, one can prove the analog of Theorem 3.7, saying that $\mathfrak{e}^{\rm gl}(k) = \mathfrak{m}(\sigma\text{-}k\text{-linked}) = \kappa_k$ is consistent (where the κ_k form an increasing sequence of regular uncountable cardinals). Furthermore, by Theorem 3.8, $\sup\{\mathfrak{e}^{\rm gl}(k);\ k \in \omega\} < \mathsf{add}(\mathcal{N})$ is consistent, and, by the previous paragraph, so is $\mathfrak{e}^{\rm gl}(k) > \mathfrak{m}(\sigma\text{-}k\text{-linked})$.

We close this section with a few questions. We have no dual result for Theorem 2.2 so far.

Question 3.11 *Is* $\mathfrak{e}_2^{\rm const} > \sup\{\mathfrak{e}_2^{\rm const}(k);\ k < \omega\}$ *consistent?*

Question 3.12 *Can* $\mathfrak{e}_2^{\rm const}$ *have countable cofinality?*

By Theorem 3.7, one of these two questions must have a positive answer. In fact, in view of the proof of Theorem 3.8, $\mathfrak{e}_2^{\rm const}$ must be

- either $\max\{\kappa_k;\ k \in \omega\}$ (in case the set has a max),
- or $\sup\{\kappa_k;\ k \in \omega\}$ or its successor (in case the set has no max)

in the model of Theorem 3.7.

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