NONREFLECTING STATIONARY SETS IN $\mathcal{P}_\kappa\lambda$

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ABSTRACT. Let κ be a regular uncountable cardinal and $\lambda \geq \kappa^+$. The principle of Stationary Reflection in $\mathcal{P}_\kappa\lambda$ has been successful in settling problems of infinitary combinatorics in the case $\kappa = \omega_1$. For $\kappa \geq \omega_2$ the principle is known to fail if λ is large enough. In this paper the principle is shown to fail for every $\lambda \geq \kappa^+$.

1. INTRODUCTION

In [6] Foreman, Magidor and Shelah introduced the following principle for a cardinal $\lambda \geq \omega_2$: If S is a stationary set in $\mathcal{P}_{\omega_1}\lambda$, $S \cap \mathcal{P}_{\omega_1}A$ is stationary in $\mathcal{P}_{\omega_1}A$ for some $\omega_1 \subset A \subset \lambda$ of size ω_1. Let us call the principle Stationary Reflection in $\mathcal{P}_{\omega_1}\lambda$. It follows from Martin's Maximum [6] and holds after a supercompact cardinal is Lévy-collapsed to ω_2 [2]. For recent applications of reflection principles for stationary sets in $\mathcal{P}_{\omega_1}\lambda$, see e.g. [3, 14, 16, 17].

What if ω_1 is replaced by a higher regular cardinal? Feng and Magidor [4] proved that Stationary Reflection in $\mathcal{P}_{\omega_2}\lambda$ fails if λ is large enough. Their argument shows in effect that Stationary Reflection in $\mathcal{P}_\kappa\lambda$ for some large enough λ implies that the club filter on κ is presaturated (see also [2]). It is known that the club filter on a successor cardinal $\geq \omega_2$ cannot be presaturated [10].

Extending the Feng–Magidor result, Foreman and Magidor [5] proved in effect that Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails if κ is regular $\geq \omega_2$ and λ is large enough. More precisely

Theorem 1. *Let κ be regular $\geq \omega_2$. Then Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda \geq 2^{\kappa^+}$.*

1991 *Mathematics Subject Classification.* 03E05.

The first author was supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. Publication 764. The second author was partially supported by Grant-in-Aid for Scientific Research (No.12640098), Ministry of Education, Science, Sports and Culture of Japan. Both authors would like to thank the referees for their careful reading of the draft and helpful suggestions for improving the presentation.

We include a proof of Theorem 1 in §4. A further example of nonreflection, which is based on PCF Theory [11] can be found in [12].

This paper shows that for $\kappa \geq \omega_2$ Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails *everywhere*:

Theorem 2. *Let κ be regular $\geq \omega_2$. Then Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda \geq \kappa^+$.*

In §3 we prove Theorem 2 in much greater generality.

2. Preliminaries

For background material we refer the reader to [7]. Throughout the paper, we use κ, λ, μ to denote an infinite cardinal. We write S^κ_λ for $\{\gamma < \lambda : \operatorname{cf}\gamma = \kappa\}$, and $[\lambda]^\mu$ for $\{x \subset \lambda : |x| = \mu\}$.

Let A be a set of ordinals. The set of limit points of A is denoted $\lim A$. It is easy to see $|\lim A| \leq |A|$. A is called σ-closed if $\gamma \in A$ for every $\gamma \in \lim A$ of cofinality ω.

Let κ be regular, $\omega_1 \leq \kappa \leq \mu \leq \lambda$ and $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$. We write $C(f)$ for $\{x \in \mathcal{P}_\kappa\lambda : f``[x]^{<\omega} \subset \mathcal{P}(x)\}$. For $x \in \mathcal{P}_\kappa\lambda$ the smallest superset of x in $C(f)$ is denoted $\operatorname{cl}_f x$. It is well-known that for every club $C \subset \mathcal{P}_\kappa\lambda$ there is $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ with $C(f) \subset C$.

Stationary Reflection in $\mathcal{P}_\kappa\lambda$ states that if S is a stationary set in $\mathcal{P}_\kappa\lambda$, $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size κ. Let S be a stationary set in $\mathcal{P}_\kappa\lambda$. S is called nonreflecting if it witnesses the failure of Stationary Reflection, i.e. $S \cap \mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for every $\kappa \subset A \subset \lambda$ of size κ. More generally S is called μ-nonreflecting if $S \cap \mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for every $\mu \subset A \subset \lambda$ of size μ. If S is a μ-nonreflecting stationary set in $\mathcal{P}_\kappa\mu^+$ and $\mu^+ \leq \lambda$, $\{x \in \mathcal{P}_\kappa\lambda : x \cap \mu^+ \in S\}$ is (easily seen to be) a μ-nonreflecting stationary set. In particular Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda \geq \kappa^+$ iff Stationary Reflection in $\mathcal{P}_\kappa\kappa^+$ fails.

3. Main Theorem

This section is devoted to the main Theorem 3 and its corollaries. We prove Theorem 3 using ideas from Nonstructure Theory [13]. Similar ideas can be found in the proof of Diamond for $\mathcal{P}_\kappa\lambda$ [10, 15].

Theorem 3. *Let κ be regular $\geq \omega_2$ and μ a cardinal $\geq \kappa$. Assume there are $\{c_\xi : \xi < \mu\} \subset \mathcal{P}_\kappa\mu$ and a stationary $T \subset \mathcal{P}_\kappa\mu$ of size μ such that if $z \in T$ and $b \in [z]^\omega$, there is $\xi \in z$ with $b \subset c_\xi$. Then $\mathcal{P}_\kappa\lambda$ has a μ-nonreflecting stationary subset for every $\lambda \geq \mu^+$.*

Proof. It suffices to give a μ-nonreflecting stationary set in $\mathcal{P}_\kappa\mu^+$.

Let $\{c_\xi : \xi < \mu\}$ and T be as above. By Solovay's theorem we have a partition of $S^\omega_{\mu^+}$ into μ disjoint stationary sets $\{S_z : z \in T\}$. For $\mu \leq \gamma < \mu^+$ fix a bijection $\pi_\gamma : \mu \to \gamma$.

Set $S = \{x \in \mathcal{P}_\kappa \mu^+ : \forall \gamma \in x - \mu (\pi_\gamma``(x \cap \mu) \subset x) \wedge x \cap \mu \in T \wedge \sup x \in S_{x \cap \mu}\}$.

Claim. *S is stationary in $\mathcal{P}_\kappa \mu^+$.*

Proof. Since $\{x \in \mathcal{P}_\kappa \mu^+ : \forall \gamma \in x - \mu (\pi_\gamma``(x \cap \mu) \subset x)\}$ is club, it suffices to show that $\{x \in \mathcal{P}_\kappa \mu^+ : x \cap \mu \in T \wedge \sup x \in S_{x \cap \mu}\}$ is stationary.

Fix $f : [\mu^+]^{<\omega} \to \mathcal{P}_\kappa \mu^+$. For $z \in T$ consider the following game $\mathcal{G}(z)$ of length ω between two players *I* and *II*:

In round n *I* chooses $\mu \leq \gamma_n < \mu^+$. Then *II* chooses $x_n \in C(f)$ with $\gamma_n < \sup x_n$. We further require $\sup x_n < \gamma_{n+1}$ and $x_n \subset x_{n+1}$. Finally *II* wins just in case $x_n \cap \mu = z$ for every $n < \omega$.

Set $T' = \{z \in T : II \text{ has no winning strategy in } \mathcal{G}(z)\}$.

Subclaim. *T' is nonstationary in $\mathcal{P}_\kappa \mu$.*

Proof. Suppose otherwise. Note that the game $\mathcal{G}(z)$ is closed for *II*, hence determined. Hence for $z \in T'$ we have a winning strategy σ_z for *I* in $\mathcal{G}(z)$. Set $D = \{\delta < \mu^+ : f``[\delta]^{<\omega} \subset \mathcal{P}_\kappa \delta\}$, which is club. By induction on $n < \omega$ we define $\beta_n \in S^\omega_{\mu^+} \cap D$ and x^z_n for $z \in T'$ so that $\langle x^z_n : n < \omega \rangle$ is a play of *II* in $\mathcal{G}(z)$ against σ_z and $\sup x^z_n = \beta_n$ for every $z \in T'$ as follows:

Assume we have β_i and $\{x^z_i : z \in T'\}$ for $i < n$ as above. Since $|T'| \leq |T| = \mu$, we have $\sup_{z \in T'} \sigma_z(\langle x^z_i : i < n \rangle) < \beta_n \in S^\omega_{\mu^+} \cap D$. Then $\beta_{n-1} = \sup x^z_{n-1} < \sigma_z(\langle x^z_i : i < n \rangle) < \beta_n$ for every $z \in T'$.

Fix $z \in T'$. Since $\sup x^z_{n-1} < \beta_n \in S^\omega_{\mu^+} \cap D$, $C^z_n = \{x \in \mathcal{P}_\kappa \beta_n \cap C(f) : x^z_{n-1} \subset x \wedge \sup x = \beta_n\}$ is club in $\mathcal{P}_\kappa \beta_n$. Let x^z_n be $\pi_{\beta_n}``z$ if $\pi_{\beta_n}``z \in C^z_n$, otherwise an element of C^z_n.

Set $\beta = \sup_{n<\omega} \beta_n$. Then $\mu \leq \sup_{z \in T'} \sigma_z(\emptyset) < \beta_0 < \beta$. Since $\beta_n \in S^\omega_{\mu^+} \cap D$ for every $n < \omega$, $C = \{x \in \mathcal{P}_\kappa \beta \cap C(f) : \forall n < \omega (\pi_{\beta_n}``(x \cap \mu) = x \cap \beta_n \wedge \sup(x \cap \beta_n) = \beta_n)\}$ is club in $\mathcal{P}_\kappa \beta$. Since T' is stationary in $\mathcal{P}_\kappa \mu$, we can take $x \in C$ so that $x \cap \mu \in T'$.

Set $z = x \cap \mu \in T'$. Since $x \in C$, we see by induction on $n < \omega$ that $\pi_{\beta_n}``z = \pi_{\beta_n}``(x \cap \mu) = x \cap \beta_n \in C^z_n$ and $x^z_n = x \cap \beta_n$. Hence $x^z_n \cap \mu = x \cap \mu = z$ for every $n < \omega$. Thus *II* wins in $\mathcal{G}(z)$ against σ_z with the play $\langle x^z_n : n < \omega \rangle$. This contradicts that σ_z is a winning strategy for *I* in $\mathcal{G}(z)$, as desired. □

Fix $z \in T - T'$ and a winning strategy τ for *II* in $\mathcal{G}(z)$. Since S_z is stationary in μ^+, we have $\mu < \gamma \in S_z$ such that $\sup \tau(s) < \gamma$ for every $s \in \gamma^{<\omega}$. Since $\operatorname{cf} \gamma = \omega$, we have γ_n inductively so that $\gamma_0 = \mu$,

$\sup \tau(\langle \gamma_i : i < n\rangle) < \gamma_n$ and $\sup_{n<\omega} \gamma_n = \gamma$. Then $\langle \gamma_n : n < \omega\rangle$ is a play of I in $\mathcal{G}(z)$ against τ.

For $n < \omega$ set $x_n = \tau(\langle \gamma_i : i \leq n\rangle)$. Then II wins in $\mathcal{G}(z)$ with the play $\langle x_n : n < \omega\rangle$. Hence $\{x_n : n < \omega\} \subset C(f)$ is increasing, $x_n \cap \mu = z$ and $\gamma_n < \sup x_n < \gamma_{n+1}$ for every $n < \omega$. Set $x = \bigcup_{n<\omega} x_n$. Then $x \in C(f)$, $x \cap \mu = z \in T$ and $\sup x = \sup_{n<\omega} \sup x_n = \sup_{n<\omega} \gamma_n = \gamma \in S_z = S_{x\cap\mu}$, as desired. $\square$

Claim. *S is μ-nonreflecting.*

Proof. Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\mu \subset A \subset \mu^+$ of size μ. Then $\{x \in \mathcal{P}_\kappa A : \forall \gamma \in x - \mu(\pi_\gamma``(x \cap \mu) \subset x)\}$ is unbounded in $\mathcal{P}_\kappa A$. Hence $\gamma = \pi_\gamma``\mu = \pi_\gamma``(A \cap \mu) \subset A$ for every $\gamma \in A - \mu$. Thus $A = \delta$ for some $\mu \leq \delta < \mu^+$.

Subclaim. cf $\delta < \kappa$.

Proof. Since $\{x \in \mathcal{P}_\kappa \delta : \pi_\delta``(x \cap \mu) = x\}$ is club, $S' = \{x \in S \cap \mathcal{P}_\kappa \delta : \pi_\delta``(x \cap \mu) = x\}$ is stationary in $\mathcal{P}_\kappa \delta$. Fix $x \in S'$. Since $\sup x \in S_{x\cap\mu} \subset S^\omega_{\mu^+}$, we have $b_x \in [x]^\omega$ with $\sup b_x = \sup x$. Since $\pi_\delta^{-1}``b_x \in [x \cap \mu]^\omega$ and $x \cap \mu \in T$, we have $\xi \in x \cap \mu$ with $\pi_\delta^{-1}``b_x \subset c_\xi$.

Now we have $\xi^* < \mu$ and a stationary $S^* \subset S'$ such that $b_x \subset \pi_\delta``c_{\xi^*}$ for every $x \in S^*$. Since $S^*$ is unbounded in $\mathcal{P}_\kappa \delta$, $\delta = \sup_{x\in S^*} \sup x = \sup_{x\in S^*} \sup b_x \leq \sup \pi_\delta``c_{\xi^*} \leq \delta$. Hence $\delta = \sup \pi_\delta``c_{\xi^*}$ has cofinality $< \kappa$. $\square$

Thus $\{x \in S \cap \mathcal{P}_\kappa \delta : \sup x = \delta\}$ is stationary in $\mathcal{P}_\kappa \delta$. Take x, y from this set so that $x \cap \mu \neq y \cap \mu$. Then $\delta = \sup x = \sup y \in S_{x\cap\mu} \cap S_{y\cap\mu}$. This contradicts $S_{x\cap\mu} \cap S_{y\cap\mu} = \emptyset$, as desired. $\square$

Therefore $\mathcal{P}_\kappa \mu^+$ has a μ-nonreflecting stationary subset. $\square$

Now Theorem 2 follows from Theorem 3 with $\mu = \kappa$: It is easy to check that the hypothesis of Theorem 3 is satisfied with $c_\xi = \xi$ for $\xi < \kappa$ and $T = S^{\omega_1}_\kappa$.

Theorem 3 with $\mu = \kappa^+$ yields the following

Corollary 1. *Let κ be regular $\geq \omega_2$. Then $\mathcal{P}_\kappa \lambda$ has a κ^+-nonreflecting stationary subset for every $\lambda \geq \kappa^{++}$.*

Proof. It suffices to check that the hypothesis of Theorem 3 is satisfied.

For $\kappa \leq \gamma < \kappa^+$ we have a club $T_\gamma \subset \mathcal{P}_\kappa \gamma$ of size κ. List the set $\bigcup_{\kappa\leq\gamma<\kappa^+} T_\gamma$ as $\{c_\xi : \xi < \kappa^+\}$. Then $D = \{\delta < \kappa^+ : \bigcup_{\kappa\leq\gamma<\delta} T_\gamma = \{c_\xi : \xi < \delta\}\}$ is club. Set $T = \{z \in \bigcup_{\kappa\leq\gamma<\kappa^+} T_\gamma : \forall b \in [z]^\omega \exists \xi \in z(b \subset c_\xi)\}$. Then $|T| \leq \kappa^+$. We show that T is stationary in $\mathcal{P}_\kappa \kappa^+$.

Fix $f : [\kappa^+]^{<\omega} \to \mathcal{P}_\kappa\kappa^+$. We have $\delta \in S^\kappa_{\kappa^+} \cap D$ with $f``[\delta]^{<\omega} \subset \mathcal{P}_\kappa\delta$. Then $T_\delta \cap C(f)$ is club in $\mathcal{P}_\kappa\delta$. Moreover $\{c_\xi : \xi < \delta\} = \bigcup_{\kappa\leq\gamma<\delta} T_\gamma$ is unbounded in $\mathcal{P}_\kappa\delta$. Hence we can build an increasing sequence $\{z_\alpha : \alpha < \omega_1\} \subset T_\delta \cap C(f)$ so that $z_\alpha \subset c_\xi$ for some $\xi \in z_{\alpha+1}$. Then $\bigcup_{\alpha<\omega_1} z_\alpha \in T \cap C(f)$, as desired. □

If $\operatorname{cf}\mu < \kappa$, $\mathcal{P}_\kappa\mu$ has no stationary subset of size μ. So Theorem 3 has nothing to say in this case. It has something to say, however, about a question of [8]:

Corollary 2. *Let κ be regular $\geq \omega_2$ and $\mu^{<\kappa} = \mu$. Then $\mathcal{P}_\kappa\lambda$ has a μ-nonreflecting stationary subset for every $\lambda \geq \mu^+$.*

Proof. Since $\mu^{<\kappa} = \mu$, we can list $\mathcal{P}_\kappa\mu$ as $\{c_\xi : \xi < \mu\}$. Then $T = \{z \in \mathcal{P}_\kappa\mu : \forall b \in [z]^\omega \exists\xi \in z(b \subset c_\xi)\}$ is stationary:

Fix $f : [\mu]^{<\omega} \to \mathcal{P}_\kappa\mu$. Build an increasing sequence $\{z_\alpha : \alpha < \omega_1\} \subset C(f)$ so that $z_\alpha = c_\xi$ for some $\xi \in z_{\alpha+1}$. Then $\bigcup_{\alpha<\omega_1} z_\alpha \in T \cap C(f)$, as desired. □

4. Proof of Theorem 1

This section presents the Foreman–Magidor example of a nonreflecting stationary set in $\mathcal{P}_\kappa\lambda$ as we understand it. Although the construction seems to work only for $\lambda \geq 2^{\kappa^+}$, the example has the virtue that the intersection with $\{x \in \mathcal{P}_\kappa\lambda : \operatorname{cf}(x \cap \kappa) = \omega\}$ is stationary [5]. In contrast our example of Theorem 2 is (easily seen to be) a subset of $\{x \in \mathcal{P}_\kappa\lambda : \operatorname{cf}(x \cap \kappa) > \omega = \operatorname{cf}\sup(x \cap \kappa^+)\}$.

Two Subclaims below are proved using ideas to show that Chang's Conjecture holds after a measurable cardinal is Lévy-collapsed to ω_2 [9] and that $\mathcal{P}_\kappa\kappa^+$ has a club subset of size $\leq (\kappa^+)^{\omega_1}$ [1] respectively.

Proof of Theorem 1. Since $\lambda \geq 2^{\kappa^+}$, we can list (possibly with repetition) the functions $: \kappa^+ \to \mathcal{P}_\kappa\kappa$ as $\{g_\xi : \xi < \lambda\}$. For $\kappa \leq \gamma < \kappa^+$ fix a bijection $\pi_\gamma : \kappa \to \gamma$. Define $h : \kappa \times (\kappa^+ - \kappa) \to \mathcal{P}_\kappa\kappa^+$ by $h(\alpha, \beta) = \lim \pi_\beta``\alpha$. Then $D = \{x \in \mathcal{P}_\kappa\lambda : \forall\xi \in x(g_\xi``(x \cap \kappa^+) \subset \mathcal{P}(x)) \wedge \forall\gamma \in x \cap (\kappa^+ - \kappa)(\pi_\gamma``(x \cap \kappa) = x \cap \gamma) \wedge h``((x \cap \kappa) \times (x \cap (\kappa^+ - \kappa))) \subset \mathcal{P}(x)\}$ is club.

Set $S = \{x \in \mathcal{P}_\kappa\lambda : \{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is nonstationary in $\kappa^+\}$.

Claim. *S is stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. Suppose otherwise. By induction on $n < \omega$ we build $f_n : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ and $\xi_n : [\lambda]^{<\omega} \to \lambda$ as follows:

Since S is nonstationary, we have f_0 with $C(f_0) \subset D - S$. Assume next we have f_n. Define ξ_n and f_{n+1} by $g_{\xi_n(a)}(\gamma) = \operatorname{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa$

and $f_{n+1}(a) = f_n(a) \cup \{\xi_n(a)\}$. Finally define $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ by $f(a) = \bigcup_{n<\omega} f_n(a)$.

Subclaim. *If $x \in C(f)$, $\{\sup(z \cap \kappa^+) : x \subset z \in C(f) \wedge z \cap \kappa = x \cap \kappa\}$ is unbounded in κ^+.*

Proof. Fix $\alpha < \kappa^+$. Since $x \in C(f) \subset C(f_0) \subset \mathcal{P}_\kappa\lambda - S$, $\{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is stationary in κ^+. Hence we have $x \subset y \in D$ such that $y \cap \kappa = x \cap \kappa$ and $\alpha < \sup(y \cap \kappa^+)$. Fix $\alpha < \gamma \in y \cap \kappa^+$. Then $z = \bigcup\{\mathrm{cl}_{f_n}(a \cup \{\gamma\}) : n < \omega \wedge a \in [x]^{<\omega}\}$ witnesses the Subclaim:

Since $\gamma \in z$, $\alpha < \gamma \leq \sup(z \cap \kappa^+)$. By the definition of f, it is easy to check $x \subset z \in C(f)$. To see $z \cap \kappa \subset x \cap \kappa$, fix $\beta \in z \cap \kappa$. Then $\beta \in \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa = g_{\xi_n(a)}(\gamma)$ for some $n < \omega$ and $a \in [x]^{<\omega}$. Since $x \in C(f)$ and $a \in [x]^{<\omega}$, $\xi_n(a) \in f(a) \subset x \subset y$. Since $\xi_n(a), \gamma \in y \in D$, $\beta \in g_{\xi_n(a)}(\gamma) \subset y \cap \kappa = x \cap \kappa$, as desired. □

For $i = 0, 1$ build an increasing sequence $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ so that $x^i_\xi \cap \kappa = x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$, $\kappa < \sup(x^0_\xi \cap \kappa^+) \leq \sup(x^1_\xi \cap \kappa^+) < \sup(x^0_{\xi+1} \cap \kappa^+)$ but $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$ as follows:

First we have $x^0_0 \in C(f)$ such that $x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$ and $\kappa < \sup(x^0_0 \cap \kappa^+)$. By the Subclaim we can take x^0_1 from $X = \{z \in C(f) : x^0_0 \subset z \wedge z \cap \kappa = x^0_0 \cap \kappa\}$ so that $\sup(x^0_1 \cap \kappa^+)$ is the κ-th element of $\{\sup(z \cap \kappa^+) : z \in X\}$. Since $x^0_1 \cap \kappa^+$ has $< \kappa$ initial segments, we have $x^1_0 \in X$ such that $\sup(x^1_0 \cap \kappa^+) < \sup(x^0_1 \cap \kappa^+)$ but $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$, as required above. The rest of the construction using the Subclaim is routine.

Set $x^i = \bigcup_{\xi<\omega_1} x^i_\xi$. Since $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ is increasing and $\kappa \geq \omega_2$, $x^i \in C(f)$. Since $x^i_\xi, x^i \in C(f) \subset D$, $x^i_\xi \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$: $x^i_\xi \cap \gamma = \pi_\gamma``(x^i_\xi \cap \kappa) = \pi_\gamma``(x^0_0 \cap \kappa) = \pi_\gamma``(x^i \cap \kappa) = x^i \cap \gamma$ for every $\gamma \in x^i_\xi \cap (\kappa^+ - \kappa)$. By the construction of x^i_ξ's, $\sup(x^0 \cap \kappa^+) = \sup_{\xi<\omega_1} \sup(x^0_\xi \cap \kappa^+) = \sup_{\xi<\omega_1} \sup(x^1_\xi \cap \kappa^+) = \sup(x^1 \cap \kappa^+) \in S^{\omega_1}_{\kappa^+}$.

Subclaim. *$x^i \cap \kappa^+$ is σ-closed.*

Proof. Fix $\gamma \in \lim(x^i \cap \kappa^+)$ of cofinality ω. Then we have $b \subset x^i \cap \kappa^+$ of order type ω with $\sup b = \gamma$. Since $\kappa < \sup(x^i \cap \kappa^+) \in S^{\omega_1}_{\kappa^+}$, we have $b \subset \beta \in x^i \cap (\kappa^+ - \kappa)$. Since $\beta \in x^i \in D$, $\pi_\beta^{-1}``(x^i \cap \beta) = x^i \cap \kappa = x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$. Since $\pi_\beta^{-1}``b \in [\pi_\beta^{-1}``(x^i \cap \beta)]^\omega$, we have $\pi_\beta^{-1}``b \subset \alpha \in x^i \cap \kappa$. Hence $b \subset \pi_\beta``\alpha$. Since $\alpha, \beta \in x^i \in D$, $\gamma = \sup b \in \lim \pi_\beta``\alpha = h(\alpha, \beta) \subset x^i$, as desired. □

Thus we have $\sup(x^0_1 \cap \kappa^+) < \gamma \in x^0 \cap x^1 \cap \kappa^+$. Since $\gamma \in x^i \in D$, $x^0 \cap \gamma = \pi_\gamma``(x^0 \cap \kappa) = \pi_\gamma``(x^0_0 \cap \kappa) = \pi_\gamma``(x^1 \cap \kappa) = x^1 \cap \gamma$. This contradicts that $x^i_\xi \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$ but $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$, as desired. □

Claim. *S is nonreflecting.*

Proof. Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size κ. Fix a bijection $\pi : \kappa \to A$. Then $\{\gamma < \kappa : \pi``\gamma \in S\}$ is stationary. Since $\{\gamma < \kappa : (\pi``\gamma) \cap \kappa = \gamma\}$ is club, their intersection T is stationary. Since $\{y \in \mathcal{P}_\kappa\lambda : \pi``(y \cap \kappa) \subset y \in D\}$ is club, $\{y \in \mathcal{P}_\kappa\lambda : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary. Hence $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary in κ^+.

Since $|T| = \kappa$, we have $\gamma \in T$ such that $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa = \gamma\}$ is stationary in $\kappa^+$. Note that $(\pi``\gamma) \cap \kappa = \gamma$ by $\gamma \in T$. Hence $\{\sup(y \cap \kappa^+) : \pi``\gamma \subset y \in D \wedge y \cap \kappa = (\pi``\gamma) \cap \kappa\}$ is stationary in κ^+. But $\pi``\gamma \in S$ by $\gamma \in T$. Contradiction. □

Therefore Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails. □

Finally we remark that the same proof goes through even if "nonstationary" is replaced by "bounded" in the definition of S.

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