CARDINAL SEQUENCES AND COHEN REAL EXTENSIONS

ISTVÁN JUHÁSZ, SAHARON SHELAH, LAJOS SOUKUP,
AND ZOLTÁN SZENTMIKLÓSSY

ABSTRACT. We show that if we add any number of Cohen reals to the ground model then, in the generic extension, a locally compact scattered space has at most $(2^{\aleph_0})^V$ many levels of size ω.

We also give a complete ZFC characterization of the cardinal sequences of regular scattered spaces. Although the classes of the regular and of the 0-dimensional scattered spaces are different, we prove that they have the same cardinal sequences.

1. INTRODUCTION

Let us start by recalling that a topological space X is called *scattered* if every non-empty subspace of X has an isolated point. Via the well-known Cantor-Bendixson analysis then X decomposes into levels, the α^{th} Cantor-Bendixson level of X will be denoted by $\mathrm{I}_\alpha(X)$. The *height of* X, $\mathrm{ht}(X)$, is the least ordinal α with $\mathrm{I}_\alpha(X) = \emptyset$. The *width of* X, $\mathrm{wd}(X)$, is defined by $\mathrm{wd}(X) = \sup\{\, |\mathrm{I}_\alpha(X)| : \alpha < \mathrm{ht}(X)\}$. Our main object of study is the *cardinal sequence* of X, denoted by $\mathrm{CS}(X)$, that is the sequence of cardinalities of the non-empty Candor-Bendixson levels of X, i.e.

$$CS(X) = \langle\ |I_\alpha(X)| : \alpha < \mathrm{ht}(X)\ \rangle.$$

The cardinality of a T_3 , in particular of a locally compact, scattered T_2 (in short: LCS) space X is at most $2^{|\mathrm{I}_0(X)|}$, hence clearly

1991 *Mathematics Subject Classification.* 54A25, 06E05, 54G12, 03E35.

Key words and phrases. locally compact scattered space, superatomic Boolean algebra, Cohen reals, cardinal sequence, regular space, 0-dimensional.

The first, third and fourth authors were supported by the Hungarian National Foundation for Scientific Research grant no. 37758 .

The second author was supported by The Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. Publication 765.

The third author was partially supported by Grant-in-Aid for JSPS Fellows No. 98259 of the Ministry of Education, Science, Sports and Culture, Japan.

$\mathrm{ht}(X) < (2^{|\mathrm{I}_0(X)|})^+$ and $|I_\alpha(X)| \le 2^{|\mathrm{I}_0(X)|}$ for each α. (Locally compact scattered spaces are closely related to superatomic boolean algebras via Stone duality and the study of their cardinal sequences was actually originated in that subject.) Thus, in particular, under CH there is no scattered T_3 space of height ω_2 and having only countably many isolated points. After I. Juhász and W. Weiss, [5, theorem 4], had proved in ZFC that for every $\alpha < \omega_2$ there is an LCS space X with $\mathrm{ht}(X) = \alpha$ and $\mathrm{wd}(X) = \omega$, it was a natural question if the existence of an LCS space of height ω_2 and width ω follows from $\neg CH$. This question was answered in the negative by W. Just who proved, [6, theorem 2.13], that if one blows up the continuum by adding Cohen reals to a model of CH then in the resulting generic extension there is no LCS space of height ω_2 and width ω. On the other hand, in their ground breaking work [1], J. Baumgartner and S. Shelah produced a model in which there is a LCS space of height ω_2 and width ω, moreover they proved in ZFC that for each $\alpha < (2^\omega)^+$ there is a scattered 0-dimensional T_2 space X with $\mathrm{ht}(X) = \alpha$ and $\mathrm{wd}(X) = \omega$. Building on the idea of the proof of this latter result, in section 3 we succeeded in giving a complete characterization of the cardinal sequences of both T_3 and zero-dimensional T_2 scattered spaces. Although the classes of the regular and of the zero-dimensional scattered spaces are different, it will turn out that they yield the same class of cardinal sequences. We should add that, with quite a bit of extra effort, in [8], J.-C. Martinez extended the former result of Baumgartner and Shelah by producing a model in which for every ordinal $\alpha < \omega_3$ there is a LCS space of height α and width ω. The question if it is consistent to have a LCS space of height ω_3 and width ω remains a big mystery.

In section 2 we strengthened the result of Just by proving, in particular, that in the same Cohen real extension no LCS space may have ω_2 many countable (non-empty) levels. It seems to be an intriguing (and natural) problem if the non-existence of an LCS space of width ω and height ω_2 implies in ZFC the above conclusion, or more generally: is any subsequence of the cardinal sequence of an LCS space again such a cardinal sequence? In connection with this problem let us remark that, (as is shown in [2] or [3]), in the side-by-side random real extension of a model of CH the combinatorial principle $\mathcal{C}^s(\omega_2)$ introduced in [4, definition 2.3] holds, consequently in such an extension there is no LCS space X of height ω_2 and width ω. In fact, by [4, theorem 4.12], $\mathcal{C}^s(\omega_2)$ implies that $\{\alpha \in \omega_2 : |I_\alpha(X)| = \omega\}$ is non-stationary in ω_2. However, we do not know if our above mentioned result, namely theorem 2.1, holds there.

The morale of our above discussion may be concisely formulated as follows: The cardinal sequences of regular or zero-dimensional scattered spaces are only subject to the trivial inequality $|X| \le 2^{|I_0(X)|}$, however those of the LCS spaces are much harder to determine, in particular, they are sensitive to the model of set theory in which we look at them.

2. Countable levels in Cohen real extensions

Let us formulate then the promised strengthening of Just's result. We note that no assumption (like CH) is made on our ground model.

Theorem 2.1. *Let us set $\kappa = (2^\omega)^+$ and add any number of Cohen reals to our ground model. Then in the resulting extension no LCS space contains a κ-sequence $\{E_\alpha : \alpha < \kappa\}$ of pairwise disjoint countable subspaces such that $\overline{E_\alpha} \supset E_\beta$ holds for all $\alpha < \beta < \kappa$. In particular, for any LCS space X we have $\big|\{\alpha : |\,\mathrm{I}_\alpha(X)| = \omega\}\big| < \kappa$.*

In fact, we shall prove a more general statement, but to formulate that we need a definition. A family of pairs (of sets) $\mathcal{D} = \big\{\langle D_0^\alpha, D_1^\alpha\rangle : \alpha \in I\big\}$ is said to be *dyadic over a set T* iff $D_0^\alpha \cap D_1^\alpha = \emptyset$ for each $\alpha \in I$ and

$$\mathcal{D}[\varepsilon] = \bigcap\{D^\alpha_{\varepsilon(\alpha)} : \alpha \in \operatorname{dom}\varepsilon\}$$

intersects T for each $\varepsilon \in \mathrm{Fn}(I,2)$. We simply say that $\mathcal{D}$ *is dyadic* iff it is dyadic for some T, i.e. $\mathcal{D}[\varepsilon] \ne \emptyset$ for each $\varepsilon \in \mathrm{Fn}(I,2)$.

Now, it is obvious that in a LCS space

- the compact open sets form a base that is closed under finite unions,
- there is no infinite dyadic system of pairs of compact sets.

Consequently, theorem 2.2 below immediately yields theorem 2.1 above.

Theorem 2.2. *Set $\kappa = (2^\omega)^+$ and add any number of Cohen reals to the ground model. Then in the resulting generic extension the following statement holds: If X is any T_2 space containing pairwise disjoint countable subspaces $\{E_\alpha : \alpha < \kappa\}$ such that $\overline{E_\alpha} \supset E_\beta$ for $\alpha < \beta < \kappa$ and $X = \overline{E_0}$ (i. e. E_0 is dense in X) , moreover, for each $x \in X$, we have fixed a neighbourhood base $\mathcal{B}(x)$ of x in X that is closed under finite unions then there is an infinite set $a \in \big[\kappa\big]^\omega$, for each $\alpha \in a$ there are disjoint finite subsets L^0_α and L^1_α of E_α, and for each $x \in L^0_\alpha \cup L^1_\alpha$ there is a basic neighbourhood $V(x) \in \mathcal{B}(x)$ such that the infinite family of pairs*

$$\Big\{\big\langle \bigcup_{x\in L^0_\alpha} V(x), \bigcup_{x\in L^1_\alpha} V(x)\big\rangle : \alpha \in a\Big\}$$

is dyadic.

This topological statement in the Cohen extension in turn will follow from a purely combinatorial one concerning certain matrices, namely theorem 2.7.

To formulate this theorem we again need some notation and definitions.

For an ordinal α the interval $[\omega\alpha, \omega\alpha+\omega)$ will be denoted by $\mathbb{I}_\alpha$.

Given two sets A and B we write $f : A \xrightarrow{p} B$ to denote that f is a partial function from A to B, i. e. a function from a subset of A into B. As usual, we let

$$\operatorname{Fn}(A,B) = \{f : |f| < \omega \text{ and } f : A \xrightarrow{p} B\}.$$

If $A \subset \mathrm{On}$ then for any partial function $f : A \xrightarrow{p} B$ we set

$$\gamma(f) = \begin{cases} \min \operatorname{dom} f & \text{if } \operatorname{dom} f \neq \emptyset, \\ \sup A & \text{if } \operatorname{dom} f = \emptyset. \end{cases}$$

We let

$$\Omega = \big\{\langle A, B\rangle \in \big[\omega\big]^{<\omega} \times \big[\omega\big]^{<\omega} : A \cap B = \emptyset\big\},$$

and for $\ell = \langle A, B\rangle \in \Omega$ we set $\pi_0(\ell) = A$ and $\pi_1(\ell) = B$.

If S and T are sets of ordinals, we denote by $\mathcal{M}(S,T)$ the family of all $S \times \omega$-matrices consisting of subsets of T, i. e. $\mathcal{A} \in \mathcal{M}(S,T)$ means that $\mathcal{A} = \langle A_{\alpha,i} : \alpha \in S, i \in \omega\rangle$, where $A_{\alpha,i} \subset T$ for each $\alpha \in S$ and $i < \omega$.

For $\mathcal{A} \in \mathcal{M}(S,T)$, $f : S \xrightarrow{p} S$, and $s : S \xrightarrow{p} \Omega$ the pair (f,s) is said to be $\mathcal{A}$-*dyadic (over U)* iff the family of pairs

$$\Big\{\Big\langle \cup\big\{A_{f(\alpha),n} : n \in \pi_0(s(\alpha))\big\}, \cup\big\{A_{f(\alpha),n} : n \in \pi_1(s(\alpha))\big\}\Big\rangle : \alpha \in \operatorname{dom} f \cap \operatorname{dom} s\Big\}.$$

is dyadic (over U). If the pair $\langle \mathrm{id}_S, s\rangle$ is $\mathcal{A}$-dyadic (over U) then s is simply called $\mathcal{A}$-*dyadic (over U)*. It is this latter notion of $\mathcal{A}$-dyadicity of a single partial function that is really important (that for pairs is only of technical significance). Hence we state below an alternative characterisation of it.

For $\mathcal{A} \in \mathcal{M}(S,T)$, $s : S \xrightarrow{p} \Omega$, and $\varepsilon \in \operatorname{Fn}(\operatorname{dom} s, 2)$ we write

$$\mathcal{A}[s,\varepsilon] = \bigcap_{\alpha \in \operatorname{dom}\varepsilon} \bigcup\{A_{\alpha,n} : n \in \pi_{\varepsilon(\alpha)}(s(\alpha))\}.$$

Observation 2.3. *If $\mathcal{A} \in \mathcal{M}(S,T)$ then $s : S \xrightarrow{p} \Omega$ is $\mathcal{A}$-dyadic over U iff $\mathcal{A}[s,\varepsilon] \cap U \neq \emptyset$ for each $\varepsilon \in \operatorname{Fn}(\operatorname{dom} s, 2)$ and*

$$\bigcup\{A_{\alpha,n} : n \in \pi_0(s(\alpha))\} \cap \bigcup\{A_{\alpha,n} : n \in \pi_1(s(\alpha))\} = \emptyset$$

for each $\alpha \in \operatorname{dom} s$.

The following easy observation will be applied later, in the proof of lemma 2.9:

Observation 2.4. *If* $g : S \xrightarrow{p} S$ *and* $s : S \xrightarrow{p} \Omega$ *satisfy* $\operatorname{dom} s \subset \operatorname{ran} g$*, and the pair* $(g, s \circ g)$ *is* $\mathcal{A}$*-dyadic over* U *then* s *is* $\mathcal{A}$*-dyadic over* U*, as well.*

Definition 2.5. Fix a cardinal κ and let $\mathcal{D} \in \mathcal{M}(\kappa, \kappa)$. For $s : \kappa \xrightarrow{p} \Omega$ we say that s is $\mathcal{D}$*-min-dyadic (m.d.)* iff s is $\mathcal{D}$-dyadic over $\mathbb{I}_{\gamma(s)}$.

Moreover, we say that the matrix $\mathcal{D}$ is *m.d.-extendible* iff for each finite $\mathcal{D}$-min-dyadic partial function $s : \kappa \xrightarrow{p} \Omega$ and for each $\gamma < \gamma(s)$ there is an $\ell \in \Omega$ such that $s \cup \{\langle \gamma, \ell \rangle\}$ is also $\mathcal{D}$-min-dyadic, i. e. $\mathcal{D}$-dyadic over $\mathbb{I}_\gamma$.

Since $\mathbb{I}_0 = \omega$, we clearly have the following.

Observation 2.6. *If* $\mathcal{D} \in \mathcal{M}(\kappa, \kappa)$ *is m.d-extendible and* $s : \kappa \xrightarrow{p} \Omega$ *is a finite* $\mathcal{D}$*-min-dyadic partial function then* s *is* $\mathcal{D}$*-dyadic over* ω*.*

Finally, a matrix $\mathcal{D} \in \mathcal{M}(\kappa, \kappa)$ will be called ω*-determined* iff $D_{\alpha,n} \cap D_{\alpha,m} \cap \omega = \emptyset$ implies $D_{\alpha,n} \cap D_{\alpha,m} = \emptyset$ whenever $\alpha < \kappa$ and $n < m < \omega$.

With this we now have all the necessary ingredients to formulate and prove the promised combinatorial statement that will be valid in any Cohen real extension.

Theorem 2.7. *Set* $\kappa = (2^\omega)^+$ *and add any number of Cohen reals to the ground model. Then in the resulting generic extension for every* ω*-determined and m.d.-extendible matrix* $\mathcal{D} \in \mathcal{M}(\kappa, \kappa)$ *there is an infinite* $\mathcal{D}$*-dyadic partial function* $h : \kappa \xrightarrow{p} \Omega$*.*

Before proving theorem 2.7, however, we show how theorem 2.2 can be deduced from it.

Proof of theorem 2.2 using theorem 2.7. We can assume without any loss of generality that $E_\alpha = \mathbb{I}_\alpha$ for each $\alpha < \kappa$ and then will define an appropriate matrix $\mathcal{D} \in \mathcal{M}(\kappa, \kappa)$.

To this end, for coding purposes, we first fix a bijection $\rho : \left[\omega\right]^2 \longrightarrow \omega$ and let $\eta : \omega \longrightarrow \omega$ and $\nu : \omega \longrightarrow \omega$ be the "co-ordinate" functions of its inverse, i. e. $k = \rho(\{\nu(k), \eta(k)\})$ and $\nu(k) < \eta(k)$ for each $k < \omega$.

Since X is T_2, for each $n < \omega$ we can simultaneously pick basic neighbourhoods $B^\alpha_n(m) \in \mathcal{B}(\omega\alpha + m)$ of the points $\omega \cdot \alpha + m \in E_\alpha = \mathbb{I}_\alpha$ for all $m < n$ such that the sets $\{B^\alpha_n(m)\colon m < n\}$ are pairwise disjoint.

Now we define $\mathcal{D} = \langle D_{\alpha,k} : \langle \alpha, k \rangle \in \kappa \times \omega \rangle \in \mathcal{M}(\kappa, \kappa)$ as follows:

$$D_{\alpha,k} = B^\alpha_{\eta(k)}(\nu(k)) \cap \kappa.$$

This matrix $\mathcal{D}$ is clearly ω-determined because $E_0 = \mathbb{I}_0 = \omega$ is dense in X. It is a bit less easy to establish the following

Claim . *$\mathcal{D}$ is also m.d.-extendible.*

Proof of the claim. Let $s : \kappa \xrightarrow{p} \Omega$ be a finite $\mathcal{D}$-min-dyadic partial function and let $\gamma < \gamma(s)$.

Since the sets $\{\mathcal{D}[s, \varepsilon] : \varepsilon \in {}^{\operatorname{dom} s}2\}$ are all open in the subspace κ and they all intersect $\mathbb{I}_{\gamma(s)}$, moreover every element of $\mathbb{I}_{\gamma(s)}$ is an accumulation point of $\mathbb{I}_\gamma$, it follows that $\mathcal{D}[s, \varepsilon] \cap \mathbb{I}_\gamma$ must be infinite for each $\varepsilon \in {}^{\operatorname{dom} s}2$. Thus we can easily pick two disjoint finite subsets A_0 and A_1 of $\mathbb{I}_\gamma$ such that every $\mathcal{D}[s, \varepsilon]$ intersects both A_0 and A_1. Let $n < \omega$ be chosen in such a way that $A_0 \cup A_1 \subset \{\omega\gamma + m : m < n\}$, and set $K_i = \{\rho\{m, n\} : m < n \wedge \omega\gamma + m \in A_i\}$ for $i < 2$. Since ϱ is one-to-one we have $K_0 \cap K_1 = \emptyset$, hence $\ell = \langle K_0, K_1 \rangle \in \Omega$, moreover

$$(\star) \qquad \Big(\bigcup_{m \in K_0} D_{\gamma,m} \Big) \cap \Big(\bigcup_{m \in K_1} D_{\gamma,m} \Big) = \emptyset$$

because the elements of the family $\{B^\gamma_n(m) : m < n\}$ are pairwise disjoint.

Now put $t = s \cup \{\langle \gamma, \ell \rangle\}$. Then for each $\varepsilon \in {}^{\operatorname{dom} t}2$ we clearly have

$$(\star\star) \qquad A_{\varepsilon(\gamma)} \cap \mathcal{D}[t, \varepsilon] \neq \emptyset,$$

hence $(\star)$ and $(\star\star)$ together yield that the extension t of s is $\mathcal{D}$-dyadic over $\mathbb{I}_\gamma = \mathbb{I}_{\gamma(t)}$. □

Thus we may apply theorem 2.7 to the matrix $\mathcal{D}$ to obtain an infinite $\mathcal{D}$-dyadic partial function $h : \kappa \xrightarrow{p} \Omega$. Set $a = \operatorname{dom} h$ and for each $\alpha \in a$ and $i < 2$ put $L^i_\alpha = \{\omega\alpha + \nu(k) : k \in \pi_{\mathrm{i}}(h(\alpha))\}$. For $x \in L^i_\alpha$ put

$$V(x) = \cup\{B^\alpha_{\eta(k)}(\nu(k)) : x = \omega\alpha + \nu(k) \text{ and } k \in \pi_{\mathrm{i}}(h(\alpha))\}.$$

Then $V(x) \in \mathcal{B}(x)$ because $\mathcal{B}(x)$ is closed under finite unions. Since for $i < 2$

$$\big(\cup\{V(x) : x \in L^i_\alpha\}\big) \cap \kappa = \cup\{D_{\alpha,k} : k \in \pi_{\mathrm{i}}(h(\alpha))\}$$

and

$$\cup\{D_{\alpha,k} : k \in \pi_0(h(\alpha))\} \cap \cup\{D_{\alpha,k} : k \in \pi_1(h(\alpha))\} = \emptyset,$$

we have

$$\big(\cup\{V(x) : x \in L^i_\alpha\}\big) \cap \big(\cup\{V(x) : x \in L^i_\alpha\}\big) = \emptyset$$

because the latter intersection is an open set which does not intersect the dense set $\mathbb{I}_0 \subset \kappa$. Hence the infinite family

$$\Big\{ \langle \bigcup_{x \in L^0_\alpha} V(x), \bigcup_{x \in L^1_\alpha} V(x) \rangle : \alpha \in a \Big\}$$

is indeed dyadic. $\square_{2.2}$

Proof of theorem 2.7. The proof will be based on the following two lemmas, 2.9 and 2.10. For these we need some more notation and a new and rather technical notion of extendibility for set matrices.

Given a set A we set

$$\mathcal{F}(A) = \{f \in \operatorname{Fn}(A, A) : f \text{ is injective and } \operatorname{dom}(f) \cap \operatorname{ran}(f) = \emptyset\}.$$

Each function $f \in \mathcal{F}(A)$ can be extended in natural way to a bijection $f^* : A \longrightarrow A$ as follows:

$$f^*(a) = \begin{cases} f(a) & \text{if } a \in \operatorname{dom} f, \\ f^{-1}(a) & \text{if } a \in \operatorname{ran} f, \\ a & \text{otherwise.} \end{cases}$$

Definition 2.8. If S and T are sets of ordinals then the matrix $\mathcal{A} \in \mathcal{M}(S, T)$ is called *nicely extendible* iff for each $f \in \mathcal{F}(S)$ there are a family $N(f) \subset \operatorname{Fn}(S, \Omega)$ and a function $K^f : N(f) \longrightarrow \left[S\right]^{\leq\omega}$ such that

(1) the pair (f, s) is $\mathcal{A}$-dyadic whenever $f \in \mathcal{F}(S)$ and $s \in N(f)$,
(2) $\emptyset \in N(f)$ for each $f \in \mathcal{F}(S)$,
(3) for $f, g \in \mathcal{F}(S)$ and $s \in N(f)$ if $f^* \restriction K^f(s) = g^* \restriction K^f(s)$ then $s \in N(g)$.
(4) for any $f \in \mathcal{F}(S)$, $s \in N(f)$ and $\alpha \in S \cap \gamma(s)$ there is $\ell \in \Omega$ such that $s \cup \{\langle \alpha, \ell \rangle\} \in N(f)$.

Clearly, this last condition (4) is what explains our terminology.

Lemma 2.9. *If $\kappa > \omega_1$ is regular and $\mathcal{A} \in \mathcal{M}(\kappa, \omega)$ is a nicely extendible matrix then there is an infinite partial function $h : \kappa \xrightarrow{p} \omega$ that is $\mathcal{A}$-dyadic .*

Proof. By induction on $n \in \omega$ we will define functions $h_0 \subset h_1 \subset \ldots h_n \subset \ldots$ from $\operatorname{Fn}(\kappa, \Omega)$ such that $|h_n| = n$ and for each $\nu \in \kappa$

$(*)^n_\nu$ there is $g \in \mathcal{F}(\kappa)$ such that $\gamma(g) > \nu$, $\operatorname{ran} g = \operatorname{dom} h_n$ and $h_n \circ g \in N(g)$.

First observe that $h_0 = \emptyset$ satisfies our requirements because, according to (2), condition $(*)^0_\nu$ holds trivially for each $\nu < \kappa$.

Next assume that the construction has been done and the induction hypothesis has been established for n. For each $\nu < \kappa$ choose a function $g_\nu \in \mathcal{F}(\kappa)$ witnessing $(*)^n_{\nu+\omega_1}$ and then write $K_\nu = K^{g_\nu}(h_n \circ g_\nu)$ and pick $\zeta_\nu \in (\nu, \nu + \omega_1) \setminus K_\nu$. Clearly the set

$$L = \{\xi \in \kappa : |\{\nu < \kappa : \xi \notin K_\nu\}| < \kappa\}$$

is countable and so we can pick $\xi_n \in \kappa \setminus (L \cup \operatorname{dom} h_n)$; then the set

$$J = \{\nu < \kappa : \xi_n \notin K_\nu\}$$

is of size κ.

Now set $g'_\nu = g_\nu \cup \{\langle \zeta_\nu, \xi_n \rangle\}$ for every $\nu \in J$. For every such ν then $\zeta_\nu, \xi_n \notin K_\nu$ implies ${g_\nu}^* \restriction K_\nu = {g'_\nu}^* \restriction K_\nu$, hence $h_n \circ g_\nu \in N(g'_\nu)$ by (3). Since $\zeta_\nu < \nu + \omega_1 < \gamma(g_\nu) = \gamma(h_n \circ g_\nu)$, we can now apply (4) to get $\ell^\nu \in \Omega$ such that $(h_n \circ g_\nu) \cup \{\langle \zeta_\nu, \ell^\nu \rangle\} \in N(g'_\nu)$.

We can then fix $\ell_n \in \Omega$ such that $J_n = \{\nu \in J : \ell^\nu = \ell_n\}$ is of size κ and let $h_{n+1} = h_n \cup \{\langle \xi_n, \ell_n \rangle\}$.

If $\nu \in J_n$ then $h_{n+1} \circ g'_\nu = (h_n \circ g_\nu) \cup \{\langle \zeta_\nu, \ell_n \rangle\} \in N(g'_\nu)$ and $\gamma(g'_\nu) > \nu$, so g'_ν witnesses $(*)^{n+1}_\nu$. But J_n is unbounded in κ, hence the inductive step is completed.

By $(*)^n_0$, for each $n < \omega$ there is g_n such that $\operatorname{dom} h_n = \operatorname{ran} g_n$ and $h_n \circ g_n \in N(g_n)$. Hence, by (1), $(g_n, h_n \circ g_n)$ is $\mathcal{A}$-dyadic, and so h_n is $\mathcal{A}$-dyadic according to observation 2.4. Consequently $h = \bigcup\{h_n : n < \omega\}$ is as required: it is $\mathcal{A}$-dyadic and infinite. $\Box_{2.9}$

Given any infinite set I we denote by $\mathcal{C}_I$ the poset $\operatorname{Fn}(I, 2)$, i.e. the standard notion of forcing that adds $|I|$ many Cohen reals.

Lemma 2.10. *Let $\kappa = (2^\omega)^+$. Then for each λ we have*

$V^{\mathcal{C}_\lambda} \models$ *If $\mathcal{D} \in \mathcal{M}(\kappa, \kappa)$ is both ω-determined and m.d.-extendible then there is $I \in [\kappa]^\kappa$ such that $\mathcal{D}^* = \langle D_{\alpha,n} \cap \omega : \langle \alpha, n \rangle \in I \times \omega \rangle$ is nicely extendible.*

Proof. Assume that

$$1_{\mathcal{C}_\lambda} \Vdash \dot{\mathcal{D}} \in \mathcal{M}(\kappa, \kappa) \text{ is m.d.-extendible.}$$

Let θ be a large enough regular cardinal and consider the structure $\mathcal{H}_\theta = \left\langle H_\theta, \in, \lhd, \kappa, \lambda, \dot{\mathcal{D}} \right\rangle$, where $H_\theta = \{x : |\operatorname{TC}(x)| < \theta\}$ and $\lhd$ is a fixed well-ordering of H_θ.

Working in V, for each $\alpha < \kappa$ choose a countable elementary submodel N_α of $\mathcal{H}_\theta$ with $\alpha \in N_\alpha$. Then there is $I \in [\kappa]^\kappa$ such that the models $\{N_\alpha : \alpha \in I\}$ are not only pairwise isomorphic but, denoting by $\sigma_{\alpha,\beta}$ the unique isomorphism between N_α and N_β, we have

(i) the family $\{N_\alpha \cap \theta : \alpha \in I\}$ forms a Δ-system with kernel Λ,
(ii) $\sigma_{\alpha,\beta}(\xi) = \xi$ for each $\xi \in \Lambda$,
(iii) $\sigma_{\alpha,\beta}(\alpha) = \beta$.

For each $\alpha < \kappa$ and $n < \omega$ let $\dot{D}_{\alpha,n}$ be the $\lhd$-minimal $\mathcal{C}_\lambda$-name of the $\langle \alpha, n \rangle^{\text{th}}$ entry of $\dot{\mathcal{D}}$. Since $\lhd$ is in $\mathcal{H}_\theta$ and $\sigma_{\alpha,\beta}(\alpha) = \beta$ we have

Claim 2.10.1. $\sigma_{\alpha,\beta}(\dot{D}_{\alpha,n}) = \dot{D}_{\beta,n}$ *for each $\alpha, \beta \in I$ and $n \in \omega$.*

Let G be any $\mathcal{C}_\lambda$-generic filter over V. We shall show that

$$V[G] \models \text{“}\mathcal{D}^* = \langle D_{\alpha,n} \cap \omega : \langle \alpha, n\rangle \in I \times \omega\rangle \text{ is nicely extendible.”}$$

For each $f \in \mathcal{F}(I)$ define the bijection $\rho_f : \lambda \longrightarrow \lambda$ as follows:

$$\rho_f(\xi) = \begin{cases} \sigma_{\alpha,f^*(\alpha)}(\xi) & \text{if } \xi \in N_\alpha \cap \lambda \text{ for some } \alpha \in I, \\ \xi & \text{otherwise.} \end{cases}$$

In a natural way ρ_f extends to an automorphism of $\mathcal{C}_\lambda$, which will be denoted by ρ_f as well. Clearly, we have

Claim 2.10.2. *If $f \in \mathcal{F}(I)$, $f(\alpha) = \beta$, $p \in \mathcal{C}_\lambda \cap N_\alpha$ then $\sigma_{\alpha,\beta}(p) = \rho_f(p)$.*

For $f \in \mathcal{F}(I)$ let $G^f = \{\rho_f^{-1}(p) : p \in G\}$ and then set

$$N(f) = \{s \in \mathrm{Fn}(I, \Omega) : s \text{ is } \dot{\mathcal{D}}[G^f]\text{-min-dyadic}\} = \\ \{s \in \mathrm{Fn}(I, \Omega) : \exists q \in G^f\ q \Vdash \text{“}s \text{ is } \dot{\mathcal{D}}\text{-min-dyadic”}\}.$$

To define K^f, for each $s \in N(f)$ pick a condition $p_s \in G$ such that

$$\rho_f^{-1}(p_s) \Vdash s \text{ is } \dot{\mathcal{D}}\text{-min-dyadic}$$

and let

$$K^f(s) = \{\alpha \in I : (N_\alpha \setminus \Lambda) \cap \operatorname{dom} p_s \neq \emptyset\}.$$

Note that $K^f(s)$ as defined above is finite, although 2.8.(3) only requires $K^f(s)$ to be countable.

To check property 2.8.(3) assume that $f, g \in \mathcal{F}(I)$ and $s \in N(f)$ with $g^* \restriction K^f(s) = f^* \restriction K^f(s)$. Then $\rho_g^{-1}(p_s) = \rho_f^{-1}(p_s)$ and so

$$\rho_g^{-1}(p_s) \Vdash s \text{ is } \dot{\mathcal{D}}\text{-min-dyadic},$$

hence s is also $\dot{\mathcal{D}}[G^g]$-min-dyadic , i.e. $s \in N(g)$.

Before checking 2.8.(1) we need one more observation.

Claim 2.10.3. *$\dot{D}_{f(\alpha),n}[G] \cap \omega = \dot{D}_{\alpha,n}[G^f] \cap \omega$ whenever $f \in \mathcal{F}(I)$, $\alpha \in \operatorname{dom} f$, and $n < \omega$.*

Proof of claim 2.10.3. Let $k \in \omega$. Then $k \in \dot{D}_{f(\alpha),n}[G]$ iff $\exists p \in G\ p \Vdash$ “$k \in \dot{D}_{f(\alpha),n}$” iff $\exists p \in G \cap N_{f(\alpha)}\ p \Vdash$ “$k \in \dot{D}_{f(\alpha),n}$” iff $\exists q \in G^f \cap N_\alpha$ $p = \sigma_{\alpha,f(\alpha)}(q) \Vdash$ “$k \in \dot{D}_{f(\alpha),n}$” iff $\exists q \in G^f \cap N_\alpha\ q \Vdash$ “$k \in \dot{D}_{\alpha,n}$” iff $\exists q \in G^f\ q \Vdash$ “$k \in \dot{D}_{\alpha,n}$” iff $k \in \dot{D}_{\alpha,n}[G^f]$. $\square_{2.10}$

Now let $f \in \mathcal{F}(I)$ and $s \in N(f)$. By the definition of $N(f)$, s is $\dot{\mathcal{D}}[G^f]$-min-dyadic and so by observation 2.6 s is $\dot{\mathcal{D}}[G^f]$-dyadic over ω. But it follows from 2.10.3, that s is $\dot{\mathcal{D}}[G^f]$-dyadic over ω if and only if the pair (f, s) is $\dot{\mathcal{D}}[G]$-dyadic over ω.

2.8.(2) is clear because $\emptyset$ is trivially $\mathcal{A}$-min-dyadic for any $\mathcal{A} \in \mathcal{M}(\kappa, \omega)$. Finally 2.8.(4) follows from the definition of $N(f)$ because $\dot{\mathcal{D}}[G^f]$ is m.d.-extendible. $\square_{2.10}$

Now, to complete the proof of theorem 2.7, first apply lemma 2.10 to get $I \in [\kappa]^\kappa$ such that

$$\mathcal{D}^* = \langle D_{\alpha,n} \cap \omega : \langle \alpha, n \rangle \in I \times \omega \rangle$$

is nicely extendible. Then applying lemma 2.9 to $\mathcal{D}^*$ we obtain an infinite $\mathcal{D}^*$-dyadic function $h : \kappa \xrightarrow{p} \Omega$. Since the matrix $\mathcal{D}$ is ω-determined the function h is $\mathcal{D}$-dyadic, as well. $\square_{2.7}$

3. Cardinal sequences of regular and 0-dimensional spaces

For any regular, scattered space X we have $|X| \le 2^{|I(X)|}$, hence $\mathrm{ht}(X) < (2^{|I(X)|})^+$ and $|I_\alpha(X)| \le 2^{|\mathrm{I}_0(X)|}$ for each α. This implies that for such a space X its cardinal sequence s satisfies $length(s) < (2^{|I(X)|})^+$ and $s(\alpha) \le 2^{s(\beta)}$ whenever $\beta < \alpha$. We shall show below that these properties of a sequence s actually characterize the cardinal sequences of regular scattered spaces.

In [1], for each $\gamma < (2^\omega)^+$, a 0-dimensional, scattered space of height γ and width ω was constructed. The next lemma generalizes that construction.

For an infinite cardinal κ, let S_κ be the following family of sequences of cardinals:

$$S_\kappa = \big\{ \langle \kappa_i : i < \delta \rangle : \delta < (2^\kappa)^+,\ \kappa_0 = \kappa \text{ and } \kappa \le \kappa_i \le 2^\kappa \text{ for each } i < \delta \ \big\}.$$

Lemma 3.1. *For any infinite cardinal κ and $s \in S_\kappa$ there is 0-dimensional scattered space X with $\mathrm{CS}(X) = s$.*

Proof. Let $s = \langle \kappa_\alpha : \alpha < \delta \rangle \in S_\kappa$. Write $X = \bigcup \big\{ \{\alpha\} \times \kappa_\alpha : \alpha < \delta \big\}$. Since $|I| \le 2^\kappa$ we can fix an independent family $\{F_x : x \in X\} \subset [\kappa]^\kappa$.

The underlying set of our space is X and and the topology τ on X is given by declaring for each $x = \langle \alpha, \xi \rangle \in X$ the set

$$U_x = \{x\} \cup (\alpha \times F_x)$$

to be clopen, i.e. $\{U_x, X \setminus U_x : x \in X\}$ is a subbase for τ.

The space X is clearly 0-dimensional and T_2.

Claim 3.1.1. *If $x = \langle \beta, \xi \rangle \in U \in \tau$ and $\alpha < \beta$ then $U \cap (\{\alpha\} \times \kappa_\alpha)$ is infinite.*

Proof of the claim. We can find disjoint sets $A, B \in \big[X \setminus \{x\}\big]^{<\omega}$ such that

$$x \in U_x \cap \bigcap_{y\in A} U_y \setminus \bigcup_{z\in B} U_z \subset U.$$

Observe that if $\langle \gamma, \xi\rangle \in A$ then $\beta < \gamma$. Thus

$$U \cap (\{\alpha\} \times \kappa_\alpha) \supset \{\alpha\} \times \Big(\bigcap_{y\in A\cup\{x\}} F_y \setminus \bigcup_{z\in B} F_z \Big),$$

and the set on the right side is infinite because $\{F_x : x \in X\}$ was chosen to be independent. $\square$

To complete our proof, by induction on $\alpha < \kappa$, we verify that $\mathrm{I}_\alpha(X) = \{\alpha\} \times \kappa_\alpha$, hence $\mathrm{CS}(X) = s$. Assume that this is true for $\nu < \alpha$. If $x \in \{\alpha\} \times \kappa_\alpha$ then

$$U_x \cap \big(X \setminus \bigcup_{\nu<\alpha} \mathrm{I}_\nu(X)\big) = \{x\},$$

hence $\{\alpha\} \times \kappa_\alpha \subset \mathrm{I}_\alpha(X)$. On the other hand, if $x = \langle \beta, \xi\rangle \in X$ with $\beta > \alpha$ and $U \in \tau$ is a neighbourhood of x, then, by the claim above, $U \cap (\{\alpha\} \times \kappa_\alpha)$ is infinite, hence x is not isolated in $X \setminus \bigcup_{\nu<\alpha} \mathrm{I}_\nu(X)$, i.e., $x \notin \mathrm{I}_\alpha(X)$. Thus $\mathrm{I}_\alpha(X) = \{\alpha\} \times \kappa_\alpha$. $\square_{3.1}$

Theorem 3.2. *For any sequence s of cardinals the following statements are equivalent*

(1) $s = CS(X)$ for some regular scattered space X,
(2) $s = CS(X)$ for some 0-dimensional scattered space X,
(3) for some natural number m there are infinite cardinals $\kappa_0 > \kappa_1 > \cdots > \kappa_{m-1}$ and for all $i < m$ sequences $s_i \in S_{\kappa_i}$ such that $s = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}$ or $s = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}{}^\frown \langle n\rangle$ for some natural number $n > 0$.

Proof.
(1)$\Longrightarrow$ (3)
By induction on j we choose ordinals $\nu_j < \mathrm{ht}(X)$ and cardinals κ_j such that $\nu_0 = 0$ and $\kappa_0 = |\,\mathrm{I}_0(X)|$, moreover, for $j > 0$ with κ_{j-1} infinite

$$\nu_j = \min\big\{\nu \le \mathrm{ht}(X) : |\,\mathrm{I}_\nu(X)| < \kappa_{j-1}\big\},$$

and $\kappa_j = |\,\mathrm{I}_{\nu_j}(X)|$. We stop when κ_m is finite. For each $j < m$ let $\delta_j = \nu_{j+1} \dot{-} \nu_j$. Then the sequence $s_j = \big\langle |\,\mathrm{I}_{\nu_j+\delta}(X)| : \delta < \delta_j\big\rangle$ is in S_{κ_j}. Thus $\mathrm{CS}(X) = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}$ provided $\kappa_m = 0$ (i.e. $\mathrm{I}_{\nu_m}(X) = \emptyset$) and $\mathrm{CS}(X) = s_0{}^\frown s_1{}^\frown \ldots {}^\frown s_{m-1}{}^\frown \langle \kappa_m\rangle$ when $0 < \kappa_m < \omega$.

(3)$\Longrightarrow$ (2)
First we prove this implication for sequences s of the form $s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{m-1}$ by induction on m. If $s \in S_{\kappa_o}$ then the statement is just lemma 3.1

Assume now that $s = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{m-1}$, where $\kappa_0 > \kappa_1 > \dots > \kappa_{m-1}$ and $s_i \in S_{\kappa_i}$ for $i < m$.

According to lemma 3.1 there is a 0-dimensional space Y with cardinal sequence s_{m-1}. Using the inductive assumption we can also fix pairwise disjoint 0-dimensional topological spaces $X_{y,n}$ for $\langle y, n\rangle \in I_0(Y) \times \omega$, each having the cardinal sequence $s' = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{m-2}$. We then define the space $Z = \langle Z, \tau\rangle$ as follows. Let

$$Z = Y \cup \bigcup\{X_{y,n} : y \in I_0(Y), n < \omega\}.$$

A set $U \subset Z$ is in τ iff

(i) $U \cap Y$ is open in Y,
(ii) $U \cap X_{y,n}$ is open in $X_{y,n}$ for each $\langle y, n\rangle \in I_0(Y) \times \omega$,
(iii) if $y \in I_0(Y) \cap U$ then there is $m < \omega$ such that $\bigcup\{X_{y,n} : m < n < \omega\} \subset U$.

If U is a clopen subset of Y and $n < \omega$ then it is easy to check that

$$Z(U, n) = U \cup \bigcup\{X_{y,m} : y \in \mathrm{I}_0(Y) \cap U, n < m < \omega\}$$

is clopen in Z. Hence

$$\mathcal{B} = \{Z(U, n) : U \subset Y \text{ is clopen}, n < \omega\} \cup \{T : T \text{ is a clopen subset of some } X_{y,n}\}$$

is a clopen base of Z and so Z is 0-dimensional.

Let $\delta' = \text{length}(s')$ and $\delta = \text{length}(s)$.

Claim 3.2.1. $\mathrm{I}_\alpha(Z) = \bigcup\{\mathrm{I}_\alpha(X_{y,n}) : \langle y, n\rangle \in \mathrm{I}_0(Y) \times \omega\}$ *for* $\alpha < \delta'$.

Proof of the claim 3.2.1. Since $X_{y,n}$ is an open subspace of Z it follows that $\mathrm{I}_\alpha(X_{y,n}) \subset \mathrm{I}_\alpha(Z)$. On the other hand,

$$Y \subset \overline{\bigcup\{\mathrm{I}_\alpha(X_{y,n}) : \langle y, n\rangle \in \mathrm{I}_0(Y) \times \omega\}}^{Z},$$

hence $Y \cap \mathrm{I}_\alpha(Z) = \emptyset$. $\square_{3.2.1}$

Since, by claim 3.2.1,

$$Z \setminus \bigcup_{\alpha<\delta'} \mathrm{I}_\alpha(Z) = Y,$$

it follows that for $\delta' \le \alpha < \delta$ we have

$$(*) \qquad \mathrm{I}_\alpha(Z) = \mathrm{I}_{\alpha \dot{-} \delta'}(Y).$$

Thus $Z = \bigcup_{\alpha<\delta} \mathrm{I}_\alpha(Z)$, hence Z is a scattered space of height δ.

If $\alpha < \delta'$ then, by claim 3.2.1,

$$|\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_0(y)| \cdot \omega \cdot s'(\alpha) = \kappa_{m-1} \cdot \omega \cdot s'(\alpha) = s'(\alpha) = s(\alpha).$$

If $\delta' \le \alpha < \delta$ then, by $(*)$, $|\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_{\alpha \dot{-} \delta'}(Y)| = s_{m-1}(\alpha \dot{-} \delta') = s(\alpha)$, consequently $\mathrm{CS}(Z) = s$.

Thus we proved the statement for sequences of the form $s_0{}^\frown \dots {}^\frown s_{m-1}$.

If $s = s_0{}^\frown \dots {}^\frown s_{m-1}{}^\frown \langle n \rangle$ then writing $s' = s_0{}^\frown \dots {}^\frown s_{m-1}$ we can first find pairwise disjoint 0-dimensional scattered spaces $X_{i,m}$, $\langle i, m\rangle \in n \times \omega$ each having cardinal sequence s'. Let

$$Z = \{x_i : i < n\} \cup \bigcup \{X_{\langle i,m \rangle} : i < n, m < \omega\}.$$

Declare a set $U \subset Z$ open iff

(i) $U \cap X_{i,m}$ is open in $X_{i,m}$ for each $\langle i, m\rangle \in n \times \omega$,
(ii) if $x_i \in U$ then there is $n_i < \omega$ such that $\bigcup\{X_{i,m} : n_i < m < \omega\} \subset U$.

Then Z is 0-dimensional, and

$$\mathrm{I}_\alpha(Z) = \begin{cases} \bigcup\{\mathrm{I}_\alpha(X_{i,m}) : i < n, m < \omega\} & \text{if } \alpha < \mathrm{length}(s'), \\ \{x_i : i < n\} & \text{if } \alpha = \mathrm{length}(s'). \end{cases}$$

Hence again Z is a scattered space with $\mathrm{CS}(Z) = s$.

(2)$\Longrightarrow$ (1) Straightforward.

$\square_{3.2}$

We leave it to the reader to verify that the sequences described in item (3) of theorem 3.2 are exactly those mentioned in the beginning of the section with the additional obvious necessary condition that all but the last term of the sequence are infinite cardinals.

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Alfréd Rényi Institute of Mathematics
Email address: juhasz@renyi.hu

The Hebrew University of Jerusalem
Email address: shelah@math.huji.ac.il

Alfréd Rényi Institute of Mathematics
Email address: soukup@renyi.hu

Eötvös University of Budapest
Email address: zoli@renyi.hu