ADDITIVITY PROPERTIES OF TOPOLOGICAL DIAGONALIZATIONS

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Abstract. In a work of Just, Miller, Scheepers and Szeptycki it was asked whether certain diagonalization properties for sequences of open covers are provably closed under taking finite or countable unions. In a recent work, Scheepers proved that one of the properties in question is closed under taking countable unions. After surveying the known results, we show that none of the remaining classes is provably closed under taking finite unions, and thus settle the problem. We also show that one of these properties is consistently (but not provably) closed under taking unions of size less than the continuum, by relating a combinatorial version of this problem to the Near Coherence of Filters (NCF) axiom, which asserts that the Rudin-Keisler ordering is downward directed.

1. Introduction

1.1. Selection principles.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be collections of covers of a space X. The following selection hypotheses have a long history for the case when the collections $\mathfrak{U}$ and $\mathfrak{V}$ are topologically significant.

$\mathsf{S}_1(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{V_n\}_{n\in\mathbb{N}}$ such that for each n $V_n\in\mathcal{U}_n$, and $\{V_n\}_{n\in\mathbb{N}}\in\mathfrak{V}$.

$\mathsf{S}_{fin}(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that each $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\mathfrak{V}$.

$\mathsf{U}_{fin}(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that for each n $\mathcal{F}_n$ is a finite (possibly

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empty) subset of $\mathcal{U}_n$, and either for some n $\cup\mathcal{F}_n = X$, or else $\{\cup\mathcal{F}_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

Assume that $\mathfrak{V} \subseteq \mathfrak{U}$. Following [29], we say that X satisfies $\binom{\mathfrak{U}}{\mathfrak{V}}$ (read: $\mathfrak{U}$ *choose* $\mathfrak{V}$) if for each cover $\mathcal{U} \in \mathfrak{U}$ there exists a subcover $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathfrak{V}$. Observe that $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$ implies $\binom{\mathfrak{U}}{\mathfrak{V}}$.

1.2. **Special covers.**

We will concentrate on spaces for which the usual induced topology has a subbase whose elements are *clopen* (both closed and open), that is, sets which are *zero-dimensional.* More specifically, we will usually work in $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$, and by *set of reals* we mean a subset of $\mathbb{P}$ (or any homeomorphic space).

Let X be a set of reals. An *ω-cover* $\mathcal{U}$ of X is a cover such that each finite subset of X is contained in some member of the cover. $\mathcal{U}$ is a *γ-cover* of X if it is infinite, and each element of X belongs to all but finitely many members of the cover. Let $\mathcal{O}$, Ω, and Γ denote the collections of countable open covers, ω-covers, and γ-covers of X, respectively, and let $\mathcal{B}, \mathcal{B}_\Omega, \mathcal{B}_\Gamma$ be the corresponding countable *Borel* covers. The diagonalization properties for these types of covers include as particular cases the properties $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$ of Menger [19], $\mathsf{U}_{fin}(\Gamma, \Gamma)$ of Hurewicz [15], $\mathsf{S}_1(\Omega, \Gamma)$ of Gerlits and Nagy (known as the γ-property) [14], $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ of Rothberger (known as the C'' property) [21], $\mathsf{S}_{fin}(\Omega, \Omega)$ of Arkhangel'skiǐ [1], and $\mathsf{S}_1(\Omega, \Omega)$ of Sakai [22]. These properties were extensively studied in the general framework in, e.g., [23, 17, 26]. Many of these properties turn out equivalent. The surviving properties appear in Figure 1, where an arrow denotes implication. To understand the implications in this diagram, use the fact that $\mathsf{S}_1(\mathcal{O}, \mathcal{O}) = \mathsf{S}_1(\Omega, \mathcal{O})$ [23]. Only two implications (the dotted ones in the diagram) remain unsettled.

In the diagram, each property appears together with its *critical cardinality*, that is, the minimal size of a set of reals which does not satisfy that property. The constants $\mathfrak{p}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{s}$, and $\mathsf{cov}(\mathcal{M})$ are the pseudo-intersection number, the unbounding number, the dominating number, the splitting number, and the covering number of the meager (first category) ideal, respectively – see, e.g., [12, 8] for the definitions of these constants.

1.3. **Additivity properties.**

One of the important questions about a class $\mathcal{J}$ of sets of reals is whether it is closed under taking countable or at least finite unions, that is, whether it is countably or finitely *additive.* A more informative task is to determine the exact cardinality κ such that a union of less than κ members of $\mathcal{J}$ belongs to $\mathcal{J}$, but a union of κ many need not. This cardinal κ is called the *additivity*

$$\begin{tikzcd}
 & & & \begin{array}{c}\mathsf{U}_{fin}(\Gamma,\Gamma)\\ \mathfrak{b}\end{array} \arrow[rr] \arrow[dr, dotted, "?"] & & \begin{array}{c}\mathsf{U}_{fin}(\Gamma,\Omega)\\ \mathfrak{d}\end{array} \arrow[rr] \arrow[dl, dotted, "?"'] & & \begin{array}{c}\mathsf{U}_{fin}(\Gamma,\mathcal{O})\\ \mathfrak{d}\end{array} \\
 & & & & \begin{array}{c}\mathsf{S}_{fin}(\Gamma,\Omega)\\ \mathfrak{d}\end{array} \arrow[ur] & & & \\
 & \begin{array}{c}\mathsf{S}_1(\Gamma,\Gamma)\\ \mathfrak{b}\end{array} \arrow[uurr] \arrow[rr] & & \begin{array}{c}\mathsf{S}_1(\Gamma,\Omega)\\ \mathfrak{d}\end{array} \arrow[ur] \arrow[rrr] & & & \begin{array}{c}\mathsf{S}_1(\Gamma,\mathcal{O})\\ \mathfrak{d}\end{array} \arrow[uur] & \\
\begin{array}{c}\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)\\ \mathfrak{b}\end{array} \arrow[ur] \arrow[rr] & & \begin{array}{c}\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Omega)\\ \mathfrak{d}\end{array} \arrow[ur] \arrow[rrr] & & & \begin{array}{c}\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B})\\ \mathfrak{d}\end{array} \arrow[ur] & & \\
 & & & & \begin{array}{c}\mathsf{S}_{fin}(\Omega,\Omega)\\ \mathfrak{d}\end{array} \arrow[uuu] & & & \\
 & & \begin{array}{c}\mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)\\ \mathfrak{d}\end{array} \arrow[urr] \arrow[uu] & & & & & \\
 & \begin{array}{c}\mathsf{S}_1(\Omega,\Gamma)\\ \mathfrak{p}\end{array} \arrow[uuuu] \arrow[rr] & & \begin{array}{c}\mathsf{S}_1(\Omega,\Omega)\\ \mathsf{cov}(\mathcal{M})\end{array} \arrow[uuuu] \arrow[uur] \arrow[rrr] & & & \begin{array}{c}\mathsf{S}_1(\mathcal{O},\mathcal{O})\\ \mathsf{cov}(\mathcal{M})\end{array} \arrow[uuuu] & \\
\begin{array}{c}\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma)\\ \mathfrak{p}\end{array} \arrow[uuuu] \arrow[ur] \arrow[rr] & & \begin{array}{c}\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)\\ \mathsf{cov}(\mathcal{M})\end{array} \arrow[ur] \arrow[uu] \arrow[rrr] & & & \begin{array}{c}\mathsf{S}_1(\mathcal{B},\mathcal{B})\\ \mathsf{cov}(\mathcal{M})\end{array} \arrow[uuuu] \arrow[ur] & &
\end{tikzcd}$$

FIGURE 1. The surviving classes

number of $\mathcal{J}$, and denoted $\mathsf{add}(\mathcal{J})$. Despite the extensive study of the classes defined by the mentioned diagonalization properties, not much was known about their additivity properties. The purpose of this paper is to clear up most of what was not known with respect to the additivity numbers of these properties. It turns out that most of the classes are *not* provably closed under taking finite unions. For some of the classes which are not provably closed under taking finite unions, we consider the consistency of their *being* closed under taking finite (and larger) unions. For $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$ and $\mathsf{U}_{fin}(\Gamma, \Omega)$ this turns out to be related to the well known NCF (Near Coherence of Filters) axiom, which asserts that the Rudin-Keisler ordering of ultrafilters is downward directed.

2. POSITIVE RESULTS

In [17] it is pointed out that the properties $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$ (*Menger's property*), $\mathsf{U}_{fin}(\Gamma, \Gamma)$ (*Hurewicz' property*), $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (*Rothberger's property*), and $\mathsf{S}_1(\Gamma, \mathcal{O})$ are closed under taking countable unions. The argument behind these assertions actually shows the following.

Proposition 2.1. *Each property of the form* $\Pi(\mathfrak{U},\mathcal{O})$ *(or* $\Pi(\mathfrak{U},\mathcal{B})$*),* $\Pi\in\{\mathsf{S}_1,\mathsf{S}_{fin},\mathsf{U}_{fin}\}$*, is closed under taking countable unions.*

Proof. Let $A_1, A_2,\dots$ be a partition of $\mathbb{N}$ into disjoint infinite sets. Assume that $X_1, X_2\dots$ satisfy $\Pi(\mathfrak{U},\mathcal{O})$. Assume that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}\in\mathfrak{U}$ are covers of $X=\bigcup_{k\in\mathbb{N}}X_k$. For each k, use this property of X_k to extract from the sequence $\{\mathcal{U}_n\}_{n\in A_k}$ the appropriate cover $\mathcal{V}_k$ of X_k. Then $\bigcup_{k\in\mathbb{N}}\mathcal{V}_k$ is the desired cover of X.

The proof for $\Pi(\mathfrak{U},\mathcal{B})$ is identical. □

We now give some more exact additivity results. Denote the critical cardinality of a class $\mathcal{J}$ of sets of reals by $\mathsf{non}(\mathcal{J})$. The *covering number* of $\mathcal{J}$, $\mathsf{cov}(\mathcal{J})$, is the minimal cardinality of a subcollection $\mathcal{F}\subseteq\mathcal{J}$ such that $\cup\mathcal{F}=\mathbb{P}$. Then $\mathsf{add}(\mathcal{J})$ is a regular cardinal, $\mathsf{add}(\mathcal{J})\le\mathrm{cf}(\mathsf{non}(\mathcal{J}))$, and $\mathsf{add}(\mathcal{J})\le\mathsf{cov}(\mathcal{J})$.

Proposition 2.2. *If* $\mathcal{I}$ *and* $\mathcal{J}$ *are collections of sets of reals such that:*

$X\in\mathcal{I}$ *if, and only if, for each Borel function* $\Psi: X\to\mathbb{P}$
$\Psi[X]\in\mathcal{J}$.

then $\mathsf{add}(\mathcal{J})\le\mathsf{add}(\mathcal{I})$.

Proof. Assume that X_i, $i\in I$, are members of $\mathcal{I}$ such that $X=\bigcup_{i\in I}X_i\notin\mathcal{I}$. Then there exists a Borel function $\Psi: X\to\mathbb{P}$ such that $\Psi[X]\notin\mathcal{J}$. But $\Psi[X]=\bigcup_{i\in I}\Psi[X_i]$. □

Using results from [26] and [30], we get from Proposition 2.2 that the following inequalities hold.

1. $\mathsf{add}(\mathsf{S}_1(\mathcal{O},\mathcal{O}))\le\mathsf{add}(\mathsf{S}_1(\mathcal{B},\mathcal{B}))\le\mathsf{cf}(\mathsf{cov}(\mathcal{M}))$,
2. $\max\{\mathsf{add}(\mathsf{S}_1(\Gamma,\Gamma)),\mathsf{add}(\mathsf{U}_{fin}(\Gamma,\Gamma))\}\le\mathsf{add}(\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma))\le\mathfrak{b}$,
3. $\max\{\mathsf{add}(\mathsf{S}_1(\Gamma,\mathcal{O})),\mathsf{add}(\mathsf{U}_{fin}(\Gamma,\mathcal{O}))\}\le\mathsf{add}(\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}))\le\mathsf{cf}(\mathfrak{d})$,
4. $\mathsf{add}(\mathsf{S}_1(\Omega,\Gamma))\le\mathsf{add}(\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma))\le\mathfrak{p}$;
5. $\max\{\mathsf{add}(\mathsf{S}_1(\Gamma,\Omega)),\mathsf{add}(\mathsf{S}_{fin}(\Gamma,\Omega)),\mathsf{add}(\mathsf{U}_{fin}(\Gamma,\Omega))\}\le$
   $\le\mathsf{add}(\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Omega))\le\mathsf{cf}(\mathfrak{d})$.

Let $\mathcal{N}$ denote the collection of null (measure zero) sets of reals. In [3] it is proved that $\mathsf{add}(\mathcal{N})\le\mathsf{add}(\mathsf{S}_1(\mathcal{O},\mathcal{O}))$. In order to get similar lower bounds on other classes, we will use combinatorial characterizations of these classes. The Baire space ${}^{\mathbb{N}}\mathbb{N}$ is assigned the product topology. Hurewicz ([16], see also Recław [20]) proved that a set of reals X satisfies $\mathsf{U}_{fin}(\Gamma,\mathcal{O})$ if, and only if, every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is not dominating. Likewise, he showed that X satisfies $\mathsf{U}_{fin}(\Gamma,\Gamma)$ if, and only if, every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded (with respect to $\le^*$). It is easy to see that a union of less than $\mathfrak{b}$ many bounded subsets of ${}^{\mathbb{N}}\mathbb{N}$ is bounded, and a union of less than $\mathfrak{b}$ many subsets of ${}^{\mathbb{N}}\mathbb{N}$ which are not dominating is not dominating.

Corollary 2.3. *The following holds:*

(1) $\mathsf{add}(\mathsf{U}_{fin}(\Gamma,\Gamma)) = \mathsf{add}(\mathsf{U}_{fin}(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)) = \mathfrak{b}$;
(2) $\mathfrak{b} \le \mathsf{add}(\mathsf{U}_{fin}(\Gamma,\mathcal{O})) \le \mathsf{add}(\mathsf{U}_{fin}(\mathcal{B}_\Gamma,\mathcal{B})) \le \mathsf{cf}(\mathfrak{d})$.

Consider an unbounded subset B of ${}^{\mathbb{N}}\mathbb{N}$ such that $|B| = \mathfrak{b}$, and define, for each $f \in B$, $Y_f = \{g \in {}^{\mathbb{N}}\mathbb{N} : f \not\le^* g\}$. Then the sets Y_f are not dominating, but $\bigcup_{f\in B} Y_f = {}^{\mathbb{N}}\mathbb{N}$: For each $g \in {}^{\mathbb{N}}\mathbb{N}$ there exists $f \in B$ such that $f \not\le^* g$, that is, $g \in Y_f$. Thus the second assertion in Corollary 2.3 cannot be strengthened in a trivial manner.

Problem 2.4. *Is* $\mathsf{add}(\mathsf{U}_{fin}(\Gamma,\mathcal{O}))$ *provably equal to* $\mathfrak{b}$*?*

Let $\mathfrak{h}$ be the *density number* [8]. Then $\aleph_1 \le \mathfrak{h}$. Recently, Scheepers [24] proved that $\mathfrak{h} \le \mathsf{add}(\mathsf{S}_1(\Gamma,\Gamma))$. Thus, $\mathfrak{h} \le \mathsf{add}(\mathsf{S}_1(\Gamma,\Gamma)) \le \mathfrak{b}$.

3. Negative results

Showing that a certain class is not closed under taking finite unions is apparently harder: All known results require axioms beyond ZFC. (This is often necessary – see Section 4.) In [13, 29] it is shown that assuming the Continuum Hypothesis, no class between $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma)$ and $\mathsf{S}_1(\Omega,\Gamma)$ is closed under taking finite unions.

The following problem is posed in [17].

Problem 3.1 ([17], Problem 5)**.** *Is any of the classes* $\mathsf{S}_1(\Gamma,\Gamma)$, $\mathsf{S}_1(\Gamma,\Omega)$, $\mathsf{S}_1(\Omega,\Omega)$, $\mathsf{S}_{fin}(\Omega,\Omega)$, $\mathsf{S}_{fin}(\Gamma,\Omega)$ *and* $\mathsf{U}_{fin}(\Gamma,\Omega)$ *closed under taking countable or finite unions?*

As mentioned in Section 2, Scheepers answered the question positively for $\mathsf{S}_1(\Gamma,\Gamma)$. We will show that if $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ (in particular, assuming the Continuum Hypothesis), then the answer is *no* for all of the remaining classes. In fact, we will show that none of the classes which lie between $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$ and $\mathsf{U}_{fin}(\Gamma,\Omega)$ is provably closed under taking finite unions.

For clarity of exposition, we will first treat the open case, and then explain how to modify the constructions in order to cover the Borel case.

3.1. The open case.

For convenience, we will work in ${}^{\mathbb{N}}\mathbb{Z}$ (with pointwise addition), which is homeomorphic to $\mathbb{P}$. The notions that we will use are topological, thus the following constructions can be translated to constructions in $\mathbb{P}$.

A collection $\mathcal{J}$ of sets of reals is *translation invariant* if for each real x and each $X \in \mathcal{J}$, $x + X \in \mathcal{J}$. $\mathcal{J}$ is *negation invariant* if for each $X \in \mathcal{J}$, $-X \in \mathcal{J}$ as well. For example, $\mathcal{M}$ and $\mathcal{N}$ are negation and translation invariant (and there are many more examples).

Lemma 3.2. *If $\mathcal{J}$ is negation and translation invariant and if X is a union of less than $\mathsf{cov}(\mathcal{J})$ many elements of $\mathcal{J}$, then for each $x \in {}^{\mathbb{N}}\mathbb{Z}$ there exist $y, z \in {}^{\mathbb{N}}\mathbb{Z} \setminus X$ such that $y + z = x$.*

Proof. $(x - X) \cup X$ is a union of less than $\mathsf{cov}(\mathcal{J})$ many elements of $\mathcal{J}$. Thus we can choose an element $y \in {}^{\mathbb{N}}\mathbb{Z} \setminus ((x - X) \cup X) = (x - {}^{\mathbb{N}}\mathbb{Z} \setminus X) \cap ({}^{\mathbb{N}}\mathbb{Z} \setminus X)$; therefore there exists $z \in {}^{\mathbb{N}}\mathbb{Z} \setminus X$ such that $x - z = y$, that is, $x = y + z$. $\square$

For a finite subset F of ${}^{\mathbb{N}}\mathbb{N}$, define $\max(F) \in {}^{\mathbb{N}}\mathbb{N}$ to be the function g such that $g(n) = \max\{f(n) : f \in F\}$ for each n. A subset Y of ${}^{\mathbb{N}}\mathbb{N}$, is *finitely-dominating* if the collection

$$\text{maxfin}(Y) := \{\max(F) : F \text{ is a finite subset of } Y\}$$

is dominating. Y is *k-dominating* if for each $g \in {}^{\mathbb{N}}\mathbb{N}$ there exists a k-element subset F of Y such that $g \leq^* \max(F)$ [7]. Clearly each k-dominating subset of ${}^{\mathbb{N}}\mathbb{N}$ is also finitely dominating. Following is the key observation for our solution of Problem 3.1.

Theorem 3.3 ([30])**.** *For a set of reals X, the following are equivalent:*

1. *X satisfies $\mathsf{U}_{fin}(\Gamma, \Omega)$;*
2. *For each continuous function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$, $\Psi[X]$ is not finitely-dominating.*

Proposition 3.4. *Assume that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. Then there exist $\mathfrak{c}$-Luzin subsets L_0 and L_1 of ${}^{\mathbb{N}}\mathbb{Z}$ satisfying $\mathsf{S}_1(\Omega, \Omega)$, such that the ($\mathfrak{c}$-Luzin) set $L_0 \cup L_1$ is 2-dominating. In particular, $L_0 \cup L_1$ does not satisfy $\mathsf{U}_{fin}(\Gamma, \Omega)$.*

Proof. This is a generalization of the constructions of [17] and [26]. Assume that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. Let $\{y_\alpha : \alpha < \mathfrak{c}\}$ enumerate ${}^{\mathbb{N}}\mathbb{Z}$; let $\{M_\alpha : \alpha < \mathfrak{c}\}$ enumerate all F_σ meager sets in ${}^{\mathbb{N}}\mathbb{Z}$ (observe that this family is cofinal in $\mathcal{M}$), and let $\{\{\mathcal{U}_n^\alpha\}_{n\in\mathbb{N}} : \alpha < \mathfrak{c}\}$ enumerate all countable sequences of countable families of open sets.

Fix a countable dense subset $Q \subseteq {}^{\mathbb{N}}\mathbb{Z}$. We construct $L_0 = \{x_\beta^0 : \beta < \mathfrak{c}\} \cup Q$ and $L_1 = \{x_\beta^1 : \beta < \mathfrak{c}\} \cup Q$ by induction on $\alpha < \mathfrak{c}$. During the construction, we make an inductive hypothesis and verify that it remains true after making the inductive step.

At stage $\alpha \geq 0$ set

$$\begin{aligned} X_\alpha^0 &= \{x_\beta^0 : \beta < \alpha\} \cup Q \\ X_\alpha^1 &= \{x_\beta^1 : \beta < \alpha\} \cup Q \end{aligned}$$

and consider the sequence $\{\mathcal{U}_n^\alpha\}_{n\in\mathbb{N}}$. For each $i < 2$, do the following. Call α *i-good* if for each n $\mathcal{U}_n^\alpha$ is an ω-cover of X_α^i. Assume that α

is i-good. Since $\mathsf{cov}(\mathcal{M}) = \mathsf{non}(\mathsf{S}_1(\Omega,\Omega))$ [17] and we assume that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, there exist elements $U_n^{\alpha,i} \in \mathcal{U}_n^\alpha$ such that $\{U_n^{\alpha,i}\}_{n\in\mathbb{N}}$ is an ω-cover of X^i_α. We make the *inductive hypothesis* that for each i-good $\beta < \alpha$, $\{U_n^{\beta,i}\}_{n\in\mathbb{N}}$ is an ω-cover of X^i_α. For each finite $F \subseteq X^i_\alpha$, and each i-good $\beta \leq \alpha$, define

$$G_i(F,\beta) = \cup\{U_n^{\beta,i} : F \subseteq U_n^{\beta,i}\}.$$

Then $Q \subseteq G_i(F,\beta)$ and thus $G_i(F,\beta)$ is open and dense.

Set

$$Y_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup \bigcup_{\substack{i<2,\, i\text{-good } \beta\leq\alpha,\\ \text{Finite } F\subseteq X^i_\alpha}} ({}^{\mathbb{N}}\mathbb{Z} \setminus G_i(F,\beta))$$

Then Y_α is a union of less than $\mathsf{cov}(\mathcal{M})$ many meager sets, thus by Lemma 3.2 we can pick $x^0_\alpha, x^1_\alpha \in {}^{\mathbb{N}}\mathbb{Z}\setminus Y_\alpha$ such that $x^0_\alpha + x^1_\alpha = y_\alpha$. To see that the inductive hypothesis is preserved, observe that for each finite $F \subseteq X^i_\alpha$ and i-good $\beta \leq \alpha$, $x^i_\alpha \in G_i(F,\beta)$ and therefore $F\cup\{x^i_\alpha\} \subseteq U_n^{\beta,i}$ for some n.

Clearly L_0 and L_1 are Luzin sets, and L_0+L_1 is a dominating family, thus $L_0 \cup L_1$ is 2-dominating, which implies by Theorem 3.3 that it does not satisfy $\mathsf{U}_{fin}(\Gamma,\Omega)$. It remains to show that L_0 and L_1 satisfy $\mathsf{S}_1(\Omega,\Omega)$.

Fix $i < 2$. Consider, for each $\beta < \mathfrak{c}$, the sequence $\{\mathcal{U}^\beta_n\}_{n\in\mathbb{N}}$. If all members of that sequence are ω-covers of L_i, then in particular they ω-cover X^i_β (that is, β is i-good). By the inductive hypothesis, $\{U_n^{\beta,i}\}_{n\in\mathbb{N}}$ is an ω-cover of X^i_α for each $\alpha < \mathfrak{c}$, and therefore an ω-cover of L_i. $\square$

3.2. **The Borel case.** We now treat the Borel case.

Theorem 3.5. *Assume that* $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. *Then there exist* $\mathfrak{c}$*-Luzin subsets* L_1 *and* L_2 *of* ${}^{\mathbb{N}}\mathbb{Z}$ *satisfying* $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$, *such that the (*$\mathfrak{c}$*-Luzin) set* $L_0 \cup L_1$ *is 2-dominating. In particular,* $L_0 \cup L_1$ *does not satisfy* $\mathsf{U}_{fin}(\Gamma,\Omega)$.

Proof. We follow the proof steps of Proposition 3.4. The major problem is that here the sets $G_i(F,\beta)$ need not be comeager. In order to overcome this, we will consider only ω-covers where these sets are guaranteed to be comeager, and make sure that it is enough to restrict attention to this special sort of ω-covers.

Definition 3.6 ([26])**.** A cover $\mathcal{U}$ of X is *ω-fat* if for each finite $F \subseteq X$ and each finite family $\mathcal{F}$ of nonempty open sets, there exists $U \in \mathcal{U}$ such that $F \subseteq U$ and for each $O \in \mathcal{F}$, $U\cap O$ is not meager. (Thus each

ω-fat cover is an ω-cover.) Let $\mathcal{B}_{\Omega_{\mathrm{fat}}}$ denote the collection of countable ω-fat Borel covers of X.

Lemma 3.7. *Assume that $\mathcal{U}$ is a countable collection of Borel sets. Then $\cup\mathcal{U}$ is comeager if, and only if, for each nonempty basic open set O there exists $U \in \mathcal{U}$ such that $U \cap O$ is not meager.*

Proof. ($\Rightarrow$) Assume that O is a nonempty basic open set. Then $\cup\mathcal{U} \cap O = \cup\{U \cap O : U \in \mathcal{U}\}$ is a countable union which is not meager. Thus there exists $U \in \mathcal{U}$ such that $U \cap O$ is not meager.

($\Leftarrow$) Set $B = \cup\mathcal{U}$. As B is Borel, it has the Baire property. Let O be an open set and M be a meager set such that $B = (O \setminus M) \cup (M \setminus O)$. For each basic open set G, $B \cap G$ is not meager, thus $O \cap G$ is not meager as well. Thus, O is open dense. As $O \setminus M \subseteq B$, we have that $\mathbb{R} \setminus B \subseteq (\mathbb{R} \setminus O) \cup M$ is meager. $\square$

Corollary 3.8. *Assume that $\mathcal{U}$ is an ω-fat cover of some set X. Then:*

1. *For each finite $F \subseteq X$ and finite family $\mathcal{F}$ of nonempty* basic *open sets, the set*
$$\cup\{U \in \mathcal{U} : F \subseteq U \text{ and for each } O \in \mathcal{F},\ U \cap O \notin \mathcal{M}\}$$
is comeager.
2. *For each element x in the intersection of all sets of this form, $\mathcal{U}$ is an ω-fat cover of $X \cup \{x\}$.*

Proof. Write
$$\mathcal{V}_{F,\mathcal{F}} = \{U \in \mathcal{U} : F \subseteq U \text{ and for each } O \in \mathcal{F},\ U \cap O \notin \mathcal{M}\}.$$

(1) Assume that G is a nonempty open set. As $\mathcal{U}$ is ω-fat and the family $\mathcal{F} \cup \{G\}$ is finite, there exists $U \in \mathcal{V}_{F,\mathcal{F}}$ such that $U \cap G$ is not meager. By Lemma 3.7, $\cup\mathcal{V}_{F,\mathcal{F}}$ is comeager.

(2) Assume that F is a finite subset of $X \cup \{x\}$ and $\mathcal{F}$ is a finite family of nonempty basic open sets. As $x \in \cup\mathcal{V}_{F\setminus\{x\},\mathcal{F}}$, there exists $U \in \mathcal{U}$ such that $x \in U$, $F \setminus \{x\} \subseteq U$ (thus $F \subseteq U$), and for each $O \in \mathcal{F}$, $U \cap O$ is not meager. $\square$

Lemma 3.9. *If $|X| < \mathsf{cov}(\mathcal{M})$, then X satisfies $\mathsf{S}_1(\mathcal{B}_{\Omega_{\mathrm{fat}}}, \mathcal{B}_{\Omega_{\mathrm{fat}}})$.*

Proof. Assume that $|X| < \mathsf{cov}(\mathcal{M})$, and let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a sequence of countable Borel ω-fat covers of X. Enumerate each cover $\mathcal{U}_n$ by $\{U^n_k\}_{k\in\mathbb{N}}$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinitely many infinite sets. For each m, let $a_m \in {}^{\mathbb{N}}\mathbb{N}$ be an increasing enumeration of A_m. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an enumeration of all finite families of nonempty basic open sets.

For each finite subset F of X and each m define a function $\Psi^m_F \in {}^{\mathbb{N}}\mathbb{N}$ by

$$\Psi^m_F(n) = \min\{k : F \subseteq U^{a_m(n)}_k \text{ and for each } O \in \mathcal{F}_m,\, U^{a_m(n)}_k \cap O \notin \mathcal{M}\}$$

Since there are less than $\mathsf{cov}(\mathcal{M})$ many functions Ψ^m_F, there exists by [2] a function $f \in {}^{\mathbb{N}}\mathbb{N}$ such that for each m and F, $\Psi^m_F(n) = f(n)$ for infinitely many n. Consequently, $\mathcal{V} = \{U^{a_m(n)}_{f(n)} : m, n \in \mathbb{N}\}$ is an ω-fat cover of X. $\square$

The following lemma justifies our focusing on ω-fat covers.

Lemma 3.10. *Assume that L is a set of reals such that for each nonempty basic open set O, $L \cap O$ is not meager. Then every countable Borel ω-cover $\mathcal{U}$ of L is an ω-fat cover of L.*

Proof. Assume that $\mathcal{U}$ is a countable collection of Borel sets which is not an ω-fat cover of L. Then there exist a finite set $F \subseteq L$ and nonempty open sets $O_1, \dots, O_k$ such that for each $U \in \mathcal{U}$ containing F, $U \cap O_i$ is meager for some i. For each $i = 1, \dots, k$ let

$$M_i = \cup\{U \in \mathcal{U} : F \subseteq U \text{ and } U \cap O_i \in \mathcal{M}\}.$$

Then $M_i \cap O_i$ is meager, thus there exists $x_i \in (L \cap O_i) \setminus M_i$. Then $F \cup \{x_1, \dots, x_k\}$ is not covered by any $U \in \mathcal{U}$. $\square$

Let ${}^{\mathbb{N}}\mathbb{Z} = \{y_\alpha : \alpha < \mathfrak{c}\}$, $\{M_\alpha : \alpha < \mathfrak{c}\}$ be all F_σ meager subsets of ${}^{\mathbb{N}}\mathbb{Z}$, and $\{\{\mathcal{U}^\alpha_n\}_{n\in\mathbb{N}} : \alpha < \mathfrak{c}\}$ be all sequences of countable families of Borel sets. Let $\{O_k : k \in \mathbb{N}\}$ and $\{\mathcal{F}_m : m \in \mathbb{N}\}$ be all nonempty basic open sets and all finite families of nonempty basic open sets, respectively, in ${}^{\mathbb{N}}\mathbb{Z}$.

We construct $L_i = \{x^i_\beta : \beta < \mathfrak{c}\}$, $i = 1, 2$, by induction on $\alpha < \mathfrak{c}$ as follows. At stage $\alpha \geq 0$ set $X^i_\alpha = \{x^i_\beta : \beta < \alpha\}$ and consider the sequence $\{\mathcal{U}^\alpha_n\}_{n\in\mathbb{N}}$. Say that α is i-good if for each n $\mathcal{U}^\alpha_n$ is an ω-fat cover of X^i_α. In this case, by Lemma 3.9 there exist elements $U^{\alpha,i}_n \in \mathcal{U}^\alpha_n$ such that $\{U^{\alpha,i}_n\}_{n\in\mathbb{N}}$ is an ω-fat cover of X^i_α. We make the inductive hypothesis that for each i-good $\beta < \alpha$, $\{U^{\beta,i}_n\}_{n\in\mathbb{N}}$ is an ω-fat cover of X^i_α. For each finite $F \subseteq X^i_\alpha$, i-good $\beta \leq \alpha$, and m define

$$G_i(F, \beta, m) = \cup\{U^{\beta,i}_n : F \subseteq U^{\beta,i}_n \text{ and for each } O \in \mathcal{F}_m,\, U^{\beta,i}_n \cap O \notin \mathcal{M}\}.$$

By Corollary 3.8(1), $G_i(F, \beta, m)$ is comeager. Set

$$Y_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup \bigcup_{\substack{i<2,\ i\text{-good } \beta \leq \alpha \\ m \in \mathbb{N},\ \text{Finite } F \subseteq X^i_\alpha}} ({}^{\mathbb{N}}\mathbb{Z} \setminus G_i(F, \beta, m)),$$

and $Y_\alpha^* = \{x \in {}^{\mathbb{N}}\mathbb{Z} : (\exists y \in Y_\alpha)\ x =^* y\}$ (where $x =^* y$ means that $x(n) = y(n)$ for all but finitely many n.) Then Y_α^* is a union of less than $\mathsf{cov}(\mathcal{M})$ many meager sets. Use Lemma 3.2 to pick $x_\alpha^0, x_\alpha^1 \in {}^{\mathbb{N}}\mathbb{Z} \setminus Y_\alpha^*$ such that $x_\alpha^0 + x_\alpha^1 = y_\alpha$. Let $k = \alpha \bmod \omega$, and change a finite initial segment of x_α^0 and x_α^1 so that they both become members of O_k. Then $x_\alpha^0, x_\alpha^1 \in O_k \setminus Y_\alpha$, and $x_\alpha^0 + x_\alpha^1 =^* y_\alpha$. By Corollary 3.8(2), the inductive hypothesis is preserved.

Thus each L_i satisfies $\mathsf{S}_1(\mathcal{B}_{\Omega_{\text{fat}}}, \mathcal{B}_{\Omega_{\text{fat}}})$ and its intersection with each nonempty basic open set has size $\mathfrak{c}$. By Lemma 3.10, $\mathcal{B}_{\Omega_{\text{fat}}} = \mathcal{B}_\Omega$ for L_i. Finally, $L_0 + L_1$ is dominating, so $L_0 \cup L_1$ is 2-dominating. $\square$

Thus, none of the remaining classes (in the open case as well as the Borel case) is provably additive.

As $\mathsf{non}(\mathsf{U}_{fin}(\Gamma, \Omega)) = \mathfrak{d}$, a natural question is whether the method of Proposition 3.4 can be generalized to work for $\mathsf{U}_{fin}(\Gamma, \Omega)$ under the weaker assumption $\mathfrak{d} = \mathfrak{c}$. But such a trial is doomed to fail: In the coming section we show that an axiom which is consistent with $\mathfrak{d} = \mathfrak{c}$ implies that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$ and $\mathsf{U}_{fin}(\Gamma, \Omega)$ are countably additive.

4. Consistency results

The fact that a property is not provably additive does not rule out the possibility that it is *consistently* additive. Consider for example the properties $\mathsf{S}_1(\Omega, \Gamma)$ and $\mathsf{S}_1(\Omega, \Omega)$. As Rothberger's property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ implies strong measure zero, Borel's Conjecture (which asserts that each strong measure zero set is countable) implies that all elements of $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ are countable, and thus all classes below $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ are closed under taking countable unions. Borel's Conjecture was proved consistent by Laver [18]. An analogue conjecture for the property $\mathsf{U}_{fin}(\Gamma, \Omega)$ is false [17, 24, 4]. Thus, another approach is needed in order to prove the consistency of $\mathsf{U}_{fin}(\Gamma, \Omega)$ being additive.

Consider Theorem 3.3. In its current form, this theorem is not enough for showing that $\mathsf{U}_{fin}(\Gamma, \Omega)$ is consistently closed under taking finite unions: Let Y_0 (respectively, Y_1) be the subset of ${}^{\mathbb{N}}\mathbb{N}$ consisting of those functions which are identically zero on the evens (respectively, odds). Then Y_0 and Y_1 are not finitely-dominating, but $Y_0 \cup Y_1$ is. Denote by ${}^{\mathbb{N}\nearrow}\mathbb{N}$ the *(strictly) increasing* elements of ${}^{\mathbb{N}}\mathbb{N}$. The following variant of the theorem characterization avoids the mentioned problem.

Corollary 4.1. *For a set of reals X, the following are equivalent:*

1. *X satisfies $\mathsf{U}_{fin}(\Gamma, \Omega)$;*
2. *For each continuous function Ψ from X to ${}^{\mathbb{N}\nearrow}\mathbb{N}$, $\Psi[X]$ is not finitely-dominating.*

Proof. $1 \Rightarrow 2$: A continuous image in ${}^{\mathbb{N}\nearrow}\mathbb{N}$ is in particular a continuous image in ${}^{\mathbb{N}}\mathbb{N}$.

$2 \Rightarrow 1$: It is enough to show that 2 implies item 2 of Theorem 3.3. Assume that $\Psi : X \to {}^{\mathbb{N}}\mathbb{N}$ is continuous, and consider the homeomorphism φ from ${}^{\mathbb{N}}\mathbb{N}$ to ${}^{\mathbb{N}\nearrow}\mathbb{N}$ defined by

$$f(n) \overset{\varphi}{\mapsto} n + f(0) + f(1) + \ldots + f(n).$$

$\varphi \circ \Psi[X]$ is a continuous image of X in ${}^{\mathbb{N}\nearrow}\mathbb{N}$, and is thus not finitely-dominating. Let $g \in {}^{\mathbb{N}}\mathbb{N}$ be a witness for that. Obviously for each $f \in {}^{\mathbb{N}}\mathbb{N}$ we have that $f(n) \leq \varphi(f)(n)$ for each n. Thus g witnesses that $\Psi[X]$ is not finitely-dominating. □

We now consider the purely combinatorial counterpart of the question whether $\mathsf{U}_{fin}(\Gamma, \Omega)$ is closed under taking finite unions. Let $\mathfrak{D}_{\text{fin}}$ denote the collection of subsets of ${}^{\mathbb{N}\nearrow}\mathbb{N}$ which are not finitely-dominating. By Corollary 4.1 we get that if $\mathfrak{D}_{\text{fin}}$ is closed under taking unions of size κ, then so is $\mathsf{U}_{fin}(\Gamma, \Omega)$ (and, therefore, also $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$). In the sequel we will show that $\mathfrak{D}_{\text{fin}}$ is closed under taking finite unions if, and only if, the Rudin-Keisler ordering is downward directed.

For natural numbers k, m, denote by $[k, m)$ the set $\{k, k+1, \ldots, m-1\}$. For $a \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ and a filter $\mathcal{F}$ on $\mathbb{N}$, let

$$\mathcal{F}/a = \{A : \bigcup_{n \in A} [a(n), a(n+1)) \in \mathcal{F}\}.$$

We say that filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathbb{N}$ are *compatible* in the *Rudin-Keisler* ordering if there exists $a \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ such that $\mathcal{F}_1/a \cup \mathcal{F}_2/a$ satisfies the finite intersection property (that is, it is a filter base).

Definition 4.2. Let *NCF (near coherence of filters)* stand for the statement that the Rudin-Keisler ordering is downward directed, that is, that each two ultrafilters on $\mathbb{N}$ are compatible.

NCF is independent of ZFC [10, 11], and has many equivalent forms and implications (e.g., [5, 6]).

Lemma 4.3. *If NCF fails, then there exist ultrafilters $\mathcal{F}_1$ and $\mathcal{F}_2$ such that for each $a \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ there exist $A_1 \in \mathcal{F}_1/a$ and $A_2 \in \mathcal{F}_2/a$ such that for all but finitely many $n \in A_1$ and $m \in A_2$, $|n - m| > 1$.*

Proof. Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are incompatible filters and let a be a function in ${}^{\mathbb{N}\nearrow}\mathbb{N}$. Define $b_0, b_1 \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ by

$$\begin{aligned} b_0(n) &= a(2n) \\ b_1(n) &= a(2n+1) \end{aligned}$$

Then there exist

$$\begin{array}{ll} X_1 \in \mathcal{F}_1/b_0 & X_2 \in \mathcal{F}_2/b_0 \\ Y_1 \in \mathcal{F}_1/b_1 & Y_2 \in \mathcal{F}_2/b_1 \end{array}$$

such that the sets $X_1 \cap X_2$ and $Y_1 \cap Y_2$ are finite. For $i = 1, 2$ let

$$\begin{aligned} \tilde{X}_i &= 2 \cdot X_i \cup (2 \cdot X_i + 1) \\ \tilde{Y}_i &= (2 \cdot Y_i + 1) \cup (2 \cdot Y_i + 2) \end{aligned}$$

Observe that $\tilde{X}_1 \cap \tilde{X}_2$ and $\tilde{Y}_1 \cap \tilde{Y}_2$ are also finite. Now,

$$\begin{aligned} \bigcup_{n \in X_i} [b_0(n), b_0(n+1)) &= \bigcup_{n \in \tilde{X}_i} [a(n), a(n+1)) \\ \bigcup_{n \in Y_i} [b_1(n), b_1(n+1)) &= \bigcup_{n \in \tilde{Y}_i} [a(n), a(n+1)) \end{aligned}$$

therefore $\tilde{X}_i, \tilde{Y}_i \in \mathcal{F}_i/a$, thus $A_i = \tilde{X}_i \cap \tilde{Y}_i \in \mathcal{F}_i/a$. If $n \in A_1$ is even, then $n, n+1 \in \tilde{X}_1$, and $n-1, n \in \tilde{Y}_1$. Thus, if n is large enough, then $n, n+1 \notin \tilde{X}_2$, and $n-1, n \notin \tilde{Y}_2$, therefore $n-1, n, n+1 \notin A_2$. The case that $n \in A_1$ is odd is similar. $\square$

Theorem 4.4. *NCF holds if, and only if, $\mathfrak{D}_{\text{fin}}$ is closed under taking finite unions.*

Proof. ($\Rightarrow$) Following the convention of [27], for $f, g \in {}^{\mathbb{N}}\mathbb{N}$ we write $[f \leq g]$ for the set $\{n : f(n) \leq g(n)\}$. Assume that $Y_1, Y_2 \in \mathfrak{D}_{\text{fin}}$, and let $g_1, g_2 \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ witness that. Then there exist filters $\mathcal{F}_1$, $\mathcal{F}_2$ such that for $i = 1, 2$ and each $f \in Y_i$, $[f \leq g_i] \in \mathcal{F}_i$. Let $a \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ be such that $a(0) = 0$, and $\mathcal{F}_1/a \cup \mathcal{F}_2/a$ has the finite intersection property. Let $h \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ be such that for each n $h(a(n)) \geq \max\{g_1(a(n+1)), g_2(a(n+1))\}$.

Let F_1 and F_2 be finite subsets of Y_1 and Y_2, respectively. Then for $i = 1, 2$, $[\max(F_i) \leq g_i] \in \mathcal{F}_i$. Let

$$B_i = \{n : [\max(F_i) \leq g_i] \cap [a(n), a(n+1)) \neq \emptyset\}.$$

Then $\bigcup_{n \in B_i} [a(n), a(n+1)) \supseteq [\max(F_i) \leq g_i] \in \mathcal{F}_i$, thus $B_i \in \mathcal{F}_i/a$, so $B = B_1 \cap B_2$ is infinite.

For each $n \in B$ let $k_i \in [\max(F_i) \leq g_i] \cap [a(n), a(n+1))$. Then $h(a(n)) \geq g_i(a(n+1)) > g_i(k_i) \geq \max(F_i)(k_i) \geq \max(F_i)(a(n))$, Thus, for each $n \in B$, $h(a(n)) \geq \max(F_1 \cup F_2)(a(n))$. Therefore, $Y_1 \cup Y_2 \in \mathfrak{D}_{\text{fin}}$.

($\Leftarrow$) For a filter $\mathcal{F}$ and $h \in {}^{\mathbb{N}\nearrow}\mathbb{N}$, define

$$Y_{\mathcal{F},h} = \{f \in {}^{\mathbb{N}\nearrow}\mathbb{N} : [f \leq h] \in \mathcal{F}\}.$$

Then $Y_{\mathcal{F},h} \in \mathfrak{D}_{\text{fin}}$.

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Lemma 4.5. *If* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are as in Lemma 4.3, and* $h(n) \geq 2n$ *for each* n*, then* $Y_{\mathcal{F}_1,h} \cup Y_{\mathcal{F}_2,h}$ *is* 2*-dominating.*

Proof. Let $g \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ be any function. Define by induction

$$\begin{array}{rcl} a(0) & = & 0 \\ a(n+1) & = & g(a(n))+1 \end{array}$$

By the assumption, there exist $A_1 \in \mathcal{F}_1/a$ and $A_2 \in \mathcal{F}_2/a$ such that for each $n \in A_1$ and $m \in A_2$, $|n-m| > 1$.

Fix $i < 2$. For each n, define

$$f_i(n) = \begin{cases} g(a(k-1)) + n - a(k-1) & n \in [a(k), a(k+1)) \text{ for } k \in A_i \\ g(a(k)) + n - a(k) & \begin{array}{l} n \in [a(k), a(k+1)) \\ \text{ where } k \notin A_i, k+1 \in A_i \end{array} \\ g(n) & \text{otherwise} \end{cases}$$

It is not difficult to verify that $f_i \in {}^{\mathbb{N}\nearrow}\mathbb{N}$.

For each $k \in A_i$ and $n \in [a(k), a(k+1))$,

$$\begin{array}{rcl} f_i(n) & = & g(a(k-1)) + n - a(k-1) \leq \\ & \leq & a(k) + n - a(k-1) \leq a(k) + n \leq 2n \leq h(n). \end{array}$$

Therefore $f_i \in Y_{\mathcal{F}_i,h}$.

For each n let k be such that $n \in [a(k), a(k+1))$. If n is large enough, then either $k, k+1 \notin A_1$, and therefore $f_1(n) = g(n)$, or else $k, k+1 \notin A_2$, and therefore $f_2(n) = g(n)$, that is, $g(n) \leq \max\{f_1(n), f_2(n)\}$. $\square$

This completes the proof of Theorem 4.4. $\square$

The following two lemmas will imply that if NCF holds, then $\mathfrak{D}_{\text{fin}}$ is closed under taking countable, and consistently much larger, unions. For each $k \in \mathbb{N}$, let $\mathfrak{D}_k$ be the collection of subsets of ${}^{\mathbb{N}}\mathbb{N}$ which are not k-dominating.

Lemma 4.6. *Assume that* $|I| < \mathfrak{b}$ *and* $Y_i \subseteq {}^{\mathbb{N}}\mathbb{N}$*,* $i \in I$*, are such that for each* $F \subseteq I$ *with* $|F| \leq k$*,* $\bigcup_{i\in F} Y_i \in \mathfrak{D}_k$*. Then* $\bigcup_{i\in I} Y_i \in \mathfrak{D}_k$*.*

Proof. For each $F \subseteq I$ with $|F| \leq k$, Let $g_F \in {}^{\mathbb{N}}\mathbb{N}$ witness that $\bigcup_{i\in F} Y_i \in \mathfrak{D}_k$. The collection $\{g_F : \text{Finite } F \subseteq I\}$ has size $< \mathfrak{b}$, therefore it is bounded, say by $h \in {}^{\mathbb{N}}\mathbb{N}$. Then h witnesses that $Y = \bigcup_{i\in I} Y_i \in \mathfrak{D}_k$: Each k-subset Z of Y is contained in $\bigcup_{i\in F} Y_i$ for a suitable k-subset F of I. Thus $\max(Z)$ cannot dominate g_F; in particular it cannot dominate h. $\square$

Let $\mathfrak{g}$ be the *groupwise density* number [7, 8].

Lemma 4.7 (Blass [7]). *Assume that* $|I| < \mathfrak{g}$ *and* $Y_i \subseteq {}^{\mathbb{N}}\mathbb{N}$, $i \in I$, *are such that for each* $F \subseteq I$ *with* $|F| \leq k$, $\bigcup_{i\in F} Y_i \in \mathfrak{D}_{2k}$. *Then* $\bigcup_{i\in I} Y_i \in \mathfrak{D}_k$.

Corollary 4.8. *If* $\mathfrak{D}_{\text{fin}}$ *is closed under taking finite unions, then it is closed under taking unions of size* $< \max\{\mathfrak{b}, \mathfrak{g}\}$.

Proof. Assume that for each $i \in I$, $Y_i \in \mathfrak{D}_{\text{fin}}$. Then for each k, and each finite $F \subseteq I$ with $|F| \leq k$, $\bigcup_{i\in F} Y_i \in \mathfrak{D}_{\text{fin}} \subseteq \mathfrak{D}_{2k} \subseteq \mathfrak{D}_k$. Thus, if $|I| < \mathfrak{b}$ (respectively, $|I| < \mathfrak{g}$), then by Lemma 4.6 (respectively, Lemma 4.7) $\bigcup_{i\in I} Y_i \in \mathfrak{D}_k$ for all k. As $\mathfrak{D}_{\text{fin}} = \bigcap_k \mathfrak{D}_k$ [30], we have that $\bigcup_{i\in I} Y_i \in \mathfrak{D}_{\text{fin}}$. □

The *ultrafilter number* $\mathfrak{u}$ is the minimal size of an ultrafilter base. It is known that $\mathfrak{u} < \mathfrak{g}$ implies NCF; the other direction is open: All the known models of NCF [10, 11] satisfy $\mathfrak{u} < \mathfrak{g}$. In particular, it is consistent that $\mathfrak{u} < \mathfrak{g}$. $\mathfrak{u} < \mathfrak{g}$ also implies that $\mathfrak{b} = \mathfrak{u} < \mathfrak{g} = \mathfrak{d} = \mathfrak{c}$ [8].

Corollary 4.9. *If* $\mathfrak{u} < \mathfrak{g}$ *holds, then* $\mathfrak{D}_{\text{fin}}$ *is closed under taking unions of size* $< \mathfrak{c}$.

We can now summarize the results of this section.

Theorem 4.10. *The following are equivalent:*

(1) *NCF,*
(2) $\mathfrak{D}_{\text{fin}}$ *is closed under taking finite unions;*
(3) $\mathfrak{D}_{\text{fin}}$ *is closed under taking unions of size* $< \max\{\mathfrak{b}, \mathfrak{g}\}$.

Corollary 4.11. (1) *If NCF holds, then*

$$\max\{\mathfrak{b}, \mathfrak{g}\} \leq \mathsf{add}(\mathsf{U}_{fin}(\Gamma, \Omega)) \leq \mathsf{add}(\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)) \leq \mathsf{cf}(\mathfrak{d}) = \mathfrak{d}.$$

(2) *If* $\mathfrak{u} < \mathfrak{g}$, *then*

$$\mathsf{add}(\mathsf{U}_{fin}(\Gamma, \Omega)) = \mathsf{add}(\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)) = \mathfrak{c} > \mathfrak{u}.$$

Proof. (1) follows from the inequalities of Section 2, Corollary 4.1, and Theorem 4.10 (together with the fact that NCF implies that $\mathfrak{d}$ is regular [5]).

(2) follows from (1), as $\mathfrak{u} < \mathfrak{g}$ implies $\mathfrak{g} = \mathfrak{c}$. □

Problem 4.12. *Is any of the classes* $\mathsf{S}_{fin}(\Omega, \Omega)$, $\mathsf{S}_1(\Gamma, \Omega)$, *and* $\mathsf{S}_{fin}(\Gamma, \Omega)$ *consistently closed under taking finite unions?*

For the Borel case there remains only one unsolved class.

Problem 4.13. *Is* $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ *consistently closed under taking finite unions?*

5. τ-COVERS

$\mathcal{U}$ is a *τ-cover* of X if it is a large cover of X (that is, each member of X is contained in infinitely many members of the cover), and for each $x, y \in X$, (at least) one of the sets $\{U \in \mathcal{U} : x \in U, y \notin U\}$ and $\{U \in \mathcal{U} : y \in U, x \notin U\}$ is finite. τ-covers are motivated by the *tower number* $\mathfrak{t}$ [28] and were incorporated into the framework of selection principles in [29]. Let T and $\mathcal{B}_\mathrm{T}$ denote the collections of countable open and Borel τ-covers of X.

By Proposition 2.1, $\mathsf{S}_1(\mathrm{T}, \mathcal{O})$ and $\mathsf{S}_1(\mathcal{B}_\mathrm{T}, \mathcal{B})$ are also closed under taking countable unions. In [29] it is shown that assuming the Continuum Hypothesis, no class between $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ and $\binom{\Omega}{\mathrm{T}}$ is closed under taking finite unions (recall that $\mathsf{S}_{fin}(\Omega, \mathrm{T})$ implies $\binom{\Omega}{\mathrm{T}}$). It is also proved there that $\mathsf{add}(\mathsf{S}_1(\mathcal{B}_\mathrm{T}, \mathcal{B}_\Gamma)) = \mathfrak{t}$. Using Scheepers' result mentioned in Section 2, we have that this also holds in the open case.

Theorem 5.1. $\mathsf{add}(\mathsf{S}_1(\mathrm{T}, \Gamma)) = \mathfrak{t}$.

Proof. Clearly $\mathsf{S}_1(\mathrm{T}, \Gamma) = \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_1(\Gamma, \Gamma)$. In [29] it is shown that $\mathsf{add}(\binom{\mathrm{T}}{\Gamma}) = \mathfrak{t}$. Thus, $\mathfrak{t} = \min\{\mathfrak{t}, \mathfrak{h}\} \leq \mathsf{add}(\mathrm{T}, \Gamma) \leq \mathsf{non}(\mathsf{S}_1(\mathrm{T}, \Gamma)) = \mathfrak{t}$. $\square$

Problem 5.2. *Is any of the properties* $\mathsf{S}_1(\mathrm{T}, \mathrm{T})$, $\mathsf{S}_{fin}(\mathrm{T}, \mathrm{T})$, $\mathsf{S}_1(\Gamma, \mathrm{T})$, $\mathsf{S}_{fin}(\Gamma, \mathrm{T})$, *and* $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$ *(or any of their Borel versions) provably (or at least consistently) closed under taking finite unions?*

Problem 5.3. *Is any of the classes* $\mathsf{S}_{fin}(\Omega, \mathrm{T})$, $\mathsf{S}_1(\mathrm{T}, \Omega)$, *and* $\mathsf{S}_{fin}(\mathrm{T}, \Omega)$ *consistently closed under taking finite unions?*

Remark 5.4. 1. It turns out that in [25], Scheepers used the Continuum Hypothesis (a stronger assumption then our $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$) to construct two sets satisfying $\mathsf{S}_1(\Omega, \Omega)$ such that their union does not satisfy $\mathsf{S}_{fin}(\Omega, \Omega)$. This is extended by our Proposition 3.4, which in turn is extended by Theorem 3.5.

2. Our paper was written in 2001, more or less the time Blass' paper [9] was written. This explains some overlaps between the results of that paper and the current one. In particular, Blass proves there a slightly weaker version of Theorem 4.10 – see Corollary 5.5, Theorem 6.5, and Theorem 6.6 in [9].

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