SPECIALISING ARONSZAJN TREES BY COUNTABLE APPROXIMATIONS

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Abstract. We show that there are proper forcings based upon countable trees of creatures that specialise a given Aronszajn tree.

0. Introduction

The main point of this work is finding forcing notions specialising an Aronszajn tree, which are creature forcings, tree-like with halving, but being based on ω_1 (the tree) rather than ω. Techniques to specialise a given Aronszajn tree are often useful for building models of the Souslin hypothesis SH, i.e. models in which there is no Souslin tree. The present work grew from attempts at showing the consistency of SH together with ♣ (see [9, I.7.1]), a question by Juhász. This stays open.

Creature forcing tries to enlarge and systemise the family of very nice forcings. There is "the book on creature forcing" [6], and for uncountable forcings the work is extended in [5, 7, 3, 8, 4] and [Sh:F977]. At first glance it cannot be applied for specialising an Aronszajn tree, because we have to add a subset of ω_1 rather than a subset of ω. Here we adopt it to ω_1. We dispense with some of the main premises made in the previous work and show new technical details. The work may also be relevant to cardinal characteristics of ${}^{\omega_1}2$, but this is left for future work.

The norm of creatures (see Definition 1.6) we shall use is natural for specialising Aronszajn trees, cf. [9, Ch. V, §6]. It is convenient that there is some $\alpha < \omega_1$ such that the union of the domains of the partial specialisation functions that are attached to any branch of the tree-like forcing condition is the initial segment of the Aronszajn tree $\mathbf{T}_{<\alpha}$, i.e. the union of the levels less than α. However, allowing that there is a finite set u such that for every branch of a given condition the union of the domains of the partial specialisations that lie on this branch is $\mathbf{T}_{<\alpha} \cup u$ is used for density arguments that show that the generic filter leads to a total specialisation function.

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1. Tree creatures

In this section we define the tree creatures which will be used later to describe the branching of the countable trees that will serve as forcing conditions. We prove three important technical properties about gluing together (Claim 1.8), about filling up (Claim 1.9) and about changing the base together with thinning out (Claim 1.10) of creatures. We shall define the forcing conditions only in the next section. They will be countable trees with finite branching, such that each node and its immediate successors in the tree are described by a creature in the sense of Definition 1.5. Roughly spoken, in our context, a creature will be an arrangement of partial specialisation functions with some side conditions.

We reserve the symbol $(T, <_T)$ for the trees in the forcing conditions, which are trees of partial specialisation functions of some given Aronszajn tree $(\mathbf{T}, <_{\mathbf{T}})$. A specialisation function is a function $f\colon \mathbf{T} \to \omega$ such that for all $s, t \in \mathbf{T}$, if $s <_{\mathbf{T}} t$, then $f(s) \neq f(t)$, see [2, p. 244].

χ stands for some sufficiently high regular cardinal, and $\mathcal{H}(\chi)$ denotes the set of all sets of hereditary cardinality less than χ. For our purpose, $\chi = (2^{\aleph_1})^+$ is enough.

Throughout this work we make the following assumption:

Hypothesis 1.1. *$\mathbf{T}$ is an Aronszajn tree ordered by $<_{\mathbf{T}}$, and for $\alpha < \omega_1$ the level α of $\mathbf{T}$ satisfies:*

$$\mathbf{T}_\alpha \subseteq [\omega\alpha, \omega\alpha + \omega).$$

The tree $\mathbf{T}$ will be fixed for the main part of the work, the analysis of $\mathbb{Q}_{\mathbf{T}}$.. Only in the end we iterate over all Aronszajn trees in the ground model and in termediate models and use $2^{\omega_1} = \omega_2$ to accomplish this. We define the following finite approximations of specialisation maps:

Definition 1.2. *For $u \subseteq \mathbf{T}$ and $n < \omega$ we let*

$$\operatorname{spec}_n(u) = \{\eta \mid \eta\colon u \to [0, n) \ \wedge\ (\eta(x) = \eta(y) \to \neg(x <_{\mathbf{T}} y))\}.$$

We let $\operatorname{spec}(u) = \bigcup_{n<\omega} \operatorname{spec}_n(u)$ *and* $\operatorname{spec} = \operatorname{spec}^{\mathbf{T}} = \bigcup\{\operatorname{spec}(u) : u \subset \mathbf{T}, u \text{ finite}\}$.

Choice 1.3. *We choose two sequences of natural numbers $\langle n_{k,i} : i < \omega\rangle$, $k = 2, 3$, such that the following growth conditions are fulfilled:*

$$n_{2,i} \leq n_{3,i}, \tag{1.1}$$

$$n_{3,i} < n_{2,i+1}. \tag{1.2}$$

The norm of an i creature will be $\leq n_{3,i}$ in all our versions of norms. In order not to say "do nothing" at many levels of the trees in the tree creature forcing, it is good to require $2^{n_{3,i}} \leq n_{2,i+1}$. However, it does not matter. Moreover, also all the computations the numbers $n_{2,i}$ and $n_{3,i}$ can be replaced by requiring just finiteness of the partial specialisations and finiteness of $\operatorname{pos}(\mathbf{c})$ *for each creature* $\mathbf{c}$ *and letting $a \subseteq \omega$ instead of $a \subseteq n_{3,i}$ in Def. 1.6, and then also items (a)(β) and (b) in Def. 1.6 of the* nor^0 *are not needed.*

We fix the $n_{j,i}$, $j = 2, 3$, $i < \omega$, for the rest of this work.

We compare with the book [6] in order to justify the use of the name "creature". However, we cannot just cite that work, because the framework developed there is not suitable for the approximation of uncountable domains $\mathbf{T}$.

Definition 1.4. *(1)* ([6, 1.1.1]) *Let* $\mathbf{H} = \bigcup_{i\in\omega} \mathbf{H}(i)$, *and let* $\mathbf{H}(i)$ *be sets. A triple* $\mathbf{c} = (\mathrm{nor}[\mathbf{c}], \mathrm{val}[\mathbf{c}], \mathrm{dis}[\mathbf{c}])$ *is a weak creature for* $\mathbf{H}$ *if the following holds:*

- *(a)* $\mathrm{nor}[\mathbf{c}] \in \mathbb{R}^{\geq 0}$.
- *(b)* *Let* $\triangleleft$ *be the strict initial segment relation.* $\mathrm{val}[\mathbf{c}]$ *is a non-empty subset of*
$$\Big\{\langle x, y\rangle \in \bigcup_{m_0<m_1<\omega} [\prod_{i<m_0} \mathbf{H}(i) \times \prod_{i<m_1} \mathbf{H}(i)] \,:\, x \triangleleft y\Big\}.$$
- *(c)* $\mathrm{dis}[\mathbf{c}] \in \mathcal{H}(\chi)$.

(2.) nor *stands for norm,* val *stands for value, and* dis *stands for distinguish.*

In our case, we drop the component dis will be called $i(\mathbf{c})$ and $k(\mathbf{c})$, two natural numbers.

As we will see in the next definition, in this work (b) of 1.4 is not fulfilled: For us val is a non-empty subset of $\{\langle x, y\rangle \in \mathrm{spec}^{\mathbf{T}} \times \mathrm{spec}^{\mathbf{T}} : x <_T y\}$ for some strict partial order $<_T$ as in Definition 2.1. Though the members of $\mathrm{spec}^{\mathbf{T}}$ are finite partial functions, they cannot be written in a natural manner with some $n \in \omega$ as a domain, since $\mathrm{spec}^{\mathbf{T}}$ is uncountable and we want to allow arbitrary finite parts. Often properness of a tree creature forcing follows from the countability of $\mathbf{H}$. Note that our analogue to $\mathbf{H}$ is not countable. In Section 3 we shall prove that the notions of forcing we introduce are proper for other reasons.

Nevertheless the creature in the next definition is a specific case for the distinction part and the value of a weak creature in the sense of 1.4 without item (1.)(b), and later we will assign a norm. As common in the work with tree creatures we write $\mathrm{pos}(\mathbf{c})$ for $\mathrm{rge}(\mathrm{val}[\mathbf{c}])$, the set of possibilities for $\mathbf{c}$.

Definition 1.5. *(1)* *A* creature *is a tuple* $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}), k(\mathbf{c}))$ *with the following properties:*

- *(a)* *The first component,* $i(\mathbf{c})$, *is called the kind of* $\mathbf{c}$ *and is just a natural number.* $\mathbf{c}$ *is an* i*-creature if* $i(\mathbf{c}) = i$.
- *(b)* *The second component,* $\eta(\mathbf{c})$, *is called the base of* $\mathbf{c}$. *We require* ($\eta(\mathbf{c}) = \emptyset$ *and* $i(\mathbf{c}) = 0$) *or* ($i(\mathbf{c}) = i > 0$ *and* $|\mathrm{dom}(\eta(\mathbf{c}))| < n_{2,i-1}$), *and* $\eta(\mathbf{c}) \in \mathrm{spec}_{n_{3,i-1}}$.
- *(c)* $\mathrm{pos}(\mathbf{c})$ *is a non-empty subset of* $\{\eta \in \mathrm{spec}_{n_{3,i}} : \eta(\mathbf{c}) \subsetneqq \eta \wedge |\mathrm{dom}(\eta)| < n_{2,i}\}$. *So we have* $\mathrm{val}(\mathbf{c}) = \{\eta(\mathbf{c})\} \times \mathrm{rge}(\mathrm{val}(\mathbf{c}))$. *That the domain is a singleton, is typical for tree-creating creatures.*
- *(d)* $k(\mathbf{c}) \in \omega \setminus \{0\}$.

(2) *The set of creatures is denoted by* K.

For a non-negative real number r we let $m = \lfloor r \rfloor$ be the largest natural number such that $m \leq r$. We let lg denote the logarithm function to the base 2. Let $\log_2(x) = \lfloor \lg(x) \rfloor$ for $x > 0$, and we set $\log_2 0 = 0$.

The following definition has ideas from [9, Ch. V, § 6] and is the most important definition in this work.

Definition 1.6. *(1) For an i-creature $\mathbf{c}$ we define $\mathrm{nor}^0(\mathbf{c})$ as the maximal natural number m such that if $a \subseteq n_{3,i}$ and $|a| \leq m$ and $B_0, \dots, B_{m-1}$ are branches of $\mathbf{T}$, then there is $\nu \in \mathrm{pos}(\mathbf{c})$ such that*

(α) $(\forall x \in (\bigcup_{\ell<k} B_\ell \cap \mathrm{dom}(\nu)) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a)$,

(β) $\dfrac{|\mathrm{dom}(\nu)|}{n_{2,i}} \leq \dfrac{1}{m}$.

(2) We define $\mathrm{nor}^1(\mathbf{c}) = \log_2(\mathrm{nor}^0(\mathbf{c}))$.

(3) In order not to fall into specific computations, we use functions f that exhibit the following properties, in order to define norms on creatures that, in contrast to nor^0 and nor^1 also use the component $k(\mathbf{c})$:

$(*)_1$ *$f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+$ is the set of strictly positive reals.*

$(*)_2$ *f fulfils the following monotonicity properties: If $n_1 \geq n_2 \geq k_2 \geq k_1$ then $f(n_1, k_1) \geq f(n_2, k_2)$.*

$(*)_3$ *(For the 2-bigness, see Claim 1.11) $f(\frac{n}{2}, k) \geq f(n, k) - 1$.*

$(*)_4$ *$n \leq k \to f(n, k) \leq 0$.*

$(*)_5$ *(For the halving property, see Definition 3.3) For all n, k: If $f(n, k) \geq 1$, then there is some $k'(n, k) = k'$ such that $k < k' < n$ and*

$$f(n, k') \geq \frac{f(n,k)}{2},$$

and for all n', if $k' < n' < n$ and $f(n', k') \geq 1$, then

$$f(n', k) \geq \frac{f(n,k)}{2}.$$

For example, $f(n, k) = \lg(\frac{n}{k})$ for $k \leq n$, and $f(n, k) = 0$ otherwise, and $k'(n, k) = \lfloor \sqrt{nk} \rfloor$, fulfil these conditions since $k' < n' < n$ and $f(n', k') \geq 1$ imply $f(n', k) = f(n', k') + f(k', k) \geq 1 + \frac{f(n,k)}{2} - 1 = \frac{f(n,k)}{2}$.

For a creature $\mathbf{c}$ we define its norm

$$\mathrm{nor}_f(\mathbf{c}) = f(\mathrm{nor}^0(\mathbf{c}), k(\mathbf{c})).$$

Note that $\mathrm{nor}^0(\mathbf{c}) \leq n_{2,i}$ for an i-creature with $i > 0$. Some of the inequalities in the conditions of Claims 1.9 and 1.10 are easy to fulfil. Most of the time the requirement (α) is the hardest one.

Remark 1.7. *Definition 1.6(1) speaks about infinitely many requirements, by ranging over all m-tuples of branches of $\mathbf{T}$. However, at a crucial point in the proof Claim 1.9 this boils down to counting the possibilities for $a \subseteq n_{3,i}$.*

The next claim shows that we can extend the possibilities of a creature and at the same time decrease the norm of the creature only by a small amount.

Claim 1.8. *Assume that*

(a) $\eta^* \in \mathrm{spec}$,

(b) $\mathbf{c}$ *is an* i*-creature with base* η^*, $\mathrm{nor}^0(\mathbf{c}) > 0$,

(c) $k^* > 0$,

(d) *for each* $\eta \in \mathrm{pos}(\mathbf{c})$ *and* $k < k^*$ *we are given* $\eta \subseteq \rho_{\eta,k} \in \mathrm{spec}_{n_{3,i}}$ *with* $|\,\mathrm{dom}(\rho_{\eta,k})| < n_{2,i}$,

(e) *for each* $\eta \in \mathrm{pos}(\mathbf{c})$, *if* $k_1 < k_2 < k^*$ *and* $x_1 \in \mathrm{dom}(\rho_{\eta,k_1}) \setminus \mathrm{dom}(\eta)$ *and* $x_2 \in \mathrm{dom}(\rho_{\eta,k_2}) \setminus \mathrm{dom}(\eta)$, *then* x_1, x_2 *are* $<_{\mathbf{T}}$*-incomparable,*

(f) $\ell^* = \max\{|\,\mathrm{dom}(\rho_{\eta,k})| \,:\, \eta \in \mathrm{rge}(\mathrm{val}(\mathbf{c})) \wedge k < k^*\}$.

Then there is an i*-creature* $\mathbf{d}$ *given by*

$$\begin{aligned} \mathrm{pos}(\mathbf{d}) &= \{\rho_{\eta,k} \,:\, k < k^*, \eta \in \mathrm{pos}(\mathbf{c})\}, \\ \eta(\mathbf{d}) &= \eta^*, \\ k(\mathbf{d}) &= k(\mathbf{c}). \end{aligned}$$

We have $\mathrm{nor}^0(\mathbf{d}) \geq m_0 \stackrel{\mathrm{def}}{=} \min\big\{\mathrm{nor}^0(\mathbf{c}), \lfloor \frac{n_{2,i}}{\ell^*} \rfloor, k^* - 1\big\}$.

Proof. First of all we are to check Definition 1.5(1). Clauses (a),(b), and (c) follow immediately from the premises of the claim. From premise (e) and from the properties of $\mathbf{c}$ it follows that $\eta(\mathbf{d}) = \eta^*$. Therefore $\mathbf{d}$ satisfies clause (d).

Now for the norm: We check clause (α) of Definition 1.6. Let branches $B_0, \dots B_{m_0-1}$ of $\mathbf{T}$ and a set $a \subseteq n_{3,i}$ be given, $|a| \leq m_0$. Since $m_0 \leq \mathrm{nor}^0(\mathbf{c})$, there is some $\eta \in \mathrm{pos}(\mathbf{c})$ such that $(\forall x \in \big(\bigcup_{\ell<m_0} B_\ell\big) \cap \mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a)$. We fix such an η. Now for each $\ell < m_0$, we let

$$w_{\eta,\ell} = \{j < k^* \,:\, \exists x \in B_\ell \cap \mathrm{dom}(\rho_{\eta,j}) \setminus \mathrm{dom}(\eta)\}.$$

Now we have that $|w_{\eta,\ell}| \leq 1$ because otherwise we would have $k_1 < k_2 < k^*$ in $w_{\eta,\ell}$ and $x_i \in B_\ell \cap \mathrm{dom}(\rho_{\eta,k_i}) \setminus \mathrm{dom}(\eta)$, $i = 1, 2$. As x_1 and x_2 are $<_{\mathbf{T}}$-comparable, this is contradicting the requirement (e) of 1.8.

Since $m_0 < k^*$, there is some $j \in k^* \setminus \bigcup_{\ell<m_0} w_{\eta,\ell}$. For such a j, $\rho_{\eta,j}$ is as required.

We check clause (β) of Definition 1.6. We take the $\rho_{\eta,j}$ as chosen above. Then we have

$$\frac{|\,\mathrm{dom}(\rho_{\eta,j})|}{n_{2,i}} \leq \frac{\ell^*}{n_{2,i}} \leq \frac{1}{\lfloor \frac{n_{2,i}}{\ell^*} \rfloor} \leq \frac{1}{m_0},$$

as $m_0 \leq \lfloor \frac{n_{2,i}}{\ell^*} \rfloor$. ⊣

Whereas the previous claim will be used only in Section 3 in the proof on properness in Claim 3.8, the following two claims will be used in the next section for density arguments in the forcings built from creatures.

Claim 1.9. *Suppose that* $\mathbf{c}$, m, k *are as follows:*

(a) $\mathbf{c}$ *is an* i*-creature,*

(b) $\mathrm{nor}^0(\mathbf{c}) = k + m$,

(c) $m \le \frac{n_{2,i}}{k} - \frac{n_{2,i}}{k+m}$, *so for example* $m \le \frac{n_{2,i}}{k(k+1)}$ *(if we do not want to have a quadratic inequality),*

(d) $x_0, \dots, x_{m-1} \in \mathbf{T}$, $1 \le m$,

(e) $n_{3,i} \ge \frac{n_{2,i}}{k+m} + m + k$.

Then there is some creature $\mathbf{d}$ *such that*

(1) $\eta(\mathbf{d}) = \eta(\mathbf{c})$,

(2) $\mathrm{pos}(\mathbf{d}) \subseteq \{\nu \in \mathrm{spec}^{\mathbf{T}} : (\exists \eta \in \mathrm{pos}(\mathbf{c}))(\eta \subseteq \nu \wedge \mathrm{dom}(\nu) = \mathrm{dom}(\eta) \cup \{x_0, \dots, x_{m-1}\})\}$,

(3) $\mathrm{nor}^0(\mathbf{d}) \ge k$.

Proof. For each $\eta \in \mathrm{pos}(\mathbf{c})$ we choose $m+k$ elements from $n_{3,i} \setminus \mathrm{rge}(\eta)$, and put them into a set B_η. By (e) this set is not empty. Note that given $k \ge 2$ and m for a sufficiently large i, (e) is automatically true, since $n_{2,i} \le n_{3,i}$. For each $a \in [\omega]^k$ choose some set $\{z_{m'} : m' < m\} \subseteq B_\eta$, $\{z_{m'} : m' < m\} \cap a = \emptyset$ such that the $z_{m'}$'s are pairwise different. Then we have a specialisation $\nu_{\eta,\bar z} = \eta \cup \{(x_{m'}, z_{m'}) : m' < m\}$. Since the $z_{m'}$ are not in $\mathrm{rge}(\eta)$ it is a specialisation. We set

$$\mathbf{d} = \{(\eta(\mathbf{c}), \nu_{\eta,\bar z}) : \eta \in \mathrm{pos}(\mathbf{c}), \bar z \in [B_\eta]^m\}.$$

Now we check the norm: Let $B_1, \dots, B_k$ be branches of $\mathbf{T}$ and let $a \subseteq \omega$, $|a| \le k$. We have to find $\nu \in \mathrm{pos}(\mathbf{d})$ such that $(\forall \ell < k)(\forall y \in \mathrm{dom}(\nu) \cap B_\ell \setminus \mathrm{dom}(\eta(\mathbf{c})))(\nu(y) \notin a)$. We add branches B_{k+i}, $i < m$, B_i containing x_i. By premise (b), we find $\eta \in \mathrm{pos}(\mathbf{c})$ such that

$$(\forall \ell < k+m)(\forall x \in \mathrm{dom}(\eta) \cap B_\ell \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a).$$

Taking $\bar z$ disjoint from a in B_η, we have $\nu_{\eta,\bar z} \in \mathrm{pos}(\mathbf{d})$ such that

$$(\forall \ell < k)(\forall x \in \mathrm{dom}(\nu_{\eta,\bar z}) \cap B_\ell \setminus \mathrm{dom}(\eta(\mathbf{c})))(\nu_{\eta,\bar z}(x) \notin a).$$

Now $\nu_{\eta,\bar z} = \nu$ is a witness for the norm also in item (β): We have $\frac{n_{2,i}}{k+m} + m \le \frac{n_{2,i}}{k}$, which follows from the premises on m. ⊣

Usually in the applications we have $m = 1$ and $k \ge 2$. Suppose we have filled up the range of the value of a creature according to one of the previous claims. Then we want that these extended functions can serve as bases for suitable creatures as well. This is provided by the next claim, which makes the previous claim almost obsolete. We need only the weakening of the previous claim that there is $\eta^* \supseteq \eta(\mathbf{c})$, $\eta^* \in \mathrm{spec}^{\mathbf{T}}_{n_{3,i}}$ and $x_0, \dots, x_{m-1} \in \mathrm{dom}(\eta)$. i from 1.9 will now appear in 1.10 as $i-1$ since the following claim speaks about in the next level of a tree built from creatures. The idea is: Each element of $\mathrm{pos}(\mathbf{c}_t)$ is a basis of $\mathbf{c}_{t'}$ for t' being a direct successor of t in a tree of creatures.

Claim 1.10. *Assume that*

(a) $\mathbf{c}$ *is an* i*-creature.*

(b) $\eta^* \supseteq \eta(\mathbf{c})$, $\eta^* \in \mathrm{spec}_{n_{3,i-1}}$ *(note that we do not suppose that* $\eta^* \in \mathrm{pos}(\mathbf{c})$*). Furthermore we assume* $|\mathrm{dom}(\eta^*)| \le n_{2,i-1}$.

(c) *We set*

$$\ell_2^* = |\operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c}))|,$$

and

$$\ell_1^* = |\{y : (\exists \nu \in \operatorname{pos}(\mathbf{c}))(y \in \operatorname{dom}(\nu) \setminus \operatorname{dom}(\eta(\mathbf{c}))) \wedge (\exists x \in \operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c})))(x <_{\mathbf{T}} y)\}|,$$

and we assume that $\ell_1^* + \ell_2^* < \operatorname{nor}^0(\mathbf{c})$.

We define $\mathbf{d}$ *by* $\eta(\mathbf{d}) = \eta^*$ *and*

$$\operatorname{pos}(\mathbf{d}) = \{\nu \cup \eta^* \,:\, \nu \in \operatorname{pos}(\mathbf{c}) \,\wedge\, \nu \cup \eta^* \in \operatorname{spec}_{n_{3,i}} \,\wedge |\operatorname{dom}(\nu \cup \eta^*)| < n_{2,i}\}.$$

Then

(α) $\mathbf{d}$ *is an* i*-creature.*

(β) $\operatorname{nor}^0(\mathbf{d}) \geq \operatorname{nor}^0(\mathbf{c}) - \ell_2^* - \ell_1^*$.

Proof. Item (α) follows from the requirements on η^* and from the estimates on the norm, see below. For item (β), we set $k = \operatorname{nor}^0(\mathbf{c}) - \ell_1^* - \ell_2^*$. We let $B_0, \dots, B_{k-1}$ be branches of $\mathbf{T}$ and $a \subseteq n_{3,i(\mathbf{c})}$, $|a| \leq k$. We set $\ell^* = \ell_1^* + \ell_2^*$. We let $\langle y_\ell \,:\, \ell < \ell_1^* \rangle$ list $Y = \{y \,:\, \exists \nu(\nu \in \operatorname{pos}(\mathbf{c}) \wedge y \in \operatorname{dom}(\nu) \wedge (\exists x)(x \in \operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c})) \wedge x \leq_{\mathbf{T}} y))\}$ without repetition. Let $B_k, \dots, B_{k+\ell_1^*-1}$ be branches of $\mathbf{T}$ such that $y_\ell \in B_{k+\ell}$ for $\ell < \ell_1^*$. Let $\langle x_\ell \,:\, \ell < \ell_2^* \rangle$ list $\operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c}))$. Take for $\ell < \ell_2^*$, $B_{k+\ell_1^*+\ell}$ such that $x_\ell \in B_{k+\ell_1^*+\ell}$. We set $a' = a \cup \{\eta^*(x_\ell) \,:\, \ell < \ell_2^*\}$. Since $\operatorname{nor}^0(\mathbf{c}) \geq k + \ell^*$ there is some $\nu \in \operatorname{pos}(\mathbf{c})$ such that $\forall x \in ((\operatorname{dom}(\nu) \setminus \operatorname{dom}(\eta(\mathbf{c}))) \cap \bigcup_{\ell < k+\ell^*} B_\ell)(\nu(x) \notin a')$. Then, if $x \notin \operatorname{dom}(\eta^*)$, $(\nu \cup \eta^*)(x) \notin a$. Moreover $|\operatorname{dom}(\nu \cup \eta^*)| \leq \frac{n_{2,i}}{k+\ell^*} + \ell_2^* \leq \frac{n_{2,i}}{k}$, if $\frac{n_{2,i}}{k}$ is sufficiently large.

We have to show that $\nu \cup \eta^*$ is a partial specialisation: Since η^* and ν are specialisation maps, we have to consider only the case $x \in \operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c}))$ and ($y \in Y$ or ($y \in \operatorname{dom}(\nu) \setminus \operatorname{dom}(\eta^*)$ and $y <_{\mathbf{T}} x$)). If $y \in Y$, then we have $\nu(y) \neq \eta^*(x_\ell)$ for all $\ell < \ell_2^*$. If $y \in \operatorname{dom}(\nu) \setminus \operatorname{dom}(\eta^*)$ and $y <_{\mathbf{T}} x$, then y is in a branch leading to some x_ℓ, $\ell < \ell_2^*$, and hence again $\nu(y) \neq \eta^*(x_\ell)$, $\ell < \ell_2^*$. ⊣

In the applications, the proofs of the density properties, ℓ_2^* will be small compared to the norm (we add ℓ_2^* points to the domain of the functions in the set of possibilities of a creature with sufficiently high norm) and $\ell_1^* \leq |u|$, were u is the set that sticks out of $\mathbf{T}_{<\alpha(p)}$ (see Definition 2.2). We will suppose that these two are small in comparison to $\operatorname{nor}^0(\mathbf{c})$, so that the premises for Claims 1.8, 1.9 and 1.10 are fulfilled.

The next claim will help to find large homogeneous subtrees of the trees built from creatures that will later be used as forcing conditions.

Claim 1.11. *(1) The 2-bigness property* [6, Definition 2.3.2]. *If* $\mathbf{c}$ *is an* i*-creature with* $\operatorname{nor}^1(\mathbf{c}) \geq k + 1$*, and* $\mathbf{c}_1$*,* $\mathbf{c}_2$ *are* i*-creatures such that* $\operatorname{val}(\mathbf{c}) = \operatorname{val}(\mathbf{c}_1) \cup \operatorname{val}(\mathbf{c}_2)$*, then* $\operatorname{nor}^1(\mathbf{c}_1) \geq k$ *or* $\operatorname{nor}^1(\mathbf{c}_2) \geq k$.

(2) If $\mathbf{c}$ *is an* i*-creature with* $\operatorname{nor}_f(\mathbf{c}) \geq k + 1$*, and* $\mathbf{c}_1$*,* $\mathbf{c}_2$ *are* i*-creatures such that* $\operatorname{val}(\mathbf{c}) = \operatorname{val}(\mathbf{c}_1) \cup \operatorname{val}(\mathbf{c}_2)$*, and* $k(\mathbf{c}_1) = k(\mathbf{c}_2) = k(\mathbf{c})$*, then* $\operatorname{nor}_f(\mathbf{c}_1) \geq k$ *or* $\operatorname{nor}_f(\mathbf{c}_2) \geq k$.

Proof. (1) We first consider nor^0. Let $j = 2^k$. We suppose that $\mathrm{nor}^0(\mathbf{c}_1) < j$ and $\mathrm{nor}^0(\mathbf{c}_2) < j$ and derive a contradiction: For $\ell = 1, 2$ let branches $B^\ell_0, \dots, B^\ell_{j-1}$ and sets $a^\ell \subseteq n_{3,i}$ exemplify this.

Let $a = a^1 \cup a^2$ and let, by $\mathrm{nor}^0(\mathbf{c}) \geq 2j$, $\eta \in \mathrm{pos}(\mathbf{c})$ be such that for all $x \in (\mathrm{dom}(\eta) \cap \bigcup_{\ell=1,2} \bigcup_{i=0}^{j-1} B^\ell_i) \setminus \mathrm{dom}(\eta(\mathbf{c}))$ we have $\eta(x) \notin a$. But then for that $\ell \in \{1, 2\}$ for which $\eta \in \mathrm{pos}(\mathbf{c}_\ell)$ we get a contradiction to $\mathrm{nor}^0(\mathbf{c}_i) < j$. Hence (1) follows for nor^1.

Since the k-components of the creatures coincide, part (2) follows from the requirements on f in Definition 1.6(4): $f(\frac{n}{2}, k) \geq f(n, k) - 1$. $\dashv$

2. Forcing with tree-creatures

Now we define a notion of forcing with ω-trees $\langle \mathbf{c}_t : t \in (T, <_T)\rangle$ as conditions. Every node t of such a tree $(T, <_T) = (\mathrm{dom}(p), <_p)$ and its immediate successors are described by a certain creature $\mathbf{c}_t$ from Definition 1.5. We have $\mathrm{basis}(\mathbf{c}_t) = t$.

First we collect some general notation about trees. The trees here are not the Aronszajn trees of the first section, but trees T of finite partial specialisation functions, ordered by $<_T$ which is a subrelation of $\subset\restriction (\mathrm{spec}^{\mathbf{T}})^2$. Some of these trees T together with a tag $\mathbf{c}_t = (i(\mathbf{c}_t), \mathrm{val}[\mathbf{c}_t], k(\mathbf{c}_t))$ at each node $t \in T$ will serve as forcing conditions. We write such tagged trees as $p = \langle \mathbf{c}_t : t \in T\rangle$ and if T' is a subtree of T then we let $p \restriction T' = \langle \mathbf{c}_t : t \in T'\rangle$.

Definition 2.1. *(1) A tree $(T, <_T)$ is a set finite or countable set T with a partial order $<_T$ such that for $t \in T$, $\{s \in T : s <_T t\}$ is a finite linear order.*

(2) We define the successors of η in T, the restriction of T to η, the splitting points of T and the maximal points of T by

$$\mathrm{suc}_T(\eta) = \{\nu \in T : \eta <_T \nu \wedge \neg(\exists \rho \in T)(\eta <_T \rho <_T \nu)\},$$
$$T^{\langle \eta \rangle} = \{\nu \in T : \eta \leq_T \nu\},$$
$$\mathrm{split}(T) = \{\eta \in T : |\mathrm{suc}_T(\eta)| \geq 2\},$$
$$\max(T) = \{\nu \in T : \neg(\exists \rho \in T)(\nu <_T \rho)\}.$$

(3) A $\mathbf{T}$-tree $(T, <_T)$ is a set $T \subseteq \mathrm{spec}^{\mathbf{T}}$, such that for any $\eta \in T$, $(\{\nu : \nu <_T \eta\}, <_T)$ is a finite linear order and such that in T there is a least element, called the root, $\mathrm{rt}(T)$. We have for $\eta, \nu \in T$: $\eta <_T \nu$ iff $\eta \subset \nu$. We shall only work with finitely branching trees.

(4) The n-th level of T is

$$T^{[n]} = \{\eta \in T : \eta \text{ has } n \text{ and not more } <_T\text{-predecessors}\}.$$

The set of all branches through T is

$$\begin{aligned}\lim(T) = \{\langle \eta_k : k < \ell\rangle : \; & \ell \leq \omega \wedge (\forall k < \ell)(\eta_k \in T^{[k]}) \\ & \wedge (\forall k < \ell - 1)(\eta_k <_T \eta_{k+1}) \\ & \wedge \neg(\exists \eta_\ell \in T)(\forall k < \ell)(\eta_k <_T \eta_\ell)\}.\end{aligned}$$

A tree is well-founded if there are no infinite branches through it.

(4) *A subset F of T is called a* front of T *if every branch of T passes through this set, and the set consists of $<_T$-incomparable elements.*

Definition 2.2. *Let $\mathbf{T}$ be an Aronszajn tree. We define a notion of forcing $Q = Q_{\mathbf{T}}$.*

(A) *$p \in Q$ iff $p = (i(p), T^p, k^p)$ has the following properties:*

  (a) *There is a set $\mathrm{dom}(p) \subseteq \mathrm{spec}^{\mathbf{T}}$ such that $T^p = T(p) = (\mathrm{dom}(p), <_p)$ is a $\mathbf{T}$-tree with ω levels, the ℓ-th level of which is denoted by $(T(p))^{[\ell]} = p^{[\ell]}$. If $\eta \in \mathrm{spec}^{\mathbf{T}}$ appears more than once in the tree* new addition *we do not identify the nodes in the different positions. Strictly speaking we have $\mathrm{dom}(p) \subseteq \mathrm{spec}^{<\omega}$ and each $\eta \in \mathrm{dom}(p)$ is just an abbreviation for $\langle \mathrm{rt}(p), \eta_1, \dots, \eta_{n-1} = \eta\rangle$ where this is the list of $<_p$-predecessors of η.* end of new

  (b) *T^p has a root, the unique element of level 0, called $\mathrm{rt}(p)$.*

  (c) *$k^p \colon \mathrm{dom}(p) \to \omega$.*

  (d) *There is $i(p) = i < \omega$ such that the following holds: For any $\ell < \omega$ and $\eta \in p^{[\ell]}$ the set*

$$\mathrm{suc}_p(\eta) = \{\nu \in p^{[\ell+1]} \,:\, \eta <_p \nu\}$$

  is $\mathrm{pos}(\mathbf{c})$ for a $(i+\ell)$-creature $\mathbf{c}$ with base η and $k(\mathbf{c}) = k^p(\eta)$. We denote this creature by $\mathbf{c}_{p,\eta}$. (So we have $i(\mathbf{c}_{p,\mathrm{rt}(p)}) = i(p)$.)

  (e) *There is $\alpha = \alpha(p) \in \omega_1$ such that the following holds: For some $k < \omega$ for every $\eta \in p^{[k]}$ there is a finite set $u_\eta \subseteq \mathbf{T} \setminus \mathbf{T}_{<\alpha}$ such that for every ω-branch $\langle \eta_\ell \,:\, \ell < \omega\rangle$ of T^p satisfying $\eta_k = \eta$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) \setminus u_\eta = \mathbf{T}_{<\alpha}$.*

  (f) *For every ω-branch $\langle \eta_\ell \,:\, \ell \in \omega\rangle$ of T^p we have $\lim_{\ell\to\omega} \mathrm{nor}_f(\mathbf{c}_{p,\eta_\ell}) = \omega$.*

(B) *The order $\leq = \leq_Q$ is given by letting $p \leq q$ (q is stronger than p, we follow the Jerusalem convention) iff $i(p) \leq i(q)$ and there is a projection $\mathrm{pr}_{q,p}$ which satisfies*

  (a) *$\mathrm{pr}_{q,p}$ is a function from $\mathrm{dom}(q)$ to $\mathrm{dom}(p)$.*

  (b) *If $\eta \in \mathrm{dom}(q)$ then $\eta \supseteq \mathrm{pr}_{q,p}(\eta)$.*

  (c) *If η_1, η_2 are both in $\mathrm{dom}(q)$ and if $\eta_1 \leq_q \eta_2$, then $\mathrm{pr}_{q,p}(\eta_1) \leq_p \mathrm{pr}_{p,q}(\eta_2)$.*

  (d) *For every $\eta \in \mathrm{dom}(q)$, $k^q(\eta) \geq k^p(\mathrm{pr}_{q,p}(\eta))$.*

  (e) *For any $\ell \in \omega$, $\eta \in q^{[\ell]} \Rightarrow \mathrm{pr}_{q,p}(\eta) \in p^{[\ell+i(q)-i(p)]}$.*

  (f) *For any $\ell \in \omega$: If $\nu \in q^{[\ell]}$ and $\rho \in q^{[\ell+1]}$ and $\nu <_q \rho$, $\mathrm{pr}_{q,p}(\nu) = \eta$, $\mathrm{pr}_{q,p}(\rho) = \tau$, then $\mathrm{dom}(\tau) \cap \mathrm{dom}(\nu) = \mathrm{dom}(\eta)$.*

Definition 2.3. *For $p \in Q$ and $\eta \in \mathrm{dom}(p)$ we let*

$$p^{\langle\eta\rangle} = p \restriction \{\rho \in \mathrm{dom}(p) \,:\, \eta \leq \rho\}.$$

Let us give some informal description of the $\leq$-relation in Q: The stronger condition's domain is via $\mathrm{pr}_{q,p}$ mapped homomorphically w.r.t. the tree orders into $\mathrm{dom}(p^{\langle \mathrm{pr}_{q,p}(\mathrm{rt}(q))\rangle})$. The projection is in general neither one-to-one nor onto. The root can grow as well. According to (b), the projection preserves the levels in the trees but for one jump in heights (the ℓ's in $p^{[\ell]}$), due to a possible lengthening of the root. The partial specialisation functions sitting on the nodes of the tree are extended (possibly by more than one extension per function) in q as to compared with the ones attached to the image under pr, but by (b) the extensions are so small and so few that it preserves the kind i of the creature given by the node and its successors, and according to (f) the new part of the domain of the extension is disjoint from the domains of the old partial specification functions living higher up in the projection of the new tree to the old tree.

Let us compare our setting with the forcings given in the book [6]: There the $\leq$-relation of the forcing is based on a sub-composition function (whose definition is not used here, because we just deal with one particular forcing notion) whose inputs are well-founded subtrees of the weaker condition. Here the extension $\mathrm{rt}(q) \setminus \mathrm{pr}_{q,p}(\mathrm{rt}(q))$ is taken from somewhere in the Aronsajn tree and not from higher up in p, indeed, by (f) this is even forbidden. On the other hand, the projections shift all the levels by the same amount $i(q) - i(p)$, and are not arbitrary finite contractions as in most of the tree creature forcings in the book [6].

Definition 2.4. *(1)* $p \in Q$ *is called* normal *iff for every ω-branch* $\langle \eta_\ell : \ell \in \omega\rangle$ *of* T^p *the sequence* $\langle \mathrm{nor}(\mathbf{c}_{p,\eta_\ell}) : \ell \in \omega\rangle$ *is non-decreasing.*

(2) $p \in Q$ *is called* smooth *iff in clause (v) of Definition 2.2 the number k is 0 and u is empty.*

(3) $p \in Q$ *is called* weakly smooth *iff in clause (v) of Definition 2.2 the number k is 0.*

(4) *For a weakly smooth p or a smooth p we let* $\alpha(p) = \bigcup\{\mathrm{dom}(\eta) : \eta \in b\}$ *for any branch b in* $(p, <_p)$.

Fact 2.5. *(1)* *Def. 2.2(f) does not only hold for ℓ and $\ell+1$ but for any finite difference of levels.*

(2) *If p is weakly smooth and $p \leq q$ and* $\eta \in \mathrm{dom}(p)$, $\nu \in \mathrm{dom}(q)$, $\eta = \mathrm{pr}_{q,p}(\nu)$ *and* $\eta <_p \tau \in \mathrm{dom}(p)$, *then* $\mathrm{dom}(\nu) \cap \mathrm{dom}(\tau) = \mathrm{dom}(\eta)$.

(3) *If $p \leq q$ and p is weakly smooth with witness u then*
$\nu \in \mathrm{dom}(q) \rightarrow \mathrm{dom}(\nu) \cap (\mathbf{T}_{<\alpha(p)} \cup u) = \mathrm{dom}(\mathrm{pr}_{q,p}(\nu))$.

Proof. (2): If p is weakly smooth, then all branches of T^p have the same union of domains, and hence it is immaterial whether ρ and ν from 2.2(f) are in the range of $\mathrm{pr}_{q,p}$ or not. (3) follows from (2). ⊣

Definition 2.6. *For $0 \leq n < \omega$ we define the partial order $\leq_n$ on Q by letting* $p \leq_n q$ *iff*

(i) $p \leq q$,

(ii) $i(p) = i(q)$,

(iii) $\mathrm{rt}(p) = \mathrm{rt}(q)$,

(iv) $(p^{[<n]}, <_p) = (q^{[<n]}, <_q)$, *and and* $p \restriction \bigcup_{\ell<n} p^{[\ell]} = q \restriction \bigcup_{\ell<n} q^{[\ell]}$,

(v) *for every projection* $\mathrm{pr}_{q,p}$, *if* $\mathrm{pr}_{q,p}(\eta) = \nu$ *then*

- $\eta = \nu$ *and* $\mathbf{c}_{q,\eta} = \mathbf{c}_{p,\nu}$
- *or* $\mathrm{nor}_f(\mathbf{c}_{q,\eta}) \geq n$.

Note that, e.g., $p \restriction p^{[0]} = q \restriction q^{[0]}$ means $\mathrm{rt}(p) = \mathrm{rt}(q)$ and $\mathbf{c}_{p,\mathrm{rt}(p)} = \mathbf{c}_{q,\mathrm{rt}(p)}$, which is a requirement on two levels in T^p and in T^q. So property (iv) says that also on the level n the two trees still coincide. Below we show that $\mathrm{nor}_f(\mathbf{c}_{q,\eta}) \geq n$ implies $\mathrm{nor}_f(\mathbf{c}_{p,\nu}) \geq n$.

We state and prove some basic properties of the notions defined above.

Claim 2.7. *(1)* *If* $p \leq q$ *and* p *is weakly smooth, then* $\mathrm{pr}_{q,p}$ *is unique.*

(2) $(Q, \leq_Q)$ *is a partial order.*

(3) *If* $p \leq q$ *and* $\mathrm{pr}_{q,p}(\eta) = \nu$, *then* $i(\mathbf{c}_{q,\eta}) = i(\mathbf{c}_{p,\nu})$.

(4) *If* $p \leq q$ *and* $\mathrm{pr}_{q,p}(\eta) = \nu$, *then* $\mathrm{nor}^0(\mathbf{c}_{q,\eta}) \leq \mathrm{nor}^0(\mathbf{c}_{p,\nu})$ *and the same holds for* nor_f.

(5) $(Q, \leq_n)$ *is a partial order.*

(6) $p \leq_{n+1} q \to p \leq_n q \to p \leq q$.

(7) *If* $\mathbf{c}$ *is an* i*-creature with* $m \leq \mathrm{nor}^0(\mathbf{c})$ *and* $\eta \in \mathrm{pos}(\mathbf{c})$, *then there is an* i*-creature* $\mathbf{c}'$ *with* $m = \mathrm{nor}^0(\mathbf{c}')$, $\eta \in \mathrm{pos}(\mathbf{c}')$ *and* $\mathrm{val}(\mathbf{c}') \subseteq \mathrm{val}(\mathbf{c})$. *The same holds for* nor_f.

(8) *For every* $p \in Q$ *there is a* $q \geq p$ *such that for all* η *and* ν

$$\begin{aligned}\mathrm{pr}_{q,p}(\eta) = \nu \to &(\mathrm{nor}^0(\mathbf{c}_{q,\eta}) = \min\{\mathrm{nor}^0(\mathbf{c}_{p,\rho}) \,:\, \nu \leq_p \rho \in \mathrm{dom}(p)\} \wedge \\ &\mathrm{nor}^0(\mathbf{c}_{q,\eta}) = \min\{\mathrm{nor}^0(\mathbf{c}_{q,\rho}) \,:\, \nu \leq_q \rho \in \mathrm{dom}(q)\})\end{aligned}$$

Hence the normal conditions are dense in $\mathbb{Q}$.

(9) *For every (not necessarily normal)* p *we have that* $\lim_{n\to\omega} \min\{\mathrm{nor}_f(\mathbf{c}_{p,\eta}) : \eta \in p^{[n]}\} = \infty$.

(10) *If* $p \in Q$ *and* $\eta \in p^{[\ell]}$ *then* $|\mathrm{dom}(\eta)| < n_{2,i(p)+\ell-1}$ *or* $\ell = 0$ *and* $i(p) = 0$ *and* $\eta = \emptyset$.

Proof. (1) By induction on ℓ we show that $\mathrm{pr}_{q,p} \restriction \bigcup_{\ell' \leq \ell} p^{[\ell']}$ is unique: It is easy to see that for weakly smooth p, $\mathrm{pr}_{q,p}(\mathrm{rt}(q))$ is the $\subseteq$-maximal element of T^p that is a subfunction of $\mathrm{rt}(q)$. By Definition 2.1(1) such a maximum exists. Then we proceed level by level in T^q, and again Definition 2.1(1) yields uniqueness of $\mathrm{pr}_{q,p}$.

(2) Given $p \leq q$ and $q \leq r$ we define $\mathrm{pr}_{r,p} = \mathrm{pr}_{q,p} \circ \mathrm{pr}_{r,q}$. It is easily seen that this function is as required.

(3) Let ℓ be such that $\eta \in q^{[\ell]}$. Then $i(\mathbf{c}_{q,\eta}) = i(q) + \ell$ and $\nu \in p^{[\ell+i(q)-i(p)]}$. Hence $i(\mathbf{c}_{p,\nu}) = i(p) + \ell + i(q) - i(p) = i(q) + \ell = i(\mathbf{c}_{q,\eta})$.

(4) Suppose $\mathrm{nor}^0(\mathbf{c}_{q,\eta}) > \mathrm{nor}^0(\mathbf{c}_{p,\nu})$. Let $m = \mathrm{nor}^0(\mathbf{c}_{q,\eta})$ and let $i = i(\mathbf{c}_{q,\eta}) = i(\mathbf{c}_{p,\nu})$. Suppose that $a \subseteq n_{3,i}$ and the branches $B_0, \dots, B_{m-1}$ of T exemplify that $\mathrm{nor}^0(\mathbf{c}_{p,\nu}) < m$. Hence for all $\tau \in \mathrm{suc}_p(\nu)$

(α) there is $x \in (\mathrm{dom}(\tau) \cap \bigcup_{\ell=0}^{k-1} B_\ell) \setminus \mathrm{dom}(\nu)$ such that $\tau(x) \in a$, or

(β) $|\,\mathrm{dom}(\tau)| > \frac{n_{2,i}}{m}$.

Then a and $B_0, \dots, B_{m-1}$ exemplify $\mathrm{nor}^0(\mathbf{c}_{q,\eta}) < m$.

(5) Suppose that $p \le_n q \le_n r$ and $\mathrm{pr}_{r,q}(\sigma) = \eta$ and $\mathrm{pr}_{q,p}(\eta) = \nu$. By (2) we have that $\mathrm{pr}_{r,p}(\sigma) = \nu$, and now (4) implies that $p \le_n r$.

(6) Obvious.

(7) We may assume that $\mathrm{nor}^0(\mathbf{c}) > m$, because otherwise $\mathbf{c}$ itself is as required. Let $\eta \in \mathrm{pos}(\mathbf{c})$. Look at

$$Y = \{\mathbf{d} : \mathbf{d} \text{ is a } i\text{-creature and } \mathrm{val}(\mathbf{d}) \neq \emptyset \text{ and } \mathrm{nor}^0(\mathbf{d}) \geq k, \eta \in \mathrm{pos}(\mathbf{d}), \text{ and } \mathrm{val}(\mathbf{d}) \subseteq \mathrm{val}(\mathbf{c})\}.$$

Since $\mathbf{c} \in Y$, it is non-empty, and it has a member $\mathbf{d}$ with a minimal number of elements. We assume towards a contradiction that $\mathrm{nor}^0(\mathbf{d}) > m$. We choose $\eta^* \in \mathrm{pos}(\mathbf{d})$. We let $\mathrm{pos}(\mathbf{d}^*) = \mathrm{pos}(\mathbf{d}) \setminus \{\eta^*\}$.

Claim: $\mathbf{d}^* \neq \emptyset$. Otherwise we choose $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))$. Now we let B_0 be a branch of T to which x belongs and set $a = \{\eta^*(x)\}$. They witness that $\mathrm{nor}^0(\mathbf{d}) \not\geq 1$, so $\mathrm{nor}^0(\mathbf{d}) = 0$, which contradicts the assumption that $\mathrm{nor}^0(\mathbf{d}) > m > 0$.

Claim: $\mathrm{nor}^0(\mathbf{d}^*) \geq m$. Otherwise there are branches $B_0, \dots, B_{m-1}$ and a set $a \subseteq n_{3,i}$ witnessing $\mathrm{nor}^0(\mathbf{d}^*) \not\geq m$. Let $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{d}))$ and let B_m be a branch such that $x \in B_m$ and set $a' = a \cup \{\eta^*(x)\}$. The $B_0, \dots, B_m$ and a' witness that $\mathrm{nor}^0(\mathbf{d}) \not\geq k+1$. Hence $\mathbf{d}^*$ is a member of Y with fewer elements than $\mathbf{d}$, contradiction.

(8) Follows from (7). We can even take $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$. First see: For no m the set $\{\eta \in p$ such that for densely (in T^p) many $\eta' \geq_p \eta$ we have that $\mathrm{nor}^0(\mathbf{c}_{p,\eta'}) < m\}$. is anywhere dense. Otherwise we can choose a branch $\langle \eta_\ell : \ell \in \omega\rangle$ such that there is some $m \in \omega$ such that for all $\ell < \omega$, $\mathrm{nor}^0(\mathbf{c}_{p,\eta_\ell}) < m$.

Now by thinning out a spanning tree between the fronts

$$F_n = \{\varrho \in T^p : (\forall \eta \geq_p \varrho)(\mathrm{nor}(\mathbf{c}_{p,\eta} \geq n) \wedge (\forall \nu < \varrho)(\exists \eta \geq \nu)(\mathrm{nor}(\mathbf{c}_{p,\eta})) < n)\}$$

we construct q with $T^q \subseteq T^p$ and the identity is a level preserving embedding.

Now we choose by induction of ℓ, $\mathrm{dom}(q_\ell) \subseteq \mathrm{dom}(p)$, such that $\mathrm{dom}(q_\ell)$ has no infinite branch and hence is finite, though we do not have a bound on its height.

First step: Say $\min\{\mathrm{nor}^0(\mathbf{c}_{p,\eta}) : \eta \in \mathrm{dom}(p)\} = m$ and it is reached in $\eta \in \mathrm{dom}(p)$. We take $q^{[0]} = \{\eta\}$.

([1]) Then we take for any $\eta' \in \mathrm{pos}(\mathbf{c}_{p,\eta})$ the $<_p$-minimal $\eta'' > \eta'$ such that $\eta'' \in \mathrm{dom}(p)$ and such that for all $\tilde{\eta} \geq_p \eta''$, if $\tilde{\eta} \in \mathrm{dom}(p)$ then $\mathrm{nor}^0(\mathbf{c}_{p,\tilde{\eta}}) \geq m+1$. By the mentioned nowhere-density result, this is possible. We put such an η'' in $q^{[\ell]}$, if it is in $p^{[\ell+i(q)-i(p)]}$.

([2]) Then we look at the ν in the branch between η and η'' in $\mathrm{dom}(p)$. If $\mathrm{nor}^0(\mathbf{c}_{p,\nu}) > m$ we take according to (8) a subcreature $\mathbf{c}'$ of $\mathbf{c}_{p,\nu})$ with norm

k that contains in its possibilities that member of $\mathrm{pos}(\mathbf{c}_{p,\nu})$ that lies on the branch. We let $\mathbf{c}' = \mathbf{c}_{q_1,\nu}$. We have to put successors to all $\nu' \in \mathrm{pos}(\mathbf{c}_{p,\nu})$ for all ν in question into $\mathrm{dom}(q_1)$. This is done as in ([1]), applied to ν instead of η. With all the ν in this subset we do the procedure in ([1]), and repeat and repeat it. In finitely many (intermediate) steps we reach a subtree $\mathrm{dom}(q_1)$ of $\mathrm{dom}(p)$ without any ω-branches such that all its leaves fulfil $\eta'' \in \mathrm{dom}(p)$ and such that for all $\tilde\eta \supseteq \eta''$, if $\tilde\eta \in \mathrm{dom}(p)$ then $\mathrm{nor}^0(\mathbf{c}_{p,\tilde\eta}) \geq k+1$, and all its nodes η fulfil $\mathrm{nor}^0(\mathbf{c}_{q_1,\eta}) \geq m$. By König's lemma, this tree $\mathrm{dom}(q_1)$ is finite.

([3]) With the leaves of $\mathrm{dom}(q_1)$ and $m+2$ instead of $m+1$, we repeat the choice procedure in ([1]) and ([2]). We do it successively for all $\ell \in \omega$, and thus get q_ℓ such that for very node $\nu \in T^{q_\ell} \setminus T^{q_{\ell-1}}$, $\mathrm{nor}(\mathbf{c}_{q_\ell,\nu}) = m + \ell$. The union of the $\mathrm{dom}(q_\ell)$, $\ell \in \omega$, is a q as desired in (9).

(9) This follows from König's lemma: Since T^p is finitely branching, there is a branch though every infinite subset.

(10) Follows from Definitions 1.5 and 2.2. ⊣

Lemma 2.8. *Let $\langle n_i : i \in \omega\rangle$ be a strictly increasing sequence of natural numbers. We assume that for every i, $q_i \leq_{n_i} q_{i+1}$, and we set $n_{-1} = 0$. Moreover we assume that the q_i are smooth conditions. Then $q = \bigcup_{i<\omega} \bigcup_{n_{i-1} \leq n < n_i} (q_i) \restriction q_i^{[n]} \in Q$ and for all i, $q \geq_{n_i} q_i$.*

Proof. Clear by the definitions. ⊣

Now we fill up the domains of the partial specialisation functions and to show that the smooth conditions are dense in $\mathbb{Q}$.

Lemma 2.9. *If $p \in Q$ and $m < \omega$ then for some smooth $q \in Q$ we have $p \leq_m q$. Moreover, if $\bigcup\{\mathrm{dom}(\eta) : \eta \in T^p\} \subseteq \mathbf{T}_{<\alpha}$ then we can demand that $\bigcup\{\mathrm{dom}(\eta) : \eta \in T^q\} = \mathbf{T}_{<\alpha}$.*

Proof. We write the proof for nor^0. In the version for nor_f we choose the minimal $k(\mathbf{c}_{q,\nu})$ that is allowed by Def 2.2(d), and thus the result follows from a slight modification of the proof for the case of nor^1.

We first use the definition of $p \in Q$: We assume $\alpha(p) < \alpha$, otherwise we can take $q = p$. By item (v) there is some $k < \omega$ for every $\eta \in p^{[k]}$ there is $u_\eta \in \mathbf{T} \setminus \mathbf{T}_{<\alpha(p)}$ such that for every ω-branch $\langle \eta_\ell : \ell < \omega\rangle$ of T^p satisfying $\eta_k = \eta$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) \setminus u_\eta = \mathbf{T}_{<\alpha(p)}$. We fix such a k and such u_η, $\eta \in p^{[k]}$. Now for each $\eta \in p^{[k]}$ separately we perform the following inductive filling up: Fix $\eta \in p^{[k]}$. Let $\{x^\eta_r : r < \omega\}$ enumerate $\mathbf{T}_{<\alpha} \setminus (\mathbf{T}_{<\alpha(p)} \cup u_\eta)$ without repetition. Since m is arbitrary, it is enough to assume that $p_{\eta,\ell} \geq_m p^{\langle\eta\rangle}$ is already found with union of the domains $= T_{<\alpha(p)} \cup u_\eta \cup \{x^\eta_r : r < \ell\}$ along each of its branches and we have to find $q_{\eta,\ell} \geq_m p_{\eta,\ell}$ such there $q_{\eta,\ell} \geq_m p_{\eta,\ell}$ such that for every b of $T(q_{\eta,\ell})$

$$\bigcup\{\mathrm{dom}(\nu) : \nu \in b\} = \mathbf{T}_{<\alpha(p)} \cup u_\eta \cup \{x^\eta_r : r \leq \ell\}.$$

Then we can apply a fusion argument, since all the conditions $p_{\eta,\ell}$, $\ell < \omega$, are weakly smooth and the union of the domains along each branch of $p_{\eta,\ell}$ is $\mathbf{T}_{<\alpha(p)} \cup u_\eta \cup \{x^\eta_r : r < \ell\}$, and thus union over all ℓ is $\mathbf{T}_{<\alpha}$.

So we aim for such a condition. We can find $n < \omega$ such that

$(*)_1$ $m \leq n$, $\ell \leq n$,

$(*)_2$ $|u_\eta| \ll n$,

$(*)_3$ for every $\nu \in p^{[\geq n]}$, we have $\mathrm{nor}^0(\mathbf{c}_{p_\ell,\nu}) \gg m + |u_\eta| + \ell + 1$,

$(*)_4$ if $\nu \in (p^{\langle \eta \rangle})^{[n]}$, $u_\eta \cup \{x^\eta_r : r < \ell\} \subseteq \mathrm{dom}(\nu)$.

For each $\nu \in (p^{\langle \eta \rangle})^{[n]}$ let

$$w^+_\nu = \{\varrho : \nu <_{p_{\eta,\ell}} \varrho \in \mathrm{dom}(p_\ell) \wedge \mathrm{nor}^0(\mathbf{c}_{p_{\eta,\ell},\varrho}) > \ell + n + \mathrm{nor}^0(\mathbf{c}_{p_{\eta,\ell},\nu})\},$$

$$w_\nu = \{\varrho \in w^+_\nu : (\nexists \zeta)(\nu <_{p_{\eta,\ell}} \zeta <_{p_{\eta,\ell}} \varrho \wedge \zeta \in w^+_\nu)\}.$$

For each $\varrho \in w_\nu$ take $\tilde{\varrho} \supseteq \varrho$ with $x^\eta_\ell \in \mathrm{dom}(\tilde{\varrho})$ as in Claim 1.9 such that $\mathrm{dom}(\tilde{\varrho}) \setminus \mathrm{dom}(\varrho) = \{x^\eta_\ell\}$. Then we have

$$(\boxtimes) \quad \begin{array}{l} |\,\mathrm{dom}(\tilde{\varrho}) \setminus \mathrm{dom}(\varrho)| \leq 1, \text{ and} \\ |\{y : (\exists \tilde{\eta} \in \mathrm{pos}(\mathbf{c}_{p_{\eta,\ell},\varrho}))(y \in \mathrm{dom}(\tilde{\eta}) \wedge x^\eta_\ell <_{p_{\eta,\ell}} y)\}| \leq |u_\eta| + \ell \ll n, \end{array}$$

since only $y \notin T_{<\alpha(p)}$ can be in the latter set.

So for each $\mathbf{c}_{p_\ell,\rho}$ we can do the operation from Claim 1.10 and get a creature as $\mathbf{d}$ there with $x^\eta_\ell \in \mathrm{dom}(\mathrm{basis}(\mathbf{d}))$, and $\mathbf{d}$ serves as $\mathbf{d}_{q_{\eta,\ell+1},\varrho}$. Then we can go on with Claim 1.10 and thin out $\mathbf{c}_{p_{\eta,\ell},\nu'}$ to $\mathbf{c}_{q_{\ell,\ell},\nu'}$ for all $\nu' >_{p_{\eta,\ell}} \varrho$ as there and after having worked through all of $T(p_{\eta,\ell})$ we let $q_{\eta,\ell} = p_{\eta,\ell+1} \geq_{\ell+m} q_{\eta,\ell}$.

Indeed the construction of $p_{\eta,\ell+1}$ can be performed so that $\mathrm{nor}^1(\mathbf{c}_{q,\eta,\ell+1,\nu}) \geq \mathrm{nor}^1(\mathbf{c}_{p_{\eta,\ell+1},\mathrm{pr}_{p_{\eta,\ell+1},p_{\eta,\ell}}(\nu)}) - 1$ and, $k(\mathbf{c}_{p_{\eta,\ell+1},\nu})$ can be chosen so that $\mathrm{nor}_f(\mathbf{c}_{p_{\eta,\ell+1},\nu}) \geq \mathrm{nor}_f(\mathbf{c}_{p_{\eta,\ell},\mathrm{pr}_{p_{\eta,\ell+1},p_{\eta,\ell}}(\nu)}) - 1$ for $\nu \in T(p_{\eta,\ell+1})$. This is accomplished by inserting the x^η_ℓ at sufficiently high-normed nodes of $T^{p_{\eta,\ell}}$. $(\boxtimes)$ says what is high enough.

By the choice of $q = \bigcup_{\eta \in p^{[k]}, \ell < \omega} p_{\eta,\ell}$, it is smooth. ⊣

The fusion lemma together with the previous lemma are usually applied in the following setting:

Conclusion 2.10. *(1) The smooth conditions are $\leq_m$-dense in $\mathbb{Q}$.*

(2) Suppose $p \in \mathbb{Q}$ is given and we are to find $q \geq p$ such that q in the intersection of countably many open dense sets. For this it is enough to find for any dense open set and any p_0 and $k^* \in \omega$ some $q \geq_{k^*} p_0$ in the dense open set.

Conclusion 2.11. *Forcing with $\mathbb{Q}$ specialises $\mathbf{T}$.*

Proof. We have to show that for any $\alpha \in \omega_1 (= \mathbf{T})$, $D(\alpha) = \{p \in \mathbb{Q} : \alpha < \alpha(p)\}$ is dense in $\mathbb{Q}$. This follows from the proof of Lemma 2.9. ⊣

3. Decisions taken by the tree creature forcing

In this section we prove that $\mathbb{Q}$ is proper and ${}^\omega\omega$-bounding. Indeed, we prove that $\mathbb{Q}$ has "continuous reading of names" (this is the property stated in Claim 3.9), which implies Axiom A (see [1]) and ω^ω-bounding. These implications are proved Sections 2.6 and 3.1 of [6].

Claim 3.1. *(1) If $p \in \mathbb{Q}$ and $\{\eta_1, \dots, \eta_n\}$ is a front of p, then $\{p^{\langle \eta_1 \rangle}, \dots, p^{\langle \eta_n \rangle}\}$ is predense above p.*

(2) If $\{\eta_1, \dots, \eta_n\}$ is a front of p and $p^{\langle \eta_\ell \rangle} \leq q_\ell \in \mathbb{Q}$ for each ℓ, then there is $q \geq p$ with $\{\eta_1, \dots, \eta_n\} \subseteq T^q$ such that for all ℓ we have that $q^{\langle \eta_\ell \rangle} = q_\ell$. Hence $\{q^{\langle \eta_\ell \rangle} : 1 \leq \ell \leq n\}$ is predense above q.

(3) If $n \in \omega$ and $\{\eta_1, \dots, \eta_r\}$ is a front of p and $p^{\langle \eta_\ell \rangle} \leq_0 q_\ell \in \mathbb{Q}$ for each ℓ and

- *for all $\ell \leq r$, $(\forall \nu \in \mathrm{dom}(q_\ell))(\mathrm{nor}(\mathbf{c}_{q_\ell,\nu}) \geq n)$*
- *for all $\nu \in p$ if $\mathrm{nor}(\mathbf{c}_{p,\nu}) < n$ then $(\exists \ell \leq r)(\nu <_p \eta_\ell)$,*

then there is $q \geq_n p$ with $\{\eta_1, \dots, \eta_r\} \subseteq T^q$ such that for all ℓ we have that $q^{\langle \eta_\ell \rangle} = q_\ell$ and $\{\eta_1, \dots, \eta_r\}$ is a front of q.

Claim 3.2. *If $p \in \mathbb{Q}$ and $X \subseteq \mathrm{dom}(p)$ is $<_p$-downwards closed, then there is some q such that*

(a) $p \leq_0 q$, and either $(\forall \ell)(q^{[\ell]} \subseteq X)$ or $(\forall^\infty \ell)(q^{[\ell]} \cap X = \emptyset)$,

(b) $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$,

(c) $k^q = k^p \restriction \mathrm{dom}(q)$,

(c) for every $\nu \in \mathrm{dom}(q)$, if $\mathbf{c}_{q,\nu} \neq \mathbf{c}_{p,\nu}$, then $\mathrm{nor}^1(\mathbf{c}_{q,\nu}) \geq \mathrm{nor}^1(\mathbf{c}_{p,\nu}) - 1$ and $\mathrm{nor}_f(\mathbf{c}_{q,\nu}) \geq \mathrm{nor}_f(f\mathbf{c}_{p,\nu}) - 1$.

Proof. We will choose $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and then let $k^q = k^p \restriction \mathrm{dom}(q)$. For each ℓ we first choose by downward induction on $j \leq \ell$ subsets a colouring $f_{\ell,j}$ of $p^{[j]}$ with two colours, 0 and 1. For $\nu \in p^{[\ell]}$ we set $f_{\ell,\ell}(\nu) = 0$ iff $\nu \in X$ and $f_{\ell,\ell}(\nu) = 1$ otherwise.

Suppose that $f_{\ell,j}$ is defined. For $\eta \in p^{[j-1]}$ we have

$$\begin{aligned}\mathrm{pos}(\mathbf{c}_{\eta,p}) =& \{\nu \in \mathrm{pos}(\mathbf{c}_{\eta,p}) : f_{\ell,j}(\nu) = 0\} \cup \\ & \{\nu \in \mathrm{pos}(\mathbf{c}_{\eta,p}) : f_{\ell,j}(\nu) = 1\}\end{aligned}$$

By Claim 1.11 there is $r \in \{0,1\}$ such that $\mathbf{c}_r = (i(\mathbf{c}_{p,\eta}), \eta, \{\nu \in \mathrm{pos}(\mathbf{c}_{\eta,p}) : f_{\ell,j}(\nu) = r\}, k(\mathbf{c}_{p,\eta}))$ has with $\mathrm{nor}^1(\mathbf{c}_r) \geq \mathrm{nor}^1(\mathbf{c}_{\eta,p}) - 1$. The same holds for nor_f. Now we colour $\eta \in p^{[j-1]}$ as follows $f_{\ell,j-1}(\eta) = r$ iff the non-minority of $\nu \in \mathrm{pos}(\mathbf{c}_{\eta,p})$ has $f_{\ell,j}(\nu) = r$, so that is, if we chose $\mathbf{c}_r$ at the place η. We work downwards until we come to the root of p and keep $f_{\ell,0}(\mathrm{rt}(p))$ in our memory.

We repeat the procedure of the downwards induction on j for larger and larger ℓ. Since X is downwards closed, we have for each η, ℓ and $j \leq \ell$ if $f_{\ell+1,j}(\eta) = 0$, then $f_{\ell,j}(\eta) = 0$.

First case: There are infinitely many ℓ such that $f_{\ell,\ell}(\mathrm{rt}(p)) = 0$. If there are infinitely many ℓ such that $f_{\ell,\ell}(\mathrm{rt}(p)) = 0$, then this holds for all ℓ. Since for each fixed ℓ there are only finitely many possible $\langle f_{\ell,j}(\eta) : \eta \in p^{[j]}, j \leq \ell\}$ and since the preimage of colour 0 shrinks with increasing ℓ we find an infinite subsequence $\langle \ell_k : k < \omega\rangle$ such that for each k for all $k' \geq k$ for all $j \leq \ell_k$, for all $\eta \in p^{[j]}$, $f(\ell_{k'}, j)(\eta) = f(\ell_k, j)(\eta) = 0$ and the we build a condition q and an infinite subset $A' \subseteq A$ such that $\forall \ell \in A'$, $q^{[\ell]} \subseteq X$. The norms drop at most by one by taking the majority, and since X is downwards closed we have the first possibility in (a). In the opposite case, there is ℓ' such that $\forall \ell \geq \ell'$, $f_{\ell,\ell}(\mathrm{rt}(p)) = 1$ and hence there is a subset of $\mathrm{dom}(p)$ such that $(\forall \ell' \geq \ell)(q^{[\ell']} \cap X = \emptyset)$. The item (b) is clear. We choose for $\eta \in \mathrm{dom}(q)$, $k^q(\mathbf{c}_{q,\eta}) = k^p(\mathbf{c}_{p,\eta})$. Then item (c) is true. Item (d) follows from our choice of q and from Claim 1.11. $\dashv$

The next claim is very similar to 3.2. We want to find $q \geq_m p$, and therefore we have to weaken the homogeneity property in item (a) of 3.2.

Claim 3.3. *If $p \in \mathbb{Q}$, $m \in \omega$, and $X \subseteq \mathrm{dom}(p)$, then there is some q such that*

- *(a)* $p \leq_m q$, *and there is a front* $\{\nu_0, \dots, \nu_{s-1}\}$ *such that* $\{\nu \in \mathrm{dom}(p) : \mathrm{nor}^0(\mathbf{c}_{p,\nu}) \leq m\} \subseteq \{\nu \in \mathrm{dom}(q) : (\exists i < s)(\nu <_q \nu_i)\}$, *and such that for all* ν_i *we have: either* $(\forall \ell)((q^{\langle \nu_i\rangle})^{[\ell]} \subseteq X$ *or* $(\forall^\infty \ell)((q^{\langle \nu_i\rangle})^{[\ell]} \cap X = \emptyset)$ *and for all* $i \leq s$, $\nu \geq \nu_i$, $\mathrm{nor}^0(\mathbf{c}_{q,\nu}) \geq m$, *The same holds for* nor_f.
- *(b)* $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ *and* $q = p \restriction \mathrm{dom}(q)$,
- *(c)* *for every* $\nu \in \mathrm{dom}(q)$, *if* $\mathbf{c}_{q,\nu} \neq \mathbf{c}_{p,\nu}$, *then* $\mathrm{nor}^1(\mathbf{c}_{q,\nu}) \geq \mathrm{nor}^1(\mathbf{c}_{p,\nu}) - 1$ *and* $\mathrm{nor}_f(\mathbf{c}_{q,\nu}) \geq \mathrm{nor}_f(\mathbf{c}_{p,\nu}) - 1$.

Proof. We choose a front of p as in (a) and repeat the proof of 3.2 for each $p^{\langle \nu_i\rangle}$. $\dashv$

Now for the first time we make use of the coordinate $k(\mathbf{c})$ of our creatures. The next lemma states that the creatures have the halving property (compare to [6, 2.2.7]).

Definition 3.4. *K has the halving property, iff there is a function* $\mathrm{half}\colon K \to K$ *with the following properties:*

- *(1)* $\mathrm{half}(\mathbf{c}) = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}), k(\mathrm{half}(\mathbf{c})))$,
- *(2)* $\mathrm{nor}_f(\mathrm{half}(\mathbf{c})) \geq \frac{\mathrm{nor}(\mathbf{c})}{2}$,
- *(3)* *if* $\mathbf{c}' = (i, \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}))$ *is an i creature and*

$$(\bullet) \qquad \textit{if } k \geq k(\mathrm{half}(\mathbf{c})) \textit{ and } \mathrm{nor}_f(i(\mathbf{c}'), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k) > 0,$$

$$\textit{then } \mathrm{nor}_f(i(\mathbf{c}'), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c})) \geq \frac{\mathrm{nor}_f(\mathbf{c})}{2}.$$

Note that equation $(\bullet)$ is a strong requirement. In our constructions we can first take $\mathbf{c}_{q,\eta}) = \mathrm{half}(\mathbf{c}_{p,\eta})$ and thus increase k^p to k^q but then continuing the construction with q, we have to fulfill $(\bullet)$ in the next construction step $\mathbf{c}'$, and this is hard, because k there is big. It turns out that we can leave out this halving step and just go on with p directly.

Lemma 3.5. *K has the halving property.*

Proof. We set $k(\mathrm{half}(\mathbf{c})) = k'(\mathrm{nor}^0(\mathbf{c}), k(\mathbf{c})) \geq k(\mathbf{c})$ with a function $k'(x,y)$ as in 1.6(4). Then we have that $\mathrm{nor}_f(\mathrm{half}(\mathbf{c})) = f(\mathrm{nor}^0(i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c})), k(\mathrm{half}(\mathbf{c}))) \geq \frac{\mathrm{nor}_f(\mathbf{c})}{2}$, by Definition 1.6(4).

If $\mathbf{c}'$ is an i-creature and $\mathrm{nor}_f(i, \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathrm{half}(\mathbf{c})) > 0$ then

$$\begin{aligned}
\mathrm{nor}_f(i(\mathbf{c}'), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}')), k(\mathbf{c})) &= f(\mathrm{nor}^0(i(\mathbf{c}'), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}')), k(\mathbf{c})) \\
&\geq f(\mathrm{nor}^0(i(\mathbf{c}'), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}')), k(\mathrm{half}(\mathbf{c}))) \\
&\quad + f(\mathrm{nor}^0(i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c})), k(\mathrm{half}(\mathbf{c}))) \\
&\geq 1 + \frac{\mathrm{nor}_f(\mathbf{c})}{2} - 1 \geq \frac{\mathrm{nor}_f(\mathbf{c})}{2}.
\end{aligned}$$

⊣

Definition 3.6. *Let $\nu_0, \nu_1 \in \mathrm{spec}$ and let $p \in \mathbb{Q}$. We say ν_0 is isomorphic to ν_1 over $\mathbf{T}_{<\alpha}$ if there is some injective partial function $f \colon \mathbf{T} \to \mathbf{T}$ such that $x <_{\mathbf{T}} y$ iff $f(x) <_{\mathbf{T}} f(y)$ and $\mathrm{dom}(\nu_0) \cup \mathbf{T}_{<\alpha} \subseteq \mathrm{dom}(f)$ and $f \restriction \mathbf{T}_{<\alpha} = id$ and $f[\mathrm{dom}(\nu_0)] = \mathrm{dom}(\nu_1)$ and $\nu_0(x) = \nu_1(f(x))$ for all $x \in \mathrm{dom}(\nu_0)$.*

Fact 3.7. *For each $\alpha < \omega_1$, there are only countably many isomorphism types for $\eta \in \mathrm{spec}$ over $\mathbf{T}_{<\alpha}$.*

In the following two claims nor means nor^1 or nor_f. Both work, since the coordinate $k(\mathbf{c}_{s,\varrho})$ for $s = p, q, r$ and various ϱ is never changed in the constructions in the proofs of Claims ref3.8 and 3.9, so that (3.2) holds for all variants. Note that nor_f is used in the definition of $\leq_n$, that is Def. 2.6.

Claim 3.8. *(3.8.A in the notes) Suppose that $p_0 \in Q$ is smooth and that $m < \omega$ and that $\underset{\sim}{\tau}$ is a Q-name of a natural number. Let $N \prec H(\chi)$ and let $N \cap \omega_1 = \delta_*$. Let $\underset{\sim}{\tau} \in N$ be a $\mathbb{Q}_{\mathbf{T}}$-name of an ordinal.*

Let $\boxplus_{q,\ell,\varrho,f}$ abbreviate the following statement:

(i) $\varrho \in q^{[\ell]}$, *and*

(ii) $|\mathrm{dom}(f)| < n_{2,i(\mathbf{c}_{q,\varrho})}/(2\,\mathrm{nor}^0(\mathbf{c}_{q,\varrho}))$,

(iii) *if there is a smooth $r \geq q$ with $\mathrm{rt}(r) = \varrho \cup f$ and $\mathrm{nor}(\mathbf{c}_{r,\sigma}) \geq m+1$ for every $\sigma \in \mathrm{dom}(r)$ and r forces a value to $\underset{\sim}{\tau}$, then $q^{\langle\varrho\rangle}$ forces a value to $\underset{\sim}{\tau}$.*

Then there is a $q \in Q$ such that

(a) $p_0 \leq_m q$,

(b) *q is smooth and $\alpha(q) = \delta_*$,*

(c) *If $f \in \mathrm{spec}^{\mathbf{T}}$ and $\mathrm{dom}(f) \cap \delta_* = \emptyset$ then for infinitely many $\ell \in \omega$ we have $\forall \varrho \in q^{[\ell]}\ \boxplus_{q,\ell,\varrho,f}$.*

Proof. Let $\langle f_i : i \in \omega \rangle$ list the possible types over $\mathbf{T}_{<\delta_*}$ of an $f \in \mathrm{spec}^{\mathbf{T}}$ such that $\mathrm{dom}(f) \cap \delta_* = \emptyset$ such that each type appears infinitely often.

Let $\langle \alpha_i : i < \omega \rangle$ be an increasing sequence of ordinals that converges to δ_*. We choose (ℓ_i, p_i) by induction on i with the following properties:

(1) $p_i \in \mathbb{Q} \cap N$, $\tau \in N_0 \prec N$,

(2) $\ell_i < \ell_{i+1}$, $\alpha(p_i) \geq \alpha_i$, p_i is smooth,

(3) $p_0 = p$ and $p_i \leq_{m+i} p_{i+1}$,

(4) $\forall \ell \geq \ell_i \forall \varrho \in p_i^{[\ell]} \operatorname{nor}(\mathbf{c}_{p_i,\varrho}) \geq m+i+8$,

(5) for any $\eta \in p_i^{[\ell_i]}$ and $\nu \in p_i^{[\ell_i+1]}$, $\nu >_{p_i} \eta$ the property $(\alpha)_{i,\eta,\nu}$ or $(\beta)_{i,\eta,\nu}$ holds, again this is an abbreviation for many properties of η and ν:

$(\alpha)_{i,\eta,\nu}$ There are only boundedly many $\gamma \in \delta_*$ such that: There are $f'_0 \in N$ and a smooth $r_0 \in N$, $r_0 \geq p_i$ such that $\operatorname{rt}(r_0) = \nu \cup f'_0$ and f'_0 and f_i realise the same type over $\mathbf{T}_{<\alpha(p_i)}$, $\operatorname{dom}(f'_0) \cap \gamma = \emptyset$, and r_0 and forces a value to τ and for all $\varrho \in \operatorname{dom}(r_0)$, $\operatorname{nor}(\mathbf{c}_{r_0,\varrho}) \geq m+i+1$ and $|f'_0| \leq n_{2,i(\mathbf{c}_{p_i,\nu})}/(2\operatorname{nor}(\mathbf{c}_{p_i,\nu}))$. Then we have $p_{i+1}^{\langle\nu\rangle} = p_i^{\langle\nu\rangle}$ and we let $\nu'_0 = \nu$.

$(\beta)_{i,\eta,\nu}$ Not $(\alpha)_{i,\eta,\nu}$. Then

$$(\forall \nu'_k \in p_{i+1}^{[\ell_i+1]})((\nu'_k >_{p_{i+1}} \eta \wedge \operatorname{pr}_{p_{i+1},p_i}(\nu'_k) = \nu) \rightarrow p_{i+1}^{\langle \nu'_k\rangle} \text{ forces a value to } \underset{\sim}{\tau}.$$

We show that there is such a sequence $\langle p_i : i < \omega\rangle$. Assume we are given p_i. Then we choose ℓ_i such that $(\forall \varrho \in p_i^{[\geq \ell_i]})(\operatorname{nor}(\mathbf{c}_{p_i,\varrho}) \geq m+i+8)$ and for every $\eta \in p_i^{[\ell_i]}$ and $\nu >_{p_i} \eta$, $\nu \in p_i^{[\ell_i+1]}$, we proceed as in case $(\alpha)_{i,\eta,\nu}$ or in case $(\beta)_{i,\eta,\nu}$. In case $(\alpha)_{i,\eta,\nu}$ there is nothing to do, so let us assume that η, ν fall under case $(\beta)_{i,\eta,\nu}$.

We construct a preliminary part of p'_{i+1} that is defined by defining $(p'_{i+1})^{\langle \nu'_k\rangle}$ along a front $\{\nu'_k : \nu > \eta, \nu \in p_i^{[\ell_i+1]}, ((\alpha)_{i,\eta,\nu}$ and $k=0$ and $\nu'_0 = \nu)$ or $((\beta)_{i,\eta,\nu}$ and $k = 0, \dots, 2^{m+i+1}-1$ and ν'_k exists$)\}$. In the stronger condition p_{i+1}, the place of ν in $<_{p_i}$ will be taken by a large finite number of $\nu'_k = \nu \cup f'_k$, k sufficiently large, in the order $<_{p_{i+1}}$ such that $\operatorname{pr}_{p_{i+1},p_i}(\nu'_k) = \nu$ for any k and $\operatorname{pr}_{p_{i+1},p_i}((p'_{i+1})^{\langle\nu'_k\rangle}) = p_i^{\langle\nu\rangle}$..

Explanation: Note, that we are allowed to lengthen roots ν of the $p_i^{\langle\nu\rangle}$ to ν'_k and can still get

$$p_{i+1}^{\langle\eta\rangle} \geq_m p_i^{\langle\eta\rangle}. \tag{3.1}$$

Of course we will only have $p_{i+1}^{\langle\nu'_k\rangle} \geq p_i^{\langle\nu\rangle}$, not even with $\geq_0$. However we get (3.1) we must be careful: $\operatorname{nor}^0(\mathbf{c}_{p_{i+1},\eta})$ might drop (also down to 0) unless we lengthen the ν in many ways to ν'_k and possibly use Claim 1.8.

By the case assumption, there are f'_k, $1 \leq k < \omega$, with the following properties

(p1) f'_k and f_i have the same type over $\mathbf{T}_{<\alpha(p_i)}$,

(p2) $\operatorname{dom}(f'_k) \cap \alpha_{k+1} = \emptyset$,

(p3) $f'_k \in N$,

(p4) $r_k \in N$ forces a value to $\underset{\sim}{\tau}$, r_k is smooth, (p5) $\mathrm{rt}(r_k) = \nu \cup f'_k$,

(p6) for all $\varrho \in \mathrm{dom}(r_0)$, $\mathrm{nor}(\mathbf{c}_{r_0,\varrho}) \geq m + i + 1$.

By a fact on uncountably many disjoint finite subsets in an Aronszajn tree, applied in N, we can have additionally

(p7) and such that for $k \neq k'$ any $t \in \mathrm{dom}(f'_k)$ and any $t' \in \mathrm{dom}(f'_{k'})$ are $\leq_{\mathbf{T}}$-incomparable.

Then we pick for each $k < \omega$ some f'_k and r_k and let $\nu'_k := \nu \cup f'_k \in (p'_{i+1})^{[\ell_i+1]}$ and $(p'_{i+1})^{\langle \nu'_k \rangle} = r_k$ and $\mathrm{pr}_{p_{i+1},p_i}(\nu'_k) = \nu$. This ends $(\beta)_{i,\eta,\nu}$. We use only $k = 0, \dots, 2^{m+i+8}$.

Thereafter we take a smooth $p_{i+1}^{\langle \nu'_k \rangle} \geq_m (p'_{i+1})^{\langle \nu'_k \rangle}$ for $k = 0$ in case $(\alpha)_{i,\beta,\nu}$ or for $k = 0, \dots, 2^{m+i+8}$ in case $(\beta)_{i,\eta,\nu}$, such that there is $\alpha'_{i+1} \in [\alpha_{i+1}, \delta_*)$ such that for all ν'_k, $\alpha(p_{i+1}^{\langle \nu'_k \rangle}) = \alpha'_{i+1}$. Gluing all the members of

$$\{p_{i+1}^{\langle \nu'_k \rangle} : (k < 2^{m+i+8} \text{ in case } (\beta)_{i,\eta,\nu}, k = 0 \text{ in case } (\alpha)_{i,\eta,\nu}),$$
$$\nu \in p_i^{[\ell_i+1]}, \nu >_{p_i} \eta, \eta \in p_i^{[\ell]}\}$$

together in a natural way finally gives p_{i+1} with $\alpha(p_{i+1}) = \alpha'_{i+1}$.

Now by Claim 1.8,

$$\mathrm{nor}(\mathbf{c}_{p_{i+1},\eta}) \geq \mathrm{nor}(\mathbf{c}_{p_i,\eta}) - 1 \text{ and } p_{i+1} \geq_{m+i} p_i. \tag{3.2}$$

Hence there is a sequence $\langle p_i : i < \omega \rangle$ with the properties (1) to (5). We let q be the fusion of the p_i.

We show that q is as desired as in the claim: Let f be given. We take ℓ_0 such that for $\varrho \in q^{[\geq \ell_0]}$ in $\boxplus_{i,\ell,\varrho,f}$ the second condition holds. Then we assume that from some $\ell' \geq \ell_0$ onwards for all $\ell \geq \ell'$, for some $\varrho \in q^{\ell}$, $\boxplus_{q,\ell,\varrho,f}$ fails. Then in some construction step f appears as f_i and $|f_i|$ fulfils (ii) of not only of $\boxplus_{q,\ell,\varrho,f}$ but also of $\boxplus_{q,\ell,\eta,f}$ for η being the direct $<_q$-predecessor of ϱ. Since by the failure of $\boxplus_{q,\ell,\varrho,f}$ for f exists an r as in the premise of the claim, also for f_i there exists an $r' \in N$, we take for r' as in (iii) of $\boxplus$. Also since f is above δ_*, within N there are cofinally many $\gamma < \delta_*$ such that there is a copy f_i of f_i with $\mathrm{dom}(f'_i) \cap \gamma = \emptyset$. So, in the inductive construction we continued as $(\beta)_{i,\eta,\rho}$. However, the construction ensures that from this step ℓ_i onwards, for all $\nu' \in p_{i+1}^{[\ell_i+1]}$ with $\nu' >_{p_{i+1}} \eta$, $p_{i+1}^{\langle \nu' \rangle}$ forces a value to $\underset{\sim}{\tau}$ and hence also $q^{\langle \nu' \rangle}$ forces a value to $\underset{\sim}{\tau}$. So ϱ is one of the ν', and this shows that $\boxplus_{q,\ell\varrho,f}$ did not fail. ⊣

The property in (b) in the next claim is a version of "continuous reading of names" that implies that $\mathbb{Q}$ is ${}^{\omega}\omega$ bounding.

Claim 3.9. *Suppose that $p_0 \in \mathbb{Q}$ is smooth and that $m < \omega$ and that $\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name of a natural number. Then there is a $q \in Q$ such that*

(a) $p_0 \leq_m q$,

(b) *for some $\ell \in \omega$ we have that for every $\eta \in q^{[\ell]}$ the condition $q^{\langle \eta \rangle}$ forces a value to $\underset{\sim}{\tau}$.*

Proof. Let $N \prec H(\chi)$ be such that $\mathbb{Q}_{\mathbf{T}}, p, \underset{\sim}{\tau} \in N$. We take $p_0' \geq_{m+1} p_0$ in the role of q from the previous claim applied to N, $\delta_* = N \cap \omega_1$ and $\underset{\sim}{\tau} \in N$ and p_0.

Then we define for $k \in \omega$,

$$X_\tau(p_0', k, m) = \big\{\rho \,:\, \rho \in \bigcup_{n \geq k} (p_0')^{[n]} \,\wedge (\exists s)$$

$$\big(p^{\langle \rho \rangle} \leq_0 s \,\wedge\, (s \text{ forces a value to } \underset{\sim}{\tau}) \wedge (\forall \nu \in T^s)(\mathrm{nor}^0(\mathbf{c}_{s,\nu}) \geq m+1)\big)\big\}.$$

For $\tilde{m} < \omega$, $r, s \in Q$, $\eta \in \mathrm{dom}(r)$, we denote the following property:

$$(*)^{\tilde{m},\eta}_{r,s} \qquad \begin{array}{l} r^{\langle \eta \rangle} \leq_0 s \,\wedge \\ \forall \nu(\eta \subseteq \nu \in \mathrm{dom}(s) \rightarrow \mathrm{nor}^0(\mathbf{c}_{s,\nu}) \geq \tilde{m}+1) \,\wedge \\ (s \text{ forces a value to } \underset{\sim}{\tau}). \end{array}$$

Note that $p^{[\ell]} \subseteq X(p_0', k, m)$ implies $\forall \eta \in (p_0')^{[\ell]} (\exists s)(*)^{m,\eta}_{p_0,s}$.

Choose

(1) k such that $\rho \in (p_0')^{[\geq k]} \rightarrow \mathrm{nor}_f(\mathbf{c}_{p_0,\rho}) > m+2$.

(2) $q \geq_{m+1} p_0'$ is chosen as in Claim 3.3 applied to p_0', the front $(p_0')^{[k]}$ and $X = \mathrm{dom}(p_0') \setminus X(p_0', k, m+1)$ which is downwards closed.

We claim that $\forall^\infty \ell \forall \eta \in q^{[\ell]} q^{\langle \eta \rangle}$ forces a value to τ. Note that p_0' has with respect to p_0 the properties from the previous claim, and also q has these properties and for η such that $(\forall \varrho \in q^{\langle \eta \rangle})(\mathrm{nor}(\mathbf{c}_{q,\eta}) \geq m+1)$, also $q^{\langle \eta \rangle}$ has the properties of the conclusion of the previous claim.

First case: In 3.3(a) we get $\forall \ell q^{[\ell]} \subseteq X$. We show that this does not happen. Suppose $\eta \in \mathrm{dom}(q)$ is such that $(\forall \varrho \in \mathrm{dom}(q^{\langle \eta \rangle})(\mathrm{nor}(\mathbf{c}_{q,\varrho}) \geq m+1)$. Then, by the definition of X, $q^{\langle \eta \rangle}$ does not force a value to τ. However, $\alpha(p_0') = \alpha(q) = \alpha(q^{\langle \eta \rangle}) = \delta_* = N \cap \omega_1$. We take any $r \geq q^{\langle \eta \rangle}$ that forces a value to τ. Without loss of generality we can assume that for all $\varrho \in \mathrm{dom}(r)$, $\mathrm{nor}(\mathbf{c}_{r,\varrho}) \geq m+1$. Then $f := \mathrm{rt}(r)) \setminus \eta$ has $\mathrm{dom}(f) \cap \delta_* = \emptyset$ by Fact 2.5 (3). We take ℓ so large that (ii) in $\forall \eta \in q^{[\ell]} \boxplus_{q,\ell,\eta,f}$ holds. Then since $q^{\langle \eta \rangle}$ has the properties of the previous claim we get there are infinitely many ℓ such that such that $(\forall \eta \in q^{[\ell]})(\boxplus_{q,\ell,\eta,f})$. So we have $q^{[\ell]} \not\subseteq X$.

Second case: In Claim 3.3(a) we get $(\forall^\infty \ell)(q^\ell \cap X = \emptyset)$. By the definition of $X(p_0', k, m) = \mathrm{dom}(p_0') \setminus X$ we are done. $\dashv$

Conclusion 3.10. *$\mathbb{Q}_{\mathbf{T}}$ is a proper ${}^\omega\omega$-bounding forcing that specialises the Aronszajn tree $\mathbf{T}$.*

Checking that the coordinate $k(\mathbf{c})$ was only used in the halving property gives:

Conclusion 3.11. *All the results, with the exception of the halving property, hold for $\mathbb{Q}_{\mathbf{T}}$ that is built like $\mathbb{Q}_{\mathbf{T}}$ but with $k(\mathbf{c}) = 1$ for all $\mathbf{c}$. Then nor^1 coincides with nor_f for $f(n,k) = \lg(\frac{n}{k})$.*

Corollary 3.12. *It is consistent relative to ZFC that there are no Souslin trees and $\mathfrak{d} = \aleph_1$ and $2^\omega = \aleph_2$.*

Proof. The preservation theorems for properness and ${}^{\omega}\omega$-bounding allow us to iterate forcings $Q = Q_{\mathbf{T}}$ with countable support, for various $\mathbf{T}$. Starting form a ground model with $2^{\aleph_1} = \aleph_2$ we can successively specialise all Aronszajn trees in the ground model and in all intermediate models of the iteration and we can interweave other ω^ω-bounding proper iterands. By the preservation theorem for ω^ω-bounding [9, Chapter 6] we thus get a model where all Aronszajn trees are special and $\mathfrak{d} = \aleph_1$ and $2^\omega = \aleph_2$. ⊣

Since ♣ and CH together imply ♢ (see [9, Fact 7.3]), SH and ♣ together imply $2^\omega \geq \aleph_2$. So our forcing is at least an attempt in the direction of showing that ♣ together with all Aronszajn trees are special is consistent relative to ZFC.

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