

## COFINALITY OF THE NONSTATIONARY IDEAL

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ABSTRACT. We show that the reduced cofinality of the nonstationary ideal  $\mathcal{NS}_\kappa$  on a regular uncountable cardinal  $\kappa$  may be less than its cofinality, where the reduced cofinality of  $\mathcal{NS}_\kappa$  is the least cardinality of any family  $\mathcal{F}$  of nonstationary subsets of  $\kappa$  such that every nonstationary subset of  $\kappa$  can be covered by less than  $\kappa$  many members of  $\mathcal{F}$ . For this we investigate connections of the various cofinalities of  $\mathcal{NS}_\kappa$  with other cardinal characteristics of  ${}^\kappa\kappa$  and we also give a property of forcing notions (called *manageability*) which is preserved in  $<\kappa$ -support iterations and which implies that the forcing notion preserves non-meagerness of subsets of  ${}^\kappa\kappa$  (and does not collapse cardinals nor changes cofinalities).

### 0. INTRODUCTION

Let  $\kappa$  be a regular uncountable cardinal. For  $C \subseteq \kappa$  and  $\gamma \leq \kappa$ , we say that  $\gamma$  is a *limit point of  $C$*  if  $\bigcup(C \cap \gamma) = \gamma > 0$ .  $C$  is *closed unbounded* if  $C$  is a cofinal subset of  $\kappa$  containing all its limit points less than  $\kappa$ . A set  $A \subseteq \kappa$  is *nonstationary* if  $A$  is disjoint from some closed unbounded subset  $C$  of  $\kappa$ . The nonstationary subsets of  $\kappa$  form an ideal on  $\kappa$  denoted by  $\mathcal{NS}_\kappa$ . The *cofinality* of this ideal,  $\text{cof}(\mathcal{NS}_\kappa)$ , is the least cardinality of a family  $\mathcal{F}$  of nonstationary subsets of  $\kappa$  such that every nonstationary subset of  $\kappa$  is contained in a member of  $\mathcal{F}$ . The *reduced cofinality* of  $\mathcal{NS}_\kappa$ ,  $\overline{\text{cof}}(\mathcal{NS}_\kappa)$ , is the least cardinality of a family  $\mathcal{F} \subseteq \mathcal{NS}_\kappa$  such that every nonstationary subset of  $\kappa$  can be covered by less than  $\kappa$  many members of  $\mathcal{F}$ . This paper addresses the question whether  $\overline{\text{cof}}(\mathcal{NS}_\kappa) = \text{cof}(\mathcal{NS}_\kappa)$ . Note that

$$\kappa^+ \leq \overline{\text{cof}}(\mathcal{NS}_\kappa) \leq \text{cof}(\mathcal{NS}_\kappa) \leq 2^\kappa,$$

so under GCH we have  $\overline{\text{cof}}(\mathcal{NS}_\kappa) = \text{cof}(\mathcal{NS}_\kappa)$ .

Let  ${}^\kappa 2$  be endowed with the  $\kappa$ -box product topology,  $2$  itself considered discrete. We say that a set  $W \subseteq {}^\kappa 2$  is  $\kappa$ -*meager* if there is a sequence  $\langle U_\alpha : \alpha < \kappa \rangle$  of dense open subsets of  ${}^\kappa 2$  such that  $W \cap \bigcap_{\alpha < \kappa} U_\alpha = \emptyset$ . The *covering number for the category* of the space  ${}^\kappa 2$ , denoted  $\text{cov}(\mathbf{M}_{\kappa, \kappa})$ , is the least cardinality of any collection  $\mathcal{X}$  of  $\kappa$ -meager subsets of  ${}^\kappa 2$  such that  $\bigcup \mathcal{X} = {}^\kappa 2$ . It is not hard to verify that

$$\text{cov}(\mathbf{M}_{\kappa, \kappa}) \leq \text{cof}(\mathcal{NS}_\kappa) \leq (\overline{\text{cof}}(\mathcal{NS}_\kappa))^{<\kappa}.$$

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It follows that if  $\overline{\text{cof}}(\mathcal{NS}_\kappa) < \text{cov}(\mathbf{M}_{\kappa,\kappa})$  and the Singular Cardinals Hypothesis holds true, then  $\text{cf}(\overline{\text{cof}}(\mathcal{NS}_\kappa)) < \kappa$  and  $\text{cof}(\mathcal{NS}_\kappa) = (\overline{\text{cof}}(\mathcal{NS}_\kappa))^+$ . We prove:

**Theorem 0.1.** *Assume GCH. Then there is a  $\kappa$ -complete,  $\kappa^+$ -cc forcing notion  $\mathbb{P}$  such that*

$$\Vdash_{\mathbb{P}} \text{“} \overline{\text{cof}}(\mathcal{NS}_\kappa) = \kappa^{+\omega} \text{ and } \text{cof}(\mathcal{NS}_\kappa) = \kappa^{+(\omega+1)} \text{”}.$$

What about the consistency of “ $\overline{\text{cof}}(\mathcal{NS}_\kappa)$  is regular and  $\overline{\text{cof}}(\mathcal{NS}_\kappa) < \text{cof}(\mathcal{NS}_\kappa)$ ”? We establish:

**Theorem 0.2.** *It is consistent, relative to the existence of a cardinal  $\nu$  such that  $o(\nu) = \nu^{++}$ , that  $\text{cof}(\mathcal{NS}_{\omega_1}) = \aleph_{\omega+1}$  and  $\text{cov}(\mathbf{M}_{\aleph_1,\aleph_1}) = \aleph_{\omega+2}$ .*

The structure of the paper is as follows. In Section 1, for each infinite cardinal  $\mu \leq \kappa$  we introduce the  $< \mu$ -cofinality  $\text{cof}^{< \mu}(\mathcal{NS}_\kappa)$  and the  $< \mu$ -dominating number  $\mathfrak{d}_\kappa^{< \mu}$ , and we show that these two numbers are equal. Section 2 is concerned with a variant of  $\mathfrak{d}_\kappa^{< \mu}$  denoted by  $\mathfrak{d}_\kappa^{\text{cl}, < \mu}$  (where cl stands for “club”). We establish that  $\mathfrak{d}_\kappa^{< \mu} = \mathfrak{d}_\kappa^{\text{cl}, < \mu}$  if  $\mu > \omega$ .

$\mathcal{NS}_\kappa$  is the smallest normal ideal on  $\kappa$ . Section 3 deals with  $\mathcal{NS}_{\kappa,\lambda}^\kappa$ , the smallest  $\kappa$ -normal ideal on  $\mathcal{P}_\kappa(\lambda)$ . We compute  $\text{cof}^{< \mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa)$  and give examples of situations when  $\text{cof}^{< \mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa) < \text{cof}(\mathcal{NS}_{\kappa,\lambda}^\kappa)$ .

In the following section we present some basic facts regarding the ideal of  $\kappa$ -meager subsets of  ${}^\kappa 2$  and its covering number.

The final three sections of the paper present the consistency results mentioned in Theorems 0.1, 0.2 above. First, in Section 5 we introduce *manageability*, a property of  $< \kappa$ -complete  $\kappa^+$ -cc forcing notions which implies preservation of non-meagerness of subsets of  ${}^\kappa \kappa$  and which can be iterated. Next, in Section 6, we define one-step forcing and verify that it has all required properties. The final section gives the applications obtained by iterating this forcing notion.

*Notation 0.3.* Our notation is rather standard and compatible with that of classical textbooks (like Jech [Jec03]). In forcing we keep the older (Cohen’s) convention that *a stronger condition is the larger one*. Some of our conventions are listed below.

- (1) For a forcing notion  $\mathbb{P}$ ,  $\Gamma_{\mathbb{P}}$  stands for the canonical  $\mathbb{P}$ -name for the generic filter in  $\mathbb{P}$ . With this one exception, all  $\mathbb{P}$ -names for objects in the extension via  $\mathbb{P}$  will be denoted with a dot above (e.g.  $\dot{\tau}$ ,  $\dot{X}$ ). The weakest element of  $\mathbb{P}$  will be denoted by  $\emptyset_{\mathbb{P}}$  (and we will always assume that there is one, and that there is no other condition equivalent to it). In iterations, if  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \dot{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$  and  $p \in \lim(\bar{\mathbb{Q}})$ , then we keep convention that  $p(\alpha) = \emptyset_{\dot{\mathbb{Q}}_\alpha}$  for  $\alpha \in \zeta^* \setminus \text{Dom}(p)$ .
- (2) Ordinal numbers will be denoted by  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi$  and also by  $i, j$  (with possible sub- and superscripts). Infinite cardinal numbers will be called  $\theta, \iota, \mu, \nu, \tau$  (with possible sub- and superscripts);  $\kappa$  is our fixed regular uncountable cardinal,  $\lambda$  will denote a fixed cardinal  $> \kappa$  (in Section 3).
- (3) By  $\chi$  we will denote a sufficiently large regular cardinal and by  $\mathcal{H}(\chi)$  the family of all sets hereditarily of size less than  $\chi$ . Moreover, we fix a well ordering  $<_\chi^*$  of  $\mathcal{H}(\chi)$ .

- (4) A bar above a letter denotes that the object considered is a sequence; usually  $\bar{X}$  will be  $\langle X_i : i < \zeta \rangle$ , where  $\zeta$  denotes the length of  $\bar{X}$ . For a set  $A$  and a cardinal  $\mu$ , the set of all sequences of members of  $A$  of length  $\mu$  (length  $< \mu$ , respectively), will be denoted by  ${}^\mu A$  ( $<{}^\mu A$ , respectively).

$$1. \text{ cof}^{<\mu}(\mathcal{NS}_\kappa)$$

- Definition 1.1.** (1) For a set  $A$  and a cardinal  $\mu$ ,  $\mathcal{P}_\mu(A) = \{a \subseteq A : |a| < \mu\}$ .  
(2) Given two infinite cardinals  $\mu \leq \tau$ ,  $u(\mu, \tau)$  is the least cardinality of a collection  $A \subseteq \mathcal{P}_\mu(\tau)$  such that  $\mathcal{P}_\mu(\tau) = \bigcup_{a \in A} \mathcal{P}(a)$ .

**Definition 1.2.** Let  $S$  be an infinite set and  $\mathcal{J}$  be an ideal on  $S$  (containing all singletons).

- (1)  $\text{cof}(\mathcal{J})$  is the least cardinality of any  $\mathcal{X} \subseteq \mathcal{J}$  such that for every  $A \in \mathcal{J}$ , there is  $B \in \mathcal{X}$  with  $A \subseteq B$ .  
(2)  $\text{add}(\mathcal{J})$  is the least cardinality of any  $\mathcal{X} \subseteq \mathcal{J}$  such that  $\bigcup \mathcal{X} \notin \mathcal{J}$ .  
(3) For an infinite cardinal  $\mu \leq \text{add}(\mathcal{J})$ ,  $\text{cof}^{<\mu}(\mathcal{J})$  is the least cardinality of a family  $\mathcal{X} \subseteq \mathcal{J}$  such that for every  $A \in \mathcal{J}$ , there is  $\mathcal{Y} \in \mathcal{P}_\mu(\mathcal{X})$  such that  $A \subseteq \bigcup \mathcal{Y}$ .  
(4) We let  $\overline{\text{cof}}(\mathcal{J}) = \text{cof}^{<\text{add}(\mathcal{J})}(\mathcal{J})$ .

The following proposition collects some trivialities.

**Proposition 1.3.** *Let  $S$  be an infinite set and  $\mathcal{J}$  be an ideal on  $S$  that contains all singletons. Then:*

- (i)  $\text{cof}^{<\omega}(\mathcal{J}) = \text{cof}(\mathcal{J})$ .  
(ii) If  $\mu, \nu$  are two infinite cardinals with  $\mu \leq \nu \leq \text{add}(\mathcal{J})$ , then  $\text{cof}^{<\nu}(\mathcal{J}) \leq \text{cof}^{<\mu}(\mathcal{J})$ .  
(iii)  $\text{cof}(\mathcal{J}) \leq u(\mu, \text{cof}^{<\mu}(\mathcal{J}))$  for every infinite cardinal  $\mu \leq \text{add}(\mathcal{J})$ .  
(iv)  $\text{add}(\mathcal{J}) \leq \overline{\text{cof}}(\mathcal{J})$ .

The following is well-known (see, e.g., Matet, Péan and Shelah [MPS16]):

**Lemma 1.4.** *Let  $\mu$  be a regular infinite cardinal. Then  $u(\mu, \mu^{+n}) = \mu^{+n}$  for every  $n < \omega$ .*

**Proposition 1.5.** *Let  $S$  be an infinite set and  $\mathcal{J}$  be an ideal on  $S$  such that  $(\text{add}(\mathcal{J}))^{+\omega} \leq \text{cof}(\mathcal{J})$ . Then  $(\text{add}(\mathcal{J}))^{+\omega} \leq \overline{\text{cof}}(\mathcal{J})$ .*

*Proof.* Use Lemma 1.4. □

With these preliminaries out of the way, we can concentrate on ideals on  $\kappa$ . If there is a family of size  $\kappa^{+\omega}$  of pairwise almost disjoint cofinal subsets of  $\kappa$ , then there is a  $\kappa$ -complete ideal  $\mathcal{J}$  on  $\kappa$  such that  $\overline{\text{cof}}(\mathcal{J}) < \text{cof}(\mathcal{J})$  (see Matet and Pawlikowski [MP03]).

**Proposition 1.6.** *Suppose  $\mathcal{J}$  is a normal ideal on  $\kappa$  and  $\kappa$  is a limit cardinal. Then  $\overline{\text{cof}}(\mathcal{J}) = \text{cof}^{<\mu}(\mathcal{J})$  for some infinite cardinal  $\mu < \kappa$ .*

*Proof.* Assume that the conclusion fails. Fix  $\mathcal{X} \subseteq \mathcal{J}$  such that  $|\mathcal{X}| = \overline{\text{cof}}(\mathcal{J})$  and

$$\mathcal{J} = \bigcup \{\mathcal{P}(\bigcup X) : X \in \mathcal{P}_\kappa(\mathcal{X})\}.$$

Set  $\mathcal{Y} = \{A \cup \beta : A \in \mathcal{X} \ \& \ \beta \in \kappa\}$ . Note that  $|\mathcal{Y}| = \overline{\text{cof}}(\mathcal{J})$ . For each infinite cardinal  $\mu < \kappa$  we may select a set  $B_\mu \in \mathcal{J}$  so that  $B_\mu \not\subseteq \bigcup Y$  for any  $Y \in \mathcal{P}_\mu(\mathcal{Y})$ .

Now let  $B$  be the set of all  $\alpha < \kappa$  such that  $\alpha \in B_\mu$  for some infinite cardinal  $\mu < \alpha$ . Since  $B \in \mathcal{J}$  (by normality of  $\mathcal{J}$ ), there must be  $X \in \mathcal{P}_\kappa(\mathcal{X})$  such that  $B \subseteq \bigcup X$ . Let  $\tau$  be any infinite cardinal such that  $|X| < \tau < \kappa$ . Then  $B_\tau \subseteq \bigcup_{A \in X} (A \cup (\tau + 1))$ , which is a contradiction.  $\square$

Arguing as in Proposition 1.6, we get:

**Proposition 1.7.** *Suppose  $\mathcal{J}$  is a  $\kappa$ -complete ideal on  $\kappa$  and  $\nu$  is an uncountable limit cardinal  $< \kappa$ . Then there is an infinite cardinal  $\mu < \nu$  such that  $\text{cof}^{<\nu}(\mathcal{J}) = \text{cof}^{<\mu}(\mathcal{J})$ . Moreover, the least such  $\mu$  is either  $\omega$ , or a successor cardinal.*

The remainder of this section is concerned with  $\text{cof}^{<\mu}(\mathcal{NS}_\kappa)$ . Let us recall the definition of the bounding number  $\mathfrak{b}_\kappa$ :

**Definition 1.8.** The *bounding number*  $\mathfrak{b}_\kappa$  is the least cardinality of any  $\mathcal{F} \subseteq {}^\kappa\kappa$  with the property that for every  $g \in {}^\kappa\kappa$ , there is  $f \in \mathcal{F}$  such that

$$|\{\alpha < \kappa : g(\alpha) \leq f(\alpha)\}| = \kappa.$$

The following is proved in Matet and Pawlikowski [MP03]:

**Proposition 1.9.** (i)  $\overline{\text{cof}}(\mathcal{NS}_\kappa) \geq \mathfrak{b}_\kappa$ .  
(ii) If  $\overline{\text{cof}}(\mathcal{NS}_\kappa) = \mathfrak{b}_\kappa$ , then  $\overline{\text{cof}}(\mathcal{NS}_\kappa) = \text{cof}(\mathcal{NS}_\kappa)$ .

**Proposition 1.10.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then*

$$\text{either } \text{cf}(\text{cof}^{<\mu}(\mathcal{NS}_\kappa)) < \mu, \text{ or } \text{cf}(\text{cof}^{<\mu}(\mathcal{NS}_\kappa)) \geq \mathfrak{b}_\kappa.$$

*Proof.* Suppose to the contrary that  $\mu \leq \text{cf}(\text{cof}^{<\mu}(\mathcal{NS}_\kappa)) = \tau < \mathfrak{b}_\kappa$ . For  $\alpha < \tau$  select  $\mathcal{X}_\alpha \subseteq \mathcal{NS}_\kappa$  so that

- (i)  $|\mathcal{X}_\alpha| < \text{cof}^{<\mu}(\mathcal{NS}_\kappa)$ ,
- (ii)  $\mathcal{X}_\beta \subseteq \mathcal{X}_\alpha$  for  $\beta < \alpha$ ,
- (iii)  $\mathcal{NS}_\kappa = \bigcup_{\alpha < \tau} \{\mathcal{P}(\bigcup X) : X \in \mathcal{P}_\mu(\bigcup \mathcal{X}_\alpha)\}$ .

For  $\alpha < \tau$ , set  $\mathcal{Y}_\alpha = \{A \cup \beta : A \in \mathcal{X}_\alpha \ \& \ \beta \in \kappa\}$  and pick  $B_\alpha \in \mathcal{NS}_\kappa$  so that  $B_\alpha \not\subseteq \bigcup Y$  for any  $Y \in \mathcal{P}_\mu(\mathcal{Y}_\alpha)$ . By a result of Balcar and Simon (see [BS89, Theorem 5.25]), there is  $B \in \mathcal{NS}_\kappa$  such that  $|B_\alpha \setminus B| < \kappa$  for every  $\alpha < \tau$ . Select  $X \in \mathcal{P}_\mu(\bigcup_{\alpha < \tau} \mathcal{X}_\alpha)$  so that  $B \subseteq \bigcup X$ . There is  $\gamma < \tau$  such that  $X \subseteq \mathcal{X}_\gamma$ . Then  $B_\gamma \subseteq \bigcup_{A \in X} (A \cup \beta)$  for some  $\beta \in \kappa$ , which is a contradiction.  $\square$

**Definition 1.11.** Let  $\tau \leq \kappa$ . A family  $\mathcal{F} \subseteq {}^\kappa\kappa$  is called

- a *dominating family* if

$$(\forall h \in {}^\kappa\kappa)(\exists f \in \mathcal{F})(\forall j < \kappa)(h(j) < f(j)),$$

- a  $<\tau$ -*dominating family* if

$$(\forall h \in {}^\kappa\kappa)(\exists F \in \mathcal{P}_\tau(\mathcal{F}))(\forall j < \kappa)(h(j) < \sup\{f(j) : f \in F\}).$$

We define dominating numbers  $\mathfrak{d}_\kappa, \mathfrak{d}_\kappa^{<\tau}$  by:

$$\begin{aligned} \mathfrak{d}_\kappa &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa \text{ is a dominating family } \}, \\ \mathfrak{d}_\kappa^{<\tau} &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa \text{ is a } <\tau\text{-dominating family } \}. \end{aligned}$$

We let  $\bar{\mathfrak{d}}_\kappa = \mathfrak{d}_\kappa^{<\kappa}$  and for an infinite cardinal  $\mu < \kappa$  we put  $\mathfrak{d}_\kappa^\mu = \mathfrak{d}_\kappa^{<\mu^+}$ .

Note that  $\mathfrak{d}_\kappa^{<\omega} = \mathfrak{d}_\kappa$ . Landver [Lan90] established that  $\text{cof}(\mathcal{NS}_\kappa) = \mathfrak{d}_\kappa$ . His result can be generalized as follows:

**Theorem 1.12.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then  $\text{cof}^{<\mu}(\mathcal{NS}_\kappa) = \mathfrak{d}_\kappa^{<\mu}$ .*

*Proof.* Set  $\tau = \text{cof}^{<\mu}(\mathcal{NS}_\kappa)$ . First we will argue that  $\mathfrak{d}_\kappa^{<\mu} \leq \tau$ . Select a family  $\mathcal{C}$  of size  $\tau$  of closed unbounded subsets of  $\kappa$  so that for every closed unbounded subset  $D$  of  $\kappa$ , there is  $X \in \mathcal{P}_\mu(\mathcal{C}) \setminus \{\emptyset\}$  with  $\bigcap X \subseteq D$ . For  $U \in \mathcal{P}_\omega(\mathcal{C}) \setminus \{\emptyset\}$  define  $f_U \in {}^\kappa\kappa$  by  $f_U(\alpha) = \min(\bigcap U \setminus (\alpha + 1))$ . Note that  $f_V(\alpha) \leq f_U(\alpha)$  whenever  $V \in \mathcal{P}(U) \setminus \{\emptyset\}$ . Now given  $g \in {}^\kappa\kappa$ , let  $D$  be the set of all limit ordinals  $\delta < \kappa$  such that  $g(\alpha) < \delta$  for every  $\alpha < \delta$ . Pick  $X \in \mathcal{P}_\mu(\mathcal{C}) \setminus \{\emptyset\}$  so that  $\bigcap X \subseteq D$ . Define  $h \in {}^\kappa\kappa$  by

$$h(\alpha) = \sup \{f_U(\alpha) : U \in \mathcal{P}_\omega(X) \setminus \{\emptyset\}\}.$$

We are going to show that  $g < h$ . Let  $\alpha < \kappa$  and  $C \in X$ . First, suppose that there is  $W \in \mathcal{P}_\omega(X) \setminus \{\emptyset\}$  such that  $h(\alpha) = f_W(\alpha)$ . Then  $h(\alpha) = f_{W \cup \{C\}}(\alpha)$  and hence  $h(\alpha) \in C$ . Next suppose that  $f_U(\alpha) < h(\alpha)$  for all  $U \in \mathcal{P}_\omega(X) \setminus \{\emptyset\}$ . Then  $h(\alpha)$  is a limit ordinal. Set  $\iota = \text{cf}(h(\alpha))$  and pick an increasing sequence  $\langle \gamma_\beta : \beta < \iota \rangle$  cofinal in  $h(\alpha)$ . For  $\beta < \iota$ , select  $T_\beta \in \mathcal{P}_\omega(X) \setminus \{\emptyset\}$  with  $\gamma_\beta < f_{T_\beta}(\alpha)$ , and set  $\delta_\beta = f_{T_\beta \cup \{C\}}(\alpha)$ . Note that  $\delta_\beta \in C$ . Obviously, the sequence  $\langle \delta_\beta : \beta < \iota \rangle$  is cofinal in  $h(\alpha)$ , and consequently  $h(\alpha) \in C$ . Thus for each  $\alpha < \kappa$ ,  $h(\alpha)$  belongs to  $\bigcap X$  and therefore to  $D$ . Since clearly  $h(\alpha) > \alpha$ , it follows that  $h(\alpha) > g(\alpha)$ .

It remains to show that  $\mathfrak{d}_\kappa^{<\mu} \geq \tau$ . Let  $\mathfrak{F}$  be the set of all strictly increasing functions from  $\kappa$  to  $\kappa$ . Select  $\mathcal{F} \subseteq \mathfrak{F}$  so that

- (a)  $|\mathcal{F}| = \mathfrak{d}_\kappa^{<\mu}$ , and
- (b) given  $g \in {}^\kappa\kappa$ , there is  $F_g \in \mathcal{P}_\mu(\mathcal{F})$  such that

$$(\forall \alpha < \kappa)(g(\alpha) < \sup\{f(\alpha) : f \in F_g\}).$$

For  $f \in \mathfrak{F}$ , let  $C_f$  be the set of all limit ordinals  $\alpha < \kappa$  such that  $f(\beta) < \alpha$  for every  $\beta < \alpha$ . Easily

$$\mathcal{NS}_\kappa = \{A \subseteq \kappa : (\exists g \in \mathfrak{F})(A \cap C_g = \emptyset)\}$$

(see, e.g., [MP03]) and (as  $\bigcap_{f \in F_g} C_f \subseteq C_g$  for every  $g \in \mathfrak{F}$ ) it follows that  $\tau \leq |\mathcal{F}|$ .  $\square$

It follows from Propositions 1.6 and 1.7 and Theorem 1.12 that to determine the value of  $\text{cof}^{<\mu}(\mathcal{NS}_\kappa)$  for every infinite cardinal  $\mu \leq \kappa$ , it suffices to compute  $\mathfrak{d}_\kappa$  and  $\mathfrak{d}_\kappa^\tau$  for every infinite cardinal  $\tau < \kappa$ .

## 2. $\mathfrak{d}_\kappa^{\text{cl}, <\mu}$

It is straightforward to check that  $\mathfrak{d}_\kappa^{<\mu}$  is the least cardinality of a family  $\mathcal{F} \subseteq {}^\kappa\kappa$  such that

$$(\forall g \in {}^\kappa\kappa)(\exists F \in \mathcal{P}_\mu(\mathcal{F}))(|\{\alpha \in \kappa : g(\alpha) \geq \sup\{f(\alpha) : f \in F\}\}| < \kappa).$$

In this section we discuss the variant that arises if we replace “has cardinality  $< \kappa$ ” by “is nonstationary”.

**Definition 2.1.** (1)  $\mathfrak{d}_\kappa^{\text{cl}}$  is the least cardinality of a family  $\mathcal{F} \subseteq {}^\kappa\kappa$  with the property that for every  $g \in {}^\kappa\kappa$ , there is  $f \in \mathcal{F}$  such that

$$\{\alpha \in \kappa : g(\alpha) \geq f(\alpha)\} \in \mathcal{NS}_\kappa.$$

- (2) For an infinite cardinal  $\mu \leq \kappa$ ,  $\mathfrak{d}_\kappa^{\text{cl}, < \mu}$  is the least cardinality of a family  $\mathcal{F} \subseteq {}^\kappa \kappa$  with the property that for every  $g \in {}^\kappa \kappa$ , there is  $F \in \mathcal{P}_\mu(\mathcal{F})$  such that

$$\{\alpha \in \kappa : g(\alpha) \geq \sup\{f(\alpha) : f \in F\}\} \in \mathcal{NS}_\kappa.$$

Note that  $\mathfrak{d}_\kappa^{\text{cl}, < \omega} = \mathfrak{d}_\kappa^{\text{cl}}$ . It is simple to check that  $\text{cf}(\mathfrak{d}_\kappa^{\text{cl}}) \geq \mathfrak{b}_\kappa$ .

**Theorem 2.2.** *For every uncountable cardinal  $\mu \leq \kappa$ ,*

$$\mathfrak{d}_\kappa^{\text{cl}, < \mu} = \mathfrak{d}_\kappa^{< \mu}.$$

Theorem 2.2 easily follows from the next two lemmas.

**Lemma 2.3.** *Let  $\mu$  be an uncountable limit cardinal  $\leq \kappa$ . Then  $\mathfrak{d}_\kappa^{\text{cl}, < \mu} = \mathfrak{d}_\kappa^{\text{cl}, < \tau}$  for some infinite cardinal  $\tau < \mu$ .*

*Proof.* The proof is similar to that of Proposition 1.7. Suppose that the conclusion fails. Fix a family  $\mathcal{F} \subseteq {}^\kappa \kappa$  such that  $|\mathcal{F}| = \mathfrak{d}_\kappa^{\text{cl}, < \mu}$  and

$$(\forall g \in {}^\kappa \kappa)(\exists F \in \mathcal{P}_\mu(\mathcal{F}))(\{\alpha \in \kappa : g(\alpha) \geq \sup\{f(\alpha) : f \in F\}\} \in \mathcal{NS}_\kappa).$$

For each infinite cardinal  $\tau < \mu$  we may select  $g_\tau \in {}^\kappa \kappa$  so that for every  $F \in \mathcal{P}_\tau(\mathcal{F})$  we have

$$\{\alpha \in \kappa : g_\tau(\alpha) \geq \sup\{f(\alpha) : f \in F\}\} \notin \mathcal{NS}_\kappa.$$

Define  $g \in {}^\kappa \kappa$  so that  $g(\alpha) \geq g_\tau(\alpha)$  for every infinite cardinal  $\tau < \mu$  such that  $\tau < \alpha$ . Now pick  $F \in \mathcal{P}_\mu(\mathcal{F})$  such that

$$\{\alpha \in \kappa : g(\alpha) \geq \sup\{f(\alpha) : f \in F\}\} \in \mathcal{NS}_\kappa.$$

Let  $\tau$  be any infinite cardinal with  $|F| < \tau < \mu$ . Obviously,  $F \in \mathcal{P}_\tau(\mathcal{F})$  and

$$\{\alpha \in \kappa : g_\tau(\alpha) \geq \sup\{f(\alpha) : f \in F\}\} \in \mathcal{NS}_\kappa,$$

a contradiction. □

To establish the following lemma, we adapt the proof of Theorem 5 in Cummings and Shelah [CS95].

**Lemma 2.4.** *Let  $\mu$  be a regular uncountable cardinal  $\leq \kappa$ . Then  $\mathfrak{d}_\kappa^{< \mu} = \mathfrak{d}_\kappa^{\text{cl}, < \mu}$ .*

*Proof.* Select a family  $\mathcal{F} \subseteq {}^\kappa \kappa$  such that

- (a) every member of  $\mathcal{F}$  is increasing,
- (b)  $|\mathcal{F}| = \mathfrak{d}_\kappa^{\text{cl}, < \mu}$ , and
- (c) for each  $g \in {}^\kappa \kappa$ , there is  $F \in \mathcal{P}_\mu(\mathcal{F})$  such that

$$\{\alpha \in \kappa : g(\alpha) \geq \sup\{f(\alpha) : f \in F\}\} \in \mathcal{NS}_\kappa.$$

We claim that the family

$$\mathcal{F}^* \stackrel{\text{def}}{=} \{f \in {}^\kappa \kappa : (\exists \alpha, \beta < \kappa)(\exists g \in \mathcal{F})(f \upharpoonright \beta \equiv \alpha \ \& \ f \upharpoonright [\beta, \kappa) = g \upharpoonright [\beta, \kappa))\}$$

is  $< \mu$ -dominating. So let  $g \in {}^\kappa \kappa$ . Stipulate that  $g_{-1} = g$ . By induction on  $n \in \omega$  choose a closed unbounded subset  $C_n$  of  $\kappa$ ,  $g_n, h_n \in {}^\kappa \kappa$  and  $F_n \in \mathcal{P}_\mu(\mathcal{F})$  so that

- (i)  $C_{n+1} \subseteq C_n$ ,
- (ii)  $g_{n-1}(\alpha) < \sup\{f(\alpha) : f \in F_n\}$  for all  $\alpha \in C_n$ ,
- (iii)  $h_n(\beta) = \min(C_n \setminus (\beta + 1))$ ,
- (iv)  $g_n(\beta) = \sup(\text{Rng}(g_{n-1} \upharpoonright (h_n(\beta) + 1)))$ .

Note that, by (iii) and (iv),  $g(\beta) \leq g_0(\beta) \leq g_1(\beta) \leq \dots$  for all  $\beta \in \kappa$ . Set  $F = \bigcup_{n \in \omega} F_n$  and  $\zeta = \sup\{\min(C_n) : n \in \omega\}$ . We are going to show that  $g(\gamma) < \sup\{f(\gamma) : f \in F\}$  whenever  $\zeta < \gamma < \kappa$ . To this end suppose that  $\zeta < \gamma < \kappa$ . By (i), there are  $m \in \omega$  and  $\xi \in \kappa$  such that  $\xi = \sup(\gamma \cap C_n)$  whenever  $m \leq n < \omega$ . By (iii),  $h_m(\xi) \geq \gamma$  and so (by (iv))  $g(\gamma) \leq g_{m-1}(\gamma) \leq g_m(\xi)$ . Since  $\gamma > \zeta$  we also have  $\gamma \cap C_{m+1} \neq \emptyset$ . Hence  $\xi \in C_{m+1}$  and consequently, by (ii),

$$g(\gamma) \leq g_m(\xi) < \sup\{f(\xi) : f \in F_{m+1}\} \leq \sup\{f(\gamma) : f \in F_{m+1}\} \leq \sup\{f(\gamma) : f \in F\}.$$

□

Theorem 2.2 implies that  $\mathfrak{d}_\kappa^\omega \leq \mathfrak{d}_\kappa^{\text{cl}} \leq \mathfrak{d}_\kappa$ . We mention that it was shown in Cummings and Shelah [CS95] that  $\mathfrak{d}_\kappa^{\text{cl}} = \mathfrak{d}_\kappa$  if  $\kappa > \beth_\omega$ .

### 3. $\text{cof}^{<\mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa)$

Throughout this section  $\lambda$  denotes a fixed cardinal  $> \kappa$ . Our object of study will be the ideal  $\mathcal{NS}_{\kappa,\lambda}^\kappa$ , a  $\mathcal{P}_\kappa(\lambda)$  version of  $\mathcal{NS}_\kappa$ .

**Definition 3.1.** For a regular uncountable cardinal  $\nu$  and a cardinal  $\tau \geq \nu$ ,  $\mathcal{J}_{\nu,\tau}$  is the set of all  $A \subseteq \mathcal{P}_\nu(\tau)$  such that for some  $a \in \mathcal{P}_\nu(\tau)$  we have  $\{b \in A : a \subseteq b\} = \emptyset$ .

It is straightforward to check that  $\mathcal{J}_{\nu,\tau}$  is a  $\nu$ -complete ideal on  $\mathcal{P}_\nu(\tau)$ .

**Definition 3.2.** (1) An ideal  $\mathcal{J}$  of  $\mathcal{P}_\kappa(\lambda)$  is  $\kappa$ -normal if given  $A \in \mathcal{J}^+$  and  $f : A \rightarrow \kappa$  such that  $f(a) \in a \cap \kappa$  for all  $a \in A$ , there is  $B \in \mathcal{J}^+ \cap \mathcal{P}(A)$  such that  $f$  is constant on  $B$ .

(2) The smallest  $\kappa$ -normal ideal on  $\mathcal{P}_\kappa(\lambda)$  containing  $\mathcal{J}_{\kappa,\lambda}$  is denoted by  $\mathcal{NS}_{\kappa,\lambda}^\kappa$ .

(3) For  $f \in {}^\kappa(\mathcal{P}_\kappa(\lambda))$  we let

$$C_f \stackrel{\text{def}}{=} \{a \in \mathcal{P}_\kappa(\lambda) : a \cap \kappa \neq \emptyset \text{ and } \bigcup_{\alpha \in a \cap \kappa} f(\alpha) \subseteq a\}.$$

The following lemma is due to Abe.

**Lemma 3.3** (Abe [Abe97]). *Let  $A \subseteq \mathcal{P}_\kappa(\lambda)$ . Then*

$$A \in \mathcal{NS}_{\kappa,\lambda}^\kappa \quad \text{if and only if} \quad (\exists f \in {}^\kappa(\mathcal{P}_\kappa(\lambda)))(A \cap C_f = \emptyset).$$

Our purpose in this section is to compute the value of  $\text{cof}^{<\mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa)$ . We will need an analogue of  $\mathfrak{d}_\kappa^{<\mu}$  defined in 3.4(1) below.

**Definition 3.4.** Let  $\mu \leq \kappa$  be an infinite cardinal.

(1)  $\mathfrak{d}_{\kappa,\lambda}^{\kappa,<\mu}$  is the least cardinality of a family  $\mathcal{X}$  of functions from  $\kappa$  to  $\mathcal{P}_\kappa(\lambda)$  with the property that

$$(\forall g \in {}^\kappa(\mathcal{P}_\kappa(\lambda)))(\exists X \in \mathcal{P}_\mu(\mathcal{X}))(\forall \alpha \in \kappa)(g(\alpha) \subseteq \bigcup_{f \in X} f(\alpha)).$$

(2)  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu)$  is the least cardinality of a family  $\mathcal{X} \subseteq \mathcal{P}_{\kappa^+}(\lambda)$  such that

$$(\forall B \in \mathcal{P}_{\kappa^+}(\lambda))(\exists X \in \mathcal{P}_\mu(\mathcal{X}))(B \subseteq \bigcup X).$$

**Theorem 3.5.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then*

$$\text{cof}^{<\mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa) = \mathfrak{d}_{\kappa,\lambda}^{\kappa,<\mu} = \max\{\mathfrak{d}_\kappa^{<\mu}, \text{cov}(\lambda, \kappa^+, \kappa^+, \mu)\}.$$

Theorem 3.5 is an immediate consequence of Lemmas 3.6–3.9 below.

**Lemma 3.6.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then*

$$\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \leq \text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa).$$

*Proof.* By 3.3 we may pick a family  $\mathcal{X} \subseteq {}^\kappa(\mathcal{P}_\kappa(\lambda))$  with the property that  $|\mathcal{X}| = \text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa)$  and for every function  $g : \kappa \rightarrow \mathcal{P}_\kappa(\lambda)$  there is  $X \in \mathcal{P}_\mu(\mathcal{X})$  such that  $\bigcap_{f \in X} C_f \subseteq C_g$ . For  $f \in \mathcal{X}$ , let  $B_f = \kappa \cup \bigcup_{\alpha < \kappa} f(\alpha) \in \mathcal{P}_{\kappa^+}(\lambda)$ .

Suppose now that  $B \in \mathcal{P}_{\kappa^+}(\lambda)$ . Pick a function  $g : \kappa \rightarrow \mathcal{P}_\kappa(\lambda)$  such that  $B \subseteq \bigcup_{\alpha < \kappa} g(\alpha)$ . There is  $X \in \mathcal{P}_\mu(\mathcal{X})$  such that  $\bigcap_{f \in X} C_f \subseteq C_g$ . We are going to show that  $B \subseteq \bigcup_{f \in X} B_f$ . To this end suppose  $\alpha < \kappa$  and let us argue that  $g(\alpha) \subseteq \bigcup_{f \in X} B_f$ .

For  $n < \omega$  let  $a_n \in \mathcal{P}_\kappa(\bigcup_{f \in X} B_f)$  be defined by

$$a_0 = \{\alpha\}, \quad \text{and} \quad a_{n+1} = a_n \cup \bigcup_{f \in X} \bigcup_{\beta \in a_n \cap \kappa} f(\beta),$$

and let  $a = \bigcup_{n < \omega} a_n$ . Then  $\alpha \in a \in \bigcap_{f \in X} C_f \subseteq C_g$  and consequently  $g(\alpha) \subseteq a \subseteq \bigcup_{f \in X} B_f$ .  $\square$

**Lemma 3.7.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then  $\mathfrak{d}_\kappa^{<\mu} \leq \text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa)$ .*

*Proof.* By Theorem 1.12, it suffices to establish that  $\text{cof}^{<\mu}(\mathcal{NS}_\kappa) \leq \text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa)$ . Let a family  $\mathcal{X} \subseteq {}^\kappa(\mathcal{P}_\kappa(\lambda))$  be such that  $|\mathcal{X}| = \text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa)$  and

$$(\forall B \in \mathcal{NS}_{\kappa, \lambda}^\kappa)(\exists X \in \mathcal{P}_\mu(\mathcal{X}) \setminus \{\emptyset\})(B \cap \bigcap_{f \in X} C_f = \emptyset).$$

For  $f \in \mathcal{X}$ , let  $Z_f$  be the set of all limit ordinals  $\alpha < \kappa$  such that

$$(\forall \beta < \alpha)(f(\beta) \cap \kappa \subseteq \alpha).$$

Plainly,  $Z_f$  is a closed unbounded subset of  $\kappa$ . Now given a closed unbounded subset  $T$  of  $\kappa$ , set  $B_T = \{a \in \mathcal{P}_\kappa(\lambda) : a \cap \kappa \not\subseteq T\}$ . A simple argument (see, e.g., [MPS16]) shows that  $B_T \in \mathcal{NS}_{\kappa, \lambda}^\kappa$ . Hence there is  $X_T \in \mathcal{P}_\mu(\mathcal{X}) \setminus \{\emptyset\}$  such that  $B_T \cap \bigcap_{f \in X_T} C_f = \emptyset$ . We will show that  $\bigcap_{f \in X_T} Z_f \subseteq T$ . Thus let  $\alpha \in \bigcap_{f \in X_T} Z_f$ . Setting  $a = \alpha \cup \bigcup_{f \in X_T} \bigcup_{\beta < \alpha} f(\beta)$ , it is easy to see that  $a \cap \kappa = \alpha$  and  $a \in \bigcap_{f \in X_T} C_f$ . It follows that  $\alpha = a \cap \kappa \in T$ .  $\square$

**Lemma 3.8.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then  $\text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa) \leq \mathfrak{d}_{\kappa, \lambda}^{\kappa, <\mu}$ .*

*Proof.* The inequality easily follows from the following observation.

Suppose  $h : \kappa \rightarrow \mathcal{P}_\kappa(\lambda)$  and  $X \in \mathcal{P}_\mu({}^\kappa(\mathcal{P}_\kappa(\lambda)))$  are such that

$$(\forall \alpha < \kappa)(h(\alpha) \subseteq \bigcup_{f \in X} f(\alpha)).$$

Then  $\bigcap_{f \in X} C_f \subseteq C_h$ .  $\square$

**Lemma 3.9.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then*

$$\mathfrak{d}_{\kappa, \lambda}^{\kappa, <\mu} \leq \max\{\mathfrak{d}_\kappa^{<\mu}, \text{cov}(\lambda, \kappa^+, \kappa^+, \mu)\}.$$



*Proof.* Fix  $\mathcal{F} \subseteq {}^\kappa\kappa$  so that  $|\mathcal{F}| = \mathfrak{d}_\kappa^{<\mu}$  and

$$(\forall h \in {}^\kappa\kappa)(\exists F \in \mathcal{P}_\mu(\mathcal{F}))(\forall \alpha < \kappa)(h(\alpha) \leq \sup\{f(\alpha) : f \in F\}).$$

Also, fix  $\mathcal{X} \subseteq \mathcal{P}_{\kappa^+}(\lambda)$  such that  $|\mathcal{X}| = \text{cov}(\lambda, \kappa^+, \kappa^+, \mu)$  and

$$\mathcal{P}_{\kappa^+}(\lambda) = \bigcup \{\mathcal{P}(\bigcup X) : X \in \mathcal{P}_\mu(\mathcal{X})\}.$$

For each  $a \in \mathcal{X}$ , select a mapping  $\varphi_a : \kappa \xrightarrow{\text{onto}} a$ . Now for  $f \in \mathcal{F}$  and  $a \in \mathcal{X}$  define  $g_{f,a} : \kappa \rightarrow \mathcal{P}_\kappa(\lambda)$  by

$$g_{f,a}(\alpha) = \{\varphi_a(\delta) : \delta < f(\alpha)\}.$$

Suppose now that  $g : \kappa \rightarrow \mathcal{P}_\kappa(\lambda)$ . By the choice of  $\mathcal{X}$ , there is  $X \in \mathcal{P}_\mu(\mathcal{X})$  such that  $\bigcup_{\alpha < \kappa} g(\alpha) \subseteq \bigcup X$ . Choose  $h \in {}^\kappa\kappa$  such that

$$(\forall \alpha < \kappa)(g(\alpha) \subseteq \bigcup_{a \in X} \{\varphi_a(\xi) : \xi < h(\alpha)\}).$$

Next pick  $F \in \mathcal{P}_\mu(\mathcal{F})$  such that  $(\forall \alpha < \kappa)(h(\alpha) \leq \sup\{f(\alpha) : f \in F\})$ . Then

$$(\forall \alpha < \kappa)(g(\alpha) \subseteq \bigcup_{f \in F} \bigcup_{a \in X} g_{f,a}(\alpha)).$$

□

Another formula worth noting is:

$$\text{cof}^{<\mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa) = \max\{\text{cof}^{<\mu}(\mathcal{NS}_\kappa), \text{cof}^{<\mu}(\mathcal{J}_{\kappa^+,\lambda})\}.$$

This identity follows from Theorems 1.12 and 3.5 and the next proposition.

**Proposition 3.10.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) = \text{cof}^{<\mu}(\mathcal{J}_{\kappa^+,\lambda})$ .*

*Proof.* The result easily follows from the following observation.

Suppose that  $\mathcal{X} \subseteq \mathcal{P}_{\kappa^+}(\lambda)$  and  $X \in \mathcal{P}_\mu(\mathcal{X}) \setminus \{\emptyset\}$ . Then

$$\bigcap_{a \in X} \{c \in \mathcal{P}_{\kappa^+}(\lambda) : a \subseteq c\} = \{c \in \mathcal{P}_{\kappa^+}(\lambda) : \bigcup X \subseteq c\},$$

and therefore for each  $b \in \mathcal{P}_{\kappa^+}(\lambda)$

$$b \subseteq \bigcup X \quad \text{if and only if} \quad \bigcap_{a \in X} \{c \in \mathcal{P}_{\kappa^+}(\lambda) : a \subseteq c\} \subseteq \{c \in \mathcal{P}_{\kappa^+}(\lambda) : b \subseteq c\}.$$

□

We next consider special cases when  $\text{cof}^{<\mu}(\mathcal{NS}_{\kappa,\lambda}^\kappa) < \text{cof}(\mathcal{NS}_{\kappa,\lambda}^\kappa)$ .

**Lemma 3.11.** *Let  $\mu$  be an infinite cardinal  $\leq \kappa$ . Then  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \geq \lambda$ .*

*Proof.* It is shown in Matet, Péan and Shelah [MPS05] that  $\overline{\text{cof}}(\mathcal{J}_{\kappa^+,\lambda}) \geq \lambda$ . Now observe that (by Proposition 3.10)  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \geq \overline{\text{cof}}(\mathcal{J}_{\kappa^+,\lambda})$ . □

**Lemma 3.12.** *Suppose  $\lambda$  is singular and  $\mu$  is a cardinal such that  $\text{cf}(\lambda) < \mu \leq \kappa$ . Then  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \leq \sup\{u(\kappa^+, \nu) : \kappa < \nu < \lambda\}$ .*

*Proof.* Let  $\langle \lambda_\xi : \xi < \text{cf}(\lambda) \rangle$  be an increasing sequence of cardinals cofinal in  $\lambda$ . Then, for every  $a \in \mathcal{P}_{\kappa^+}(\lambda)$ ,  $a = \bigcup_{\xi < \text{cf}(\lambda)} a \cap \lambda_\xi$ . The desired inequality follows. □

**Proposition 3.13.** *Let  $\mu$  be an uncountable cardinal  $\leq \kappa$ . Then*

$$\text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \kappa^{+\omega}}^\kappa) = \max\{\mathfrak{d}_\kappa^{<\mu}, \kappa^{+\omega}\}.$$

*Proof.* By Lemmas 1.4, 3.11 and 3.12 we have

$$\text{cov}(\kappa^{+\omega}, \kappa^+, \kappa^+, \mu) = \kappa^{+\omega},$$

so the result follows from Theorem 3.5.  $\square$

Thus, if  $\mathfrak{d}_\kappa^{<\omega_1} \leq \kappa^{+\omega}$ , then

$$\text{cof}^{<\omega_1}(\mathcal{NS}_{\kappa, \kappa^{+\omega}}^\kappa) < \text{cof}(\mathcal{NS}_{\kappa, \kappa^{+\omega}}^\kappa).$$

**Lemma 3.14.** *Assume the Singular Cardinals Hypothesis. If  $\lambda \geq 2^\kappa$ , then*

$$u(\kappa^+, \lambda) = \begin{cases} \lambda^+ & \text{if } \text{cf}(\lambda) \leq \kappa, \\ \lambda & \text{otherwise.} \end{cases}$$

*Proof.* Plainly,  $\lambda \leq u(\kappa^+, \lambda) \leq \lambda^\kappa$ . It follows immediately that  $u(\kappa^+, \lambda) = \lambda$  if  $\text{cf}(\lambda) > \kappa$ . For the other case, use the well-known fact (see, e.g., [MPS16]) that  $\text{cf}(u(\kappa^+, \lambda)) \geq \kappa^+$ .  $\square$

**Proposition 3.15.** *Assume the Singular Cardinals Hypothesis. If  $\lambda \geq 2^\kappa$  and  $\aleph_0 \leq \mu \leq \kappa$ , then*

$$\text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa) = \begin{cases} \lambda^+ & \text{if } \mu \leq \text{cf}(\lambda) \leq \kappa, \\ \lambda & \text{otherwise.} \end{cases}$$

*Proof.* By Lemma 3.11,  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \geq \lambda \geq \mathfrak{d}_\kappa \geq \mathfrak{d}_\kappa^{<\mu}$ , so by Theorem 3.5

$$\text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa) = \text{cov}(\lambda, \kappa^+, \kappa^+, \mu).$$

CASE:  $\text{cf}(\lambda) > \kappa$ .

By Lemmas 3.11 and 3.14 we have  $\lambda \leq \text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \leq u(\kappa^+, \lambda) \leq \lambda$ , and hence  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) = \lambda$ .

CASE:  $\mu \leq \text{cf}(\lambda) \leq \kappa$ .

By Lemma 3.14 we know that

$$\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \leq u(\kappa^+, \lambda) \leq \lambda^+ \quad \text{and} \quad \lambda^+ \leq u(\kappa^+, \lambda) \leq (\text{cov}(\lambda, \kappa^+, \kappa^+, \mu))^{<\mu}.$$

Since  $\lambda^{<\mu} = \lambda$ , it follows that  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) = \lambda^+$ .

CASE:  $\text{cf}(\lambda) < \mu$ .

By Lemmas 3.11, 3.12 and 3.14 we have

$$\lambda \leq \text{cov}(\lambda, \kappa^+, \kappa^+, \mu) \leq \sup\{u(\kappa^+, \nu) : \kappa < \nu < \lambda\} \leq \lambda,$$

and consequently  $\text{cov}(\lambda, \kappa^+, \kappa^+, \mu) = \lambda$ .  $\square$

Thus, if the Singular Cardinals Hypothesis holds,  $\mu \leq \kappa$  and  $\lambda \geq 2^\kappa$ , then

$$\text{cof}^{<\mu}(\mathcal{NS}_{\kappa, \lambda}^\kappa) < \text{cof}(\mathcal{NS}_{\kappa, \lambda}^\kappa) \quad \text{if and only if} \quad \text{cf}(\lambda) < \mu.$$

4.  $\text{cov}(\mathbf{M}_{\kappa,\kappa})$ 

Let us recall some basic facts and definitions related to the combinatorics of the  $\kappa$ -meager ideal  $\mathbf{M}_{\kappa,\kappa}$  on  ${}^\kappa 2$ .

- Definition 4.1.** (1) *The Baire number*  $n(X)$  of a topological space  $X$  (also called *the Novak number of*  $X$ ) is the least number of nowhere dense subsets of  $X$  needed to cover  $X$ .
- (2) For a topological space  $X$  and a cardinal  $\mu$ , the  $(<\mu)$ -complete ideal of subsets of  $X$  generated by nowhere dense subsets of  $X$  is denoted by  $\mathbf{M}_{<\mu}(X)$ ;  $\mathbf{M}_{<\mu^+}(X)$  will be also denoted by  $\mathbf{M}_\mu(X)$ . The ideal  $\mathbf{M}_\mu(X)$  is *the ideal of  $\mu$ -meager subsets of*  $X$ .
- (3) The space  ${}^\kappa \kappa$  (respectively  ${}^\kappa 2$ ) is endowed with the topology obtained by taking as basic open sets  $\emptyset$  and  $O_s$  for  $s \in {}^{<\kappa} \kappa$  (respectively  $s \in {}^{<\kappa} 2$ ), where  $O_s = \{f \in {}^\kappa \kappa : s \subseteq f\}$  (respectively  $O_s = \{f \in {}^\kappa 2 : s \subseteq f\}$ ).
- (4) The ideals of  $\kappa$ -meager subsets of  ${}^\kappa \kappa$ ,  ${}^\kappa 2$  are denoted by  $\mathbf{M}_{\kappa,\kappa}^\kappa$  and  $\mathbf{M}_{\kappa,\kappa}$ , respectively.

*Remark 4.2.* (1) Clearly, for a topological space  $X$ ,  $n(X)$  is the least number of open dense subsets of  $X$  with empty intersection. If  $\mu \leq n(X)$ , then  $\mathbf{M}_{<\mu}(X)$  is a proper ideal (i.e.,  $X \notin \mathbf{M}_{<\mu}(X)$ ).

- (2) Following the tradition of the set theory of the reals, we may consider the covering number  $\text{cov}(\mathbf{M}_{<\mu}(X))$  of the ideal  $\mathbf{M}_{<\mu}(X)$ :

$$\text{cov}(\mathbf{M}_{<\mu}(X)) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{M}_{<\mu}(X) \text{ \& } \bigcup \mathcal{A} = X\}.$$

By the definition,  $n(X) = \text{cov}(\mathbf{M}_{<\aleph_0}(X))$ . But also for every  $\mu < n(X)$  we have  $\text{cov}(\mathbf{M}_{<\mu}(X)) = n(X)$ ; also  $\text{cov}(\mathbf{M}_{<n(X)}(X)) = \text{cf}(n(X))$ .

- (3) Plainly,  $n({}^\kappa \kappa) > \kappa$  and  $n({}^\kappa 2) > \kappa$  (remember,  $\kappa$  is assumed to be regular).

**Lemma 4.3.** *Suppose that*  $X$  *is a topological space,  $\mu < n(X)$ , and*  $Y_\alpha$  *are open subsets of*  $X$  *(for*  $\alpha < \mu$ *).* *Assume also that*  $Y = \bigcap_{\alpha < \mu} Y_\alpha$  *is dense in*  $X$ . *Then, if*  $Y$  *is equipped with the subspace topology,  $n(Y) = n(X)$ .*

*Proof.* Let  $U_\beta$  (for  $\beta < n(X)$ ) be open dense subsets of  $X$  such that  $\bigcap_{\beta < n(X)} U_\beta = \emptyset$ .

Then  $U_\beta \cap Y$  are open dense subsets of  $Y$  (remember  $Y$  is dense) and  $\bigcap_{\beta < n(X)} (U_\beta \cap Y) = \emptyset$ . This shows that  $n(Y) \leq n(X)$ .

Now, let  $V_\beta \subseteq Y$  (for  $\beta < n(Y)$ ) be open dense subsets of  $Y$  such that  $\bigcap_{\beta < n(Y)} V_\beta = \emptyset$ . Take open subsets  $U_\beta$  of  $X$  such that  $V_\beta = U_\beta \cap Y$  – clearly  $U_\beta$ 's are dense in  $X$  (as  $Y$  is so). Then  $\emptyset = \bigcap_{\beta < n(Y)} (U_\beta \cap Y) = \bigcap_{\beta < n(Y)} U_\beta \cap \bigcap_{\alpha < \mu} Y_\alpha$ , and hence  $n(X) \leq n(Y) + \mu$  and therefore  $n(X) \leq n(Y)$ .  $\square$

**Proposition 4.4.**  $\text{cov}(\mathbf{M}_{\kappa,\kappa}^\kappa) = n({}^\kappa \kappa) = n({}^\kappa 2) = \text{cov}(\mathbf{M}_{\kappa,\kappa})$ .

*Proof.* For  $s \in {}^{<\kappa} 2$  and  $\alpha < \kappa$  let  $F(s, \alpha) \in {}^{<\kappa} 2$  be such that  $\text{lh}(F(s, \alpha)) = \text{lh}(s) + \alpha + 1$  and

$$F(s, \alpha) \upharpoonright \text{lh}(s) = s, \quad F(s, \alpha) \upharpoonright [\text{lh}(s), \text{lh}(s) + \alpha) \equiv 1, \quad \text{and} \quad F(s, \alpha)(\text{lh}(s) + \alpha) = 0.$$

Now, let  $\pi : {}^{<\kappa} \kappa \rightarrow {}^{<\kappa} 2$  be such that

- $\pi(\langle \rangle) = \langle \rangle$ ,  $\pi(s \frown \langle \alpha \rangle) = F(\pi(s), \alpha)$  for  $s \in {}^{<\kappa}\kappa$ , and
  - if  $\langle s_\zeta : \zeta < \xi \rangle \subseteq {}^{<\kappa}\kappa$  is  $\triangleleft$ -increasing,  $\xi < \kappa$ ,  $s = \bigcup_{\zeta < \xi} s_\zeta$ ,
- then  $\pi(s) = \bigcup_{\zeta < \xi} \pi(s_\zeta)$ .

Then  $\pi$  induces a one-to-one mapping  $\pi^* : {}^\kappa\kappa \longrightarrow {}^\kappa 2 : \eta \mapsto \bigcup_{\zeta < \kappa} \pi(\eta \upharpoonright \zeta)$ . The range of  $\pi^*$  is

$$\text{Rng}(\pi^*) = \{\rho \in {}^\kappa 2 : (\forall \alpha < \kappa)(\exists \beta < \kappa)(\alpha < \beta \ \& \ \rho(\beta) = 0)\}.$$

Plainly,  $\text{Rng}(\pi^*)$  is the intersection of  $\kappa$  many open dense subsets of  ${}^\kappa 2$ . Moreover,  $\pi^*$  is a homeomorphism from  ${}^\kappa\kappa$  onto  $\text{Rng}(\pi^*)$ . Therefore, using Lemma 4.3, we get  $n({}^\kappa\kappa) = n(\text{Rng}(\pi^*)) = n({}^\kappa 2)$ . The rest should be clear (remember Remark 4.2(2,3)).  $\square$

**Proposition 4.5.**  $\text{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{d}_\kappa$ .

**Definition 4.6.**  $\mathbb{C}_{\mu,\kappa}$  is the forcing notion for adding  $\mu$  Cohen functions in  ${}^\kappa\kappa$  with  $<\kappa$ -support. Thus a condition in  $\mathbb{C}_{\mu,\kappa}$  is a function  $q$  such that

$$\text{Dom}(q) \subseteq \mu \times \kappa, \quad \text{Rng}(q) \subseteq \kappa \quad \text{and} \quad |q| < \kappa.$$

The order of  $\mathbb{C}_{\mu,\kappa}$  is the inclusion.

**Proposition 4.7.** *Assume  $2^{<\kappa} = \kappa < \mu$ . Then  $\Vdash_{\mathbb{C}_{\mu,\kappa}} \text{“cov}(\mathbf{M}_{\kappa,\kappa}) \geq \mu \text{”}$ .*

## 5. MANAGEABLE FORCING NOTIONS

In this section we introduce a property of forcing notions which is crucial for the consistency results presented later:  $(\theta, \mu, \kappa)$ -manageability. This property has three ingredients: an iterable variant of  $\kappa^+$ -cc (see Definition 5.1),  $\kappa$ -completeness and a special property implying preservation of non-meagerness of subsets of  ${}^\kappa\kappa$  (see Proposition 5.9). Since later we will work with  $<\kappa$ -support iterations, we also prove a suitable preservation theorem (see Theorem 5.11).

**Throughout the section we will assume that our fixed (uncountable) regular cardinal  $\kappa$  satisfies  $2^{<\kappa} = \kappa$  (so also  $\kappa^{<\kappa} = \kappa$ ).**

**Definition 5.1** (See Shelah [She92, Definition 1.1] and [She00, Definition 7]). Let  $\mathbb{P}$  be a forcing notion, and  $\varepsilon < \kappa$  be a limit ordinal.

- (1) We define a game  $\mathfrak{D}_{\varepsilon,\kappa}^{\text{cc}}(\mathbb{P})$  of two players, Player I and Player II. A play lasts  $\varepsilon$  steps, and at each stage  $\alpha < \varepsilon$  of the play  $\bar{q}^\alpha, \bar{p}^\alpha, \varphi^\alpha$  are chosen so that:

- $\bar{q}^0 = \langle \emptyset_{\mathbb{P}} : i < \kappa^+ \rangle$ ,  $\varphi^0 : \kappa^+ \longrightarrow \kappa^+ : i \mapsto 0$ ;
- If  $\alpha > 0$ , then Player I picks  $\bar{q}^\alpha, \varphi^\alpha$  such that
  - (i)  $\bar{q}^\alpha = \langle q_i^\alpha : i < \kappa^+ \rangle \subseteq \mathbb{P}$  satisfies
 
$$(\forall \beta < \alpha)(\forall i < \kappa^+)(p_i^\beta \leq q_i^\alpha),$$
  - (ii)  $\varphi^\alpha : \kappa^+ \longrightarrow \kappa^+$  is regressive, i.e.,  $(\forall i < \kappa^+)(\varphi^\alpha(i) < 1 + i)$ ;
- Player II answers choosing a sequence  $\bar{p}^\alpha = \langle p_i^\alpha : i < \kappa^+ \rangle \subseteq \mathbb{P}$  such that  $(\forall i < \kappa^+)(q_i^\alpha \leq p_i^\alpha)$ .

If at some stage of the game Player I does not have any legal move, then he loses. If the game lasted  $\varepsilon$  steps, Player I wins a play  $\langle \bar{q}^\alpha, \bar{p}^\alpha, \varphi^\alpha : \alpha < \varepsilon \rangle$

if there is a club  $C$  of  $\kappa^+$  such that for each distinct members  $i, j$  of  $C$  satisfying  $\text{cf}(i) = \text{cf}(j) = \kappa$  and  $(\forall \alpha < \varepsilon)(\varphi^\alpha(i) = \varphi^\alpha(j))$ , the set

$$\{p_i^\alpha : \alpha < \varepsilon\} \cup \{p_j^\alpha : \alpha < \varepsilon\}$$

has an upper bound in  $\mathbb{P}$ .

- (2) The forcing notion  $\mathbb{P}$  satisfies condition  $(*)_\kappa^\varepsilon$  if Player I has a winning strategy in the game  $\mathfrak{D}_{\varepsilon, \kappa}^{\text{cc}}(\mathbb{P})$ .

*Remark 5.2.* Condition  $(*)_\kappa^\varepsilon$  is a strong version of  $\kappa^+$ -cc (easily, if  $\varepsilon < \kappa$  is limit,  $\kappa^\varepsilon = \kappa$ , and  $\mathbb{P}$  satisfies  $(*)_\kappa^\varepsilon$ , then  $\mathbb{P}$  satisfies  $\kappa^+$ -cc). This condition was used in a number of papers, e.g., to obtain a series of consistency results on partition relations; see Shelah and Stanley [SS82], [SS86], Shelah [She78], [She92], [She00]. Its primary use comes from the fact that it is preserved in  $<\kappa$ -support iterations.

**Proposition 5.3** (See Shelah [She92, Iteration Lemma 1.3] and [She00, Theorem 35]). *Let  $\varepsilon < \kappa$  be a limit ordinal,  $\kappa = \kappa^{<\kappa}$ . Suppose that  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \gamma \rangle$  is a  $<\kappa$ -support iteration such that for each  $\xi < \gamma$*

$$\Vdash_{\mathbb{P}_\xi} \text{“} \dot{\mathbb{Q}}_\xi \text{ satisfies } (*)_\kappa^\varepsilon \text{”}.$$

*Then  $\mathbb{P}_\gamma$  satisfies  $(*)_\kappa^\varepsilon$ .*

**Definition 5.4.** A forcing notion  $\mathbb{P}$  is  $<\theta$ -complete if every  $\leq_{\mathbb{P}}$ -increasing chain of length less than  $\theta$  has an upper bound in  $\mathbb{P}$ . It is  $<\theta$ -lub-complete if every  $\leq_{\mathbb{P}}$ -increasing chain of length less than  $\theta$  has a least upper bound in  $\mathbb{P}$ .

**Definition 5.5.** Let  $\theta$  and  $\mu$  be cardinals such that  $\theta < \kappa$  and  $\mu^{<\kappa} = \mu$ . Let  $\mathbb{P}$  be a  $<\theta^+$ -lub-complete forcing notion.

- (1) A model  $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$  is  $(\mathbb{P}, \kappa, \mu)$ -relevant if  $\mathbb{P}, \mu \in N$ ,  $\mu \subseteq N$ ,  $|N| = \mu$  and  $<^\kappa N \subseteq N$ .
- (2) For a  $(\mathbb{P}, \kappa, \mu)$ -relevant model  $N$  we define a game  $\mathfrak{D}^m(N, \theta, \mathbb{P})$  of two players, He and She, as follows. A play lasts  $\theta$  moves, and in the  $i^{\text{th}}$  move conditions  $p_i, q_i \in \mathbb{P}$  are chosen so that:
  - $q_i \in N \cap \mathbb{P}$ ,  $q_i \leq p_i$ ,
  - $(\forall j < i)(q_j \leq q_i \ \& \ p_j \leq p_i)$ ,
  - She chooses  $p_i, q_i$  if  $i$  is odd, He picks  $p_i, q_i$  if  $i$  is even.

She wins the play  $\langle q_i, p_i : i < \theta \rangle$  whenever

*if  $p_\theta$  is a least upper bound of  $\langle p_i : i < \theta \rangle$ , and  $q_\theta$  is a least upper bound of  $\langle q_i : i < \theta \rangle$ ,*

*then  $(\forall q \in N \cap \mathbb{P})(q_\theta \leq q \Rightarrow q, p_\theta \text{ are compatible})$ .*

- (3) The forcing notion  $\mathbb{P}$  is *weakly  $(\theta, \mu, \kappa)$ -manageable* if (it is  $<\theta^+$ -lub-complete and) there is an  $x \in \mathcal{H}(\chi)$  (called a *witness*) such that for every  $(\mathbb{P}, \kappa, \mu)$ -relevant model  $N \prec \mathcal{H}(\chi)$  with  $x \in N$ , She has a winning strategy in the game  $\mathfrak{D}^m(N, \theta, \mathbb{P})$ .
- (4) The forcing notion  $\mathbb{P}$  is  $(\theta, \mu, \kappa)$ -manageable if it is  $<\kappa$ -complete, weakly  $(\theta, \mu, \kappa)$ -manageable, and satisfies the condition  $(*)_\kappa^\theta$ .

*Remark 5.6.* Suppose that  $\mathbb{P}$  is  $<\theta^+$ -lub-complete and  $N$  is  $(\mathbb{P}, \kappa, \mu)$ -relevant. Then both players have always legal moves in the game  $\mathfrak{D}^m(N, \theta, \mathbb{P})$ . Moreover, if  $\langle q_i, p_i : i < \theta \rangle$  is a (legal) play of  $\mathfrak{D}^m(N, \theta, \mathbb{P})$ , then there are least upper bounds  $q_\theta \in N \cap \mathbb{P}$  of  $\langle q_i : i < \theta \rangle$ , and  $p_\theta \in \mathbb{P}$  of  $\langle p_i : i < \theta \rangle$  (and  $q_\theta \leq p_\theta$ ).

**Definition 5.7.** Let  $N$  be a  $(\mathbb{P}, \kappa, \mu)$ -relevant model, and let  $q \in N \cap \mathbb{P}$ ,  $p \in \mathbb{P}$  be such that  $q \leq p$ . We say that a pair  $(q^*, p^*)$  is an  $N$ -cover for  $(q, p)$ , if

- $q \leq q^* \in N \cap \mathbb{P}$ ,  $p \leq p^* \in \mathbb{P}$ ,  $q^* \leq p^*$ , and
- every condition  $q' \in N \cap \mathbb{P}$  stronger than  $q^*$  is compatible with  $p^*$ .

**Lemma 5.8.** *Suppose that  $\mathbb{P}$  is a  $<\theta^+$ -lub-complete forcing notion,  $N$  is a  $(\mathbb{P}, \kappa, \mu)$ -relevant model, and She has a winning strategy in the game  $\mathfrak{D}^m(N, \theta, \mathbb{P})$ . Then:*

- (1) *For all conditions  $q \in N \cap \mathbb{P}$  and  $p \in \mathbb{P}$  such that  $q \leq p$ , there is an  $N$ -cover  $(q^*, p^*)$  for  $(q, p)$ .*
- (2)  *$N \cap \mathbb{P} \triangleleft \mathbb{P}$ .*

*Proof.* 1) Consider a play  $\langle q_i, p_i : i < \theta \rangle$  of  $\mathfrak{D}^m(N, \theta, \mathbb{P})$  in which He starts with  $q_0 = q$ ,  $p_0 = p$ , and then he always plays the  $<_{\chi}^*$ -first legal moves, and She uses her winning strategy. Let  $q^* \in N \cap \mathbb{P}$ ,  $p^* \in \mathbb{P}$  be least upper bounds of  $\langle q_i : i < \theta \rangle$ ,  $\langle p_i : i < \theta \rangle$ , respectively. Plainly, as She won the play, the pair  $(q^*, p^*)$  is an  $N$ -cover for  $(q, p)$ .

2) Suppose that  $\mathcal{A} \subseteq N \cap \mathbb{P}$  is a maximal antichain in  $N \cap \mathbb{P}$ , but  $p \in \mathbb{P}$  is incompatible with all members of  $\mathcal{A}$ . Let  $(q^*, p^*)$  be an  $N$ -cover for  $(\emptyset_{\mathbb{P}}, p)$ . The condition  $q^*$  is compatible with some  $q \in \mathcal{A}$ , so let  $q^+ \in N \cap \mathbb{P}$  be such that  $q^+ \geq q^*$ ,  $q^+ \geq q \in \mathcal{A}$ . By the choice of  $(q^*, p^*)$  we know that the conditions  $q^+$  and  $p^*$  are compatible, and hence  $q$  and  $p$  are compatible, a contradiction.

The rest follows from the elementarity of  $N$ .  $\square$

**Proposition 5.9.** *Assume  $\theta < \kappa \leq \mu = \mu^{<\kappa} < \tau$ . Suppose that a set  $Y \subseteq {}^{\kappa}\kappa$  cannot be covered by the union of less than  $\tau$  nowhere dense subsets of  ${}^{\kappa}\kappa$ , and  $\mathbb{P}$  is a weakly  $(\theta, \mu, \kappa)$ -manageable forcing notion not collapsing cardinals. Then*

$$\Vdash_{\mathbb{P}} \text{“} Y \text{ is not the union of } < \tau \text{ nowhere dense subsets of } {}^{\kappa}\kappa \text{”}.$$

*Proof.* Let  $\mathbb{P}$  be weakly  $(\theta, \mu, \kappa)$ -manageable with a witness  $x \in \mathcal{H}(\chi)$ . Suppose toward contradiction that a condition  $q \in \mathbb{P}$  is such that

$$q \Vdash \text{“} Y \text{ is the union of } < \tau \text{ nowhere dense subsets of } {}^{\kappa}\kappa \text{”}.$$

Passing to a stronger condition if needed, we may assume that for some  $\iota < \tau$  and  $\mathbb{P}$ -names  $\dot{A}_{\xi}$  (for  $\xi < \iota$ ) we have:

- $q \Vdash \text{“} \dot{A}_{\xi} \subseteq {}^{<\kappa}\kappa \ \& \ (\forall s \in {}^{<\kappa}\kappa)(\exists t \in \dot{A}_{\xi})(s \subseteq t) \text{”}$  and
- $q \Vdash \text{“} (\forall y \in Y)(\exists \xi < \iota)(\forall t \in \dot{A}_{\xi})(t \not\subseteq y) \text{”}.$

For each  $\zeta < \iota$  pick a  $(\mathbb{P}, \kappa, \mu)$ -relevant model  $N_{\zeta} \prec (\mathcal{H}(\chi), \in, <_{\chi}^*)$  such that  $q, \langle \dot{A}_{\xi} : \xi < \iota \rangle, x, \zeta \in N_{\zeta}$ . Then  $|\bigcup_{\zeta < \iota} N_{\zeta}| = \iota \cdot \mu < \tau$ , so we may pick a  $y \in Y$  such that  $y \in \mathcal{O}$  for all open dense subsets  $\mathcal{O}$  of  ${}^{\kappa}\kappa$  from  $\bigcup_{\zeta < \iota} N_{\zeta}$ . By our assumptions, there are  $\xi < \iota$  and  $p \geq q$  such that

$$p \Vdash \text{“} (\forall t \in \dot{A}_{\xi})(t \not\subseteq y) \text{”}.$$

Let  $(q^*, p^*)$  be an  $N_{\xi}$ -cover for  $(q, p)$  (there is one by Lemma 5.8(1)). Put

$$A = \{s \in {}^{<\kappa}\kappa : (\exists q' \geq q^*)(q' \Vdash s \in \dot{A}_{\xi})\}.$$

Clearly  $A \in N_{\xi}$ ,  $A \subseteq N_{\xi}$ , and  $\mathcal{O} = \bigcup_{s \in A} O_s \in N_{\xi}$  is an open dense subset of  ${}^{\kappa}\kappa$ .

Hence  $s \subseteq y$  for some  $s \in A$ . Let  $q' \in N_{\xi} \cap \mathbb{P}$  be a condition stronger than  $q^*$  and

such that  $q' \Vdash s \in \dot{A}_\xi$ . The condition  $q'$  is compatible with  $p^*$ , and so with  $p$ . Take a condition  $q^+$  stronger than both  $q'$  and  $p$ . Then

$$q^+ \Vdash "s \in \dot{A}_\xi \ \& \ s \subseteq y" \quad \text{and} \quad q^+ \Vdash "(\forall t \in \dot{A}_\xi)(t \not\subseteq y)",$$

a contradiction.  $\square$

**Corollary 5.10.** *Suppose that  $\theta < \kappa \leq \mu = \mu^{<\kappa}$  and  $\text{cov}(\mathbf{M}_{\kappa,\kappa}) > \mu$ . Let  $\mathbb{P}$  be a  $(\theta, \mu, \kappa)$ -manageable forcing notion. Then*

$$\Vdash_{\mathbb{P}} "(\text{cov}(\mathbf{M}_{\kappa,\kappa}))^{\mathbf{V}} \leq \text{cov}(\mathbf{M}_{\kappa,\kappa})".$$

*Proof.* Remembering Proposition 4.4, apply Proposition 5.9 to  $\tau = \text{cov}(\mathbf{M}_{\kappa,\kappa}) = \text{cov}(\mathbf{M}_{\kappa,\kappa}^\kappa)$  and  $Y = {}^\kappa\kappa$  to get

$$\Vdash_{\mathbb{P}} "({}^\kappa\kappa)^{\mathbf{V}} \text{ is not the union of } < \tau \text{ nowhere dense sets} ".$$

But this clearly implies  $\Vdash_{\mathbb{P}} "\tau \leq \text{cov}(\mathbf{M}_{\kappa,\kappa}) = \text{cov}(\mathbf{M}_{\kappa,\kappa}^\kappa)". \square$

**Theorem 5.11.** *Assume that  $\theta < \kappa \leq \mu = \mu^{<\kappa}$ . Let  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \gamma \rangle$  be  $<\kappa$ -support iteration such that for each  $\xi < \gamma$*

$$\Vdash_{\mathbb{P}_\xi} "\dot{\mathbb{Q}}_\xi \text{ is } (\theta, \mu, \kappa)\text{-manageable} ".$$

*Then  $\mathbb{P}_\gamma$  is  $(\theta, \mu, \kappa)$ -manageable.*

*Proof.* Let  $\theta, \kappa, \mu$  and  $\bar{\mathbb{Q}}$  be as in the assumptions of the theorem.

First note that the limits of  $<\kappa$ -support iterations of  $<\kappa$ -complete forcing notions satisfying the condition  $(*)_\kappa^\theta$  are  $<\kappa$ -complete  $\kappa^+$ -cc (as  $\kappa^{<\kappa} = \kappa$ ; remember Proposition 5.3). Therefore no such iteration collapses cardinals nor changes cofinalities nor adds sequences of ordinals of length  $< \kappa$ . Hence the assumed properties of  $\theta, \kappa$  and  $\mu$  hold in all intermediate extensions  $\mathbf{V}^{\mathbb{P}_\xi}$  and our assumption on  $\dot{\mathbb{Q}}_\xi$ 's is meaningful.

Plainly,  $\mathbb{P}_\gamma$  is  $<\kappa$ -complete,  $<\theta^+$ -lub-complete and satisfies condition  $(*)_\kappa^\theta$ . We have to show that  $\mathbb{P}_\gamma$  is weakly  $(\theta, \mu, \kappa)$ -manageable.

For  $\xi < \gamma$  let  $\dot{x}_\xi$  be a  $\mathbb{P}_\xi$ -name for a witness for  $\dot{\mathbb{Q}}_\xi$  being weakly manageable and let  $\bar{x} = \langle \dot{x}_\xi : \xi < \gamma \rangle$ . Suppose that  $N \prec (\mathcal{H}(\chi), \in, <_\chi^*)$  is a  $(\mathbb{P}_\gamma, \kappa, \mu)$ -relevant model such that  $(\bar{x}, \bar{\mathbb{Q}}) \in N$ .

Since each  $\mathbb{P}_\xi$  is  $<\kappa$ -complete and satisfies the  $\kappa^+$ -cc (and  $\kappa + 1 \subseteq N$ ) we know that if  $\xi \in N \cap \gamma$  and  $G_\xi \subseteq \mathbb{P}_\xi$  is generic over  $\mathbf{V}$ , then in  $\mathbf{V}[G_\xi]$  we have:

$$N[G_\xi] \cap \mathbf{V} = N \quad \text{and} \quad N[G_\xi] \prec (\mathcal{H}(\chi), \in, <_\chi^*)^{\mathbf{V}[G_\xi]} \quad \text{and} \quad <^\kappa N[G_\xi] \subseteq N[G_\xi].$$

Since clearly  $\dot{\mathbb{Q}}_\xi^{G_\xi} \in N[G_\xi]$ , we conclude that  $N[G_\xi]$  is  $(\dot{\mathbb{Q}}_\xi^{G_\xi}, \kappa, \mu)$ -relevant, and  $\dot{x}_\xi^{G_\xi} \in N[G_\xi]$ . Therefore, She has a winning strategy in the game  $\mathfrak{D}^m(N[G_\xi], \theta, \dot{\mathbb{Q}}_\xi^{G_\xi})$ . Let  $\dot{\text{st}}_\xi$  be a  $\mathbb{P}_\xi$ -name for such a strategy. We may assume that the strategy  $\dot{\text{st}}_\xi$  is such that

$$(*) \text{ if } i < \theta \text{ is even and } q_i = p_i = \dot{\emptyset}_{\dot{\mathbb{Q}}_\xi},$$

$$\text{then } \dot{\text{st}}_\xi \text{ instructs Her to play } q_{i+1} = p_{i+1} = \dot{\emptyset}_{\dot{\mathbb{Q}}_\xi}.$$

We define a strategy  $\text{st}$  for Her in the game  $\mathfrak{D}^m(N, \theta, \mathbb{P}_\gamma)$  as follows. At an odd stage  $i < \theta$  of the game, the strategy  $\text{st}$  first instructs Her to choose (side) conditions  $q_i^-, p_i^- \in \mathbb{P}_\gamma$  and only then pick conditions  $q_i \in N \cap \mathbb{P}_\gamma$  and  $p_i \in \mathbb{P}_\gamma$  which are to be played. These conditions will be chosen so that if  $\langle q_j, p_j : j < i \rangle$  is a legal play

of  $\mathfrak{D}^m(N, \theta, \mathbb{P}_\gamma)$  in which She uses st, and  $q_j^-, p_j^-$  are the side conditions picked by her (for odd  $j \leq i$ ), then

- ( $\alpha$ ) <sub>$i$</sub>   $\text{Dom}(q_i) = \text{Dom}(q_i^-) = \text{Dom}(p_{i-1}) \cap N$ ,  $\text{Dom}(p_i^-) = \text{Dom}(p_{i-1})$ ,
- ( $\beta$ ) <sub>$i$</sub>   $p_{i-1} \leq p_i^- \leq p_i$ ,  $q_{i-1} \leq q_i^- \leq p_i^-$ ,  $q_{i-1} \leq q_i \leq p_i$ ,
- ( $\gamma$ ) <sub>$i$</sub>  letting  $(q_j^*, p_j^*)$  be  $(q_j^-, p_j^-)$  if  $j \leq i$  is odd and  $(q_j, p_j)$  if  $j < i$  is even, for every  $\xi \in \text{Dom}(q_i)$  we have

$$p_i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ } q_i(\xi) = q_i^-(\xi) \text{ and the sequence } \langle q_j^*(\xi), p_j^*(\xi) : j \leq i \rangle \text{ is a legal play of } \mathfrak{D}^m(N[\Gamma_{\mathbb{P}_\xi}], \theta, \mathbb{Q}_\xi) \text{ in which She uses the strategy } \dot{\text{st}}_\xi \text{ ”.}$$

So suppose that  $i < \theta$  is odd,  $\langle q_j, p_j : j < i \rangle$  is a partial play of  $\mathfrak{D}^m(N, \theta, \mathbb{P}_\gamma)$  in which She uses st (and the side conditions for odd  $j < i$  are  $q_j^-, p_j^-$ ), and the clauses ( $\alpha$ ) <sub>$j$</sub> –( $\gamma$ ) <sub>$j$</sub>  hold for all odd  $j < i$ . Let  $(q_j^*, p_j^*)$  be  $(q_j^-, p_j^-)$  if  $j < i$  is odd and  $(q_j, p_j)$  if  $j < i$  is even.

We first declare that  $\text{Dom}(q_i^-) = \text{Dom}(p_{i-1}) \cap N$ ,  $\text{Dom}(p_i^-) = \text{Dom}(p_{i-1})$  and  $p_i^-(\zeta) = p_{i-1}(\zeta)$  for all  $\zeta \in \text{Dom}(p_i^-) \setminus N$ . Next, by induction on  $\xi \in \text{Dom}(q_i^-)$  we define  $q_i^-(\xi), p_i^-(\xi)$ . So suppose that  $\xi \in \text{Dom}(q_i^-)$  and  $q_i^- \upharpoonright \xi, p_i^- \upharpoonright \xi$  have been defined so that  $q_{i-1} \upharpoonright \xi \leq q_i^- \upharpoonright \xi \leq p_i^- \upharpoonright \xi$  and  $p_{i-1} \upharpoonright \xi \leq p_i^- \upharpoonright \xi$ . Then, by clauses ( $\gamma$ ) <sub>$j$</sub> ,

$$p_i^- \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ the sequence } \langle q_j^*(\xi), p_j^*(\xi) : j < i \rangle \text{ is a legal play of } \mathfrak{D}^m(N[\Gamma_{\mathbb{P}_\xi}], \theta, \mathbb{Q}_\xi) \text{ in which She uses the strategy } \dot{\text{st}}_\xi \text{ ”.}$$

(Remember our assumption (\*) on  $\dot{\text{st}}_\xi$  and our convention regarding  $\emptyset_{\mathbb{P}}$  stated in Notation 0.3(1).) Let  $q_i^-(\xi)$  and  $p_i^-(\xi)$  be  $\mathbb{P}_\xi$ -names for members of  $\mathbb{Q}_\xi$  such that

$$q_i^- \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ } q_i^-(\xi) \in N[\Gamma_{\mathbb{P}_\xi}] \text{ \& } q_{i-1}(\xi) \leq q_i^-(\xi) \text{ ”,}$$

and

$$p_i^- \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ } (q_i^-(\xi), p_i^-(\xi)) \text{ is what } \dot{\text{st}}_\xi \text{ tells Her to play as the answer to } \langle q_j^*(\xi), p_j^*(\xi) : j < i \rangle \text{ ”.}$$

(So  $q_i^-(\xi)$  is a name for a member of  $N[\Gamma_{\mathbb{P}_\xi}]$ , but it does not have to be from  $N$ .) This completes the definition of  $q_i^-, p_i^- \in \mathbb{P}_\gamma$ . Now we use the fact that  $\mathbb{P}_\gamma$  is  $<\kappa$ -complete and  $|\text{Dom}(q_i^-)| < \kappa$  to pick a condition  $p_i \in \mathbb{P}_\gamma$  stronger than  $p_i^-$  and names  $\dot{\tau}_\xi \in N$  (for  $\xi \in \text{Dom}(q_i^-)$ ) such that  $p_i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ } q_i^-(\xi) = \dot{\tau}_\xi \text{ ”}$ . Since  $<^\kappa N \subseteq N$ , the sequence  $\langle \dot{\tau}_\xi : \xi \in \text{Dom}(q_i^-) \rangle$  is in  $N$ . Hence we may find a condition  $q_i \in N \cap \mathbb{P}_\gamma$  such that

- $\text{Dom}(q_i) = \text{Dom}(q_i^-)$ , and
- for each  $\xi \in \text{Dom}(q_i)$ ,

$$\Vdash_{\mathbb{P}_\xi} \text{ “ if } \dot{\tau}_\xi \geq q_{i-1}(\xi), \text{ then } q_i(\xi) = \dot{\tau}_\xi, \text{ otherwise } q_i(\xi) = q_{i-1}(\xi) \text{ ”.}$$

(For definiteness we pick the  $<^*_\chi$ -first  $p_i, q_i$  as above.) It should be clear that  $q_i^-, q_i, p_i^-, p_i$  satisfy conditions ( $\alpha$ ) <sub>$i$</sub> –( $\gamma$ ) <sub>$i$</sub> . This finishes the description of the strategy st. Let us argue that it is a winning strategy for Her.

To this end suppose that  $\langle q_i, p_i : i < \theta \rangle$  is the result of a play of  $\mathfrak{D}^m(N, \theta, \mathbb{P}_\gamma)$  in which She uses st. Let  $q_\theta, p_\theta \in \mathbb{P}_\gamma$  be least upper bounds of  $\langle q_i : i < \theta \rangle, \langle p_i : i < \theta \rangle$ , respectively. Then for every  $\xi \in \text{Dom}(p_\theta)$  we have

$$p_\theta \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ } p_\theta(\xi) \text{ is a least upper bound of } \langle p_i(\xi) : i < \theta \rangle \text{ ”.}$$

We may also assume that  $\text{Dom}(q_\theta) = \bigcup_{i < \theta} \text{Dom}(q_i) = \text{Dom}(p_\theta) \cap N$ .



Let  $q \in N \cap \mathbb{P}_\gamma$  be a condition stronger than  $q_\theta$  (and thus stronger than all  $q_i$  for  $i < \theta$ ). We define a condition  $p \in \mathbb{P}_\gamma$  as follows. First, we declare that  $\text{Dom}(p) = \text{Dom}(q) \cup \text{Dom}(p_\theta)$ , and  $p(\xi) = q(\xi)$  for  $\xi \in \text{Dom}(q) \setminus \text{Dom}(p_\theta)$ , and  $p(\xi) = p_\theta(\xi)$  for  $\xi \in \text{Dom}(p_\theta) \setminus \text{Dom}(q) = \text{Dom}(p_\theta) \setminus N$ . Now suppose that  $\xi \in \text{Dom}(q) \cap \text{Dom}(p_\theta)$  and we have already defined  $p \upharpoonright \xi$  so that  $q \upharpoonright \xi \leq p \upharpoonright \xi$  and  $p_\theta \upharpoonright \xi \leq p \upharpoonright \xi$ . Then, by our choices,

$$p \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{ “ the sequence } \langle q_j(\xi), p_j^*(\xi) : j < \theta \rangle \text{ is a legal play of } \\ \mathfrak{D}^m(N[\Gamma_{\mathbb{P}_\xi}], \theta, \dot{\mathbb{Q}}_\xi) \text{ in which She uses the strategy } \text{st}_\xi, \text{ and } \\ q(\xi) \in N[\Gamma_{\mathbb{P}_\xi}] \text{ is stronger than all } q_j(\xi) \text{ for } j < \theta \text{ ”.}$$

(Above,  $p_j^*$  are as in the definition of  $\text{st}$ : either  $p_j$  or  $p_j^-$ , depending on the parity of  $j$ .) Consequently,

$$p \upharpoonright \xi \Vdash \text{ “ } q(\xi) \text{ and } p_\theta(\xi) \text{ are compatible ”,}$$

so we may pick a  $\mathbb{P}_\xi$ -name  $p(\xi)$  for a condition in  $\dot{\mathbb{Q}}_\xi$  such that

$$p \upharpoonright \xi \Vdash \text{ “ } q(\xi) \leq p(\xi) \text{ and } p_\theta(\xi) \leq p(\xi) \text{ ”.}$$

This completes the choice of  $p \in \mathbb{P}_\gamma$ . Plainly,  $p$  is an upper bound of  $q$  and  $p_\theta$  showing that they are compatible.  $\square$

*Remark 5.12.* Note that

if  $\mathbb{P}$  is weakly  $(\theta, \mu, \kappa)$ -manageable,  
then it satisfies the  $\mu^+$ -cc.

Hence we may use a slight modification of the proof of 5.11 to show (by induction on  $\gamma$ ) that

if  $\theta < \kappa = \kappa^{<\kappa}$ ,  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \gamma \rangle$  is a  $(<\kappa)$ -support iteration of  $<\kappa$ -complete weakly  $(\theta, \kappa, \kappa)$ -manageable forcing notions, then  $\mathbb{P}_\gamma$  is weakly  $(\theta, \kappa, \kappa)$ -manageable and  $\kappa$ -complete (and thus also  $\kappa^+$ -cc).

## 6. THE ONE-STEP FORCING

In this section we introduce a forcing notion  $\mathbb{Q}$  for adding a small family of functions in  ${}^\kappa\kappa$  which  $\tau$ -dominates  ${}^\kappa\kappa \cap \mathbf{V}$ . Iterating this type of forcing notions we will get models with  $\mathfrak{d}_\kappa^\tau$  small. Our forcing is (of course) manageable for suitable parameters, and thus it preserves non-meagerness of subsets of  $\kappa$ . Throughout this section we assume the following.

- Context 6.1.* (i)  $\tau = \text{cf}(\tau) < \text{cf}(\kappa) = \kappa = 2^{<\kappa}$ ,  
(ii)  $\bar{\mu} = \langle \mu_\alpha : \alpha < \tau \rangle$  is an increasing sequence of regular cardinals,  $\kappa \leq \mu_0$ ,  
(iii)  $|\prod_{\alpha < \tau} \mu_\alpha| = 2^\kappa$  and  $\pi : \prod_{\alpha < \tau} \mu_\alpha \rightarrow {}^\kappa\kappa$  is a bijection.

We will write  $\pi_\eta$  for  $\pi(\eta)$ . Also for a set  $u \subseteq \prod_{\alpha < \tau} \mu_\alpha$  we let

$$T(u) \stackrel{\text{def}}{=} \{\eta \upharpoonright \alpha : \alpha < \tau \ \& \ \eta \in u\}$$

**Definition 6.2.** (1) We define a forcing notion  $\mathbb{Q} = \mathbb{Q}(\pi, \bar{\mu}, \kappa)$  as follows.

**A condition** in  $\mathbb{Q}$  is a tuple  $p = (i, u, \bar{f}, g) = (i^p, u^p, \bar{f}^p, g^p)$  such that

- (a)  $i < \kappa$ ,  $u \in \mathcal{P}_\kappa(\prod_{\alpha < \tau} \mu_\alpha)$ ,  
(b)  $\bar{f} = \langle f_\sigma : \sigma \in T(u) \rangle$  and  $f_\sigma : i \rightarrow \kappa$  for  $\sigma \in T(u)$ ,

(c)  $g : u \longrightarrow i + 1$ , and if  $\eta \in u$ ,  $g(\eta) \leq j < i$ , then

$$\pi_\eta(j) < \sup\{f_{\eta \upharpoonright \alpha}(j) : \alpha < \tau\}.$$

**The order of  $\mathbb{Q}$**  is such that for  $p, q \in \mathbb{Q}$  we have

$p \leq q$  if and only if

$$i^p \leq i^q, u^p \subseteq u^q, g^p \subseteq g^q \text{ and } f_\sigma^p \subseteq f_\sigma^q \text{ for } \sigma \in T(u^p).$$

(2) For a set  $U \subseteq \prod_{\alpha < \tau} \mu_\alpha$  we let  $\mathbb{Q} \upharpoonright U = \{p \in \mathbb{Q} : u^p \subseteq U\}$ , and for a condition

$q \in \mathbb{Q}$  we put

$$q \upharpoonright U = (i^q, u^q \cap U, \bar{f}^q \upharpoonright T(u^q \cap U), g^q \upharpoonright (u^q \cap U)).$$

**Proposition 6.3.** (1)  $\mathbb{Q}$  is a  $< \kappa$ -lub-complete forcing notion of size  $2^\kappa$ .

(2) Let  $U \subseteq \prod_{\alpha < \tau} \mu_\alpha$  be of size  $\leq \kappa$ . Then  $|\mathbb{Q} \upharpoonright U| \leq \kappa$ .

*Proof.* 1) Plainly,  $(\mathbb{Q}, \leq)$  is a partial order of size  $2^\kappa$ . To prove the completeness suppose that  $\langle p_\xi : \xi < \xi^* \rangle$  is an  $\leq$ -increasing sequence of members of  $\mathbb{Q}$  and  $\xi^* < \kappa$ . Put

$$i^q = \sup_{\xi < \xi^*} i^{p_\xi}, \quad u^q = \bigcup_{\xi < \xi^*} u^{p_\xi}, \quad g^q = \bigcup_{\xi < \xi^*} g^{p_\xi}$$

and  $f_\sigma^q = \bigcup \{f_\sigma^{p_\xi} : \xi < \xi^* \text{ \& } \sigma \in T(u^{p_\xi})\}$  for  $\sigma \in T(u^q)$ . Clearly  $q = (i^q, u^q, \bar{f}^q, g^q) \in \mathbb{Q}$  is the least upper bound of  $\langle p_\xi : \xi < \xi^* \rangle$ .

2) Should be clear. □

**Proposition 6.4.** The forcing notion  $\mathbb{Q}$  satisfies the condition  $(*)_\kappa^\varepsilon$  (see 5.1(2)) for any limit ordinal  $\varepsilon < \kappa$ .

*Proof.* Let  $\varepsilon < \kappa$  be a limit ordinal. To give the winning strategy for Player I in the game  $\mathfrak{D}_{\varepsilon, \kappa}^{\text{cc}}(\mathbb{Q})$  we need two technical observations.

**Claim 6.4.1.** If  $p, q \in \mathbb{Q}$  are such that  $i^p = i^q$  and  $g^p \upharpoonright (u^p \cap u^q) = g^q \upharpoonright (u^p \cap u^q)$  and  $f_\sigma^p = f_\sigma^q$  for  $\sigma \in T(u^p) \cap T(u^q)$ , then the conditions  $p, q$  have a least upper bound.

*Proof of the Claim.* Let  $i^r = i^p = i^q$ ,  $u^r = u^p \cup u^q$ ,  $g^r = g^p \cup g^q$  and

$$f_\sigma^r = \begin{cases} f_\sigma^p & \text{if } \sigma \in T(u^p) \\ f_\sigma^q & \text{if } \sigma \in T(u^q). \end{cases}$$

Then  $r = (i^r, u^r, \bar{f}^r, g^r) \in \mathbb{Q}$  is the least upper bound of  $p, q$ . □

**Claim 6.4.2.** Suppose  $\bar{q} = \langle q_j : j < \kappa^+ \rangle \subseteq \mathbb{Q}$ . Then there is a regressive function  $\varphi_{\bar{q}} : \kappa^+ \longrightarrow \kappa^+$  such that

if  $j < j' < \kappa^+$ ,  $\text{cf}(j) = \text{cf}(j') = \kappa$  and  $\varphi_{\bar{q}}(j) = \varphi_{\bar{q}}(j')$ ,

then  $i^{q_j} = i^{q_{j'}}$ , and  $g^{q_j} \upharpoonright (u^{q_j} \cap u^{q_{j'}}) = g^{q_{j'}} \upharpoonright (u^{q_j} \cap u^{q_{j'}})$ , and  $f_\sigma^{q_j} = f_\sigma^{q_{j'}}$  for  $\sigma \in T(u^{q_j}) \cap T(u^{q_{j'}})$ .

*Proof of the Claim.* Take a sequence  $\langle \eta_\xi : \xi < \kappa^+ \rangle \subseteq \prod_{\alpha < \tau} \mu_\alpha$  such that for each  $j < \kappa^+$  of cofinality  $\kappa$  and an  $\alpha < j$  we have  $u^{q_\alpha} \subseteq \{\eta_\xi : \xi < j\}$ . Let  $U = \{\eta_\xi : \xi < \kappa^+\}$  and  $U_j = \{\eta_\xi : \xi < j\}$  for  $j < \kappa^+$ . By 6.3(2) we know that  $|\mathbb{Q} \upharpoonright U_j| \leq \kappa$  (for  $j < \kappa^+$ ) and  $|\mathbb{Q} \upharpoonright U| \leq \kappa^+$ , and hence we may pick a mapping  $\psi_0 : \kappa^+ \longrightarrow \mathbb{Q} \upharpoonright U$  such that

$$(\forall j < \kappa^+) (\text{cf}(j) = \kappa \Rightarrow \text{Rng}(\psi_0 \upharpoonright j) = \mathbb{Q} \upharpoonright U_j).$$

Also, for  $j < \kappa^+$ , let  $F(U_j)$  be the set

$$\{\bar{f} = \langle f_\sigma : \sigma \in \text{Dom}(\bar{f}) \rangle : \text{Dom}(\bar{f}) \in \mathcal{P}_\kappa(T(U_j)) \text{ \& } (\forall \sigma \in \text{Dom}(\bar{f}))(f_\sigma \in {}^{< \kappa} \kappa)\},$$

and  $F(U) = \bigcup_{j < \kappa^+} F(U_j)$ . Note that  $|F(U_j)| \leq \kappa$  and  $|F(U)| \leq \kappa^+$ . Choose a function  $\psi_1 : \kappa^+ \rightarrow F(U)$  such that

$$(\forall j < \kappa^+) (\text{cf}(j) = \kappa \Rightarrow \text{Rng}(\psi_1 \upharpoonright j) = F(U_j)).$$

Finally, let  $c : \kappa^+ \times \kappa^+ \rightarrow \kappa^+$  be a bijection such that

$$(\forall j < \kappa^+) (\text{cf}(j) = \kappa \Rightarrow \text{Rng}(c \upharpoonright (j \times j)) = j).$$

Now let  $\varphi_{\bar{q}} : \kappa^+ \rightarrow \kappa^+$  be a regressive function such that for  $j < \kappa^+$  of cofinality  $\kappa$  we have

$$\varphi_{\bar{q}}(j) = c(\min\{\alpha < \kappa^+ : \psi_0(\alpha) = q_j \upharpoonright U_j\}, \min\{\alpha < \kappa^+ : \psi_1(\alpha) = \bar{f}^{q_j} \upharpoonright T(U_j)\}).$$

Easily,  $\varphi_{\bar{q}}$  is as required.  $\square$

Now we may complete the proof of Proposition 6.4. Consider the following strategy st for Player I in the game  $\mathfrak{D}_{\varepsilon, \kappa}^{\text{cc}}(\mathbb{Q})$ . Suppose that the players arrived at stage  $\alpha > 0$  of the play and they have already constructed a sequence  $\langle \bar{q}^\beta, \bar{p}^\beta, \varphi^\beta : \beta < \alpha \rangle$ . Then, for each  $j < \kappa^+$ , the sequence  $\langle p_j^\beta : \beta < \alpha \rangle$  is increasing, so Player I can take its least upper bound  $q_j^\alpha$ . This determines  $\bar{q}^\alpha$  played by Player I; the function  $\varphi^\alpha$  played at this stage is the  $\varphi_{\bar{q}^\alpha}$  given by Claim 6.4.2.

One easily verifies that the strategy st is a winning one (remember Claim 6.4.1).  $\square$

**Theorem 6.5.** *Suppose  $\theta$  and  $\iota$  are cardinals such that  $\theta = \text{cf}(\theta) < \kappa \leq \iota = \iota^{< \kappa}$ . Then the forcing notion  $\mathbb{Q}$  is  $(\theta, \iota, \kappa)$ -manageable.*

*Proof.* For each  $\sigma \in \bigcup_{\alpha < \tau} \prod_{\beta < \alpha} \mu_\beta$  fix a sequence  $\eta_\sigma \in \prod_{\alpha < \tau} \mu_\alpha$  such that  $\sigma \subseteq \eta_\sigma$ . Let  $\bar{\eta} = \langle \eta_\sigma : \sigma \in \bigcup_{\alpha < \tau} \prod_{\beta < \alpha} \mu_\beta \rangle$ .

Suppose that  $N$  is a  $(\mathbb{Q}, \kappa, \iota)$ -relevant model such that  $(\bar{\eta}, \bar{\mu}, \pi) \in N$ .

For a condition  $p \in \mathbb{Q}$  we define conditions  $\text{cl}_N^+(p) = q$  and  $\text{cl}_N^-(p) = r$  by

- $i^r = i^q = i^p$ ,
- $u^r = (u^p \cap N) \cup \{\eta_\sigma : \sigma \in T(u^p) \cap N\}$ ,  $u^q = u^p \cup \{\eta_\sigma : \sigma \in T(u^p) \cap N\}$ ,
- $f_\sigma^r = f_\sigma^p$  for  $\sigma \in T(u^p) \cap N$ ,  $f_\sigma^q = f_\sigma^p$  for  $\sigma \in T(u^p)$ , and  $f_\sigma^q(j) = f_\sigma^r(j) = i^p$  for  $\sigma \in T(u^q) \setminus T(u^p)$ ,  $j < i^p$ ,
- $g^r(\eta) = g^p(\eta)$  for  $\eta \in u^p \cap N$  and  $g^r(\eta) = i^p$  for  $\eta \in u^r \setminus u^p$ ;  
 $g^q(\eta) = g^p(\eta)$  for  $\eta \in u^p$  and  $g^q(\eta) = i^p$  for  $\eta \in u^q \setminus u^p$ .

Plainly,  $\text{cl}_N^+(p), \text{cl}_N^-(p)$  are conditions in  $\mathbb{Q}$  and  $\text{cl}_N^-(p)$  belongs to  $N$  (remember  ${}^{< \kappa} N \subseteq N$ ). If  $p \in N$  then also  $\text{cl}_N^+(p) \in N$ .

**Claim 6.5.1.** *Suppose that  $p \in \mathbb{Q}$ ,  $q \in N \cap \mathbb{Q}$  are such that  $q \leq p$ . Then*

- (1)  $\text{cl}_N^+(q) = \text{cl}_N^-(q)$ ,  $q \leq \text{cl}_N^-(p) \leq \text{cl}_N^+(p)$ , and  $p \leq \text{cl}_N^+(p)$ ,
- (2) if  $q' \in N \cap \mathbb{Q}$  is stronger than  $\text{cl}_N^-(p)$ , then  $q'$  and  $p$  are compatible,
- (3) if  $p' \in \mathbb{P}$  is stronger than  $\text{cl}_N^+(p)$ , then  $\text{cl}_N^-(p) \leq \text{cl}_N^-(p')$ .

*Proof of the Claim.* 1) Just check.

2) Suppose  $\text{cl}_N^-(p) \leq q' \in N \cap \mathbb{Q}$ . Put

$$i^r = i^{q'}, \quad u^r = u^{q'} \cup u^p, \quad g^r(\eta) = \begin{cases} g^{q'}(\eta) & \text{if } \eta \in u^{q'}, \\ g^p(\eta) & \text{if } \eta \in u^p \setminus u^{q'} \end{cases} \quad \text{and:}$$

if  $\sigma \in T(u^{q'})$ , then  $f_\sigma^r = f_\sigma^{q'}$ , and if  $\sigma \in T(u^p) \setminus T(u^{q'})$ , then

$$f_\sigma^p \subseteq f_\sigma^r \quad \text{and} \quad f_\sigma^r(j) = \sup\{\pi_\eta(j) : \sigma \subseteq \eta \in u^p\} + 1 \quad \text{for } i^p \leq j < i^{q'}.$$

Note that if  $\eta \in u^p \setminus u^{q'}$ , then for some  $\alpha < \tau$  we have  $\eta \upharpoonright \alpha \notin N$  (so  $\eta \upharpoonright \alpha \notin T(u^{q'})$ ). Hence we may easily verify that  $r = (i^r, u^r, \bar{f}^r, g^r) \in \mathbb{Q}$  and clearly  $r$  is stronger than  $q'$ . To check that it is also stronger than  $p$  it is enough to note that:

if  $\eta \in u^p \cap u^{q'}$ , then ( $\eta \in u^{\text{cl}_N^-(p)}$  and hence)  $g^{q'}(\eta) = g^{\text{cl}_N^-(p)}(\eta) = g^p(\eta)$ , and

if  $\sigma \in T(u^p) \cap T(u^{q'})$ , then ( $\sigma \in T(u^{\text{cl}_N^-(p)})$  and hence)  $f_\sigma^p = f_\sigma^{\text{cl}_N^-(p)} \subseteq f_\sigma^{q'}$ .

3) Note that if  $\text{cl}_N^+(p) \leq p'$ , then

$$(u^p \cap N) \cup \{\eta_\sigma : \sigma \in T(u^p) \cap N\} \subseteq u^{p'} \cap N,$$

so checking the conditions for  $\text{cl}_N^-(p) \leq \text{cl}_N^-(p')$  is pretty straightforward.  $\square$

**Claim 6.5.2.** *Suppose that a sequence  $\langle p_\zeta : \zeta < \zeta^* \rangle \subseteq \mathbb{Q}$  is increasing,  $\zeta^* < \kappa$  is a limit ordinal, and  $\text{cl}_N^+(p_\zeta) = p_{\zeta+1}$  for all even  $\zeta < \zeta^*$ . Let  $p^*$  be the least upper bound of  $\langle p_\zeta : \zeta < \zeta^* \rangle$ . Then  $\text{cl}_N^-(p^*)$  is the least upper bound of  $\langle \text{cl}_N^-(p_\zeta) : \zeta < \zeta^* \rangle$ .*

*Proof of the Claim.* It follows from Claim 6.5.1(3) that  $\text{cl}_N^-(p_\zeta) \leq \text{cl}_N^-(p^*)$  (for  $\zeta < \zeta^*$ ). To show that  $\text{cl}_N^-(p^*)$  is actually the least upper bound it is enough to note that

$$i^{\text{cl}_N^-(p^*)} = i^{p^*} = \sup\{i^{p_\zeta} : \zeta < \zeta^*\} = \sup\{i^{\text{cl}_N^-(p_\zeta)} : \zeta < \zeta^*\},$$

and

$$\begin{aligned} u^{p^*} \cap N &= \bigcup\{u^{p_{\zeta+1}} \cap N : \zeta < \zeta^* \text{ \& } \zeta \text{ even}\} = \bigcup\{u^{\text{cl}_N^-(p_\zeta)} : \zeta < \zeta^* \text{ \& } \zeta \text{ even}\}, \\ \{\eta_\sigma : \sigma \in T(u^{p^*}) \cap N\} &= \{\eta_\sigma : \sigma \in T(u^{p_\zeta}) \cap N \text{ \& } \zeta < \zeta^*\} \subseteq \bigcup\{u^{\text{cl}_N^-(p_\zeta)} : \zeta < \zeta^*\}, \end{aligned}$$

so  $u^{\text{cl}_N^-(p^*)} = u^{p^*} \cap N = \bigcup\{u^{\text{cl}_N^-(p_\zeta)} : \zeta < \zeta^*\}$ .  $\square$

Now we may describe a strategy st for Her in the game  $\mathfrak{D}^m(N, \theta, \mathbb{Q})$ . Suppose that  $i < \theta$  is even and  $(q_i, p_i)$  is His move at this stage of the play (so  $q_i \in N \cap \mathbb{P}$ ,  $q_i \leq p_i \in \mathbb{P}$ ). Then st instructs Her to play  $q_{i+1} = \text{cl}_N^-(p_i)$  and  $p_{i+1} = \text{cl}_N^+(p_i)$ . It follows from Claim 6.5.1(1) that  $(q_{i+1}, p_{i+1})$  is a legal move. It follows from Claims 6.5.2 and 6.5.1(2) that the strategy st is a winning one.

Thus we have shown that  $\mathbb{Q}$  is weakly  $(\theta, \iota, \kappa)$ -manageable (remember 6.3(1)). The rest follows from Propositions 6.3 and 6.4.  $\square$

**Definition 6.6.** We define  $\mathbb{Q}$ -names  $\dot{f}_\sigma$  (for  $\sigma \in \bigcup_{\alpha < \tau} \prod_{\beta < \alpha} \mu_\beta$ ) and  $\dot{g}$  by

$$\begin{aligned} \Vdash_{\mathbb{Q}} \quad & \text{“ } \dot{f}_\sigma = \bigcup\{f_\sigma^p : p \in \Gamma_{\mathbb{Q}} \text{ \& } \sigma \in T(u^p)\} \text{ ”}, \\ \Vdash_{\mathbb{Q}} \quad & \text{“ } \dot{g} = \bigcup\{g^p : p \in \Gamma_{\mathbb{Q}}\} \text{ ”}. \end{aligned}$$

**Proposition 6.7.** (1)  $\Vdash_{\mathbb{Q}} \text{ “ } \dot{g} : \prod_{\alpha < \tau} \mu_\alpha \longrightarrow \kappa \text{ ”}$ .

(2) For each  $\sigma \in \bigcup_{\alpha < \tau} \prod_{\beta < \alpha} \mu_\beta$  we have  $\Vdash_{\mathbb{Q}} \text{ “ } \dot{f}_\sigma : \kappa \longrightarrow \kappa \text{ ”}$ .

(3) For each  $\eta \in \prod_{\alpha < \tau} \mu_\alpha$ ,

$$\Vdash_{\mathbb{Q}} \text{ “ } (\forall j < \kappa) (\dot{g}(\eta) \leq j \Rightarrow \pi_\eta(j) < \sup\{\dot{f}_{\eta \upharpoonright \alpha}(j) : \alpha < \tau\}) \text{ ”}.$$

*Proof.* For  $\eta \in \prod_{\alpha < \tau} \mu_\alpha$  and  $i < \kappa$  let

$$\mathcal{I}_\eta = \{p \in \mathbb{Q} : \eta \in u^p\} \quad \text{and} \quad \mathcal{I}^i = \{p \in \mathbb{Q} : i < i^p\}.$$

We claim that these are open dense subsets of  $\mathbb{Q}$ . First, suppose  $\eta \notin u^p$ ,  $p \in \mathbb{Q}$  and let  $i^r = i^p$ ,  $u^r = u^p \cup \{\eta\}$ ,  $g^p \subseteq g^r$ ,  $g^r(\eta) = i^r$ ,  $f_\sigma^r = f_\sigma^p$  for  $\sigma \in T(u^p)$  and  $f_{\eta \upharpoonright \alpha}^r(j) = 1$  if  $\eta \upharpoonright \alpha \notin T(u^p)$ ,  $\alpha < \tau$ . Then  $r \in \mathcal{I}_\eta$  is stronger than  $p$ . (Thus the sets  $\mathcal{I}_\eta$  are dense.)

Now suppose that  $p \in \mathbb{P}$  is such that  $i^p \leq i < \kappa$ . Put  $i^r = i + 1$ ,  $u^r = u^p$ ,  $g^r = g^p$  and for  $\sigma \in T(u^r)$  let  $f_\sigma^r \supseteq f_\sigma^p$  be such that  $(\text{Dom}(f_\sigma^r) = i^r)$  and for  $j \in i^r \setminus i^p$  we have  $f_\sigma^r(j) = \sup\{\pi_\eta(j) : \eta \in u^r\} + 1$ . This way we have defined a condition  $r \in \mathbb{Q}$  stronger than  $p$  and such that  $r \in \mathcal{I}^i$ . (Thus the sets  $\mathcal{I}^i$  are dense.)

Using the above observation and the definition of the order of  $\mathbb{Q}$  one easily justifies (1) and (2). (Note also that, as  $\mathbb{Q}$  is  $<\kappa$ -complete,  $\Vdash_{\mathbb{Q}} \prod_{\alpha < \tau} \mu_\alpha = (\prod_{\alpha < \tau} \mu_\alpha)^{\mathbf{V}}$ .) Then (3) follows immediately once you note that

$$p \Vdash \text{“} g^p \subseteq \dot{g} \ \& \ f_\sigma^p \subseteq \dot{f}_\sigma \text{”},$$

(for  $\sigma \in T(u^p)$ ,  $p \in \mathbb{Q}$ ); remember Definition 6.2(1)(c).  $\square$

## 7. THE MODELS

**Theorem 7.1.** *Assume that*

- (a)  $\kappa = 2^{<\kappa}$ ,
- (b)  $\mu$  is a cardinal such that  $\text{cf}(\mu) < \kappa < \mu < \mu^{\text{cf}(\mu)} = 2^\kappa$ ,
- (c) there is an increasing sequence  $\bar{\mu} = \langle \mu_\alpha : \alpha < \text{cf}(\mu) \rangle$  of regular cardinals such that

$$(\forall \alpha < \text{cf}(\mu)) (\kappa \leq \mu_\alpha \leq (\mu_\alpha)^{\text{cf}(\mu)} \leq \mu_{\alpha+1}) \quad \text{and} \quad \mu = \sup\{\mu_\alpha : \alpha < \text{cf}(\mu)\}.$$

Then there is a forcing notion  $\mathbb{P}$  such that:

- (i)  $\mathbb{P}$  has a dense subset of size  $2^\kappa$ ,
- (ii)  $\mathbb{P}$  is  $(\theta, \iota, \kappa)$ -manageable for all cardinals  $\theta, \iota$  satisfying  $\text{cf}(\theta) = \theta < \kappa \leq \iota = \iota^{<\kappa}$ ,
- (iii)  $\Vdash_{\mathbb{P}} \mathfrak{d}_\kappa^{\text{cf}(\mu)} \leq \mu$ ,
- (iv) if  $\text{cov}(\mathbf{M}_{\kappa, \kappa}) > \mu$ , then  $\Vdash_{\mathbb{P}} \mathfrak{d}_\kappa^{\text{cf}(\mu)} = \mu < (\text{cov}(\mathbf{M}_{\kappa, \kappa}))^{\mathbf{V}} \leq \text{cov}(\mathbf{M}_{\kappa, \kappa})$ .

*Proof.* Assume  $\kappa, \mu, \bar{\mu} = \langle \mu_\alpha : \alpha < \text{cf}(\mu) \rangle$  are as in the assumptions (a)–(c). Note that then also  $\prod_{\alpha < \text{cf}(\mu)} \mu_\alpha = 2^\kappa$  (by Tarski’s theorem).

The forcing notion  $\mathbb{P}$  is built as the limit of a  $<\kappa$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \kappa^+ \rangle$ . The names  $\dot{\mathbb{Q}}_\xi$  are defined by induction on  $\xi < \kappa^+$  so that

- ( $\alpha$ )  $\mathbb{P}_\xi$  has a dense subset of size  $2^\kappa$ ,

and for all cardinals  $\theta, \iota$  satisfying  $\text{cf}(\theta) = \theta < \kappa \leq \iota = \iota^{<\kappa}$ ,

- ( $\beta$ ) the forcing notion  $\mathbb{P}_\xi$  is  $(\theta, \iota, \kappa)$ -manageable and
- ( $\gamma$ )  $\Vdash_{\mathbb{P}_\xi} \text{“} \dot{\mathbb{Q}}_\xi \text{ is } (\theta, \iota, \kappa)\text{-manageable”}$ .

So suppose that  $\mathbb{P}_\xi$  is already defined (and clauses ( $\alpha$ ), ( $\beta$ ) hold). Then  $\mathbb{P}_\xi$  is  $<\kappa$ -complete  $\kappa^+$ -cc, and hence the properties of  $\kappa, \mu$  and  $\bar{\mu}$  stated in (a)–(c) hold in

$\mathbf{V}^{\mathbb{P}^\xi}$ . Take a  $\mathbb{P}_\xi$ -name  $\dot{\pi}_\xi$  such that

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \dot{\pi}_\xi : \prod_{\alpha < \text{cf}(\mu)} \mu_\alpha \longrightarrow {}^\kappa \kappa \text{ is a bijection ”,}$$

and let  $\dot{\mathbb{Q}}_\xi$  be a  $\mathbb{P}_\xi$ -name for the forcing notion  $\mathbb{Q}(\dot{\pi}_\xi, \bar{\mu}, \kappa)$ . Then clause  $(\gamma)$  holds (remember Theorem 6.5).

It follows from Proposition 6.3(1) that the demand  $(\alpha)$  is preserved at successor stages and it is preserved at limits  $\leq \kappa^+$  by the support we use. By Theorem 5.11, the clause  $(\beta)$  holds for each  $\mathbb{P}_\xi$  ( $\xi \leq \kappa^+$ ). So our  $\mathbb{P} = \mathbb{P}_{\kappa^+}$  satisfies (i)+(ii).

For  $\xi < \kappa^+$  let  $\dot{f}_\sigma^\xi$  (for  $\sigma \in \bigcup_{\alpha < \text{cf}(\mu)} \prod_{\beta < \alpha} \mu_\beta$ ) and  $\dot{g}^\xi$  be  $\mathbb{P}_{\xi+1}$ -names for functions added by  $\mathbb{Q}(\dot{\pi}_\xi, \bar{\mu}, \kappa)$  (see Definition 6.6). Then the family

$$\mathcal{F} = \{\dot{f}_\sigma^\xi : \xi < \kappa^+ \ \& \ \sigma \in \bigcup_{\alpha < \text{cf}(\mu)} \prod_{\beta < \alpha} \mu_\beta\}$$

is of size  $\mu$ . Since  $\mathbb{P}$  is  $\kappa^+$ -cc, for each  $\mathbb{P}$ -name  $\dot{h}$  for a member of  ${}^\kappa \kappa$  there are  $\xi < \kappa^+$  and a  $\mathbb{P}_\xi$ -name  $\dot{h}^*$  such that  $\Vdash_{\mathbb{P}} \text{“ } \dot{h} = \dot{h}^* \text{ ”}$ . Thus using Proposition 6.7(3) we get

$$\Vdash_{\mathbb{P}} \text{“ } (\exists j^* < \kappa) (\exists F \in \mathcal{P}_{\text{cf}(\mu)^+}(\mathcal{F})) (\forall j < \kappa) (j^* \leq j \Rightarrow \dot{h}(j) < \sup\{f(j) : f \in F\}) \text{ ”}.$$

Now we easily conclude that demand (iii) holds.

To show (iv) let us assume  $\text{cov}(\mathbf{M}_{\kappa, \kappa}) > \mu$ . The forcing notion  $\mathbb{P}$  is  $(\aleph_0, \kappa, \kappa)$ -manageable, so by Corollary 5.10 and Proposition 4.5 we have

$$\Vdash_{\mathbb{P}} \text{“ } \mu < (\text{cov}(\mathbf{M}_{\kappa, \kappa}))^{\mathbf{V}} \leq \text{cov}(\mathbf{M}_{\kappa, \kappa}) \leq \mathfrak{d}_\kappa \text{ ”}.$$

By (iii) we know  $\Vdash_{\mathbb{P}} \text{“ } \mathfrak{d}_\kappa^{\text{cf}(\mu)} \leq \mu \text{ ”}$ , but as for each  $\alpha < \text{cf}(\mu)$ :

$$\Vdash_{\mathbb{P}} \text{“ } (\mu_\alpha)^{\text{cf}(\mu)} \leq \mu_{\alpha+1} < \mu < \mathfrak{d}_\kappa \text{ ”},$$

we immediately get that  $\Vdash_{\mathbb{P}} \text{“ } \mathfrak{d}_\kappa^{\text{cf}(\mu)} = \mu \text{ ”}$  (remember  $\mathfrak{d}_\kappa \leq (\mathfrak{d}_\kappa^{\text{cf}(\mu)})^{\text{cf}(\mu)}$ ).  $\square$

**Corollary 7.2.** *Assume GCH. Then there a  $\kappa$ -complete  $\kappa^+$ -cc forcing notion  $\mathbb{P}^*$  such that*

$$\Vdash_{\mathbb{P}^*} \text{“ } \bar{\mathfrak{d}}_\kappa = \mathfrak{d}_\kappa^{\aleph_0} = \kappa^{+\omega} \text{ and } \text{cov}(\mathbf{M}_{\kappa, \kappa}) = 2^\kappa = \kappa^{+(\omega+1)} \text{ ”}.$$

*Proof.* Let  $\mathbb{P}_0 = \mathbb{C}_{\kappa^{+(\omega+1)}, \kappa}$  be the forcing adding  $\kappa^{+(\omega+1)}$  many Cohen functions in  ${}^\kappa \kappa$  (with  $< \kappa$ -support). Note that

$$\Vdash_{\mathbb{P}_0} \text{“ } \kappa, \mu = \kappa^{+\omega} \text{ and } \bar{\mu} = \langle \kappa^{+n} : n < \omega \rangle \text{ are as in Theorem 7.1(a)-(c) ”}.$$

Therefore we have a  $\mathbb{P}_0$ -name  $\dot{\mathbb{P}}$  for a forcing notion satisfying 7.1(i)-(iv). (Note that  $\Vdash_{\mathbb{P}_0} \text{“ } \text{cov}(\mathbf{M}_{\kappa, \kappa}) = \kappa^{+(\omega+1)} \text{ ”}$ , so the assumption of 7.1(iv) holds.) Let  $\mathbb{P}^* = \mathbb{P}_0 * \dot{\mathbb{P}}$ .  $\square$

Note that Theorem 0.1 follows from Corollary 7.2, Theorem 1.12, and the fact that  $\text{cov}(\mathbf{M}_{\kappa, \kappa}) \leq \text{cof}(\mathcal{NS}_\kappa)$ .

**Theorem 7.3.** *Assume that (a)-(c) of Theorem 7.1 hold and*

(d)  $\nu$  is a regular cardinal such that  $\mu < \nu < 2^\kappa$ .

*Then there is a forcing notion  $\mathbb{P}_\nu$  satisfying (i)+(ii) of Theorem 7.1 and*

(iii) $^+$   $\Vdash_{\mathbb{P}_\nu} \text{“ } \mathfrak{d}_\kappa^\tau = \nu \text{ for every cardinal } \tau \text{ satisfying } \text{cf}(\mu) \leq \tau < \kappa \text{ ”}$ ,

(iv) $^-$   $\Vdash_{\mathbb{P}_\nu} \text{“ } (\text{cov}(\mathbf{M}_{\kappa, \kappa}))^{\mathbf{V}} \leq \text{cov}(\mathbf{M}_{\kappa, \kappa}) \text{ ”}$ .

*Proof.* The forcing notion  $\mathbb{P}_\nu$  is the limit of  $<\kappa$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \nu \rangle$ , where  $\dot{\mathbb{Q}}_\xi$  are defined as in the proof of Theorem 7.1 (so the only difference is the length of the iteration). As there,  $\mathbb{P}_\nu$  satisfies (i)+(ii) and  $\Vdash_{\mathbb{P}_\nu} \text{“} \mathfrak{d}_\kappa^{\text{cf}(\mu)} \leq \nu \text{”}$ . Since  $\text{cov}(\mathbf{M}_{\kappa,\kappa}) > \kappa$  and  $\mathbb{P}_\nu$  is  $(\aleph_0, \kappa, \kappa)$ -manageable, we get (iv)<sup>-</sup> (by Corollary 5.10). To show that (iii) <sub>$\nu$</sub> <sup>+</sup> holds, suppose that  $\dot{\mathcal{F}}$  is a  $\mathbb{P}_\nu$ -name for a family of functions in  ${}^\kappa\kappa$  of size  $< \nu$ . Then  $\dot{\mathcal{F}}$  is essentially a  $\mathbb{P}_\xi$ -name for some  $\xi < \nu$ . Since  $\mathbb{P}_{\xi+\kappa}$  adds a subset of  $\kappa$  which is Cohen over  $\mathbf{V}^{\mathbb{P}_\xi}$ ,  $({}^\kappa\kappa)^{\mathbf{V}^{\mathbb{P}_\xi}}$  is not a dominating family in  $({}^\kappa\kappa)^{\mathbf{V}^{\mathbb{P}_\nu}}$ , and hence for any  $\tau < \kappa$

$$\Vdash_{\mathbb{P}_\nu} \text{“} \dot{\mathcal{F}} \text{ is not } \tau\text{-dominating”}$$

(remember that  $\mathbb{P}_\nu$  is  $<\kappa$ -complete).  $\square$

Now, Theorem 0.2 follows from Theorem 1.12 and the following corollary.

**Corollary 7.4.** *It is consistent, relative to the existence of a cardinal  $\nu$  such that  $o(\nu) = \nu^{++}$ , that  $\bar{\aleph}_{\aleph_1} = \aleph_{\omega+1}$  and  $\text{cov}(\mathbf{M}_{\aleph_1, \aleph_1}) = \aleph_{\omega+2}$ .*

*Proof.* Gitik and Woodin (see Gitik [Git89] also Gitik and Merimovich [GM97]) constructed a model of “ $2^{\aleph_n} < \aleph_\omega$  for every  $n < \omega$ ,  $2^{\aleph_0} = \aleph_1$  and  $2^{\aleph_\omega} = \aleph_{\omega+2}$ ” from  $o(\nu) = \nu^{++}$ . Add  $\aleph_{\omega+2}$  Cohen subsets of  $\aleph_1$  (with countable support) to that model and then apply Theorem 7.3 (with  $\kappa = \aleph_1$ ,  $\mu = \aleph_\omega$  and  $\nu = \aleph_{\omega+1}$ ).  $\square$

**Theorem 7.5.** *Assume that*

- (a)  $\kappa = \text{cf}(\kappa) = 2^{<\kappa}$ ,  $n < \omega$
- (b)  $\mu_0, \mu_1, \dots, \mu_n$  are cardinals such that
 
$$\mu_0 > \mu_1 > \dots > \mu_n > \kappa \quad \text{and} \quad \text{cf}(\mu_0) < \text{cf}(\mu_1) < \dots < \text{cf}(\mu_n) < \kappa,$$
- (c)  $(\mu_\ell)^{\text{cf}(\mu_\ell)} = 2^\kappa$  for  $\ell = 0, \dots, n$ ,
- (d) for  $\ell = 0, \dots, n$ , there is an increasing sequence  $\bar{\mu}^\ell = \langle \mu_\alpha^\ell : \alpha < \text{cf}(\mu_\ell) \rangle$  of regular cardinals such that
 
$$(\forall \alpha < \text{cf}(\mu_\ell)) (\kappa \leq \mu_\alpha^\ell = (\mu_\alpha^\ell)^{\text{cf}(\mu_\ell)}) \quad \text{and} \quad \mu_\ell = \sup\{\mu_\alpha^\ell : \alpha < \text{cf}(\mu_\ell)\}.$$
- (e)  $\text{cov}(\mathbf{M}_{\kappa,\kappa}) = 2^\kappa$ .

Then there is a forcing notion  $\mathbb{P}$  such that

- (i)  $\mathbb{P}$  has a dense subset of size  $2^\kappa$ ,
- (ii)  $\mathbb{P}$  is  $(\theta, \iota, \kappa)$ -manageable for all cardinals  $\theta, \iota$  satisfying  $\text{cf}(\theta) = \theta < \kappa \leq \iota = \iota^{<\kappa}$ ,
- (iii)  $\Vdash_{\mathbb{P}} \text{“} \mathfrak{d}_\kappa^{\text{cf}(\mu_\ell)} = \mu_\ell \text{ for } \ell = 0, \dots, n \text{ and } \text{cov}(\mathbf{M}_{\kappa,\kappa}) = 2^\kappa \text{”}$ .

*Proof.* Let  $A_0, \dots, A_n$  be a partition of  $\kappa^+$  into sets of size  $\kappa^+$ . The forcing notion  $\mathbb{P}$  is the limit of a  $<\kappa$ -support iteration  $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \kappa^+ \rangle$  defined like in the proof of Theorem 7.1, but

- if  $\xi \in A_\ell$ , then  $\dot{\pi}_\xi$  is a  $\mathbb{P}_\xi$ -name for a bijection from  $\prod_{\alpha < \tau_\ell} \mu_\alpha^\ell$  onto  ${}^\kappa\kappa$ , and  $\dot{\mathbb{Q}}_\xi$  is  $\mathbb{Q}(\dot{\pi}_\xi, \bar{\mu}^\ell, \kappa)$ .

We argue that  $\mathbb{P}$  has the required properties similarly as in Theorem 7.1.  $\square$

*Remark 7.6.* Of course the assumption (e) in Theorem 7.5 is not very important: we may start with adding  $2^\kappa$  many Cohen subsets of  $\kappa$ .

**Corollary 7.7.** *It is consistent, relative to the existence of a strong cardinal, that*

$$\mathfrak{d}_{\aleph_2}^{\aleph_1} = \aleph_{\omega_1}, \quad \mathfrak{d}_{\aleph_2}^{\aleph_0} = \aleph_{\omega_1+\omega} \quad \text{and} \quad \text{cov}(\mathbf{M}_{\aleph_2, \aleph_2}) = \aleph_{\omega_1+\omega+1}.$$

*Proof.* As was pointed out to us by Moti Gitik, by work of Magidor [Mag77], Merimovich [Mer03], and Segal [Seg96], it is consistent relative to a strong cardinal that

- (a)  $2^{\aleph_1} = \aleph_2$ ,  $\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1}$ , and
- (b) for every  $\alpha < \omega_1$ ,

$$\aleph_{\alpha+\omega+1}^{\aleph_1} = \aleph_{\alpha+\omega+1} \quad \text{and} \quad 2^{\aleph_{\alpha+\omega+1}} = \aleph_{\alpha+\omega+2},$$

and

- (c)  $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1+\omega}^{\aleph_0} = \aleph_{\omega_1+\omega+1}$ .

After adding  $\aleph_{\omega_1+\omega+1}$  Cohen subsets of  $\aleph_2$  (with  $<\aleph_2$  support) to a model of (a)–(c) we will get a model in which the assumptions of Theorem 7.5 are satisfied for  $\kappa = \aleph_2$ ,  $n = 2$ ,  $\mu_0 = \aleph_{\omega_1+\omega}$ ,  $\mu_1 = \aleph_{\omega_1}$ .  $\square$

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