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SH820

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Abstract. We deal with the existence of universal members in a given cardinality for several classes. First we deal with classes of Abelian groups, specifically with the existence of universal members in cardinalities which are strong limit singular of countable cofinality or $\lambda = \lambda^{\aleph_0}$. We use versions of being reduced replacing $\mathbb{Q}$ by a subring (defined by a sequence $\bar{t}$) and get quite accurate results for existence of universal in a cardinal, for embedding and for pure embeddings. Second, we deal with (variants of) the oak property (from a work of Džamonja and the author), a property of complete first order theories, sufficient for the non-existence of universal models under suitable cardinal assumptions. Third, we prove that the oak property holds for the class of groups (naturally interpreted, so for quantifier free formulas) and deal more with the existence of universals.

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Annotated Content

§ 0. Introduction

On the existence of universal structures see Kojman-Shelah [KS92] and history there, and a more recent survey Džamonja [Mir05]. Of course, a complete first order theory T has a universal model in λ for "elementary embeddings" when $\lambda = 2^{<\lambda} > |T|$; this is true also for similar classes, i.e. for a.e.c. with amalgamation and the JEP and LST number $< \lambda$. The question which interests us is whether there are additional cases (mainly for elementary classes and more generally a.e.c. as above). But here we deal with some specific classes and embeddability notion.

Now §1 deals mainly with Abelian groups; it continues Kojman-Shelah [KS95] and [She96b], [She97] and [She01]. The second section deals with the class of groups; it continues Usvyatsov-Shelah [SU06] but does not rely on it. The third section deals with the oak property continuing Džamonja-Shelah [DS06], dealing with the case of singular cardinals.

The second section deals with the class of all groups, certainly an important one. Is this class complicated? Under several yard-sticks it certainly is: its first order theory is undecidable, etc., and it has the quantifier-free order property (even the class of (universal) locally finite groups, has this property, see Macintyre-Shelah [MS76]) and by [SU06] it has the SOP_3 (3-strong order property). But this does not exclude positive answers for other interpretations. By [SU06] it has the NSOP_4 (4-strong non-order property), however we do not know much about this family of classes (though we have hopes).

A recent relevant work is [She16], [S$^+$a], giving new sufficient conditions for "no universal", in particular for groups and hopefully [S$^+$b] on classes of Abelian groups.

Here we consider the oak property from Džamonja-Shelah [DS06], a relative of the tree property, (hence the name). We prove that the class of groups has the oak property, hence it follows that in some cardinals it has no universal member.

There is reasonable evidence for the class of linear orders being complicated, practically maximal for the universal spectrum problem, see [KS92].

So a specific conclusion is:

Conclusion 0.1. *1) The class of groups has the oak property, see Definition 2.1.*
2) If λ satisfies, e.g., $(*)$ below <u>*then*</u> *there is no universal group of cardinality λ when:*

$(*)$ (a) $\kappa = \mathrm{cf}(\mu) < \mu$
(b) $\lambda = \mu^{++} < \mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu)$
(c) $\alpha < \mu \Rightarrow |\alpha|^\kappa < \mu$.

Proof. 1) By 2.2.
2) By part (1) and [DS06], more exactly by 3.1. $\square_{0.1}$

In §3 we deal with the oak property per se, continuing [DS06], showing non-existence of universal in singular cardinals and dealing with a weaker relative, the weak oak property.

Concerning the first section note that strong limit singular cardinal λ is a case where it is easier to have a universal model, particularly when λ has cofinality $\aleph_0$. So the canonical case seems to be $\beth_\omega$. Examples of such positive (= existence) results are

(a) [She73, Th.3.1,p.266], where it is proved that:
if λ is strong limit singular, then
$\{G : G$ a graph with $\le \lambda$ nodes each of valency $< \lambda\}$ has a universal member under embedding onto induced subgraphs

(b) Grossberg-Shelah [GS83, §5,Corollary 27,pgs.301-302]:

(α) if λ is "large enough" then similar results hold for quite general classes (e.g. locally finite groups) where large enough means: λ (is strong limit, of cofinality $\aleph_0$ and) is above a compact cardinal (which is quite large).
More specifically,

(β) if μ is strong limit of cofinality $\aleph_0$ above a compact cardinal κ and, e.g., the class $\mathfrak{K}$ is the class of models of $T \subseteq \mathbb{L}_{\kappa,\aleph_0}, |T| < \mu$ partially ordered by $\prec_{\mathbb{L}_{\kappa,\omega}}$, then we can split $\mathfrak{K}$ to $\le 2^{|T|+\kappa}$ classes each having a universal model of cardinality μ under $\prec_{\mathbb{L}_{\kappa,\aleph_0}}$-embeddings.

See more in [KSV15] generalizing the so-called special models. Claim 1.16 below continues this, i.e., it deals with strong limit cardinal $\mu > \mathrm{cf}(\mu) = \aleph_0$, compared to [GS83] omitting the set theoretic assumption on compact cardinal at the expense of strengthening the model theoretic assumption.

There are natural examples where this can be applied; e.g. the class of torsion free Abelian groups G which are reduced (i.e., we cannot embed the rational into G), but the order is $G_1 \le_{\langle n!:n<\omega\rangle} G_2$ which means $G_1 \subseteq G_2$ but G_1 is closed inside G_2 under the $\mathbb{Z}$-adic metric; so also G_2/G_1 is reduced. The application of 1.16 to such classes is in 1.14(1)(2). Earlier in 1.2 we prove related positive results for the easier cases of complete members (for λ satisfying $\lambda = \lambda^{\aleph_0}$ or λ the limit of such cardinals).

We also get some negative results, i.e., non-existence of universal members in 1.7(2), 1.11. We deal more generally with $K^{\mathrm{rtf}}_{\bar{t}}$, the reduced torsion free Abelian group G such that for no $x \in G, x \ne 0$ and x is divisible by $t_{<n} = \prod\limits_{\ell<n} t_\ell$ for every n. We sort out the existence of universal members of cardinality $\lambda = \lambda^{\aleph_0}$ for $K^{\mathrm{rtf}}_{\bar{t},\lambda}$ under embeddings and under pure embeddings, getting complete (but different) answers for $\lambda = \lambda^{\aleph_0}$.

Recall that classes of Abelian groups are related to the classes of trees with $\omega+1$ levels. The parallel of "Abelian groups under pure embedding" is the case of such trees, in fact, non-existence of universals for Abelian groups under pure embedding implies the non-existence of such universal trees.

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Notation 0.2. a 1) For a set $A, |A|$ is its cardinality but for a structure M its cardinality is $\|M\|$ while its universe is $|M|$; this apply e.g. to groups.
2) $\bar{t}$ will denote an ω-sequence of natural numbers ≥ 2.
3) We use G, H for groups, M, N for general models.
4) Let $\mathfrak{k}$ denote a pair $(K_{\mathfrak{k}}, \le_{\mathfrak{k}})$, we may say a class $\mathfrak{k}$ instead of a pair, where:

(a) $K_{\mathfrak{k}}$ is a class of $\tau_{\mathfrak{k}}$-structures
(b) $\le_{\mathfrak{k}}$ is a partial order on $K_{\mathfrak{k}}$ such that $M \le_{\mathfrak{k}} N \Rightarrow M \subseteq N$
(c) both $K_{\mathfrak{k}}$ and $\le_{\mathfrak{k}}$ are closed under isomorphisms.

4A) We say $f : M \to N$ is a $\leq_{\mathfrak{k}}$-embedding when f is an isomorphism from M onto some $M_1 \leq_{\mathfrak{k}} N$.
5) If T is a first order theory then Mod_T is the pair $(\mathrm{mod}_T, \leq_T)$ where mod_T is the class of models of T and $\leq_T$ is: $\prec$ if T is complete, $\subseteq$ if T is not complete.
6) We may write T instead of Mod_T, e.g. in Definition 0.3 below.

Definition 0.3. 1) For a class $\mathfrak{k}$ and a cardinal λ, a set $\{M_i : i < i^*\}$ of models from $\mathfrak{k}$, are <u>jointly universal</u> for λ <u>when</u> for every $N \in K_{\mathfrak{k}}$ of size λ, there is an $i < i^*$ and an $\leq_{\mathfrak{k}}$-embedding of N into M_i.
2) For $\mathfrak{k}$ and λ as above, let (if $\mu = \lambda$ we may omit μ)

$$\begin{aligned} \mathrm{univ}(\mathfrak{k}, \mu, \lambda) := \quad & \min\{|\mathscr{M}| : \mathscr{M} \text{ is a family of members of } K_{\mathfrak{k}} \text{ each} \\ & \text{of cardinality } \leq \mu \text{ which is jointly universal} \\ & \text{for models of } \mathfrak{k} \text{ of size } \lambda\}. \end{aligned}$$

Remark 0.4. To help understanding Definition 0.3, note that $\mathrm{univ}(T, \lambda) = 1$ iff there is a universal model of T of size λ. Note that some of the classes we consider are not abstract elementary classes. Some have "weak failure" say $\mathbb{Z}$-adically complete free Abelian free groups which are torsion free, if $M_n \leq M_{n+1}$ then $\bigcup\limits_n M_n$ is not necessarily complete. We can take a completion; more seriously for some $\mathfrak{k}$ and the M_n's there are contradictory completions.

Recall

Definition 0.5. For an ideal J on a set A and a set B let $\mathbf{U}_J(B) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P}$ is a family of subsets of B, each of cardinality $\leq |A|$ such that for every function f from A into B for some $u \in \mathscr{P}$ we have $\{a \in A : f(a) \in u\} \in J^+\}$. Clearly only $|B|$ matters so we normally write $\mathbf{U}_J(\lambda)$ (see on it [She00]).

§ 1. More on Abelian groups

Earlier versions of this section originally was part of [She01] and earlier of [She97], but as the papers were too long, it was delayed.

Remark 1.1. Inspite of all cases dealt with in [She97], there are still some "missing" cardinals (see discussion in [She01, §0]). Concerning λ singular satisfying $2^{\aleph_0} < \mu^+ < \lambda < \mu^{\aleph_0}$, clearly [She01, 2.8=2.7t,3.14=3.12t], [She94] indicates that at least for most such cardinals there is no universal: as if $\chi \in (\mu^+, \lambda)$ is regular, then $\mathrm{cov}(\lambda, \chi^+, \chi^+, \chi) < \mu^{\aleph_0}$.

Let us mention concerning positive results on Case 1 (from [She01, §0]), see Definition 1.3 below. (See Fuchs [Fuc73] on Abelian groups).

Claim 1.2. *1) If $\lambda = \lambda^{\aleph_0}$ __then__ in the class $(K^{\mathrm{rtf}}_\lambda, \le_{\mathrm{pr}})$, defined in 1.3(5) below there is a universal member, in fact it is homogeneous universal.*
2) If $\lambda = \sum\limits_{n<\omega} \lambda_n$ and $\aleph_0 \le \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$ __then__ in $(\mathfrak{K}^{\mathrm{rtf}}_\lambda, \le_{\mathrm{pr}})$ there is a universal member (the parallel of special models for first order theories).
3) $(K^{\mathrm{rtf}}, \le_{\mathrm{pr}})$ has the amalgamation and JEP; is an a.e.c. (see [She09]*) and is stable in λ if $\lambda = \lambda^{\aleph_0}$.*

We shall prove 1.2 below, but first

Definition 1.3. 1) K^{tf}_λ is the class of torsion-free Abelian groups of cardinality λ. Let $K^{\mathrm{tf}} = \cup\{K^{\mathrm{tf}}_\lambda : \lambda \text{ a cardinal}\}$ and similarly $K^{\mathrm{tf}}_{\le\lambda}$.
1A) $K^{\mathrm{rtf}}_{\bar t,\lambda}$ is the class of $G \in K^{\mathrm{tf}}_\lambda$ such that there is no $x \in G\backslash\{0\}$ divisible by $\prod\limits_{\ell<k} t_\ell$ for every $k < \omega$ recalling 0.2(2).
1B) Let $K^{\mathrm{rtf}}_{\bar t} = \cup\{K^{\mathrm{rtf}}_{\bar t,\lambda} : \lambda \text{ a cardinal}\}$.
1C) $G \in K^{\mathrm{rtf}}_{\bar t}$ is called $\bar t$-complete __when__ every Cauchy sequence under $d_{\bar t}$ in G has a limit where $d_{\bar t}$ is defined in 1.3(3) below.
2) Let

- (a) $\mathfrak{T} = \{\bar t : \bar t = \langle t_n : n < \omega\rangle, 2 \le t_n \in \mathbb{N}\}$,
- (b) we call $\bar t \in \mathfrak{T}$ full when $(\forall k \ge 2)(\exists n)[k \text{ divide } \prod\limits_{\ell<n} t_\ell]$, equivalently $(\forall n)(\exists m)[m > n \wedge n | \prod\limits_{\ell=n}^{m} t_\ell]$, equivalently, every prime p, divide infinitely many t_n's
- (c) we call $\bar t \in \mathfrak{T}$ explicitly weakly full __when__ for every prime p, either p divide no t_n or it divides infinitely many t_n
- (d) we say G is $\bar t$-divisible __when__ every $x \in G$ is divisible by $\prod\limits_{\ell<n} t_\ell$ for every n
- (e) we call $\bar t \in \mathfrak{T}$ weakly full __when__ for some $n(*)$ the sequence $\langle t_{n(*)+n} : n < \omega\rangle$ is explicitly weakly full.

3) For $G \in K^{\mathrm{rtf}}_{\bar t,\lambda}$ let $G^{[\bar t]}$ be the $d_{\bar t}$-completion of G __where__ $d_{\bar t} = d_{\bar t}[G]$ is the metric defined by $d_{\bar t}(x,y) = \inf\{2^{-k} : \prod\limits_{\ell<k} t_\ell \text{ divides } x - y \text{ in the Abelian group } G\}$, justify by 1.4(3), pedantically "the d_t-completion" is determined only up to isormophism over G.
4) Let $K^{\mathrm{crtf}}_{\bar t,\lambda}$ be the class of $G \in K^{\mathrm{rtf}}_{\bar t,\lambda}$ which are $\bar t$-complete (i.e. $G = G^{[\bar t]}$).
5) For those classes, $\le$ means being a subgroup and $\le_{\mathrm{pr}}$ means being a pure subgroup.

6) We say $\bar{t}, \bar{s} \in \mathfrak{T}$ are equivalent when $K^{\rm rtf}_{\bar{t}} = K^{\rm rtf}_{\bar{s}}$.

Observation 1.4. *1) $\bar{t}$ is full iff $\bar{t}$ is equivalent to $\langle n! : n \in \mathbb{N}\rangle$ iff for every power of prime m, for some n, m divides $\prod\limits_{\ell<n} t_\ell$.*
2) If $\bar{t}$ is full then every $G \in K^{\rm tf}$ can be represented in fact uniquely as the direct sum $G_1 + G_2$ where G_1 is divisible, $G_2 \in K^{\rm rtf}_{\bar{t}}$.
3) For $G \in K^{\rm rtf}_{\bar{t}}, d_{\bar{t}}$ is a metric on G.
4) If $G \in K^{\rm rtf}_{\bar{t}}$ then there is G', called the $\bar{t}$-completion of G, such that

- (a) $G \le_{\rm pr} G' \in K^{\rm rtf}_{\bar{t}}$
- (b) G' *is $\bar{t}$-complete*
- (c) G *is dense in G' by the metric $d_{\bar{t}}$*
- (d) *if G'' satisfies (a),(b),(c) then G'', G' are isomorphic over G.*

5) $\bar{t}, \bar{s} \in \mathfrak{T}$ are equivalent when for some k, ℓ we have

- $t_{k+n} = t_{\ell+n}$ *for every n or at least*
- *for some m_*, for every $m \ge m_*$ there is n such that $\prod\limits_{i=m_*}^{m} t_{k+i}$ divide $\prod\limits_{i<n} s_{\ell+i}$ and $\prod\limits_{\ell<m} s_{\ell+i}$ divides $\prod\limits_{i<n} t_{\ell+i}$.*

6) For members of $\mathfrak{T}$ being full and being weakly full are preserved by equivalence.

Proof. Proof of 1.4:
Should be clear. □

Proof. Proof of 1.2
Let $t_n = n!$ and let $\bar{t} = \langle t_n : n < \omega\rangle$.
The point is that clearly

- (a) (α) for $G \in K^{\rm rtf}_{\bar{t}}, G \le_{\rm pr} G^{[\bar{t}]} \in K^{\rm rtf}_{\bar{t}}$ and $G^{[\bar{t}]}$ has cardinality $\le \|G\|^{\aleph_0}$ and $G^{[\bar{t}]}$ is $d_{\bar{t}}$-complete, remember $G^{[\bar{t}]}$ is the $d_{\bar{t}}$-completion of G, it is unique up to isomorphism over G
  - (β) if $G_1 \le_{\rm pr} G_2$ then $G_1^{[\bar{t}]} \le_{\rm pr} G_2^{[\bar{t}]}$, more pedantically: if $G_1 \le_{\rm pr} G_2 \le_{\rm pr} G_3$ and G_3 is $\bar{t}$-complete then $G_1^{[\bar{t}]}$ can be (purely) embedded into G_3 over G_1.

Recall $K^{\rm crtf}_{\bar{t}}$ is the class of $d_{\bar{t}}$-complete $G \in K^{\rm rtf}_{\bar{t}}$.
Easily:

- (b) $(K^{\rm crtf}_{\bar{t}}, \le_{\rm pr})$ has amalgamation, the joint embedding property and the LST (= Löwenheim-Skolem-Tarski) property down to λ for any $\lambda = \lambda^{\aleph_0}$
- (c) if $G' \le_{\rm pr} G''$ are from $K^{\rm ctrf}$ then we can find $\le_{\rm pr}$-increasing sequence $\langle G_\alpha : \alpha \le \alpha(*)\rangle$ of members of $K^{\rm crtf}$ such that
  - (α) $G' = G_0, G'' = G_{\alpha(*)}$
  - (β) $x_\alpha \in G_{\alpha+1} \backslash G_\alpha$
  - (γ) $G_{\alpha+1}$ is the $\bar{t}$-completion of the pure closure of $G_\alpha \oplus \mathbb{Z}x_\alpha$ inside $G_{\alpha+1}$

(δ) for α limit, G_α is the $\bar{t}$-completion of $\cup\{G_\beta : \beta < \alpha\}$ inside G'', note that if $\mathrm{cf}(\alpha) > \aleph_0$ then the union is $\bar{t}$-complete.

(d) if $\lambda = \lambda^{\aleph_0}$ then for each $G \in K^{\mathrm{crtf}}_{\bar{t},\leq\lambda}$, we can find $\langle (G_i, x_j) : i \leq \lambda^{\aleph_0}, j < \lambda^{\aleph_0}\rangle$ such that:

(α) $G_0 = G, G_i$ is $\leq_{\mathrm{pr}}$-increasing continuous,

(β) $x_i \in G_{i+1} \in K^{\mathrm{crtf}}_{\bar{t},\lambda}$

(γ) letting G'_i be the pure closure of $G + \mathbb{Z}x_i$ inside $G_* = \cup\{G_j : j < \lambda^{\aleph_0}\}$, we have $G_{i+1} = G_i \bigoplus_G G'_i$

(δ) if $G \leq_{\mathrm{pr}} G', x \in G' \in K^{\mathrm{crtf}}_{\bar{t},\lambda}$ and G' is the $\bar{t}$-completion of the pure closure of $G + \mathbb{Z}x$ inside G' <u>then</u> we can find $i < \lambda^{\aleph_0}$ and a pure embedding h of G' into $G_{i+1}, h \restriction G =$ the identity, $h(x) = x_i$ (so $h''(G_i) \leq_{\mathrm{pr}} G$), in fact, h is onto G'_i.

(e) if λ, G are as in clause (d) then we can find $G_* = \cup\{G_i : i < \lambda^{\aleph_0}\}$ such that

(α) $G \leq_{\mathrm{pr}} G_* \in K^{\mathrm{rtf}}_{\lambda^{\aleph_0}}$

(β) if $G \leq_{\mathrm{pr}} G' \in K^{\mathrm{rtf}}_{\lambda^{\aleph_0}}$ then G' can be purely embedded into G_* over G

(γ)
- $\langle G_i : i < \lambda^{\aleph_0}\rangle$ is a $\leq_{\mathrm{pr}}$-increasing continuous sequence of members of $K^{\mathrm{rtf}}_{\lambda^{\aleph_0}}$
- $G_0 = G$
- for limit $\delta < \lambda^{\aleph_0}, G_{\delta+1}$ is the $\bar{t}$-completion of G_δ
- for non-limit $\alpha < \lambda^{\aleph_0}$, the pair $(G_\alpha, G_{\alpha+1})$ is like (G, G_*) in clause (d)

(f) if for $i = 1, 2, G_\ell \in K^{\mathrm{ctrf}}_{\bar{t},\lambda}$ and $\langle G^\ell_i : i < \lambda^{\aleph_0}\rangle, G^\ell_*$ are as in clause (d) or as in clause (e) and π is an isomorphism from G_1 onto G_2 <u>then</u> there is an isomorphism π^+ from G^1_* onto G^2_* extending π

(g) if $\lambda = \Sigma\{\lambda_n : n < \omega\}, \lambda_n = \lambda_n^{\aleph_0} < \lambda_{n+1}$ and $G \in K^{\mathrm{rtf}}_{\leq\lambda}$ then we can find G', G'_n such that

(α) $G \leq_{\mathrm{pr}} G' \in K^{\mathrm{rtf}}_\lambda$

(β) $G'_n \in K^{\mathrm{crtf}}_{\lambda_n}$

(γ) $G'_n \leq_{\mathrm{pr}} G'_{n+1}$; moreover there is a sequence $\langle G'_{n,i} : i < \lambda_n^{\aleph_0}\rangle$ as in (e) for G'_n such that $G'_{n+1} = \cup\{G'_{n,i} : i < \lambda_n^{\aleph_0}\}$

(δ) $G' = \cup\{G'_n : n < \omega\}$.

(h) give λ, λ_n as in (g), if G', G'' are as G' is in (g) <u>then</u> G', G'' are isomorphic

(i) moreover if λ, λ_n are as in clause (g) and $H \in K^{\mathrm{rtf}}_{\leq\lambda}$ <u>then</u> H can be purely embedded into G'; (and if $H \supseteq G$ then even embedded over G).

The results now follow. $\square_{1.2}$

In 1.7(2) below we prove that there is no universal in $\lambda = \lambda^{\aleph_0}$, using [Sheb, Th.1.1], for the reader's convenience we quote the special case used.

Fact 1.5. For any λ and X, a set of cardinality $\leq \lambda$ or just $\leq \lambda^{\aleph_0}$ <u>then</u> we can find a sequence $\bar{f} = \langle f_\eta : \eta \in {}^\omega\lambda\rangle$ such that:

(a) f_η is a function from $\{\eta \restriction n : n < \omega\}$ into X

(b) if f is a function from ${}^{\omega >}\lambda$ to X <u>then</u> for some $\eta \in {}^\omega\lambda$ we have $f_\eta \subseteq f$.

Remark 1.6. 1) Concerning 1.5, see [Sheb, 1.5].
2) We use 1.5 mainly for $\lambda = \lambda^{\aleph_0}$.

Claim 1.7. *Assume $\bar t \in \mathfrak{T}$ is not full.*
1) $(K^{\rm rtf}_{\bar t}, \le_{\rm pr})$ fails amalgamation.
2) If $\lambda = \lambda^{\aleph_0}$ <u>then</u> in $(K^{\rm rtf}_{\bar t,\lambda}, \le_{\rm pr})$ there is no universal member, even for the $\aleph_1$-free ones.

Remark 1.8. Note that 1.2, 1.7(2) are not contradictory as the former deals with full $\bar t$ and the latter wtih non-full ones.

Proof. Let p be a prime witnessing $\bar t$ is not full, i.e. n_* is well defined where $n_* = \min\{n : p$ divide no t_m with $m \ge n\}$, by 1.4(5) without loss of generality $n_* = 0$.

Let $t_{<n} := \prod\limits_{\ell<n} t_\ell$ so $t_{<0} = 1$.

We now choose a^1_n, a^0_n by induction on n such that

$(*)_1$ (a) $a^1_n, a^0_n \in \mathbb{Z}$
(b) $a^1_n = a^0_n \mod t_{\le n}$
(c) $a^\ell_n = a^\ell_m \mod pt_{\le n}$ if $n = m+1$
(d) $a^1_n \ne a^0_n \mod p$ if $n = 0$.

[Why we can choose? For $n = 0$ clearly $t_{<0} = 1$ hence $a^1_n = 1, a^0_n = t_0$ are as required.

For $n = m+1$ let $a^0_n = a^0_m, a^1_n = a^1_m - pt_{<m}b_n$ for b_n chosen below. So clause $(*)_1(c)$ holds for $\ell = 0$ trivially and for $\ell = 1$ obviously. Also clause $(*)_1(b)$ means $(a^1_m - a^0_n) = pt_{<n}b_n, \mod t_{\le n}$. By the induction hypothesis $b'_m = (a^1_m - a^0_n)/t_{<n} \in \mathbb{Z}$ so the $(*)_1(b)$ means $b'_n = pt_n \mod t_n$; as $p | t_n$ there is such b_m.

Lastly, $(*)_7(d)$ holds obviously by $(*)_2(b)$ and $(*)_1(d)$ for $n = 0$.]

Choose

$(*)_2$ (a) t'_n is pt_n if $n = 0$ and is t_n if $n > 0$
(b) $t'_{<n} = \prod\limits_{k<n} t'_k$ and $t'_{\le n} = t'_{<(n+1)}$
(c) $c^\ell_n \in \mathbb{Z}$ are chosen such that $\sum\limits_{m \le n} t'_{<m} c^\ell_m = a^\ell_n$.

[Why we can choose? Just choose c^ℓ_n by induction on n.

For $n = 0$ let $c^\ell_n = a^\ell_n$.

For $n = m+1$ let $c^\ell_n = (a^\ell_n - a^\ell_m)/t'_{\le m}$ which belongs to $\mathbb{Z}$ by $(*)_1(c)$, now check the equation
$\sum\limits_{i \le n} t'_{<i} c^\ell_i = (\sum\limits_{i<n} t'_i c^\ell_i) + t'_{<n} c^\ell_n = a^\ell_m + t'_{\le m} c^\ell_n$ which by the choice above is equal to a a^ℓ_n as required.]

For every $S \subseteq {}^\omega\lambda$ we let G_S the Abelian group generated by $\{x_\eta : \eta \in {}^{\omega >}\lambda\} \cup \{y_{\eta,n} : \eta \in {}^\omega\lambda$ and $n < \omega\}$ freely except the equations:

$(*)_3$ $t'_n y_{\eta,n+1} = y_{\eta,n} - c^\ell_n x_{\langle\rangle} - x_{\eta\restriction n}$ <u>if</u> $n < \omega$ and $\eta \in S \Rightarrow \ell = 1$ and $\eta \notin S \Rightarrow \ell = 0$.

Let

$(*)_4$ for $n \in {}^\omega\lambda$ let

(a) $G_\eta = \Sigma\{\mathbb{Z}x_{\eta\restriction n} : n < \omega\} \subseteq G_S$

(b) $G_{S,\eta} = \Sigma\{\mathbb{Z}x_{\eta\restriction n} : n < \omega\} + \Sigma\{\mathbb{Z}y_{\eta,n} : n < \omega\} \subseteq G_S$.

Easily

$(*)_5$ if $S \subseteq {}^\omega\lambda$ then

(a) $G_S \in K^{\rm rtf}_{\bar{t},\lambda^{\aleph_0}}$ is $\aleph_1$-free

(b) $\eta \in {}^\omega\lambda \Rightarrow G_\eta \le_{\rm pr} G_{S,n} \le_{\rm pr} G_S$.

Now

$\boxplus$ if $S_0, S_1 \subseteq {}^\omega\lambda, \eta \in S_1 \backslash S_0$ <u>then</u> G_{S_0}, G_{S_1} and even $G_{S_0,\eta}, G_{S_1,\eta}$ cannot be $\le_{\rm pr}$-amalgamated over G_η.

Why? Toward contradiction assume $G_\eta \le_{\rm pr} H \in K^{\rm rtf}_{\bar{t}}$ and π_ℓ is a pure embedding of G_{S_ℓ} into H over G_η, for $\ell = 0, 1$.

Let $z_n = \pi_1(y_{\eta,n}) - \pi_0(y_{\eta,n})$ for any n and let $\pi = \pi_0{\restriction}G_\eta = \pi_1{\restriction}G_n$.

For any n clearly for $\ell = 1, 2$

$\bullet_1$ $G_\ell \models t'_{\le n}y'_{\eta,n+1} = y_{\eta,0} - (\sum\limits_{m\le n} t'_{<m}c^\ell_m)x_{\langle\rangle} + \sum\limits_{m\le n} t'_{<m}x_{\eta\restriction m}$.

So applying π_ℓ on the equation recalling $(*)_2(c)$ we have

$\bullet_2$ $H \models \pi_\ell(t'_{\le n}y_{\eta,n+1}) = \pi_\ell(y_{\eta,0}) - a^\ell_n\pi(x_{\langle\rangle}) - \sum\limits_{m\le n} t'_{<m}\pi(x_{\eta\restriction n})$.

Subtracting the equation in $\bullet_2$ for $\ell = 0, 1$ recalling the choice of z_0, z_n

$\bullet_3$ $H \models t'_{\le n}z_{n+1} = z_0 - (a^1_n - a^0_n)\pi(x_{\langle\rangle})$.

But $t'_{\le n}$ and $a^1_n - a^0_n$ are divisible by $t_{\le n}$ in $\mathbb{Z}$ (by $(*)_2(a),(b)$ and $(*)_1(c)$ respectively) hence

$\bullet_4$ z_0 is divisible by $t_{\le n}$ in H.

As this holds for every n and $H \in K^{\rm rtf}_{\bar{t}}$ we get

$\bullet_5$ $z_0 = 0_H$.

So $H \models t'_{\le n}z_{n+1} = z_0 - (a^1_n - a^0_n)\pi(x_{\langle\rangle})$ and for $n = 1$ we get $H \models t'_0z_1 = z_0 - (a^1_0 - a^0_0)\pi(x_{\langle\rangle})$, but in $\mathbb{Z}$ we have $p|t'_0$ and $p\dagger(a^1_0 - a^1_1)$ and $z_0 = 0$ so p divides $x_{\langle\rangle}$ in H, contradiction to purity.

This is enough for part (1), for part (2) we apply the simple black box of [Sheb, Th.1.1], i.e. 1.5. In details assume $G_* \in K^{\rm rtf}_\lambda$ and let $\bar{f} = \langle f_\eta : \eta \in {}^\omega\lambda\rangle$ be as in Fact 1.5 for $X = G_*$.

Define S as the set of $\eta \in {}^\omega\lambda$ such that:

- there is no pure embedding g_* of $G_{\{\eta\},\eta}$ into G_* such that $n < \omega \Rightarrow g_*(x_{\eta\restriction m}) = f_\eta(\eta{\restriction}n)$.

Now $G_S \in K^{\rm rtf}_{\bar t,\lambda}$ so it is enough to prove that G_S is not purely embeddable into G_*. Toward contradiction assume g is a pure embedding of G_S into G_* and let $f : {}^{\omega >}\lambda \to X = G_*$ be $f(\eta) = g(x_\eta)$. By the choice of $\bar f$ there is $\eta \in {}^\omega\lambda$ such that $f_\eta \subseteq f$. If $\eta \in S$ then $g{\restriction}G_{S,\eta} = g{\restriction}G_{\{\eta\},\eta}$ witness that $\eta \notin S$ by the definition of S.

So necessarily $\eta \in {}^\omega\lambda \backslash S$, hence there is g_* as forbidden in the definition of S. Let $g_0 = g{\restriction}G_{S,\eta}$. Easily this contradicts $\boxplus$. $\square_{1.7}$

Remark 1.9. 1) See more in [She87b, Ch.II,§3] = [Shed].
2) This holds also for $K^{{\rm rs}(p)}_\lambda$ the class of reduced separable Abelian p-groups see 1.15.

We may wonder: what if we ask about $(K^{\rm rtf}_{\bar t,\lambda}, \le)$, i.e. the embedding is not necessarily pure.

Claim 1.10. *Assume $\bar t \in \mathfrak{T}$ is weakly full so for some n_* we have: if a prime p divises some $t_n, n \ge n_*$ then it divides infinitely many t_n's, call this set of primes* $\mathbf{P}$.
1) If $\lambda = \lambda^{\aleph_0}$ then $(K^{\rm rtf}_{\bar t,\lambda}, \le)$ has a universal member.
2) If $\lambda = \sum\limits_n \lambda_n, \lambda_n = (\lambda_n)^{\aleph_0}$ for every n <u>then</u> $(K^{\rm rtf}_{\bar t,\lambda}, \le)$ has a universal member.
3) Let R be the subring of $\mathbb{Q}$ generated by $\{1\} \cup \{1/p : p$ a prime $\notin \mathbf{P}\}$. Then for every $G \in K^{\rm rtf}_{\bar t,\lambda}$ there is $H \in K^{\rm trf}_{\bar t,\lambda}$ extending G which is p-divisible for every prime $p \notin \mathbf{P}$. Hence H can be considered an R-module.
4) For the class of R-modules into which $\mathbb{Q}_R$ cannot be embedded the results of 1),2) holds replacing $\aleph_0$ by $|R| + \aleph_0$ <u>when</u> R is an integral domain which is not a field, $\mathbb{Q}_R$, its ring of quotients.

Proof. 1),2) By 4) and 3).
3) Easy.
4) The proof is like the proof for full $\bar t$. $\square_{1.10}$

Still leaves some $\bar t$'s open.

Claim 1.11. *Assume $\bar t \in \mathfrak{T}$ is not weakly full hence $\mathbf{P} := \{p : p$ a prime dividing some t_n's but only finitely many t_n's$\}$ is infinite (this is the negation of the conditions from 1.10).*
If $\lambda = \lambda^{\aleph_0}$ then $(K^{\rm rtf}_{\bar t,\lambda}, \le)$ has no universal member.

Proof. By 1.4(5) without loss of generality

$(*)_1$ (a) there are distinct primes p_n such that: $p_k \big| t_n$ <u>iff</u> $k = n$
(b) $(p_k)^{\ell(k)}$ divide t_k but $(p_k)^{\ell(k)+1}$ does not, so $\ell(k) \ge 1$.

Let $t_{<n} = \prod\limits_{\ell<n} t_\ell$, so $t_{<0} = 1$ and let $t'_n = t_n p_n^{\ell(n)}$ and $t'_{<n} = \prod\limits_{\ell<n} t'_\ell$ and $t''_n = p_n^{\ell(n)}$ and $t''_{<n} = \prod\limits_{\ell<n} t''_\ell$; let $(t_{\le n}, t'_{\le n}, t''_{\le n}) = (t_{<(n+1)}, t'_{<(n+1)}, t''_{<(n+1)})$.

We now choose $a^1_n, a^0_n \in \mathbb{Z}$ by induction on n such that:

$(*)_2$ (a) $a^1_n, a^0_n \in \mathbb{Z}$
(b) $a^1_n = a^0_n \mod t'_{<m}$
(c) $a^\ell_n = a^\ell_m \mod t'_{\le m}$
(d) if $k < n$ then $a^1_n \ne a^0_n \mod (p_k)^{\ell(k)+1}$.

[Why possible? First, for $n=0$ let $(a^1_n, a^0_n) = (p_0, t_0, t_0)$ so $a^1_n - a^0_n$ is divisible by t_0 but not by $p_n^{\ell(n)+1}$. Second, assume $n = m+1$ and (a^1_m, a^0_m) have been chosen. As $t_{\le m}(t_{\le n}$ and $k \le m \Rightarrow p_k \pm (t_{\le n}/t_{\le m})$ we can find (b^1_m, b^0_m) such that $b^\ell_m = a^\ell_m$ mod $t^*_{\le m}$ for $\ell = 0,1$ and $b^1_m = b^0_m$ mod $t^*_{\le n}$. Clearly requirement (a),(b),(c) holds and (d) for $k < m$. Let $(a^1_n, a^0_n) = (a^1_n + t^*_{\le m} \cdot t_n, a^0_n)$, now check.]

$(*)_3$ choose c^1_n, c^0_n by induction on n such that for $\ell = 0,1$ we have $\sum\limits_{m \le n} t'_{<m} c^\ell_n = a^\ell_n$.

[Why possible? For $n=0$ trivial for $n+1$ note the c^ℓ_{n+1} appear with coefficient 1.]

Next for every $S \subseteq {}^{\omega>}\lambda$ we choose an Abelian group G_S, it is generated by $\{x_\eta : \eta \in {}^{\omega>}\lambda\} \cup \{y_{\eta,n} : \eta \in {}^\omega\lambda \text{ and } n < \omega\} \cup \{x^*_n : n < \omega\}$ freely except the equations

$(*)_4$ (a) $(t_n/p_n^{\ell(n)})x^*_{n+1} = x^*_n$ and $x_{\langle\rangle} = x^*_0$

(b) $t'_n y_{\eta,n+1} = y_{\eta,n} - c^\ell_n x_{\langle\rangle} + x_{\eta\restriction n}$ when $n < \omega, \ell < 2$ and $\ell = 1 \Leftrightarrow \eta \in S$.

Also

$(*)_5$ (a) for $\eta \in {}^\omega\lambda$ let $G_\eta = \sum\limits_n \mathbb{Z}x_{\eta\restriction n} + \sum\limits_n \mathbb{Z}x^*_n$

(b) for $S \subseteq {}^\omega\lambda, \eta \in {}^\omega\lambda$ let $G_{S,\eta}$ be the following subgroup of $G_S : G_\eta + \Sigma\mathbb{Z}y_{\eta\restriction n}$.

Easily

$(*)_6$ (a) if $S \subseteq {}^\omega\lambda$ and $\eta \in {}^\omega\lambda$ then $G_\eta, G_S, G_{S,n} \in K^{\mathrm{rtf}}_{\bar t,\lambda^{\aleph_0}}$

(b) $G_\eta \le_{\mathrm{pr}} G_{S,\eta} \le_{\mathrm{pr}} G_S$.

[Why? Note that $\bigcup\limits_n \mathbb{Z}_n x^*_n \in K^{\mathrm{rtf}}_{\bar t}$ because for every $n, x^*_0 = x_{\langle\rangle}$ is not divisible by p_n.]

Now

$\boxplus$ if $S_0, S_1 \subseteq {}^\omega\lambda$ and $\eta \in S_1 \backslash S_0$ <u>then</u> G_{S_0}, G_{S_1} and even $G_{S_0,\eta}, G_{S_1,\eta}$ cannot be amalgamated over G_η in $(K^{\mathrm{rtf}}_{\bar t_*}, \le)$.

We continue as in the proof of 1.7, getting $\pi_1, \pi_0, \pi, \eta, z_n$ and proving that for every n

- $H \models t'_{\le n} z_{n+1} = z_0 - (a^1_n - a^0_n)\pi(x_{\langle\rangle})$.

But $t'_{\le n}$ is divisible by $t_{\le n}$ and $(a^1_n - a^0_n)$ is divisible by $t''_{\le n}$ (in $\mathbb{Z}$) and $x_{\langle\rangle}$ is divisible by $t_{\le n}/t''_{\le n}$ hence $(a^1_n - a^0_n)x_{\langle\rangle}$ is divisible by $t_{\le n}$, hence $(a^1_n - a^0_n)\pi(x_{\langle\rangle})$ is divisible by $t_{\le n}$. So by the equation above $z_0 \in t_{<n}H$ for every n. As $H \in K^{\mathrm{rtf}}_{\bar t}$ it follows that $z_0 = 0$.

Hence for every n

- $H \models (a^1_n - a^0_n)\pi(x_{\langle\rangle}) = -t'_{\le n} z_n$.

Now $p_n^{\ell(n)+\ell(n)}$ divide $t'_{\le n}$ and $p_n^{\ell(n)+1}$ does not divide $(a^1_{n+1} - a^0_{n+1})$ by $(*)_2(d)$ hence in $H, p_n^{\ell(n)}$ divide $\pi(x_{\langle\rangle})$. As also each $t'_{\le n}/\prod\limits_{k\le n} p_k^{\ell(k)}$ divide it, clearly $\pi(x_{\langle\rangle})$ contradict $G_* \in K^{\rm rtf}_{\bar t,\lambda}$. $\square_{1.11}$

We may wonder whether the existence result of 1.2 holds for a stronger embeddability notion. A natural candidate is

Definition 1.12. Let $G_0 \le_{\bar t} G_1$ if: G_0, G_1 are Abelian groups on which $\|-\|_{\bar t}$ is a norm, $G_0 \le_{\rm pr} G_1$ and G_0 is a $d_{\bar t}$-closed subset of G_1 (but G_ℓ is not necessarily $\bar t$-complete!).

Observation 1.13. *1) $(K^{\rm rtf}_{\bar t}, \le_{\bar t})$ is an a.e.c. except smoothness with LST number $2^{\aleph_0}$.*
2) If $A \subseteq G \in K^{\rm rtf}_{\bar t}$ then for some $G' \le_{\bar t} G, A \subseteq G', |G'| = (|A| + \aleph_0)^{\aleph_0}$.
3) If $G_1 \le_{\bar t} G_2$ then $G_1 \le_{\rm pr} G_2$.

We prove below that for μ strong limit of cofinality $\aleph_0$ the answer is positive, i.e. there is a universal member for $(K^{\rm rtf}_{\bar t,\lambda}, \le_{\bar t})$, but for cardinals like $\beth^+_\omega < (\beth_\omega)^{\aleph_0}$ the question on the existence of universals remain open.

Fact 1.14. Assume λ is strong limit and $\aleph_0 = {\rm cf}(\lambda) < \lambda$.
1) There is a universal member in $(K^{\rm rtf}_{\bar t,\lambda}, <_{\bar t})$ where $\bar t = \langle t_\ell : \ell < \omega\rangle \in \mathfrak{T}$, hence also in $(K^{\rm rtf}_{\bar t,\lambda}, \le_{\rm pr})$ and $(K^{\rm rtf}_{\bar t,\lambda}, \subseteq)$.
2) For a prime number p, similarly for $(K^{{\rm rs}(p)}_\lambda, \le_{\langle p:\ell<\omega\rangle})$, see Definition 1.15 below.

Definition 1.15. For a prime number p, and cardinal λ we let $K^{{\rm rs}(p)}_\lambda$ be the class of Abelian p-groups which are reduced and separable of cardinality λ.

Proof. Let K be the class and $\le_*$ the partial order. Let $\lambda_n < \lambda_{n+1} < \lambda = \sum\limits_n \lambda_n$ and $2^{\lambda_n} < \lambda_{n+1}$. The idea in both cases is to analyze $M \in K_\lambda$ as the union of increasing chain $\langle M_n : n < \omega\rangle, M_n \prec_{\mathbb{L}_{\lambda^+_n,\lambda^+_n}} M, \|M_n\| = 2^{\lambda_n}$.

Specifically, we shall apply 1.16, 1.18 below with:

$$\mathfrak{K} = K^{\rm rtf}$$

$$\mu_n = (2^{\lambda_n})^+$$

$$\le_1 \text{ is} : M_1 \le_1 M_2 \text{ \underline{iff} } (M_1, M_2 \in K \text{ and}) \ M_1 \le_* M_2$$

$$\le_2 \text{ is} : M_1 \le_2 M_2 \text{ \underline{iff} } \begin{array}{l} M_1 \le_1 M_2 \text{ and} \\ M_1 \prec_{\mathbb{L}_{\aleph_1,\aleph_1}} M_2, \text{ or just} : \\ \text{if } G_1 \subseteq M_1, G_1 \subseteq G_2 \subseteq M_2, \\ \text{and } G_2 \text{ is countable then there is an} \\ \le_1\text{-embedding } h \text{ of } G_2 \text{ into } M_1 \text{ over } G_1. \end{array}$$

We should check the conditions in 1.16 which we postpone.
We shall finish the proof after 1.18 below. $\square_{1.14}$

Claim 1.16. *Assume*

(*a*) *K is a class of models of a fixed vocabulary closed under isomorphism, $K_\lambda \neq \emptyset$*

(*b*) $\lambda = \sum\limits_{n<\omega} \mu_n, \mu_n < \mu_{n+1}, 2^{\mu_n} < \mu_{n+1}, \mu_n$ *is regular and the vocabulary of K has cardinality $< \mu_0$.*

(*c*) *$\leq_1$ is a partial order on K, (so $M \leq_1 M$) preserved under isomorphisms, and if $\langle M_i : i < \delta\rangle$ is $\leq_1$-increasing and continuous then $M_\delta = \bigcup\limits_{i<\delta} M_i \in K$ and $i < \delta \Rightarrow M_i \leq_1 M_\delta$ (so $(K, \leq_1)$ satisfies a quite weak version of a.e.c. see* [Shea] = [She87a]*)*

(*d*)

  (*α*) *$\leq_2$ is a two-place relation on K, preserved under isomorphisms*

  (*β*) *[weak LST] if $M \in K_\lambda$ then we can find $\langle M_n : n < \omega\rangle$ such that: $M_n \in K_{<\mu_n}, M_n <_2 M_{n+1}$ and $M = \bigcup\limits_{M<\omega} M_n$*

(e) *[non-symmetric amalgamation] if $M_0 \in K_{<\mu_n}, M_0 \leq_1 M_1 \in K_{<\mu_{n+2}}, N^1 \leq_2 N^2 \in K_{<\mu_{n+1}}, h^1$ an isomorphism from M_0 onto N^1, <u>then</u> we can find $M_2 \in K_{<\mu_{(n+2)}}$ such that $M_1 \leq_1 M_2$ and there is an embedding h^2 of N^2 into M extending h^1 satisfying $h(N^2) \leq_1 M_2$.*

<u>Then</u> we can find $\langle M^\alpha_n : n \leq \omega\rangle$ for $\alpha < 2^{<\mu_0}$ such that:

(*α*) $M^\alpha_n \in K_{<\mu_n}, M^\alpha_n \leq_1 M^\alpha_{n+1}, M^\alpha_\omega = \bigcup\limits_{n<\omega} M^\alpha_n$

(*β*) *if $M \in K_\lambda$ and the sequence $\langle M_n : n < \omega\rangle$ is as in clause (d)(β) <u>then</u>[1] for some $\alpha < 2^{<\mu_0}$ we can find an embedding h of M into M^α_ω satisfying $h(M_n) \leq_1 M^\alpha_{n+2}$ (if $\mathfrak{K} = (K, \leq_1)$ is an a.e.c. we get that h is a $\leq_{\mathfrak{K}}$-embedding of M into M^α_ω).*

Proof. Let

$$K'_0 = \big\{M : \begin{array}{l} M \in K \text{ has universe an ordinal} \\ < \mu_0, \text{ and there is } \langle M_n : n < \omega\rangle \text{ as in clause } (d)(\beta) \\ \text{with } M_0 \cong M\big\}. \end{array}$$

Clearly K'_0 has cardinality $\leq 2^{<\mu_0}$, and let us list it as $\langle M^\alpha_0 : \alpha < \alpha^*\rangle$ with $\alpha^* \leq 2^{<\mu_0}$. We now choose, for each $\alpha < \alpha^*$, by induction on $n < \omega$, M^α_n such that:

(*i*) $M^\alpha_n \in \mathfrak{K}$ has universe an ordinal $< \mu_n$

(*ii*) $M^\alpha_n \leq_1 M^\alpha_{n+1}$

(*iii*) if $N^1 \leq_2 N^2, N^1 \in K_{<\mu_n}, N^2 \in K_{<\mu_{n+1}}, h^1$ is an embedding of N^1 into M^α_{n+1} satisfying $h^1(N^1) \leq_1 M^\alpha_{n+1}$ <u>then</u> we can find h^2, an embedding of N^2 into M^α_{n+2} extending h^1 such that $h^2(N^2) \leq_1 M^\alpha_{n+2}$.

For $n = 0, 1$ we do not have much to do. (If $n = 0$ use M^α_0; if $n = 1$ let $\langle M_n : n < \omega\rangle$ be as in clause (c), $M_0 \cong M^\alpha_0$ and use M^α_1 such that $(M_1, M_0) \cong (M^\alpha_1, M^\alpha_0)$). Assume M^α_{n+1} has been defined, and we shall define M^α_{n+2}, let $\{(h^1_{n,\zeta}, N^1_{n,\zeta}, N^2_{n,\zeta}) : \zeta < \zeta^*_n\}$ list the cases of clause (iii) that need to be taken care of, with the set of elements of $N^2_{n,\zeta}$ being an ordinal. Without loss of generality $\zeta^*_n \leq 2^{<\mu_{n+1}}$ by

[1]Of course, we can omit $\langle M^\alpha_n : n \leq \omega\rangle$ when $\|M^\alpha_\omega\| < \lambda$.

cardinality consideration. We shall choose $\langle N_{n+1,\zeta} : \zeta \le \zeta^*_n\rangle$ which is $\le_1$-increasing continuous satisfying $N_{n+1,\zeta} \in K_{<\mu_{n+2}}$. We choose $N_{n+1,\zeta}$ by induction on ζ. Let $N_{n+1,0} = M^\alpha_{n+1}$, for ζ limit let $N_{n+1,\zeta} = \bigcup\limits_{\xi<\zeta} N_{n+1,\xi}$ and use clause (c) of the assumption.

Lastly, for $\zeta = \xi+1$ use clause (e) of the assumption with $h^1_{n,\zeta}(N^1_{n,\xi}), N_{n+1,\xi}, N^1_{n,\xi}, N^2_{n,\xi}$, $h^1_{n,\xi}, N_{n+1,\xi+1}$ here standing for $M_0, M_1, N^1, N^2, h^1, h^2, M_2$ there.

Having carried the induction on $\zeta \le \zeta^*_n$ we let $M^\alpha_{n+2} = N_{n+1,\zeta^*_\alpha}$; so we have carried the induction on n.

Having chosen the $\langle\langle M^\alpha_n : n < \omega\rangle : \alpha < 2^{<\mu_0}\rangle$ let $M^\alpha_\omega = \cup\{M^\alpha_n : n < \omega\}$ hence by clause (c) of the assumption, $M^\alpha_\omega \in K$ and $n < \omega \Rightarrow M^\alpha_n \le_1 M^\alpha_\omega$. Clearly clause ($\alpha$) of the desired conclusion is satisfied. For clause (β) let $M \in K_\lambda$. By clause (d) of the assumption we can find a sequence $\langle M_n : n < \omega\rangle$ such that $M_n \in K_{<\mu_n}, M_n \le_2 M_{n+1}$ and $M = \cup\{M_n : n < \omega\}$. By the choice of $\langle M^\alpha_0 : \alpha < 2^{<\mu_0}\rangle$ there is $\alpha < 2^{<\mu_0}$ such that $M_0 \cong M^\alpha_0$, and let h_0 be an isomorphism from M_0 onto M^α_0. Now by induction on $n < \omega$ we choose h_n, an embedding of M_n into M^α_{n+1} such that $h_n(M_n) \le_1 M^\alpha_{n+1}$ and $h_n \subseteq h_{n+1}$. For $n = 0$ this has already been done as $h_0(M_0) = M^\alpha_0 \le_1 M^\alpha_1$. For $n + 1$ we use clause (iii).

Lastly, $h = \cup\{h_n : n < \omega\}$ is an embedding of M into M^α_ω as required. $\square_{1.16}$

Remark 1.17. 1) We can choose $\langle M^\alpha_0 : \alpha < \alpha^*\rangle$ just to represent $\mathfrak{K}_{<\mu_0}$, and similarly later (and so ignore the "with the universe being an ordinal").
2) Actually, the family of $\langle M_n : n < \omega\rangle$ as in clause (c) such that M_n has set of elements an ordinal, forms a tree T with ω levels with the n-th level having $\le 2^{<\mu_n}$ members, and we can use some amalgamations of it (so weakening the assumptions on $\le_1$). This gives a variant of 1.16.
3) We can put into the axiomatization the stronger version of (d) from 1.16 proved in the proof of 1.14 so we can weaken (β) of 1.18 below.
4) E.g., in (d) we can add $M_n <_* M$ and so weaken clause (β) of 1.16.

Conclusion 1.18. *1) In 1.16 we can add* $\bigwedge\limits_n \bigwedge\limits_\alpha [M^\alpha_n = M^0_n]$ *provided that:*

- (f) *there is* $M_* \in K_{<\lambda}$ *such that every* $M \in K_{<\mu_0}$ *can be* $\le_1$*-embeddable into* M_*.

2) In 1.16 there is in K_λ *a universal member under* $\le_1$*-embedding if in addition we add to the assumptions of 1.16:*

- $(f)^+$ *as in part (1)*
- (g) *if* $M_n \le_1 M_{n+1}, N_n \le_1 M_n, N_n \le_2 N_{n+1}$ *and* $M_n \in K_{<\mu_{n+2}}$ *and* $N_n \in K_{<\mu_{n+1}}$ *for* $n < \omega$ *then* $\bigcup\limits_{n<\omega} N_n \le_1 \bigcup\limits_{n<\omega} M_n$.

Proof. Easy. $\square_{1.18}$

Remark 1.19. 1) In 1.18(2) we can weaken clause (f) to:

- (f)$'$ there is $M_* \in K_{<\lambda}$ as there.

2) This holds for 1.18(1) as in 1.16 in clause (β) we replace M^α_{n+2} by M^α_{n+k} when $\|M_*\| < \mu_k$.

Continuation of the proof of 1.14

We have to check the demands in 1.18 and 1.16.

The least trivial clause to check is clause (e).

Clause (e): (non-symmetric amalgamation)

Without loss of generality $h_1 =$ the identity, $N^1 \cap M_1 = M_0 = N_0$. Just take the free amalgamation $M = N^1 *_{M_0} M_1$ (in the variety of Abelian groups) and note that naturally $M_1 \le_1 M$.

* * *

Discussion 1.20. 1) Can we in 1.16, 1.18 replace $\mathrm{cf}(\lambda) = \aleph_0$, by $\mathrm{cf}(\lambda) = \theta > \aleph_0$? If increasing union of chains in $K_{<\lambda}$ of length $< \theta$ behaves nicely then yes, with no real problem.

More elaborately

- (i) in 1.16(c), we get $\langle M_\varepsilon : \varepsilon < \theta\rangle$ such that $M_\varepsilon \in K_{<\mu_\varepsilon}, \langle M_\varepsilon : \varepsilon < \theta\rangle$ is $\subseteq$-increasing continuous, $M_\varepsilon <_2 M_{\varepsilon+1}, M = \cup\{M_\varepsilon : \varepsilon < \theta\}$
- (ii) we add: if $\langle M_i : i \le \delta\rangle$ is $\le_1$-increasing continuous, $M_i \in K_{<\lambda}$ and $i < \delta \Rightarrow M_i \le_1 N$ then $M_\delta \le_i N$.

Otherwise we seem to be lost.

2) Suppose $\lambda = \sum\limits_{n<\omega} \lambda_n, \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$, and $\mu < \lambda_0, \lambda < 2^\mu$ (i.e., called Case 6b in [She01, §0]). For $\bar{t} \in \mathfrak{T}$ which is not weakly full, is there a universal member in $(\mathfrak{K}^{\mathrm{rtf}}_{\bar{t},\lambda}, <_{\bar{t}})$?

Assume $\mathbf{V} \models$ "$\mu = \mu^{<\mu}, \mu < \chi$" and $\mathbb{P}$ is the forcing notion of adding χ Cohen subsets to μ (that is $\mathbb{P} = \{f : f$ a partial function from χ to $2, |\mathrm{Dom}(f)| < \mu\}$ ordered by inclusion). So we have in $\mathbf{V}^{\mathbb{P}} : \lambda < \lambda^{\aleph_0}$ and $\mu < \lambda < \chi \Rightarrow$ in $(K^{\mathrm{rtf}}_{\bar{t},\lambda}, \le_{\bar{t}})$ there is no universal member. The proof is easy; so consistently the answer is no.

Maybe continuing [Shec, §2] = [Shear, Ch.III,§2] we can get consistency of the existence.

3) Now if $\lambda = \lambda^{\aleph_0}$ then in $(K^{\aleph_1\text{-free}}_\lambda, \subseteq)$ there is no universal member; see [Sheb] = [Shear, Ch.IV], [She01] because amalgamation fails badly. Putting together those results clearly there are few cardinals which are candidates for consistency of existence. In (2), if there is a regular $\lambda' \in (\mu, \lambda)$ with $\mathrm{cov}(\lambda, \lambda^+, \lambda^+, \lambda') < 2^\mu$ then contradict 1.2.

4) Considering consistency of existence of universal in (2), it is natural to try to combine the independent results in [Sheb] = [Shear, Ch.IV] and [DS04].

§ 2. The class of Groups

We know ([SU06]) that the class of groups has NSOP_4 and SOP_3 (from [She96a, §2]). We shall prove a result on the place of the class of groups in the model theoretic classification. We know that it falls on "the complicated side" for some division: of course is unstable. Now we prove that it has the oak property (see on it [DS06]). This is formally not well defined as the definition there was for complete first order theories. But its meaning (and "no universal" consequences) are clear in a more general context, see below. Amenability is a condition on a theory (or class) which gives sufficient condition for existence of somewhat universal structures and in suitable models of set theory (see [DS04]), the class of groups fail it because by [She16] essentially, it has no universal in λ when $\lambda = \mu^+, \mu = \mu^{<\mu}$, forcing contradiction the results on amenable elementary classes in [DS04].

Definition 2.1. 1) A theory T is said to satisfy the oak property as exhibited by (or just by) a formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ <u>when</u> for any λ, κ there are $\bar{b}_\eta(\eta \in {}^{\kappa >}\lambda)$ and $\bar{c}_\nu(\nu \in {}^{\kappa}\lambda)$ and $\bar{a}_i(i < \kappa)$ in some model $\mathfrak{C}$ of T such that

- (a) $\eta \triangleleft \nu$ and $\nu \in {}^{\kappa}\lambda$ then $\mathfrak{C} \models \varphi[\bar{a}_{\ell g(\eta)}, \bar{b}_\eta, \bar{c}_\nu]$
- (b) if $\eta \in {}^{\kappa >}\lambda$ and $\eta\hat{\ }\langle\alpha\rangle \trianglelefteq \nu_1 \in {}^{\kappa}\lambda$ and $\eta\hat{\ }\langle\beta\rangle \trianglelefteq \nu_2 \in {}^{\kappa}\lambda$, while $\alpha \neq \beta$ and $i > \ell g(\eta)$, <u>then</u> $\neg\exists \bar{y}[\varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_1}) \wedge \varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_2})]$

and in addition φ satisfies

- (c) $\varphi(\bar{x}, \bar{y}_1, \bar{z}) \wedge \varphi(\bar{x}, \bar{y}_2, \bar{z})$ implies $\bar{y}_1 = \bar{y}_2$ in any model of T.

2) A theory T has the Δ-oak property if it is exhibited by some $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \Delta$.

Claim 2.2. *The class of groups has the oak property by some quantifier free formula.*

Remark 2.3. The original proof goes as follows.

Let $w(x, y)$ be a complicated enough word, say of length $k^* = 100$, see demands below.

For cardinals κ, λ let $G = G_{\lambda,\kappa}$ be defined as follows:

Let G be the group generated by $\{x_i : i < \kappa\} \cup \{y_\eta : \eta \in {}^{\kappa >}\lambda\} \cup \{z_\nu : \nu \in {}^{\kappa}\lambda\}$ freely except the set of equations

$$\Gamma = \{y_{\nu \restriction i} = w(z_\nu, x_i) : \nu \in {}^{\kappa}\lambda, i < \kappa\}.$$

Clearly it suffices to show that

- $(*)_1$ if $\nu \in {}^{\kappa}\lambda, i < \kappa$ and $\rho \in {}^{i}\lambda \backslash \{\nu \restriction i\}$ then $G \models$ "$y_\rho \neq w(z_\nu, x_i)$".

Now

- $(*)_2$ each word $y^{-1}_{\nu \restriction i} w(z_\nu, x_i)$ is so-called cyclically reduced, i.e. both $w_1 = y^{-1}_{\nu \restriction i} w(z_v, x_i)$ and $w_2 = w(z_v, x_i) y^{-1}_{\nu \restriction i}$ are reduced, i.e. we do not have a generator and its inverse in adjacent places
- $(**)$ for any two such words or cyclical permutations of them which are not equal, any common segment has length $< k^*/6$.

Explanation and why this is enough see [LS77], no point to elaborate as this is not used.

But we prefer to use the more ad-hoc but accessible proof.

Proof. Proof of 2.2 Let $G = G_0$ be the group generated by

$$Y = \{x_i : i < \kappa\} \cup \{z_\nu : \nu \in {}^\kappa\mu\}$$

freely except (recalling $[xy] = xyx^{-1}y^{-1}$, the commutator) the set of equations $\Gamma_2 = \{[z_\nu, x_i] = [z_\eta, x_i] : i < \kappa, \nu \in {}^\kappa\lambda, \eta \in {}^\kappa\lambda$ satisfy $\nu \restriction i = \eta \restriction i\}$. So for $i < \kappa, \rho \in {}^i\lambda$ we can choose $y_\rho \in G$ such that $\eta \in {}^\kappa\lambda, \eta \restriction i = \rho \Rightarrow y_\rho = [z_\eta, x_i]$. Let G_1 be the group generated by the set Y freely, let h be the homomorphism from G_1 onto G mapping the members of Y to themselves, (using Abelian groups no two members of Y are identified in G_1). Let $N = \text{Kernel}(h)$.

Clearly it suffices to prove

$(*)_1$ in $G = G_1/N$, if $\nu, \eta \in {}^\kappa\lambda$ and $i < \kappa$ then $[z_\nu, x_i] = [z_\eta, x_i] \Leftrightarrow \nu \restriction i = \eta \restriction i$.

The implication $\Leftarrow$ holds trivially. For the other direction let $j < \kappa$ and $\eta, \nu \in {}^\kappa\lambda$ be such that $\eta \restriction j \neq \nu \restriction j$ and we shall prove that $G \models$ "$y_{\eta\restriction j} \neq y_{\nu\restriction j}$".

Let N_1 be the normal subgroup of G_1 generated by

$$(*)_2 \qquad \begin{aligned} X_* = \{x_i : i < \kappa \text{ and } i \neq j\} \quad & \cup\{z_\rho : \rho \in {}^\kappa\lambda \text{ and } \rho \restriction j \notin \{\eta \restriction j, \nu \restriction j\}\} \\ & \cup\{z_\rho z_\eta^{-1} : \rho \in {}^\kappa\lambda \text{ and } \rho \restriction j = \eta \restriction j\} \\ & \cup\{z_\rho z_\nu^{-1} : \rho \in {}^\kappa\lambda \text{ and } \rho \restriction j = \nu \restriction j\}. \end{aligned}$$

Clearly by inspection N_1 includes N. Let $N_0 = h(N_1)$, clearly N_1 is a normal subgroup of G_1 and h induces a homomorphism $\hat{h}$ from G_1/N_1 onto G_0/N_0. Now looking at the members of $X_*, G_1/N_1$ is generated by $\{x_i\} \cup \{z_\eta, z_\nu\}$. Checking the equations in Γ_2 clearly G_1/N_1 is generated by $\{x_i\} \cup \{z_\eta, z_\nu\}$ freely, hence $G_1/N_1 \models$ "$[z_\eta, x_i] \neq [z_\nu, x_i]$" which means $[z_\eta, x_i]^{-1}[z_\nu, x_i] \notin N_1$ hence $\notin N$. So recalling the choice of G in $(*)_1$ we have $G \models$ "$y_{\eta\restriction j} \neq y_{\nu\restriction j}$" as required. $\square_{2.2}$

§ 3. More On the oak property

We can in the "no universal" results in [DS06] deal also with the case of singular cardinals. We also note that the so called weak oak property suffices.

Claim 3.1. *We have* $\mathrm{univ}(\lambda_1, T) \ge \lambda_2$ *when:*

(a) T *is a complete first order theory with the oak property,* $\mathfrak{K} = (\mathrm{Mod}_T, \prec)$
(b) (i) $\kappa = \mathrm{cf}(\mu) \le \sigma < \mu < \lambda = \mathrm{cf}(\lambda) < \lambda_1 \le \lambda_2$
  (ii) $\kappa \le \sigma \le \lambda_1, |T| \le \lambda_2$
  (iii) $\mathrm{cf}([\mu]^\kappa, \subseteq) \ge \lambda_2$
(c) (i) $S \subseteq \lambda$ *is stationary*
  (ii) $\bar{C} = \langle C_\delta : \delta \in S\rangle, C_\delta \subseteq \delta, \mathrm{otp}(C_\delta) = \mu, S \subseteq \lambda$
  (iii) $J =: \{A \subseteq \lambda$*: for some club* E *of* $\lambda, \delta \in S \cap A \Rightarrow C_\delta \nsubseteq E\}$
  (iv) $\lambda \notin J$ *and* $\alpha < \lambda \Rightarrow \lambda > |\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}(C_\delta), \delta \in S\}|$,
(d) $\mathbf{U}_J(\lambda_1) < \lambda_2$
(e) *for some* $\mathscr{P}_1, \mathscr{P}_2$ *we have*
  (i) $\mathscr{P}_1 \subseteq [\lambda_1]^\kappa, \mathscr{P}_2 \subseteq [\sigma]^\kappa$
  (ii) *if* $g : \sigma \to \lambda_1$ *is one to one then for some* $X \in \mathscr{P}_2$*, we have* $\{g(i) : i \in X\} \in \mathscr{P}_1$
  (iii) $|\mathscr{P}_1| < \lambda_2$
  (iv) $|\mathscr{P}_2| \le \lambda_1$.

Remark 3.2. 1) We can in 3.1 replace clause (a) by

(a)′ $\mathfrak{k}$ is an a.e.c. which has the φ-oak property, see Definition 2.1 and $\mathrm{LST}(\mathfrak{k}) \le \lambda_2$.

2) The proof also gives $\mathrm{univ}(\lambda, \lambda_1, T) \ge \lambda_2$.

Recall

Definition 3.3. Assume $T, \lambda, \mu, S, \bar{C}$ are as in Claim 3.1, see (a),(c).
1) For $\bar{N} = \langle N_\gamma : \gamma < \lambda\rangle$ an elementary-increasing continuous sequence of models of T of size $< \lambda$ and for $a, c \in N_\lambda = \bigcup\limits_{\gamma<\lambda} N_\gamma$ and $\delta \in S$, we let $\mathrm{inv}_{\varphi,\bar{N}}(c, C_\delta, a) = \{\zeta < \mu$: there is $b \in N_{\alpha(\delta,\zeta+2)} \setminus N_{\alpha(\delta,\zeta+1)}$ such that $N_\lambda \models \varphi[a, b, c])\}$.
2) For $\delta, \bar{N}$ as above and a set $A \subseteq N_\lambda$, let $\mathrm{inv}^A_{\varphi,\bar{N}}(c, C_\delta) = \bigcup\{\mathrm{inv}_{\varphi,\bar{N}}(c, C_\delta, a) : a \in A\}$.

Proof. Step A: Assume toward a contradiction $\theta =: \mathrm{univ}(\lambda_1, T) < \lambda_2$, so let $\langle N^*_j : j < \theta\rangle$ exemplify this and let $\theta_1 = \theta + |\mathscr{P}_1| + |\mathscr{P}_2| + |T| + \mathbf{U}_J(\lambda_1)$ hence $\theta_1 < \lambda_2$.
Without loss of generality the universe of N^*_j is λ_1.

Step B: By the definition of $\mathbf{U}_J(\lambda_1)$ there is $\mathscr{A}$ such that:

(a) $\mathscr{A} \subseteq [\lambda_1]^\lambda$
(b) $|\mathscr{A}| \le \mathbf{U}_J(\lambda_1)$
(c) if $f : \lambda \to \lambda_1$ then for some $A \in \mathscr{A}$ we have $\{\delta \in S : f(\delta) \in A\} \ne \emptyset \mod J$.

For each $X \in \mathscr{P}_1, j < \theta$ and $A \in \mathscr{A}$ let $M_{j,X,A}$ be an elementary submodel of N^*_j of cardinality λ which includes $X \cup A \subseteq \lambda_1$, and let $\bar{M}_{j,X,A} = \langle M_{j,X,A,\varepsilon} : \varepsilon < \lambda\rangle$ be a filtration of $M_{j,X,A}$.

Lastly, consider

$$\mathscr{B} = \{\mathrm{inv}^X_{\bar{M}_{j,X,A}}(a, C_\delta) : j < \theta, X \in \mathscr{P}_1, A \in \mathscr{A}, \delta \in S \text{ and } a \in M_{j,X,A}\}.$$

Step C: Easily we have $|\mathscr{B}| \leq \theta_1 < \lambda_2$ and $\mathscr{B} \subseteq [\mu]^\kappa$, hence there is $B^* \in [\mu]^\kappa \backslash \mathscr{B}$. Without loss of generality $\mathrm{otp}(B) = \kappa$, each $\alpha \in B$ is a successor ordinal.

[Why? Let $h : \mu \to \mu$ be such that $(\forall \beta < \mu)(\exists^\mu \alpha < \mu)(h(\alpha+1) = \beta)$ and let $\mathscr{B}' = \{\{h(\beta) : \beta \in B\} : B \in \mathscr{B}\}$, so $|\mathscr{B}'| \leq |\mathscr{B}|$ hence we can choose $B' \in [\mu]^\kappa \backslash \mathscr{B}'$. Let $\langle \beta_i : i < \kappa\rangle$ list B' and by induction on $i < \kappa$ choose $\alpha_i < \mu$ which is $> \bigcup\limits_{j<i} \alpha_j$ and satisfies $h(\alpha_i + 1) = \beta_i$. So $\{\alpha_i + 1 : i < \kappa\}$ is as required.]

Let $\langle \alpha^*_i : i < \kappa\rangle$ list B in increasing order. For $\delta \in S$ and $i < \kappa$ let $\alpha_{\delta,i}$ be the α^*_i-th member of C_δ. Now for $\delta \in S$ and $j \leq \kappa$ let $\nu_\delta = \langle \alpha_{\delta,i} : i < \kappa\rangle$.

Now let M^* be a λ^+-saturated model of T, in which $a_i (i < \sigma), b_\eta (\eta \in {}^{\kappa>}(\lambda_2)), c_\nu\ (\nu \in {}^\kappa(\lambda_2)), \varphi$ are as in the definition of the oak property and for each $Y \in \mathscr{P}_2$, choose $\langle N_{Y,\varepsilon} : \varepsilon < \lambda\rangle, \langle \bar{c}_{Y,\varepsilon,\delta} : \delta \in S\rangle$ such that:

- (a) $N_{Y,\varepsilon}$ is increasing continuous with ε
- (b) $N_{Y,\varepsilon}$ has cardinality $< \lambda$ for $\varepsilon < \lambda$
- (c) $\bar{a}_j \in N_{Y,0}$ for $j \in Y$
- (d) $b_{\nu_\delta \restriction (i+1)} \in N_{Y,\nu_\delta(i)+1}$ for $\delta \in S, i < \kappa$
- (e) $\bar{c}_{\nu_\delta} \in N_{Y,\delta+1}$ for $\delta \in S$
- (f) if $i < \kappa, j \in Y, \mathrm{otp}(j \cap Y) = i$ and $\delta \in S$ then $N_{Y,\delta+1} \models \varphi[\bar{a}_j, \bar{b}_{\nu_\delta \restriction (i+1)}, \bar{c}_{\nu_\delta}]$.

As $|\mathscr{P}_2| \leq \lambda_1$ we can choose $N \prec M^*, \|N\| = \lambda_1$ such that $\{a_i : i < \sigma\} \cup \cup\{N_{Y,\varepsilon} : Y \in \mathscr{P}_2, \varepsilon < \lambda\} \subseteq N$.

Step D: By our choice of $\langle N^*_j : j < \theta\rangle$, there is $j(*) < \theta$ and elementary embedding $f : N \to N^*_j$. By an assumption there is $Y \in \mathscr{P}_2$ such that $X := \{f(\bar{a}_i) : i \in Y\} \in \mathscr{P}_1$. Also by the choice of $\mathscr{A}$ there is $A \in \mathscr{A}$ such that $\{\delta \in S : f(\bar{c}_{Y,\delta}) \in A\} \neq \emptyset$ mod J.

Now we can finish (note that we use here again the last clause in the definition of the oak property). $\square_{3.1}$

Definition 3.4. 1) The formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ has the weak oak property in T (a first order complete theory) when: as in Definition 2.1 omitting clause (c), (equivalently in [DS06, 1.8]).
2) A complete first order theory T has the weak oak property when some $\varphi(\bar{x}, \bar{y}, \bar{z})$ has it in T.
3) For non-complete first order property T (or class $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$) we mean φ is quantifier-free.

The weak oak property is sufficient for many results on $\mathrm{univ}(\lambda, T) \geq \lambda_2$ because of

Claim 3.5. *Assume*

- (a) *T has the weak oak property, $|T| \leq \lambda = \mathrm{cf}(\lambda)$*
- (b) *$\bar{C} = \langle C_\delta : \delta \in S\rangle, J$ are as in clause (c) of 3.1*

(c) $\kappa = \mathrm{cf}(\mu) < \sigma < \mu < \lambda = \mathrm{cf}(\lambda)$ *and* $\mathscr{P} \subseteq \{u \subseteq \sigma : \mathrm{otp}(u) = \kappa\}$ *has cardinality* $\leq \lambda$.

Then for each $B^* \subseteq \mu$ *of order type* κ, T *has a model* N^* *of cardinality* λ *and sequence* $\langle a_i : i < \sigma\rangle$ *of members of* N^* *satisfying:*

⊛ *if* N *is a model of* T *of cardinality* λ *with filtration* $\bar{N} = \langle N_\alpha : \alpha < \lambda\rangle$ *and* f *is an elementary embedding of* N^* *into* N *then for every increasing sequence* $\bar{\varepsilon} = \langle \varepsilon(i) : i < \kappa\rangle$ *enumerating in increasing order some* $u \in \mathscr{P}$ *we have*

$$\{\delta \in S : \quad \text{for some } a \in N^* \text{ we have} \\ B^* = \mathrm{inv}^{\{f(a_{\varepsilon(i)}:i<\kappa\}}_{\varphi,\bar{N}}(C_\delta, a) = S \mod J.$$

Proof. Without loss of generality some $\varphi = \varphi(x,y,z)$ witness T has the weak oak property (as we can replace T by such T' with $\mathrm{univ}(\lambda,T) = \mathrm{univ}(\lambda,T')$).

As usual, there is $N^* \models T$ with filtration $\bar{N}^* = \langle N^*_i : i < \lambda\rangle$ and $I \subseteq {}^{\kappa>}\lambda$ of cardinality $\lambda, \langle a_i : i < \kappa\rangle, \langle b_\eta : \eta \in \mathscr{T}\rangle$ and $\nu_\delta \in {}^\kappa(C_\delta) \cap \lim_\kappa(T)$ for $\delta \in S$ and $\langle c_{\nu_\delta} : \delta \in S\rangle$ such that

(a) $\langle a_i : i < \kappa\rangle, \langle b_\eta : \eta \in \mathscr{T}\rangle, \langle c_{\nu_\delta} : \delta \in S\rangle$ are as in the Definition 3.4
(b) $\mathrm{otp}(\nu_\delta(i) \cap C_\delta) =$ (the i-th member of B^*) $+ 1$.

So let $N, \langle N_\varepsilon : \varepsilon < \lambda\rangle, f$ be as in the assumption of ⊛ of the claim. Without loss of generality the universes of N^* and of N are λ.

Let

$$E_* = \{\delta < \lambda : \delta \text{ limit}, f''(\delta) = \delta, |N_\delta| = \delta = |N^*_\delta| \text{ and } (N_\delta, N^*_\delta, f) \prec (N, N^*, f)\}$$

it is a club of λ. For each $i < \sigma$ let

$$W_i = \{\alpha : \quad \text{for some } \delta \in S, \alpha \in C_\delta \subseteq E, \nu_\delta(i) > \alpha, \\ \text{but } \varphi(f(a_i), y, f(c_{\nu_\delta})) \text{ is satisfied (in } N) \\ \text{by some } b \in N_\alpha\}.$$

Now

⊛ W_i is not stationary.

[Why? Otherwise let $\mathfrak{B} \prec (\mathscr{H}(\lambda^+), \in, <^*)$ be such that $\bar{N}, \bar{N}^*, a_i$ (and even $\langle a_j : j < \sigma\rangle$ and $\mathscr{P}$ but not used) and $\langle b_\eta : \eta \in \mathscr{T}\rangle, \langle c_{\nu_\delta} : \delta \in S\rangle$ belong to $\mathfrak{B}$ and $\mathfrak{B} \cap \lambda = \alpha \in W_i$ and assume $b \in \mathfrak{B} \cap \alpha, N \models \varphi(f(a_i), b, f(c_{\nu_\delta}))$. So there is $\delta(*) \in S \cap \delta$ such that $N \models \varphi[f(a_1), b, f(c_{\nu_{\delta(*)}})$. But $\nu_\delta(i) \geq \alpha > \nu_{\delta(*)}(i)$ hence $\varphi(a_i, y, c_{\nu_\delta}), \varphi(a_i, y, c_{\nu_{\delta'}})$ are incompatible (in N^*) hence their images by f are incompatible in N by b satisfies both, a contradiction, so W_i is not stationary.]

So there is a club E^* of λ included in E_* and disjoint to W_i for each $i < \sigma$. So there is $\delta \in S$ such that $C_\delta \subseteq E^*$ and we get contradiction as earlier. $\square_{3.5}$

Question 3.6. Can we combine 3.1, 3.5?

(For many singular λ_1's, certainly yes).

References

[DS04] Mirna Džamonja and Saharon Shelah, *On the existence of universal models*, Arch. Math. Logic **43** (2004), no. 7, 901–936, arXiv: math/9805149. MR 2096141

[DS06] ______, *On properties of theories which preclude the existence of universal models*, Ann. Pure Appl. Logic **139** (2006), no. 1-3, 280–302, arXiv: math/0009078. MR 2206258

[Fuc73] Laszlo Fuchs, *Infinite abelian groups*, vol. I, II, Academic Press, New York, 1970, 1973.

[GS83] Rami P. Grossberg and Saharon Shelah, *On universal locally finite groups*, Israel J. Math. **44** (1983), no. 4, 289–302. MR 710234

[KS92] Menachem Kojman and Saharon Shelah, *Nonexistence of universal orders in many cardinals*, J. Symbolic Logic **57** (1992), no. 3, 875–891, arXiv: math/9209201. MR 1187454

[KS95] ______, *Universal abelian groups*, Israel J. Math. **92** (1995), no. 1-3, 113–124, arXiv: math/9409207. MR 1357747

[KSV15] Juliette Cara Kennedy, Saharon Shelah, and Jouko A. Väänänen, *Regular ultrapowers at regular cardinals*, Notre Dame J. Form. Log. **56** (2015), no. 3, 417–428, arXiv: 1307.6396. MR 3373611

[LS77] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 89, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[Mir05] Džamonja Mirna, *Club guessing and the universal models*, Notre Dame J. Formal Logic **46** (2005), 283–300.

[MS76] Angus J. Macintyre and Saharon Shelah, *Uncountable universal locally finite groups*, J. Algebra **43** (1976), no. 1, 168–175. MR 0439625

[S$^+$a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F1425.

[S$^+$b] ______, *Tba*, In preparation. Preliminary number: Sh:F1436.

[Shea] Saharon Shelah, *Abstract elementary classes near* $\aleph_1$, arXiv: 0705.4137 Ch. I of [Sh:h].

[Sheb] ______, *Black Boxes*, arXiv: 0812.0656 Ch. IV of The Non-Structure Theory" book [Sh:e].

[Shec] ______, *General non-structure theory and constructing from linear orders*, arXiv: 1011.3576 Ch. III of The Non-Structure Theory" book [Sh:e].

[Shed] ______, *Universal Classes: Axiomatic Framework [Sh:h]*, Ch. V (B) of [Sh:i].

[She73] ______, *Notes on combinatorial set theory*, Israel J. Math. **14** (1973), 262–277. MR 0327522

[She87a] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 419–497. MR 1033034

[She87b] ______, *Universal classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 264–418. MR 1033033

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She96a] ______, *Toward classifying unstable theories*, Ann. Pure Appl. Logic **80** (1996), no. 3, 229–255, arXiv: math/9508205. MR 1402297

[She96b] ______, *Universal in* $(<\lambda)$*-stable abelian group*, Math. Japon. **44** (1996), no. 1, 1–9, arXiv: math/9509225. MR 1402794

[She97] ______, *Non-existence of universals for classes like reduced torsion free abelian groups under embeddings which are not necessarily pure*, Advances in algebra and model theory (Essen, 1994; Dresden, 1995), Algebra Logic Appl., vol. 9, Gordon and Breach, Amsterdam, 1997, arXiv: math/9609217, pp. 229–286. MR 1683540

[She00] ______, *Applications of PCF theory*, J. Symbolic Logic **65** (2000), no. 4, 1624–1674, arXiv: math/9804155. MR 1812172

[She01] ______, *Non-existence of universal members in classes of abelian groups*, J. Group Theory **4** (2001), no. 2, 169–191, arXiv: math/9808139. MR 1812323

[She09] ______, *Classification theory for abstract elementary classes*, Studies in Logic (London), vol. 18, College Publications, London, 2009. MR 2643267

[She16] ______, *No universal group in a cardinal*, Forum Math. **28** (2016), no. 3, 573–585, arXiv: 1311.4997. MR 3510831

[Shear] ______, *Non-structure theory*, Oxford University Press, to appear.

[SU06] Saharon Shelah and Alexander Usvyatsov, *Banach spaces and groups—order properties and universal models*, Israel J. Math. **152** (2006), 245–270, arXiv: math/0303325. MR 2214463

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