

A definable nonstandard model of the reals

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Abstract

We prove, in **ZFC**, the existence of a definable, countably saturated elementary extension of the reals.

Introduction

It seems that it has been taken for granted that there is no distinguished, definable nonstandard model of the reals. (This means a countably saturated elementary extension of the reals.) Of course if $\mathbf{V} = \mathbf{L}$ then there is such an extension (just take the first one in the sense of the canonical well-ordering of \mathbf{L}), but we mean the existence provably in **ZFC**. There were good reasons for this: without Choice we cannot prove the existence of *any* elementary extension of the reals containing an infinitely large integer.^{1 2} Still there is one.

Theorem 1 (ZFC). *There exists a definable, countably saturated extension ${}^*\mathbb{R}$ of the reals \mathbb{R} , elementary in the sense of the language containing a symbol for every finitary relation on \mathbb{R} .*

The problem of the existence of a definable proper elementary extension of \mathbb{R} was communicated to one of the authors (Kanovei) by V. A. Uspensky.

A somewhat different, but related problem of *unique existence* of a nonstandard real line ${}^*\mathbb{R}$ has been widely discussed by specialists in nonstandard analysis.³ Keisler notes in [3, § 11] that, for any cardinal κ , either inaccessible or satisfying $2^\kappa = \kappa^+$, there exists unique, up to isomorphism, κ -saturated nonstandard real line ${}^*\mathbb{R}$ of cardinality κ , which means that a reasonable level of uniqueness modulo isomorphism can be

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¹In fact, from any nonstandard integer we can define a non-principal ultrafilter on \mathbb{N} , even a Lebesgue non-measurable set of reals [4], yet it is consistent with **ZF** (even plus Dependent Choices) that there are no such ultrafilters as well as non-measurable subsets of \mathbb{R} [5].

²It is worth to be mentioned that definable nonstandard elementary extensions of \mathbb{N} do exist in **ZF**. For instance, such a model can be obtained in the form of the ultrapower F/U , where F is the set of all arithmetically definable functions $f : \mathbb{N} \rightarrow \mathbb{N}$ while U is a non-principal ultrafilter in the algebra A of all arithmetically definable sets $X \subseteq \mathbb{N}$.

³“What is needed is an underlying set theory which proves the unique existence of the hyperreal number system [...]” (Keisler [3, p. 229]).

achieved, say, under GCH. Theorem 1 provides a countably saturated nonstandard real line ${}^*\mathbb{R}$, unique in absolute sense by virtue of a concrete definable construction in **ZFC**. A certain modification of this example also admits a reasonable model-theoretic characterization up to isomorphism (see Section 4).

The proof of Theorem 1 is a combination of several known arguments. First of all (and this is the key idea), arrange all non-principal ultrafilters over \mathbb{N} in a linear order A , where each ultrafilter appears repetitiously as D_a , $a \in A$. Although A is not a well-ordering, we can apply the iterated ultrapower construction in the sense of [1, 6.5] (which is “a finite support iteration” in the forcing nomenclature), to obtain an ultrafilter D in the algebra of all sets $X \subseteq \mathbb{N}^A$ concentrated on a finite number of axes \mathbb{N} . To define a D -ultrapower of \mathbb{R} , the set F of all functions $f : \mathbb{N}^A \rightarrow \mathbb{R}$, also concentrated on a finite number of axes \mathbb{N} , is considered. The ultrapower F/D is OD, that is, ordinal-definable, actually, definable by an explicit construction in **ZFC**, hence, we obtain an OD proper elementary extension of \mathbb{R} . Iterating the D -ultrapower construction ω_1 times in a more ordinary manner, i. e., with direct limits at limit steps, we obtain a definable countably saturated extension.

To make the exposition self-contained and available for a reader with only fragmentary knowledge of ultrapowers, we reproduce several well-known arguments instead of giving references to manuals.

1 The ultrafilter

As usual, \mathfrak{c} is the cardinality of the continuum.

Ultrafilters on \mathbb{N} hardly admit any definable linear ordering, but maps $a : \mathfrak{c} \rightarrow \mathcal{P}(\mathbb{N})$, whose ranges are ultrafilters, readily do. Let A consist of all maps $a : \mathfrak{c} \rightarrow \mathcal{P}(\mathbb{N})$ such that the set $D_a = \text{ran } a = \{a(\xi) : \xi < \mathfrak{c}\}$ is an ultrafilter on \mathbb{N} . The set A is ordered lexicographically: $a <_{\text{lex}} b$ means that there exists $\xi < \mathfrak{c}$ such that $a \upharpoonright \xi = b \upharpoonright \xi$ and $a(\xi) < b(\xi)$ in the sense of the lexicographical linear order $<$ on $\mathcal{P}(\mathbb{N})$ (in the sense of the identification of any $u \subseteq \mathbb{N}$ with its characteristic function).

For any set u , \mathbb{N}^u denotes the set of all maps $f : u \rightarrow \mathbb{N}$.

Suppose that $u \subseteq v \subseteq A$.

If $X \subseteq \mathbb{N}^v$ then put $X \downarrow u = \{x \upharpoonright u : x \in X\}$.

If $Y \subseteq \mathbb{N}^u$ then put $Y \uparrow v = \{x \in \mathbb{N}^v : x \upharpoonright u \in Y\}$.

We say that a set $X \subseteq \mathbb{N}^A$ is *concentrated* on $u \subseteq A$, if $X = (X \downarrow u) \uparrow A$; in other words, this means the following:

$$\forall x, y \in \mathbb{N}^A (x \upharpoonright u = y \upharpoonright u \implies (x \in X \iff y \in X)). \quad (*)$$

We say that X is a *set of finite support*, if it is concentrated on a finite set $u \subseteq A$. The collection \mathcal{X} of all sets $X \subseteq \mathbb{N}^A$ of finite support is closed under unions, intersections, complements, and differences, i. e., it is an algebra of subsets of \mathbb{N}^A . Note that if $(*)$ holds for finite sets $u, v \subseteq A$ then it also holds for $u \cap v$. (If $x \upharpoonright (u \cap v) = y \upharpoonright (u \cap v)$ then consider $z \in \mathbb{N}^A$ such that $z \upharpoonright u = x \upharpoonright u$ and $z \upharpoonright v = y \upharpoonright v$.) It follows that for any $X \in \mathcal{X}$ there is a least finite $u = ||X|| \subseteq A$ satisfying $(*)$.

In the remainder, if U is any subset of $\mathcal{P}(I)$, where I is a given set, then $U i \Phi(i)$ (*generalized quantifier*) means that the set $\{i \in I : \Phi(i)\}$ belongs to U .

The following definition realizes the idea of a finite iteration of ultrafilters. Suppose that $u = a_1 < \dots < a_n \subseteq A$ is a finite set. We put

$$\begin{aligned} D_u &= \{X \subseteq \mathbb{N}^u : D_{a_n} k_n \dots D_{a_2} k_2 D_{a_1} k_1 (\langle k_1, k_2, \dots, k_n \rangle \in X)\}; \\ D &= \{X \in \mathcal{X} : X \downarrow \|X\| \in D_{\|X\|}\}. \end{aligned}$$

The following is quite clear.

Proposition 2. (i) D_u is an ultrafilter on \mathbb{N}^u ;

(ii) if $u \subseteq v \subseteq A$, v finite, $X \subseteq \mathbb{N}^u$, then $X \in D_u$ iff $X \uparrow v \in D_v$;

(iii) $D \subseteq \mathcal{X}$ is an ultrafilter in the algebra \mathcal{X} ;

(iv) if $X \in \mathcal{X}$, $u \subseteq A$ finite, and $\|X\| \subseteq u$, then $X \in D \iff X \downarrow u \in D_u$. \square

2 The ultrapower

To match the nature of the algebra \mathcal{X} of sets $X \subseteq \mathbb{N}^A$ of finite support, we consider the family F of all $f : \mathbb{N}^A \rightarrow \mathbb{R}$, concentrated on some finite set $u \subseteq A$, in the sense that

$$\forall x, y \in \mathbb{N}^A (x \upharpoonright u = y \upharpoonright u \implies f(x) = f(y)). \quad (\dagger)$$

As above, for any $f \in F$ there exists a least finite $u = \|f\| \subseteq A$ satisfying (\dagger) .

Let \mathcal{R} be the set of all finitary relations on \mathbb{R} . For any n -ary relation $E \in \mathcal{R}$ and any $f_1, \dots, f_n \in F$, define

$$E^D(f_1, \dots, f_n) \iff D x \in \mathbb{N}^A E(f_1(x), \dots, f_n(x)).$$

The set $X = \{x \in \mathbb{N}^A : E(f_1(x), \dots, f_n(x))\}$ is obviously concentrated on $u = \|f_1\| \cup \dots \cup \|f_n\|$, hence, it belongs to \mathcal{X} , and $\|X\| \subseteq u = \|f_1\| \cup \dots \cup \|f_n\|$.

In particular, $f =^D g$ means that $D x \in \mathbb{N}^A (f(x) = g(x))$. The following is clear:

Proposition 3. $=^D$ is an equivalence relation on F , and any relation on F of the form E^D is $=^D$ -invariant. \square

Put $[f]_D = \{g \in F : f =^D g\}$, and ${}^*\mathbb{R} = F/D = \{[f]_D : f \in F\}$. For any n -ary ($n \geq 1$) relation $E \in \mathcal{R}$, let *E be the relation on ${}^*\mathbb{R}$ defined as follows:

$${}^*E([f_1]_D, \dots, [f_n]_D) \text{ iff } E^D(f_1, \dots, f_n) \text{ iff } D x \in \mathbb{N}^A E(f_1(x), \dots, f_n(x)).$$

The independence on the choice of representatives in the classes $[f_i]_D$ follows from Proposition 3. Put ${}^*\mathcal{R} = \{{}^*E : E \in \mathcal{R}\}$. Finally, for any $r \in \mathbb{R}$ we put ${}^*r = [c_r]_D$, where $c_r \in F$ satisfies $c_r(x) = r$, $\forall x$.

Let \mathcal{L} be the first-order language containing a symbol E for any relation $E \in \mathcal{R}$. Then $\langle \mathbb{R}; \mathcal{R} \rangle$ and $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle$ are \mathcal{L} -structures.

Theorem 4. The map $r \mapsto {}^*r$ is an elementary embedding (in the sense of the language \mathcal{L}) of the structure $\langle \mathbb{R}; \mathcal{R} \rangle$ into $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle$.

Proof. This is a routine modification of the ordinary argument. By $\mathcal{L}[F]$ we denote the extension of \mathcal{L} by functions $f \in F$ used as parameters. It does not have a direct semantics, but if φ is a formula of $\mathcal{L}[F]$ and $x \in \mathbb{N}^A$ then $\varphi[x]$ will denote the formula obtained by the substitution of $f(x)$ for any $f \in F$ which occurs in φ . Thus, $\varphi[x]$ is an \mathcal{L} -formula with parameters in \mathbb{R} .

Lemma 5 (Łoś). *For any closed $\mathcal{L}[F]$ -formula $\varphi(f_1, \dots, f_n)$ (all parameters $f_i \in F$ indicated), we have:*

$$\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \varphi([f_1]_D, \dots, [f_n]_D) \iff D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(f_1, \dots, f_n)[x]).$$

Proof. We argue by induction on the logic complexity of φ . For φ an atomic relation $E(f_1, \dots, f_n)$, the result follows by the definition of *E . The only notable induction step is \exists in the direction \Leftarrow . Suppose that φ is $\exists y \psi(y, f_1, \dots, f_n)$, and

$$D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(f_1, \dots, f_n)[x]), \quad \text{that is,} \quad D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \exists y \psi(y, f_1, \dots, f_n)[x]).$$

Obviously there exists a function $f \in F$, concentrated on $u = \|f_1\| \cup \dots \cup \|f_n\|$, such that, for any $x \in \mathbb{N}^A$, if there exists a real y satisfying $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(y, f_1, \dots, f_n)[x]$, then $y = f(x)$ also satisfies this formula, i. e., $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, \dots, f_n)[x]$. Formally,

$$\forall x \in \mathbb{N}^A (\exists y \in \mathbb{R} (\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(y, f_1, \dots, f_n)[x]) \implies \langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, \dots, f_n)[x]).$$

This implies $D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, \dots, f_n)[x])$. Then, by the inductive assumption, $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \psi([f]_D, [f_1]_D, \dots, [f_n]_D)$, hence $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \varphi([f_1]_D, \dots, [f_n]_D)$, as required.

□ (Lemma)

To accomplish the proof of Theorem 4, consider a closed \mathcal{L} -formula $\varphi(r_1, \dots, r_n)$ with parameters $r_1, \dots, r_n \in \mathbb{R}$. We have to prove the equivalence

$$\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(r_1, \dots, r_n) \iff \langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \varphi({}^*r_1, \dots, {}^*r_n).$$

Let $f_i = c_{r_i}$, thus, $f_i \in F$ and $f_i(x) = r_i, \forall x$. Obviously $\varphi(f_1, \dots, f_n)[x]$ coincides with $\varphi(r_1, \dots, r_n)$ for any $x \in \mathbb{N}^A$, hence $\varphi(r_1, \dots, r_n)$ is equivalent to $D x \varphi(f_1, \dots, f_n)[x]$. On the other hand, by definition, ${}^*r_i = [f_i]_D$. Now the result follows by Lemma 5. □

3 The iteration

Theorem 4 yields a definable proper elementary extension $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle$ of the structure $\langle \mathbb{R}; \mathcal{R} \rangle$. Yet this extension is not countably saturated due to the fact that the ultrapower ${}^*\mathbb{R}$ was defined with maps concentrated on finite sets $u \subseteq A$ only. To fix this problem, we iterate the extension used above ω_1 -many times.

Suppose that $\langle M; \mathcal{M} \rangle$ is an \mathcal{L} -structure, so that \mathcal{M} consists of finitary relations on a set M , and for any $E \in \mathcal{R}$ there is a relation $E^{\mathcal{M}} \in \mathcal{M}$ of the same arity, associated with E . Let F_M be the set of all maps $f : \mathbb{N}^A \rightarrow M$ concentrated on finite sets $u \subseteq A$. The structure $F_M/D = \langle {}^*M; {}^*\mathcal{M} \rangle$, defined as in Section 2, but with the modified F , will be called *the D -ultrapower* of $\langle M; \mathcal{M} \rangle$. Theorem 4 remains true in this general setting: the map $x \mapsto {}^*x$ ($x \in M$) is an elementary embedding of $\langle M; \mathcal{M} \rangle$ in $\langle {}^*M; {}^*\mathcal{M} \rangle$.

We define a sequence of \mathcal{L} -structures $\langle M_\alpha; \mathcal{M}_\alpha \rangle$, $\alpha \leq \omega_1$, together with a system of elementary embeddings $e_{\alpha\beta} : \langle M_\alpha; \mathcal{M}_\alpha \rangle \rightarrow \langle M_\beta; \mathcal{M}_\beta \rangle$, $\alpha < \beta \leq \omega_1$, so that

- (i) $\langle M_0; \mathcal{M}_0 \rangle = \langle \mathbb{R}; \mathcal{R} \rangle$;
- (ii) $\langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle$ is the D -ultrapower of $\langle M_\alpha; \mathcal{M}_\alpha \rangle$, that is, $\langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle = F_\alpha/D$, where $F_\alpha = F_{M_\alpha}$ consists of all functions $f : \mathbb{N}^A \rightarrow M_\alpha$ concentrated on finite sets $u \subseteq A$. In addition, $e_{\alpha, \alpha+1}$ is the associated *-embedding $\langle M_\alpha; \mathcal{M}_\alpha \rangle \rightarrow \langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle$, while $e_{\gamma, \alpha+1} = e_{\alpha, \alpha+1} \circ e_{\gamma\alpha}$ for any $\gamma < \alpha$ (in other words, $e_{\gamma, \alpha+1}(x) = e_{\alpha, \alpha+1}(e_{\gamma\alpha}(x))$ for all $x \in M_\alpha$);
- (iii) if $\lambda \leq \omega_1$ is a limit ordinal then $\langle M_\lambda; \mathcal{M}_\lambda \rangle$ is the direct limit of the structures $\langle M_\alpha; \mathcal{M}_\alpha \rangle$, $\alpha < \lambda$. This can be achieved by the following steps:
 - (a) M_λ is defined as the set of all pairs $\langle \alpha, x \rangle$ such that $x \in M_\alpha$ and $x \notin \text{ran } e_{\gamma\alpha}$ for all $\gamma < \alpha$.
 - (b) If $E \in \mathcal{R}$ is an n -ary relation symbol then we define an n -ary relation E_λ on M_λ as follows. Suppose that $\mathbf{x}_i = \langle \alpha_i, x_i \rangle \in M_\lambda$ for $i = 1, \dots, n$. Let $\alpha = \sup \{\alpha_1, \dots, \alpha_n\}$ and $z_i = e_{\alpha_i, \alpha}(x_i)$ for every i , so that $\alpha_i \leq \alpha < \lambda$ and $z_i \in M_\alpha$. (Note that if $\alpha_i = \alpha$ then $e_{\alpha_i, \alpha}$ is the identity.) Define $E_\lambda(\mathbf{x}_1, \dots, \mathbf{x}_n)$ iff $\langle M_\alpha; \mathcal{M}_\alpha \rangle \models E(z_1, \dots, z_n)$.
 - (c) Put $\mathcal{M}_\lambda = \{E_\lambda : E \in \mathcal{R}\}$ – then $\langle M_\lambda; \mathcal{M}_\lambda \rangle$ is an \mathcal{L} -structure.
 - (d) Define an embedding $e_{\alpha\lambda} : M_\alpha \rightarrow M_\lambda$ ($\alpha < \lambda$) as follows. Consider any $x \in M_\alpha$. If there is a least $\gamma < \alpha$ such that there exists an element $y \in M_\gamma$ with $x = e_{\gamma\alpha}(y)$ then let $e_{\alpha\lambda}(x) = \langle \gamma, y \rangle$. Otherwise put $e_{\alpha\lambda}(x) = \langle \alpha, x \rangle$.

A routine verification of the following is left to the reader.

Proposition 6. *If $\alpha < \beta \leq \omega_1$ then $e_{\alpha\beta}$ is an elementary embedding of $\langle M_\alpha; \mathcal{M}_\alpha \rangle$ to $\langle M_\beta; \mathcal{M}_\beta \rangle$. \square*

Note that the construction of the sequence of models $\langle M_\alpha; \mathcal{M}_\alpha \rangle$ is definable, hence, so is the last member $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$ of the sequence. It remains to prove that the \mathcal{L} -structure $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$ is countably saturated.

This is also a simple argument. Suppose that, for any k , $\varphi_k(p_k, x)$ is an \mathcal{L} -formula with a single parameter $p_k \in M_{\omega_1}$ (the case of many parameters does not essentially differ from the case of one parameter), and there exists an element $x_k \in M_{\omega_1}$ such that $\bigwedge_{i \leq k} \varphi_i(p_i, x_k)$ is true in $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$ – in other words, we have $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \models \varphi_i(p_i, x_k)$ whenever $k \geq i$. Fix an ordinal $\gamma < \omega_1$ such that for any k, i there exist (then obviously unique) $y_k, q_i \in M_\gamma$ with $x_k = e_{\gamma\omega_1}(y_k)$ and $p_i = e_{\gamma\omega_1}(q_i)$. Then $\varphi_i(q_i, y_k)$ is true in $\langle M_\gamma; \mathcal{M}_\gamma \rangle$ whenever $k \geq i$.

Fix $a \in A$ such that D_a is a non-principal ultrafilter, that is, all cofinite subsets of \mathbb{N} belong to D_a . Consider the structure $\langle M_{\gamma+1}; \mathcal{M}_{\gamma+1} \rangle$ as the D -ultrapower of $\langle M_\gamma; \mathcal{M}_\gamma \rangle$. The corresponding set F_γ consists of all functions $f : \mathbb{N}^A \rightarrow M_\gamma$ concentrated on finite sets $u \subseteq A$. In particular, the map $f(x) = y_k$ whenever $x(a) = k$ belongs to F_γ . As any set of the form $\{k : k \geq i\}$ belongs to D_a , we have $D_a k$ ($\langle M_\gamma; \mathcal{M}_\gamma \rangle \models \varphi_i(q_i, y_k)$), that is, $D x \in \mathbb{N}^A$ ($\langle M_\gamma; \mathcal{M}_\gamma \rangle \models \varphi_i(q_i, f)[x]$), for any $i \in \mathbb{N}$. It follows, by Lemma 5, that $\varphi_i(*q_i, \mathbf{y})$ holds in $\langle M_{\gamma+1}; \mathcal{M}_{\gamma+1} \rangle$ for any i , where $*q_i = e_{\gamma, \gamma+1}(q_i) \in M_{\gamma+1}$ while $\mathbf{y} = [f]_D \in M_{\gamma+1}$ is the D -equivalence class of f in F_γ . Put $\mathbf{x} = e_{\gamma+1, \omega_1}(\mathbf{y})$; then $\varphi_i(p_i, \mathbf{x})$ is true in $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$ for any i because obviously $p_i = e_{\gamma+1, \omega_1}(*q_i)$, $\forall i$.

\square (Theorem 1)

4 Varia

By appropriate modifications of the constructions, the following can be achieved:

1. For any given infinite cardinal κ , a κ -saturated elementary extension of \mathbb{R} , definable with κ as the only parameter of definition.
2. A *special* elementary extension of \mathbb{R} , of as large cardinality as desired. For instance, take, in stage α of the construction considered in Section 3, ultrafilters on \mathfrak{I}_α . Then the result will be a definable special structure of cardinality \beth_{ω_1} . Recall that special models of equal cardinality are isomorphic [1, Theorem 5.1.17]. Therefore, such a modification admits an explicit model-theoretical characterization up to isomorphism.
3. A class-size definable elementary extension of \mathbb{R} , κ -saturated for any cardinal κ .
4. A class-size definable elementary extension of the whole set universe, κ -saturated for any cardinal κ . (Note that this cannot be strengthened to Ord-saturation, i. e., saturation with respect to all class-size families. For instance, Ord^M-saturated elementary extensions of a minimal transitive model $M \models \mathbf{ZFC}$, definable in M , do not exist — see [2, Theorem 2.8].)

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