ALMOST ISOMETRIC EMBEDDINGS OF METRIC SPACES

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Abstract. We investigate a relations of *almost isometric embedding* and *almost isometry* between metric spaces and prove that with respect to these relations:

1. There is a countable universal metric space.
2. There may exist fewer than continuum separable metric spaces on $\aleph_1$ so that every separable metric space is almost isometrically embedded into one of them when the continuum hypothesis fails.
3. There is no collection of fewer than continuum metric spaces of cardinality $\aleph_2$ so that every ultra-metric space of cardinality $\aleph_2$ is almost isometrically embedded into one of them if $\aleph_2 < 2^{\aleph_0}$.

We also prove that various spaces X satisfy that if a space X is almost isometric to X than Y is isometric to X.

1. Introduction

We this paper we investigate a relation between metric spaces that we call "almost isometric embedding", and the notion of similarity associated with it, "almost isometry". The notion of almost isometric embedding is a weakening of the notion of isometric embedding, with respect to which there exists a countable universal metric space which is unique up to almost isometry. On the other hand, almost isometry is a sufficiently strong notion to allow many important metric spaces to maintain their isometric identity: any space which is almost isometric to, e.g., $\mathbb{R}^d$ is in fact isometric to $\mathbb{R}^d$.

We begin by examining two properties of separable metric spaces: almost isometric uniqueness of a countable dense set and almost isometric uniqueness of the whole space. It turns out that both properties are satisfied by many well-known metric spaces. This is done in Section 3.

In the rest of the paper we investigate whether analogs of well-known set-theoretic and model-theoretic results about embeddability in various categories (see [1, 14, 12, 10, 8]) hold for almost isometric embeddability in the classes of separable and of not necessarily separable metric spaces. Forcing is used only in Section 5 and uncountable combinatorics appears in Section 6. Other than that, all proofs in the paper are elementary.

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Baumgartner has shown long ago the consistency of "all $\aleph_1$-dense sets of $\mathbb{R}$ are order isomorphic", all natural analogs of this statement are false; in Section 4 it is seen that the statement "all $\aleph_1$-dense subsets of X are almost isometric," is, in contrast, false in a strong sense for $X = \mathbb{R}$, $X = \mathbb{R}^d$ and $X = \mathbb{U}$, Uryson's universal metric space. In each of these spaces there are $2^{\aleph_0}$ pairwise incomparable subspaces with respect to almost isometric embedding.

On the other hand, Section 5 shows that it is consistent that fewer than continuum $\aleph_1$-dense subsets of $\mathbb{U}$ suffice to almost isometrically embed every such set, which is a partial analog to what was proved in [14, 15, 16] for linear orders and graphs.

Finally, Section 6 handles the category of not necessarily separable spaces. We prove that the relation of almost isometric embedding between metric spaces of regular cardinality $\aleph_2$ or higher admits a representation as set inclusion over sets of reals, similarly to what is known for linear orderings, models of stable-unsuperstable theories, certain groups and certain infinite graphs [10, 11, 9, 8, 3]. A consequence of this is that, in contrast to the separable case, it is impossible to have fewer than continuum metric spaces of cardinality $\aleph_2$ so that every metric space on $\aleph_2$ is almost isometrically embedded into one of them if the continuum is larger than $\aleph_2$.

The results in Sections 3 and 6 were proved by the first author and the results in Sections 4 and 5 were proved by the second author.

2. Basic definitions and some preliminaries

Cantor proved that the order type of $(\mathbb{Q}, <)$ is characterized among all countable ordertypes as being dense and with no end-points. Thus, any countable dense subset of $\mathbb{R}$ is order isomorphic to $\mathbb{Q}$. Consider countable dense subsets of $\mathbb{R}$ with the usual metric. Not all of them are isometric, as $\mathbb{Q}$ and $\pi\mathbb{Q}$ show. In fact there are $2^{\aleph_0}$ countable dense subsets of $\mathbb{R}$ no two of which are comparable with respect to isometric embedding, since the set of distances which occur in a metric space is an isometry invariant which is preserved under isometric embeddings.

We introduce now relations of similarity and embeddability which are quite close to isometry and isometric embedding, but with respect to which the set of distances in a space is not preserved.

Definition 1. (1) *A map $f : X \to Y$, between metric spaces satisfies the Lipschitz condition with constant $\lambda > 0$ if for all $x_1, x_2 \in X$ it holds that $d(f(x_1), f(x_2)) < \lambda d(x_1, x_2)$.*
(2) *Two metric spaces X and Y are* almost isometric *if for each $\lambda > 1$ there is a homeomorphism $f : X \to Y$ such that f and f^{-1} satisfy the Lipschitz condition with constant λ.*
(3) *X is* almost isometrically embedded *in Y if for all $\lambda > 1$ there is an injection $f : X \to Y$ so that f and f^{-1} satisfy the Lipschitz condition with constant λ.*

Let us call, for simplicity, an injection f so that f and f^{-1} satisfy the λ-Lipschitz condition, λ-bi-Lipchitz. Observe that we use a strict inequality in the definition of the Lipschitz condition. We also note that X and Y are almost isometric if and only if the *Lipschitz distance* between X and Y is 0. The Lipschitz distance is a well known semi-metric on metric spaces (see [4]).

It is important to notice that one does not require in (2) that the injections f for different λ have the same range.

The graph of the function $f(x) = 1/x$ for $x > 1$ and the ray $\{(x, 0) : x > 0\}$ are each almost isometrically embedded in the other but are not almost isometric. In the infinite dimensional Hilbert space it is not hard to find two closed subsets which are almost isometric but not isometric as follows: fix an orthonormal basis $\{v_n : n \in \mathbb{N}\}$ and fix a partition $\mathbb{Q} = A_1 \cup A_2$ to two (disjoint) dense sets and let $A_i = \{r^i_n : n \in \mathbb{N}\}$ for $i = 1, 2$. Let $X_i = \bigcup_{n\in\mathbb{N}}[0, q_n v_n]$. Now X_1, X_2 are closed (and connected) subsets of the Hilbert space which are not isometric, but are almost isometric because A_1, A_2 are.

Almost isometry and almost isometric embedding can be viewed as isomorphisms and monomorphisms of a category as follows. Let $\mathcal{M}$ be the category in which the objects are metric spaces and the morphisms are defined as follows: a morphism from A to B is a sequence $\vec{f} = \langle f_n : n \in n\rangle$ where for each n, $f_n : A \to B$ satisfies the Lipchitz condition with a constant λ_n so that $\lim_n \lambda_n = 1$. The identity morphism $\vec{\mathrm{id}}_A$ is the constant sequence $\langle \mathrm{id}_A : n \in \mathbb{N}\rangle$ and the composition law is $\vec{g} \circ \vec{f} = \langle g_n \circ f_n : n \in \mathbb{N}\rangle$.

A morphism $\overline{f}$ in this category is invertible if and only if each f_n is invertible and its inverse satisfies the Lipchitz condition for some θ_n so that $\lim_n \theta_n = 1$, or, equivalently, $\vec{f} \in \hom(A, B)$ is an isomorphism if and only if f_n satisfies a bi-Lipschitz condition with a constant λ_n so that $\lim_n \lambda_n = 1$. Thus, A and B are isomorphic in this category if and only if for all $\lambda > 1$ there is a λ-bi-Lipchitz homeomorphism between A and B.

In Section 3 and in Section 5 below we shall use the following two simple facts:

Fact 2. *Suppose X is a nonempty finite set, $E \subseteq X^2$ symmetric and reflexive, and (X, E) is a connected graph. Suppose that $f : E \to \mathbb{R}^+$ is a symmetric function such that $f(x, y) = 0 \iff x = y$ for all $(x, y) \in E$. Let $\sum_{i<d} f(x_i, x_{i+1})$ be the* length *of a path $(x_0, x_1, \dots, x_d)$ in (X, E). Define $d(x, y)$ as the length of the shortest path from x to y in (X, E). Then d is a metric on X. If, furthermore, for all $(x, y) \in \mathrm{dom} f$ it holds that (x, y) is the shortest path from x to y, then d extends f.*

Fact 3. *Suppose $X = \{x_0, \dots, x_{n-1}, x_n\}$ and $Y = \{y_0, \dots, y_{n-1}\}$ are metric spaces and that the mapping $x_i \mapsto y_i$ for all $i < n$ is θ-bi-Lipschitz, $\theta > 1$. Then there is a metric extension of $Y \cup \{y_n\}$ of Y, $y_n \notin Y$, so that the extended mapping $x_i \mapsto y_i$ for $i \leq n$ is θ-bi-Lipschitz.*

Proof. Fix $1 < \lambda < \theta$ so that the mapping $x_i \mapsto y_i$ is λ-bi-Lipschitz. Add a new point y_n to Y and define $d^*(y_n, y_i) = \lambda d(x_n, x_i)$ for all $i < n$, $d^*(y_n, y_n) = 0$. Let d' on $Y \cup \{y_n\}$ be the shortest path metric obtained from $d \cup d^*$.

Let us verify that d' extends the given metric d on Y. $d(y_i, y_j) \leq \lambda d(x_i, x_j) \leq \lambda(d(x_n, x_i) + d(x_n, x_j)) = d^*(y_n, y_i) + d^*(y_n, y_j)$, so (y_i, y_j) is the shortest path from y_i to y_i.

Now let us verify that extending the mapping by $x_n \mapsto y_n$ yields a λ-bi-Lipschitz map. The path (y_n, y_i) has length $\lambda d(x_n, x_i)$ with respect to $d \cup d^*$, hence the shortest path cannot be longer, and $d'(y_n, y_i) \leq \lambda d(x_n, x_i)$. Suppose now the shortest path from y_n to y_i is (y_n, y_j, y_i). Then $d'(y_j, y_i) \geq d(x_j, x_i)/\lambda$ and certainly $d^*(y_n, y_j) > d(x_n, x_j)$, so $d'(y_n, y_i) \geq (d(x_n, x_j) + d(x_j, x_i))/\lambda \geq d(x_n, x_i)/\lambda$ as required. □

Definition 4. *Let $LAut(X)$ be the group of all auto-homeomorphisms of X which are λ-bi-Lipschitz for some $\lambda > 1$. Let $LAut_\lambda(X) = \{f \in LAut(X) : f$ is λ-bi-Lipschitz$\}$.*

X is almost ultrahomogeneous *if every finite λ-bi-Lipschitz map $f : A \to B$ between finite subsets of X extends to a λ-bi-Lipschitz autohomeomorphism.*

3. Almost-isometry uniqueness and countable dense sets

Definition 5. *A metric space X is* almost-isometry unique *if every metric space Y which is almost isometric to X is isometric to X.*

In this section we shall prove that various metric spaces are almost-isometry unique and prove that the Uryson space $\mathbb{U}$ has a unique countable dense set up to almost isometry.

Theorem 6. *Suppose X satisfies:*

1. *all closed bounded balls in X are compact;*
2. *there is $x_0 \in X$ and $r > 0$ so that for all $y \in X$ and all $\lambda > 1$ there is a λ-bi-Lipschitz auto-homeomorphism of X so that $d(f(y), x_0) < r$.*

Then X is almost-isometry unique.

Proof. Suppose X, $x_0 \in X$ and $r > 0$ are as stated, and suppose Y is a metric space and $f_n : Y \to X$ is a $(1 + 1/n)$-bi-Lipschitz homeomorphism for all $n > 0$. Fix some $y_0 \in Y$. By following each f_n by a bi-Lipschitz autohomeomorphism of X, we may assume, by condition 1, that $d(f_n(y_0), x_0) < r$ for all n.

Condition 1 implies that X is separable and complete, and since Y is homeomorphic to X, Y is also separable. Fix a countable dense set $A \subseteq Y$. Since for each $a \in A$ it holds that $d(f_n(a), x_0)$ is bounded by some L_a for all n, condition 2 implies that there is a converging subsequence $\langle f_n(a) : n \in D_a \rangle$ and, since A is countable, diagonalization allows us to assume that $f_n(a)$ converges for every $a \in A$ to a point we denote by $f(a)$. The function

f we defined on A is clearly an isometry, and hence can be extended to an isometry on Y. It can be verified that $f_n(y)$ converges pointwise to $f(y)$ for all $y \in Y$. Since each f_n is onto X, necessarily also f is onto X. Thus X is isometric to Y. □

Corollary 7. *For each $d \in \mathbb{N}$, $\mathbb{R}^d$ and $\mathbb{H}^d$ are almost-isometry unique.*

Theorem 8 (Hrusak, Zamora-Aviles [5])**.** *Any two countable dense subsets of $\mathbb{R}^n$ are almost isometric. Any two countable dense subsets of the separable infinite dimensional Hilbert space are almost isometric.*

If one regards a fixed $\mathbb{R}^d$ as a universe of metric spaces, namely considers only the subsets of $\mathbb{R}^d$, then among the countable ones there is a universal element with respect to almost isometric embedding, which is unique up to almost isometry:

Corollary 9. *Every dense subset of $\mathbb{R}^d$ is almost-isometry universal in the class of countable subspaces of $\mathbb{R}^d$.*

Proof. Let $B \subseteq \mathbb{R}^d$ be countable and extend B to a countable dense $A' \subseteq \mathbb{R}^d$. Since A' and A are almost isometric, B is almost isometrically embedable into A. □

3.1. **The Uryson space.** Uryson's universal separable metric space $\mathbb{U}$ is characterized up to isometry by separability and the following property:

Definition 10 (**Extension property**)**.** *A metric space X satisfies the* extension property *if for every finite $F = \{x_0, \dots, x_{n-1}, x_n\}$, every isometry $f : \{x_0, \dots, x_{n-1}\} \to X$ can be extended to an isometry $\hat{f} : F \to X$.*

Separability together with the extension property easily imply the isometric uniqueness of $\mathbb{U}$ as well as the fact that Every separable metric space is isometric to a subspace of $\mathbb{U}$ and that $\mathbb{U}$ is ultrahomogeneous, namely, every isometry between finite subspaces of $\mathbb{U}$ extends to an auto-isometry of $\mathbb{U}$. This property of $\mathbb{U}$ was recently used to determine the Borel complexity of the isometry relation on polish spaces [6, 7].

Definition 11 (**Almost extension property**)**.** *A metric space X satisfied the* almost extension property *if for every finite space $F = \{x_0, \dots, x_{n-1}, x_n\}$ and $\lambda > 1$, every λ-bi-Lipschitz $f : \{x_0, \dots, x_{n-1}\} \to X$ can be extended to a λ-bi-Lipschitz $\hat{f} : F \to X$.*

Claim 12. *Suppose that $A \subseteq \mathbb{U}$ is dense in $\mathbb{U}$. Then A satisfies the almost extension property.*

Proof. Suppose $f : \{x_1, \dots, x_{n-1}\} \to A$ is λ-bi-Lipschitz. By Lemma 3 there is a metric extension $\operatorname{ran} f \cup \{y_n\}$ so that $f \cup \{(x_n, y_n)\}$ is λ-bi-Lipschitz. By the extension property of $\mathbb{U}$, we may assume that $y_n \in \mathbb{U}$. Now replace y_n by a sufficiently close $y'_n \in A$ so that $f \cup \{(x_n, y'_n)\}$ is λ-bi-Lipschitz. □

A standard back and forth argument shows:

Fact 13. *Any two countable metric spaces that satisfy the almost extension property are almost isometric.*

Therefore we have proved:

Theorem 14. *Any two countable dense subsets of* $\mathbb{U}$ *are almost isometric.*

A *type* p over a metric space X is a function $p : X \to \mathbb{R}^+$ so that in some metric extension $X \cup \{y\}$ it holds that $d(y, x) = p(x)$ for all $x \in X$. A point $y \in Y$ *realizes* a type p over a subset $X \subseteq Y$ if $p(x) = d(y, x)$ for all $x \in X$. The extension property of $\mathbb{U}$ is equivalent to the property that every type over a finite subset of $\mathbb{U}$ is realized in $\mathbb{U}$.

Theorem 15 (Uryson)**.** *If a countable metric space* A *satisfies the almost extension property then its completion* $\bar{A}$ *satisfies the extension property and is therefore isometric to* $\mathbb{U}$.

Proof. Let $X \subseteq \bar{A}$ be a finite subset and let p a metric type over X. Given $\varepsilon > 0$, find $\lambda > 1$ sufficiently close to 1 so that for all $x \in X$ it holds that $\lambda p(x) - p(x) < \varepsilon/2$ and $p(x) - p(x)/\lambda < \varepsilon/2$ and find, for each $x \in X$ some $x' \in A$ with $d(x', x) < \varepsilon/2$ and sufficiently small so that the map $x \mapsto x'$ is λ-bi-Lipschitz. By the almost extension property of A there is some $y \in A$ so that $d(y, x) < \varepsilon$. Thus we have shown that for all finite $X \subseteq \bar{A}$ and type p over X, for every $\varepsilon > 0$ there is some point $y \in A$ so that $|d(y, x) - p(x)| < \varepsilon$ for all $x \in X$.

Suppose now that $X \subseteq \bar{A}$ is finite and that p is some type over X. Suppose $\varepsilon > 0$ is small, and that $y \in A$ satisfies that $|d(y, x) - p(x)| < \varepsilon$ for all $x \in X$. Extend the type p to $X \cup y$ by putting $p(y) = 2\varepsilon$ (since $|d(y, x) - p(x)| < \varepsilon$, this is indeed a type). Using the previous fact, find $z \in A$ that realizes p up to $\varepsilon/100$ and satisfies $d(y, z) < 2\varepsilon$.

Iterating the previous paragraph, one gets a Cauchy sequence $(y_n)_n \subseteq A$ so that for all $x \in X$ it holds $d(y_n, x) \to p(x)$. The limit of the sequence satisfies p in $\bar{A}$. □

We now have:

Fact 16. *A countable metric space is isometric to a dense subset of* $\mathbb{U}$ *if and only if it satisfies the almost extension property.*

Theorem 17. *The Uryson space* $\mathbb{U}$ *is almost isometry unique.*

Proof. Suppose X is almost isometric to $\mathbb{U}$. Fix a countable dense $A \subseteq \mathbb{U}$ and a countable dense $B \subseteq X$. For every $\lambda > 1$, B is λ-bi-Lipschitz homeomorphic to *some* countable dense subset of $\mathbb{U}$, so by Theorem 14 it is λ^2-bi-Lipschitz homeomorphic to A. So A and B are almost isometric. This shows that B has the almost extension property. By Uryson's theorem, $\bar{B} = Y$ is isometric to $\mathbb{U}$. □

Let us construct now, for completeness of presentation, some countable dense subset of $\mathbb{U}$.

Let $\langle A_n : n \in \mathbb{N} \rangle$ be an increasing sequence of finite metric spaces so that:

(1) all distances in A_n are rational numbers.
(2) for every function $p : A_n \to \mathbb{Q}^n$ which satisfies the triangle inequality ($p(x_1) + d(x_1, x_2) \geq p(x_2)$ and $p(x_1) + p(x_2) \geq d(x_1, x_2)$ for all $x_1, x_2 \in A_n$) *and* satisfies that $p(x) \leq n + 1$ is a rational with denominator $\leq n + 1$ there is $y \in A_{n+1}$ so that $p(x) = d(y, x)$ for all $x \in A_n$.

Such a sequence obviously exits. A_0 can be taken as a singleton. To obtain A_{n+1} from A_n one adds, for each of the finitely many distance functions p as above a new point that realizes p, and then sets the distance $d(y_1, y_1)$ between two new points to be $\min\{d(y_1, x) + d(y_2, x) : x \in A_n\}$. Let $\mathbb{A} := \bigcup_n A_n$. To see that $\mathbb{A}$ satisfies the almost extension property, one only needs to verify that every type p over a finite rational X can be arbitrarily approximated by a rational type (we leave that to the reader).

This construction, also due to Uryson, shows that the Uryson space has a dense rational subspace. (Another construction of $\mathbb{U}$ can be found in [4]). This is a natural place to recall:

Problem 18 (Erdős)**.** *Is there a dense rational subspace of* $\mathbb{R}^2$*?*

Problem 19. *Is it true that for a separable and homogeneous complete metric space all countable dense subsets are almost isometric if and only if the space is almost-isometry unique?*

4. Almost-isometric embeeddability between $\aleph_1$-dense sets

Definition 20. *A subset A of a metric space X is* $\aleph_1$-dense *if for every nonempty open ball $u \subseteq X$ it holds that $|A \cup u| = \aleph_1$.*

Among the early achievement of the technique of forcing a place of honor is occupied by Baumgartner's proof of the consistency of "all $\aleph_1$-dense sets of reals are order isomorphic" [1]. Early on, Sierpinski proved that there are $2^{2^{\aleph_0}}$ order-incomparable continuum-dense subsets of $\mathbb{R}$, hence Baumgrtner's consistency result necessitates the failure of CH.

Today it is known that Baumgartner's result follows from forcing axioms like PFA and Martin's Maximum, and also from Woodin's axiom $(*)$ [18].

In our context, one can inquire about the consistency two natural analogs of the statement whose consistency was established by Baumgartner. First, is it consistent that all $\aleph_1$-dense subsets of $\mathbb{R}$ are almost isometric? Since every bi-Lipschitz homeomorphism between two dense subset of $\mathbb{R}$ is either order preserving or order inverting, this statement strengthens a slight relaxation of Baugartner's consistent statement. Second, since $\mathbb{U}$ in the category of metric spaces has the role $\mathbb{R}$ has in the category of separable linear orders (it is the universal separable object), is it consistent that all $\aleph_1$-dense subsets of $\mathbb{U}$ are almost isometric?

The answer to both questions is negative.

4.1. **Perfect incomparable subsets in the cantor space.** For every infinite $A \subseteq 2^{\mathbb{N}}$ let $T_A \subseteq 2^{<\mathbb{N}}$ be defined (inductively) as the tree T which contains the empty sequence and for every $\eta \in T$ contains $\eta\frown 0, \eta\frown 1$ if $|\eta| \in A$, and contains only $\eta\frown 0$ if $|\eta| \notin A$. Let D_A be the set of all infinite branches through T_A. The set of positive distances occurring in D_A, namely $\{d(x,y) : x, y \in D_A, x \neq y\}$ is equal to $\{1/2^n : n \in A\}$.

Let d_3 be the metric on $2^{\mathbb{N}}$ defined by $d_3(\eta,\nu) := 1/3^{\Delta)\eta,\nu)}$. Observe that the natural isomorphism between $2^{\mathbb{N}}$ and the standard "middle-third" cantor set is a bi-Lipschitz map when $2^{\mathbb{N}}$ is taken with d_3.

For infinite sets $A, B \subseteq N$ let us define the following condition:

$(*)$ $|A| = |B| = \aleph_0$ *and for every* n *there is* k *so that for all* $a \in A, b \in B$, *if* $a, b > k$ *then* $a/b > n$ *or* $b/a > n$.

Lemma 21. *Suppose* $A, B \subseteq \mathbb{N}$ *satisfy* $(*)$*. Then for every infinite set* $X \subseteq D_A$ *and every function* $f : X \to B$*, for every* $n > 1$ *there are distinct* $x, y \in X$ *so that either* $d(f(x), f(y))/d(x,y) > n$ *or* $d(f(x), f(y))/d(x,y) < 1/n$.

Proof. Let $x, y \in X$ be chosen so that $d(x,y) > 0$ is sufficiently small. □

The lemma assures a strong form of bi-Lipschitz incomparability: no infinite subset of one of the spaces D_A, D_B can be bi-Lipschitz embedable into the other space, if A, B satisfy $(*)$.

Let $\mathcal{F}$ be an almost disjoint family over $\mathbb{N}$, namely, a family of infinite subsets of $\mathbb{N}$ with finite pairwise intersections. Replacing each $A \in \mathcal{F}$ by $\{n^2 : n \in A\}$, one obtains a family of sets that pairwise satisfy condition $(*)$. Since there is an almost disjoint family of size $2^{\aleph_0}$ over $\mathbb{N}$, there is a family $\mathcal{F}$ of subsets of $\mathbb{N}$ whose members satisfy $(*)$ pairwise, and therefore $\{D_A : A \in \mathcal{F}\}$ is a family of pairwise bi-Lipschitz incomparable subspaces of $(2^{\mathbb{N}}, d_3)$ of size $2^{\aleph_0}$.

Theorem 22. *Suppose* X *is a separable metric space and that for some* $K \geq 1$*, for every open ball* u *in* X *there is a (nonempty) open subset of* $(2^\omega, d_3)$ *which is* K*-bi-Lipschitz embedable into* u*. Then there are* $2^{\aleph_0}$ *pairwise bi-Lipschitz incomparable* $\aleph_1$*-dense subsets of* X*.*

In particular, in every separable Hilbert space (finite or infinite dimensional) and in $\mathbb{U}$ *there are* $2^{\aleph_0}$ *pairwise bi-Lipschitz incomparable* $\aleph_1$*-dense subsets.*

Proof. Fix a family $\{D_\alpha : \alpha < 2^{\aleph_0}\}$ of pairwise bi-Lipschitz incomparable perfect subspaces of $2^{\mathbb{N}}$. For each α, fix an $\aleph_1$-dense $D^*_\alpha \subseteq D_\alpha$.

Let $\langle u_n : n \in \mathbb{N}\rangle$ enumerate a basis for the topology of X (say all balls of rational radi with centers in some fixed countable dense set). For each $\alpha < 2^{\aleph_0}$, for each n, fix a K-bi-Lipschitz embedding of a nonempty open subset of D^*_α into u_n, and call the image of the embedding $E^*_{\alpha,n}$. Let $Y_\alpha = \bigcup_n E^*\alpha, n$. Y_α is thus an $\aleph_1$-dense subset of X.

Suppose $f : Y_\alpha \to Y_\beta$ is any function, $\alpha, \beta < 2^{\aleph_0}$ distinct, and $K > 1$ is arbitrary. There is $l \in \mathbb{N}$ so that $f^{-1}[E^*_{\beta,l}] \cap E^*_{\alpha,0}$ is uncountable, hence infinite. Therefore, by Lemma 21, there are $x, y \in Y_\alpha$ so that $d(f(x), f(y))/d(x, y)$ violates n-bi-Lipschitz. We conclude that Y_α, Y_β are bi-Lipschitz incomparable.

The second part of the theorem follows now from the fact that every separable metric space embeds isometrically into $\mathbb{U}$ and from the observation above that $(2^{\mathbb{N}}, d_3)$ has a bi-Lipschitz embedding into $\mathbb{R}$. □

5. Consistency results for separable spaces on $\aleph_1 < 2^{\aleph_0}$

Let $(\mathcal{M}^{sep}_{\aleph_1}, \leq)$ denote the set of all (isometry types of) separable metric spaces whose cardinality is $\aleph_1$, quasi-ordered by almost isometric embeddability and let $(\mathcal{M}_{\aleph_1}, \leq)$ denote the set of all (isometry types of) metric spaces whose cardinality is $\aleph_1$, quasi-ordered similarly.

Let $\mathrm{cf}(\mathcal{M}_{\aleph_1}, \leq)$ denote the *cofinality* of this quasi-ordered set: the least cardinality of $D \subseteq \mathcal{M}^{sep}_{\aleph_1}$ with the property that for every $M \in \mathcal{M}^{sep}_{\aleph_1}$ there is $N \in D$ so that $M \leq N$. The statement "$\mathrm{cf}(\mathcal{M}^{sep}_{\aleph_1}, \leq) = 1$" means that there is a single $\aleph_1$-dense subset of $\mathbb{U}$ in which every $\aleph_1$-dense subset of $\mathbb{U}$ is almost isometrically embedded, or, equivalently, that there is a *universal* separable metric space of size $\aleph_1$ for almost isometric embeddings.

In the previous section it was shown that there are $2^{\aleph_0}$ pairwise incomparable elements — an anti-chain — in this quasi ordering, with each of elements being an $\aleph_1$-dense subset of $\mathbb{U}$. This in itself does not rule-out the possibility that a universal separable metric space of size $\aleph_1$ exists for almost isometric embeddings. In fact, if CH holds, $\mathbb{U}$ itself is such a set.

What can one expect if CH fails? There has been a fairly extensive study of the problem of universality in $\aleph_1 < 2^{\aleph_0}$. On the one hand, it is fairly routine to produce models in which there are no universal objects in cardinality $\aleph_1$ in every reasonable class of structures (linear orders, graphs, etc) [12, 10] and here too it is easy to find models in which neither a separable metric space nor a general metric space exist in cardinality $\aleph_1$ (see below).

On the other hand, it has been shown that universal linear orderings may exists in $\aleph_1 < 2^{\aleph_0}$ [14], that universal graphs may exist in a prescribed regular $\lambda < 2^{\aleph_0}$ [15, 16] and more generally, that uninversal relational theories with certain amalgamation properties may have universal models in regular uncountable $\lambda < 2^{\aleph_0}$ [12].

We do not know whether it is consistent to have a universal separable metric space of size $\aleph_1 < 2^{\aleph_0}$ for almost isometric embeddings, but we shall prove a weaker statement: the consistency that $\aleph_1 < 2^{\aleph_0}$ and that $\mathrm{cf}(\mathcal{M}^{sep}_{\aleph_1}, \leq)$ is smaller than the continuum.

We begin by relating separable to nonseparable spaces:

Theorem 23. $\mathrm{cf}(\mathcal{M}^{sep}_{\leq\aleph_n}, \leq) \leq \mathrm{cf}(\mathcal{M}_{\aleph_n}, \leq)$ *for all* n.

Proof. Suppose $M = (\omega_1, d)$ is a metric space. For every ordinal $\alpha < \omega_1$ denote the closure of α in (M, d) by X_α is a separable space and is therefore isometric to some $Y_\alpha \subseteq \mathbb{U}$. Let $N(M) = \bigcup_{\alpha<\omega_1} Y_\alpha$. $N(M)$ is a subspace of $\mathbb{U}$ whose cardinality is $\leq \aleph_1$.

Suppose X is any separable subspace of M, and fix a countable dense $A \subseteq X$. There is some $\alpha < \omega_1$ so that $A \subseteq \alpha$, hence $X \subseteq X_\alpha$ and is thus isometrically embedded in $Y_\alpha \subseteq N$. In other words, there is a single subspace of $\mathbb{U}$ into which all *separable* subspaces of M are isometrically embedded.

Suppose now that $\mathrm{cf}(\mathcal{M}_{\aleph_1}, \leq) = \kappa$ and fix $D \subseteq \mathcal{M}_{\aleph_1}$ of cardinality κ so that for all $N \in \mathcal{M}_{\aleph_1}$ there is $M \in D$ so that $N \leq D$. For each $M \in D$ let $N(M) \subseteq \mathbb{U}$ be chosen as above. We claim that $\{N(M) : M \in D\}$ demonstrates that $\mathrm{cf}(\mathcal{M}^{sep}_{\leq\aleph_1}, \leq) \leq \kappa$. Suppose that X is a separable metric space of cardinality $\aleph_1$. Then X is almost isometrically embedded into M for some $M \in D$. Since every separable subspace of M is isometric to a subspace of $N(M)$, it follows that X is almost isometrically embedded in $N(M)$.

Simple induction on n shows that for every n there is a collection $\mathcal{F}_n$ of $\aleph_n$ many countable subsets of ω_n with the property that every countable subset of ω_n is contained in one of them. Working with $\mathcal{F}_n$ instead of the collection of initial segments of ω_1 gives that $\mathrm{cf}(\mathcal{M}^{sep}_{\leq\aleph_n}, \leq) \leq \mathrm{cf}(\mathcal{M}_{\aleph_n}, \leq)$. $\square$

Theorem 24. *After adding $\lambda \geq \aleph_2$ Cohen reals to a universe V of set theory,* $\mathrm{cf}(\mathcal{M}^{sep}_{\aleph_1}, \leq) \geq \lambda$.

Proof. View adding λ Cohen reals as an iteration. Let $\theta < \lambda$ be given. For any family $\{A_\alpha : \alpha < \theta\}$ of $\aleph_1$-dense subsets of $\mathbb{U}$ in the extension it may be assumed, by using θ of the Cohen reals, that $A_\alpha \in V$ for all $\alpha < \theta$. Let $X = \mathbb{Q} \cup \{r_i : i < \omega_1\}$ be a metric subspace of $\mathbb{R}$, where each r_i, for $i < \omega_1$ is one of the Cohen reals. We argue that X cannot be almost isometrically embedded into any A_α. Suppose to the contrary that $f : X \to A_\alpha$ is a bi-Lipschitz embedding. By using countably many of the Cohen reals, we may assume that $f \restriction Q \in V$. If $f(r_0) \in A_\alpha$ and $f \restriction Q$ are both in V, so is r_0 — contradiction. $\square$

For the next consistency result we need the following consistency result:

Theorem 25 ([2])**.** *For every regular $\lambda > \aleph_0$ and regular $\theta > \lambda^+$ there is a model V of set theory in which $2^{\aleph_0} \geq \theta$ and there is a family $\{A_\alpha : \alpha < \theta\}$ of subsets of λ, each A_α of cardinality λ and $|A_\alpha \cap A_\beta| < \aleph_0$ for all $\alpha < \beta < \theta$.*

We now state and prove the consistency for $\lambda = \aleph_1$, for simplicity. Then we extend it to a general regular $\lambda > \aleph_0$.

Theorem 26. *It is consistent that $2^{\aleph_0} = \aleph_3$ and that there are $\aleph_2$ separable metric spaces on ω_1 such that every separable metric space is almost isometrically embedded into one of them.*

The model of set theory which demonstrates this consistency is obtained as a forcing extension of a ground model which satisfies $2^{\aleph_0} = \aleph_3$ and there are $\aleph_3$ $\aleph_1$-subsets of $\aleph_1$ with finite pairwise intersections. Such a model exists by [2]. Then the forcing extension is obtained via a ccc finite support iteration of length $\aleph_2$. In each step $\zeta < \omega_2$ a single new separable metric space M_ζ of cardinality $\aleph_1$ is forced together with almost isometric embeddings of all $\aleph_1$-dense subsets of $\mathbb{U}$ that V_α knows. At the end of the iteration, every $\aleph_1$ dense subset is almost isometrically embedded into one of the spaces M_ζ that were forced.

Let $\{A_\alpha : \alpha < \omega_3\}$ be a collection of subsets of ω_1, each or cardinality $\aleph_1$ and for all $\alpha < \beta < \omega_3$ it holds that $A_\alpha \cap A_\beta$ is finite. For each $\alpha < \omega_3$ fix a partition $A_\alpha = \bigcup_{i<\omega_1} A_{\alpha,i}$ to $\aleph_1$ parts, each of cardinality $\aleph_0$.

Fix an enumeration $\langle d_\alpha : \alpha < \omega_3\rangle$ of all metrics d on ω_1 with respect to which (ω_1, d) is a separable metric space and every interval $(\alpha, \alpha+\omega)$ is dense in it. Since every metric space of cardinality ω_1 can be well ordered in ordertype ω_1 so that every interval $(\alpha, \alpha+\omega)$ is a dense set, this list contains ω_3 isometric copies of every separable metric space of cardinality $\aleph_1$.

We define now the forcing notion Q. A condition $p \in q$ is an ordered quintuple $p = \langle w^p, u^p, d^p, \overline{f}^p, \overline{\varepsilon}^p\rangle$ where:

(1) w^p is a finite subset of ω_3 (intuitively — the set of metric spaces (ω_1, d_α) which the condition handles)
(2) $u^p \subseteq \omega_2$ is finite and d^p is a metric over u^p. (u^p, d^p) is a finite approximation to the space $M = (\omega_1, d)$ which Q introduces.
(3) $\overline{\varepsilon}^p = \langle \varepsilon^p_\alpha : \alpha \in w^p\rangle$ is a sequence of rational numbers from $(0, 1)$.
(4) $\overline{f} = \langle f_\alpha : \alpha \in w^p\rangle$ is a sequence of finite function $f_\alpha : (\omega_1, d_\alpha) \to (u^p, d^p)$ that satisfy:
 (a) $f^p_\alpha(i) \in A_{\alpha,i}$ for each $i \in \mathrm{dom} f^p_\alpha$;
 (b) each f^p_α is $(1 + \varepsilon^p)$-bi-Lipschitz.

The order relation is: $p \leq q$ (q extends p) iff $w^p \subseteq w^q$, (u^p, d^p) is a subspace of (u^q, d^q), and for all $\alpha \in w^p$, $\varepsilon^p = \varepsilon^q$ and $f^p_\alpha \subseteq f^q_\alpha$.

Informally, a condition p provides finite approximations of $(1 + \varepsilon^p_\alpha)$-bi-Lipschitz embeddings of (ω_1, d_α) for finitely many $\alpha < \omega_3$ into a finite space (u^p, d^p) which approximates (ω_1, d).

Lemma 27 (Density). *For every $p \in Q$:*

(1) *For every $j \in \omega_1 \setminus u^p$ and a metric type t over (u^p, d^p) there is a condition $q \geq p$ so that $j \in u^q$ and j realizes t over u^p in (u^q, d^q).*
(2) *For every $\alpha \in \omega_3 \setminus w^p$ and $\delta > 0$ there is a condition $q \geq p$ so that $\alpha \in w^q$ and $\varepsilon^p_\alpha < \delta$.*
(3) *For every $\alpha \in w^p$ and $i \in \omega_1 \setminus \mathrm{dom} f^p_\alpha$ there is a condition $q \geq p$ so that $i \in \mathrm{dom} f^q_\alpha$*

Proof. To prove (1) simply extend (u^p, d^p) to a metric space (u^q, d^q) which contains j and in which j realizes t over u^p, leaving everything else in p unchanged.

For (2) define $w^q = w^p \cup \{\alpha\}$ and $\varepsilon^p_\alpha = \varepsilon$ for a rational $\varepsilon < \delta$.

For (3) suppose $i \notin \mathrm{dom} f^p_\alpha$. Fix some $x \in A_{\alpha,i} \setminus u^p$ and fix some $1 < \lambda < 1 + \varepsilon^p_\alpha$ so that $d_\alpha(j,k)/\lambda \le d^t(j,k) \le \lambda d_\alpha(j,k)$ for all distinct $j, j \in \mathrm{dom} f^p_\alpha$. Let $d^*(x, f^p_\alpha(j)) = r_j$ for some rational $\lambda d_\alpha(i,j) \le r_j < (1+\varepsilon^p_\alpha) d_\alpha(i,j)$, and let $d^*(x,x) = 0$. Let d be the shortest path metric obtained from $d^t \cup d^*$. This is obviously a rational metric and as in the proof of Fact 3, it follows that this metric extends d^t and that $f^p_\alpha \cup \{(i,x)\}$ is $(1+\varepsilon^p_t)$-bi-Lipschitz into $u^t \cup \{x\}$ with the extended metric. □

Lemma 28. *In V^Q there is a separable metric space $M = (\omega_1, d)$ so that for every separable metric space $(\omega_1, d') \in V$ and $\delta > 0$ there is a $(1+\delta)$-bi-Lipschitz embedding in V^Q of (ω_1, d') into M.*

Proof. Let $d = \bigcup_{p \in G} d^p$ where G is a V-generic filter of Q. By (1) in the density Lemma, d is a metric on ω_1 and, furthermore, for every given $i < \omega_1$ the interval $(i, i+\omega)$ is dense in (ω_1, d).

Given any metric space (ω_1, d') and a condition p, find some $\alpha \in \omega_3 \setminus w^p$ so that (ω_1, d_α) is isometric to (ω_1, d') and apply (2) to find a stronger condition q so that $\alpha \in w^q$ and $\varepsilon^q < \delta$. For every $i < \omega_1$ there is a stronger condition q' so that $i \in \mathrm{dom} f^{q'}_\alpha$. Thus, the set of conditions which force a partial $(1+\delta)$-bi-Lipschitz embedding from $(\omega_1, \delta_\alpha)$ into (ω_1, d) which includes a prescribed $i < \omega_1$ in its domain is dense; therefore Q forces a $(1+\delta)$-bi-Lipschitz embedding of (ω_1, d') into (ω_1, d). □

Lemma 29. *Every antichain in Q is countable.*

Proof. Before plunging into the details, let us sketch shortly the main idea of the proof. While combining two arbitrary finite λ-bi-Lipschitz embeddings into a single one is not generally possible, when the domains are sufficiently "close" to each other, that is, have very small Hausdorff distance, it is possible. The main point in the proof is that separability of the spaces $(\omega_1, \delta_\alpha)$ for $\alpha < \omega_3$ implies that among any $\aleph_1$ finite disjoint subsets of (ω_1, d_α) there are two which are very small perturbations of each other, and therefore have a common extension.

Suppose $\{p_\zeta : \alpha < \omega_1\} \subseteq Q$ is a set of conditions. Applying the Δ-system lemma the pigeon hole principle a few times, we may assume (after replacing the set of conditions by a subset and re-enumerating) that:

(1) $|u^{p_\zeta}| = n$ for all $\zeta < \omega_1$ for some fixed n and $\{u_\zeta : \zeta < \omega_1\}$ is a Δ-system with root u. Denote $u^{p_\zeta} = u \cup u_\zeta$ (so $\zeta < \xi < \omega_1 \Rightarrow u_\zeta \cap u_\xi = \emptyset$).
(2) $d^{p_\zeta} \restriction u = d$ for some fixed (rational) metric d.
(3) Denote by $g_{\zeta,\xi}$ the order preserving map from u_ζ onto u_ξ. Then $\mathrm{id}_u \cup g_{\zeta,\xi}$ is an isometry between u^{p_ζ} and u^{p_ξ}. This means that u_ζ and u_ξ are isometric and that for every $x \in u_\zeta$ and $y \in u$, $d^{p_\zeta}(x,y) = d^{p_\xi}(g_{\zeta,\xi}(x), y)$.
(4) $\{w^{p_\zeta} : \zeta < \omega_1\}$ is a Δ-system with root w.
(5) $\varepsilon^{p_\zeta}_\alpha = \varepsilon_\alpha$ for some fixed ε_α for every $\alpha \in w$ and $\zeta < \omega_1$.

(6) For every $\alpha \in w$, $\{\mathrm{dom} f_\alpha^{p_\zeta} : \zeta < \omega_1\}$ is a Δ system with root r_α and $|\mathrm{dom} f_\alpha^{p_\zeta}|$ is fixed.
(7) $f_\alpha^{p_\zeta} \restriction r_\alpha$ is fixed (may be assumed since $f_\alpha^{p_\zeta}(i) \in A_{\alpha,i}$ for all $i \in r$ and $A_{\alpha,i}$ is countable).
(8) Denote $\mathrm{dom} f_\alpha^{p_\zeta} = r_\alpha \cup s_\alpha^\zeta$ for $\alpha \in w$; then $f_\alpha^{p_\zeta}(s_\alpha^\zeta) \cap u = \emptyset$.
(9) For all $\alpha \in w$ and $\zeta, \xi < \omega_1$, $g_{\zeta,\xi}[f_\alpha^{p_\zeta}[s_\alpha^\zeta]] = f_\alpha^{p_\xi}[s_\alpha^\xi]$. Denote $h_\alpha^{\zeta,\xi} = f_\alpha^{p_\zeta} \circ h_{\zeta,\xi} \circ (f_\alpha^{p_\xi})^{-1}$. Thus $h_\alpha^{\zeta,\xi} : s_\alpha^\zeta \to s_\alpha^\xi$ and $f_\alpha^{p_\xi}(h_\alpha^{\zeta,\xi}(x)) = f_\alpha^{p_\zeta}(i)$ for all $i \in s_\alpha^z$.
(10) For all $\alpha, \beta \in w$ and $\zeta < \omega_1$ it holds that $\mathrm{ran}(f_\alpha^\zeta) \cap \mathrm{ran}(f_\beta^\zeta) \subseteq u$; here we use the fact that $\mathrm{ran} f_\alpha^\zeta \subseteq A_\alpha, \mathrm{ran}(f_\beta^\zeta) \subseteq A_\beta$ and $|A_\alpha \cap A_\beta| < \aleph_0$.

Every (ω_1, d_α) is separable, and in a separable metric space every subset of size $\aleph_1$ may be thinned out to a subset of the same size in which each points is a point of complete accumulation of the set, that is, every neighborhood of a point from the set contains $\aleph_1$ points from the set. Therefore we may also assume:

(11) for every $\alpha \in w$ and $\zeta < \omega$, every $i \in s_\alpha^\zeta$ is a point of complete accumulation in (ω_1, d_α) of $\{h_\alpha^{\zeta,\xi}(i) : \xi < \omega_1\}$.

We shall find now two conditions p_ζ, p_ξ, $\zeta < \xi < \omega_1$ and a condition $t \in Q$ which extends both p_ζ and p_ξ.

Fix some $\zeta < \omega_1$ and define $\delta_0 = \min\{d^{p_\zeta}(x, y) : x \neq y \in u \cup u_\zeta\}$. Next, for $\alpha \in w$ let $i, j \in \mathrm{dom} f_\alpha^{p_\zeta}$ be any pair of points. It holds that

$$d_\alpha(i,j)/(1 + \varepsilon_\zeta^{p_\zeta}) < d^{p_\zeta}(f_\zeta^{p_\zeta}(i), f_\zeta^{p_\zeta}(j)) < (1 + \varepsilon_\zeta^{p_\zeta}) d(i,j) \tag{1}$$

Which, denoting for simplicity $\lambda = (1 + \varepsilon_\alpha^{p_\zeta})$, $a = d_\alpha(i,j)$ and $b = d^{p_\zeta}(f_\alpha^{p_\zeta}(i), f_\zeta^{p_\zeta}(j))$, is re-written as

$$a/\lambda < b < \lambda a \tag{2}$$

There is a sufficiently small $\delta > 0$, depending on a and b, such that whenever $|b - b'| < \delta$ and $|a - a'| < \delta$, it holds that

$$a'/\lambda < b' < \lambda a' \tag{3}$$

Since w and each $\mathrm{dom} f_\alpha^{p_\zeta}$ for $\alpha \in w$ are finite, we can fix $\delta_1 > 0$ which is sufficiently small for (3) to hold for all $\alpha \in w$ and $i, j \in \mathrm{dom} f_\alpha^{p_\zeta}$.

Let $\delta = \min\{\delta_0/100, \delta_1/2\}$.

By condition (11) above, find $\zeta < \xi < \omega_1$ so that for each $\alpha \in w$ and $i \in s_\alpha^\zeta$ it holds that $d_\alpha(i, h_\alpha^{\zeta,\xi}(i)) < \delta$.

Let $u^t = u \cup u_\zeta \cup u_\xi$. We define now a metric d^t on u^t as follows. First, for each $\alpha \in w$ and $x = f_\alpha^{p_\zeta}(i) \in f_\alpha^{p_\zeta}[s_\alpha^\zeta]$ let $y = g_{\zeta,\xi}(x)$ and define $d^*(x, y) = r$, r a rational number which satisfies $1/(1 + \varepsilon_\alpha^p) d_\alpha(i,j) < r < (1 + \varepsilon_\alpha^p) d_\alpha(i,j)$, where $j = h_\alpha^{\zeta,\xi}(i)$. Since for $\alpha, \beta \in w$ the sets $f_\alpha^{p_\zeta}[s_\alpha^\zeta]$ and $f_\alpha^{p_\zeta}[s_\beta^\xi]$ are disjoint, and similarly for ξ, d^* is well defined, namely, at most one α is involved in defining the distance r. In fact, any two d^* edges are vertex disjoint.

Now $(u \cup u_\zeta \cup u_\xi, d \cup d^*)$ is a connected weighted graph. Let d^t be the shortest-path metric on $u \cup u_\zeta \cup u_\xi$ obtained from $d^{p_\zeta} \cup d^{p_\xi} \cup d^*$. It is obviously a rational metric, as all distances in $d^{p_\zeta} \cup d^{p_\xi} \cup d^*$ are rational.

Let us verify that d^t extends $d^{p_\zeta} \cup d^{p_\xi} \cup d^*$. Suppose $x \in u_\zeta, y \in u_\xi$ and $d^*(x,y)$ is defined. Any path from x to y other than (x,y) must contain some edge with a distance in $(u^{p_\zeta}, d^{p_\zeta})$, and all those distances are much larger than $d^*(x,y)$, so (x,y) is the shortest path from x to u. Suppose now that $x, y \in u^{p_\zeta}$. So (x,y) is the shortest path among all paths that lie in u^{p_ζ} and the path of minimal length from x to y among all paths that contain at least one d^* edge is necessarily the path (x, x', y', y) where $x' = g_{\zeta,\xi}(x)$ and $y' = g_{\zeta,\xi}(y)$ (since (u^{p_ξ}, d^{p_ξ}) is a metric space). Since $d^{p_\zeta}(x,y) = d^{p_\xi}(x', y')$ by condition (3), the length of this path is larger than $d^{p_\zeta}(x,y)$.

Let $f^t_\alpha = f^{p_\zeta}_\zeta \cup f^{p_\xi}_\zeta$, where we formally take f^p_α to be the empty function if $\alpha \notin w^p$.

Let $w^t = w^{p_\zeta} \cup w^{p_\xi}$ and let $\varepsilon^t_\alpha = \max\{\varepsilon^{p_\zeta}_\alpha, \varepsilon^{p_\xi}_\alpha\}$ where ε^p_α is taken as 0 if $\alpha \notin w^p$ (recall that $\varepsilon^{p_\zeta}_\alpha = \varepsilon^{p_\xi}_\alpha$ for $\alpha \in w$).

Now t is defined, and extends both p_ζ and p_ξ; one only needs to verify that $t \in Q$. For that, we need to verify that each f^t_α is $(1+\varepsilon^t_\alpha)$-bi-Lipschitz. For $\alpha \notin w$ this is trivial. Suppose $\alpha \in w$ and $i, h \in \mathrm{dom} f^t_\alpha = r_\alpha \cup s_\zeta \cup s_\xi$. The only case to check is when $i \in s^\zeta_\alpha$ and $j \in s^\xi_\alpha$. If $h^{\zeta,\xi}_\alpha(i) = j$ then this is taken care of by the choice of $d^t(f^{p_\zeta}_\alpha(i), f^{p_\xi}_\alpha(j)) = d^*(f^{p_\zeta}_\alpha(i), f^{p_\xi}_\alpha(j))$.

We are left with the main case: $j \neq h^{\zeta,\xi}_\alpha(i)$. Let $x = f^{p_\zeta}_\alpha(i)$, $y = f^{p_\xi}_\alpha(j)$ and denote $a' = d_\alpha(i,j)$. Let $j' \in x^\zeta_\alpha$ be such that $h^{\zeta,\xi}_\alpha(j') = j$ and let $y' = g^{-1}_{\zeta,\xi}(y)$ (so $f^{p_\zeta}_\zeta(j') = y'$).

Denote $b = d^t(x, y')$, $a = d_\alpha(i, j')$. We have that $a/\lambda < b < \lambda a$, and need to prove $a'/\lambda < b' < \lambda b'$.

It holds that $d_\alpha(j, j') < \delta \le \delta_1/2$, hence $d^t(y, y') < \delta_1$. By the triangle inequality in u^t, we have that $|b - b'| < \delta_1$. On the other hand, by the triangle inequality in (ω_1, d_α), we have that $|a' - a| < \delta < \delta_1$. Thus, by the choice of δ_1 so that (3) holds if (2) holds and $|a - a'| < \delta_1$, $|b - b'| < \delta_1$, we have that $a'/\lambda < b' < \lambda a'$, as required. □

Let $P = \langle P_\beta, Q_\beta : \beta < \omega_2 \rangle$ be a finite support iteration of length ω_2 in which each factor Q_β is the forcing notion we defined above, in V^{P_β}. Since each Q_β satisfies the ccc, the whole iteration satisfies the ccc and no cardinals or cofinalities are collapsed in the way — in particular the collection $\{A_\alpha : \alpha < \omega_3\}$ required for the definition of Q is preserved.

Since every metric d on ω_1 appears in some intermediate stage, the universe V^P satisfies that there is a collection of ω_2 separable metrics on ω_1 so that every separable metric on ω_1 is almost isometrically embedded into one of them.

There is no particular property of ω_1 that was required in the proof. Also, Baumgartner's result holds for other cardinals. We have proved then:

Theorem 30. *Let $\lambda > \aleph_0$ be a regular cardinal. It is consistent that $2^{\aleph_0} > \lambda^+$ and that there are λ^+ separable metrics on λ such that every separable metric on λ is almost-isometrically embedded into one of them.*

In the next section we shall see that separability is essential for this consistency result.

6. Non-separable spaces below the continuum

Now we show that the consistency proved in the previous Section for separable metric spaces of regular cardinality $\lambda < 2^{\aleph_0}$ is not possible for metric spaces in general if $\lambda > \aleph_1$. Even a weaker fact fails: there cannot be fewer than continuum metric spaces on λ so that every metric space is bi-Lipschit embedable into one of them if $\aleph_1 < \lambda < 2^{\aleph_0}$ and λ is regular.

Theorem 31. *If $\aleph_1 < \lambda < 2^{\aleph_0}$ then for every $\kappa < 2^{\aleph_0}$ and metric spaces $\{(\lambda, d_i) : i < \kappa\}$ there exists an ultra-metric space (λ, d) that is not bi-Lipschitz embedable into (λ, d_i) for all $i < \kappa$. In particular there is no single metric space (ω_2, d) into which every ultra-metric space of cardinality λ is bi-Lipschitz embedded.*

Proof. Let $\lambda > \aleph_1$ be a regular cardinal. Let $S^\lambda_0 = \{\delta < \lambda : \mathrm{cf}\delta = \lambda\}$, the stationary subset of λ of countably cofinal elements. For a regular $\lambda > \aleph_1$ we may fix a club guessing sequence $\overline{C} = \langle c_\alpha : \delta \in S^\lambda_0\rangle$ [13, 10]:

(1) $c_\delta \subseteq \delta = \sup c_\delta$ and $\mathrm{otp} c_\delta = \omega$ for all $\delta \in S^\lambda_0$;
(2) For every club $E \subseteq \lambda$ the set $S(E) = \{\delta \in S^\lambda_0 : c_\delta \subseteq E\}$ is stationary.

For each $\delta \in S^\lambda_0$ let $\langle \alpha^\delta_n : n < \omega\rangle$ be the increasing enumeration of c_δ.

Let (λ, d) be a given metric space. Let X_α denote the subspace $\{\beta : \beta < \alpha\}$. Let $d(\beta, X_\alpha)$ denote the distance of β from x_α, that is, the infimum of $d(\beta, \gamma)$ for all $\gamma < \alpha$. Thus $\langle d(\beta, X_{\alpha_n}) : n < \omega\rangle$ is a weakly decreasing sequence of positive real numbers.

Lemma 32. *Suppose $f : \lambda \to \lambda$ is a bi-Lipschitz embedding of (λ, d_1) in (λ, d_2) with constant $K \geq 1$. Then there is a club $E \subseteq \lambda$ such that: for all $\alpha \in E$ and $\beta > \alpha$ it holds that $f(\beta) > \alpha$ and*

$$(2K)^{-1} d_1(\beta, X_\alpha) \leq d_2(f(\beta), X_\alpha) \leq K d_1(\beta, X_\alpha) \tag{4}$$

Proof. Consider the structure $\mathcal{M} = (\lambda; d_1, d_2, f, \langle P_q : q \in \mathbb{Q}^+\rangle)$ where P_q is a binary predicate so that $\mathcal{M} \models P_q(\beta_1, \beta_2)$ iff $d_2(\beta_1, \beta_2) < q$.

Let $E = \{\alpha < \lambda : M{\restriction}X_\alpha \prec M\}$. Then $E \subseteq \lambda$ is a club. Suppose that $\alpha \in E$. Then X_α is closed under f and f^{-1} and therefore $f(\beta) > \alpha$. The inequality $d_2(f(\beta), X_\alpha) \leq K d_1(\beta, X_\alpha)$ is clear because X_α is preserved under f. For the other inequality suppose that $\gamma \in X_\alpha$ is arbitrary and let $\varepsilon := d_2(f(\beta), \gamma)$. Let $q > \varepsilon$ be an arbitrary rational number. Now $\mathcal{M} \models P_q(f(\beta), \gamma)$ and therefore, by $\mathcal{M}{\restriction}X_\alpha \prec \mathcal{M}$, there exists some $\beta' \in X_\alpha$ so that $\mathcal{M} \models P_q((f(\beta'), \gamma)$. Thus

$$d_2(f(\beta), f(\beta')) \leq d_2(f(\beta), \gamma) + d_2(\gamma, f(\beta')) < 2q$$

Since $K^{-1}d_1(\beta,\beta') \le d_2(f(\beta),f(\beta'))$ it follows that $K^{-1}d_1(\beta,X_\alpha) < 2q$, and since $q > \varepsilon$ is an arbitrary rational, it follows that $(2K)^{-1}d_1(\beta,X_\alpha) \le d_2(f(\beta),X_\alpha)$. □

Suppose (λ,d_i) are metric spaces for $i = 0,1$, that $K \ge 1$ and that $f:\lambda\to\lambda$ is a K-bi-Lipschitz embedding of (λ,d_1) in (λ,d_2). Let $E\subseteq\lambda$ be a club as guaranteed by the previous lemma. Suppose $\delta\in S(E)$, $\beta>\delta$ and $\langle\alpha_n : n<\omega\rangle$ is the increasing enumeration of c_δ. Let $\varepsilon_n = d_1(\beta,X_{\alpha_n})$ and let $\varepsilon'_n = d_2(f(\beta),X_{\alpha_n})$. From (4) it follows that for each n,

$$(2K^2)^{-1}\varepsilon_n/\varepsilon_{n+1} \le \varepsilon'_n/\varepsilon'_{n+1} \le 2K^2\varepsilon_n/\varepsilon_{n+1} \tag{5}$$

For a subset $A\subseteq\omega$, $\delta\in S^2_0$, $\beta>\delta$ a metric d over ω_2 and an integer $K\ge 1$ we write $\Theta_d(\beta,\delta,A,K)$ if $\varepsilon_n/\varepsilon_{n+1} = 1$ or $\varepsilon_n/\varepsilon_{n+1} > 4K^4$ for all n and $A = \{n : \varepsilon_n/\varepsilon_{n+1} > 4K^4\}$.

Write $\Phi_d(\beta,\delta,A,K)$ if $A = \{n : \varepsilon_m/\varepsilon_{n+1} > 2K^2\}$. From (5) it follows:

Lemma 33. *Suppose $f:\omega_2\to\omega_2$ is a K-bi-Lipschitz embedding of (ω_2,d_1) in (ω_2,d_2). Then there is a club $E\subseteq\omega_2$ such that for all $\delta\in S(E)$, $A\subseteq\omega$ and $\beta>\delta$ it holds that $f(\beta)>\delta$ and $\Phi_{d_1}(\beta,\delta,A,K)\Rightarrow\Theta_{d_2}(f(\beta),\delta,A,K)$.*

Lemma 34. *For every infinite $A\subseteq\omega$ there is an ultra-metric space (ω_2,d) and a club $E\subseteq\omega_2$ so that for all $\delta\in S(E)$ and integer $K\ge 1$ there exists $\beta>\delta$ with $\Phi_d(\beta,\delta,A,K)$.*

Proof. Suppose $A\subseteq\omega$ is infinite and let $\langle a_n : n<\omega\rangle$ be the increasing enumeration of A. Let $(\omega_2)^\omega$ be the tree of all ω-sequences over ω_2 and let d be the metric so that for distinct $\eta_1,\eta_2\in(\omega_2)^\omega$, $d(\eta_1,\eta_2) = 1/(n+1)$, where n is the length of the largest common initial segment of η_1,η_2 (or, equivalently, $d(\eta_1,\eta_2) = 1/(|\eta_1\cap\eta_2|+1)$). Every *finite* sequence $t:n\to\omega_2$ determines a basic clopen ball of radius $1/(n+1)$ in $((\omega_2)^\omega,d)$, which is $B_t = \{\eta\in(\omega_2)^\omega : t\subseteq\eta\}$.

By induction on $\alpha<\omega_2$ define an increasing and continuous chain of subsets $X_\alpha\subseteq(\omega_2)^\omega$ so that:

(1) $|X_\alpha| = \omega_1$ for all $\alpha<\omega_2$
(2) $X_\alpha\subseteq X_{\alpha+1}$ and $X_\alpha = \bigcup_{\beta<\alpha}$ if $\alpha<\omega_2$ is limit.
(3) For every $\nu\in X_\alpha$ and $k<\omega$ there exists $\eta\in X_{\alpha+1}\setminus X_\alpha$ so that $\nu\restriction k\subseteq\eta$ but $B_{\nu\restriction(n+1)}\cap X_\alpha = \emptyset$.
(4) If $\alpha = \delta\in S^2_\alpha$, $\langle\alpha_n : n<\omega\rangle$ is the increasing enumeration of c_δ, then for every integer $K\ge 1$ a sequence ν_K defined as follows. Let $b_n = (4K^2)^{n+1}$. Define an increasing sequence of finite sequences t_n with $|t_n| = b_n$ by induction on n as follows. $t_0 = \langle\rangle$. Suppose that t_n is defined and $B_{t_n}\cap X_{\alpha_{a_n}}\ne\emptyset$. Choose $\eta\in X_{\alpha_{a_n}+1}\cap B_{t_n}$ so that $B_{\eta\restriction(b_{n+1})}\cap X_{\alpha_{a_n}} = \emptyset$. Now let $t_{n+1} = \eta\restriction(b_{n+1})$.
Finally, let $\nu_K = \bigcup_n t_n$. Put ν_k in $X_{\delta+1}$ for each integer $K\ge 1$.

There is no problem to define X_α for all $\alpha<\omega_2$. Let $X = \bigcup_{\alpha<\omega_2} X_\alpha$. Fix a 1-1 onto function $F:X\to\omega_2$ and let d be the metric on ω_2 which makes

F an isometry. Observe that for some club $E \subseteq \omega_2$ it holds that $F[X_\alpha] = \alpha$ for all $\alpha \in E$. If $c_\delta \subseteq E$ let ν_K be the sequence we put in $X_{\alpha+1}$ in clause (4) of the inductive definition and let $\beta_K = F(\nu_K)$. We leave it to the reader to verify that $\Phi_d(\beta_K, \delta, A, K)$. $\square$

The proof of the Theorem follows from both lemmas. Suppose $\kappa < 2^{\aleph_0}$ and d_i is a metric on ω_2 for each $i < \kappa$. The set $\{A \subseteq \omega : (\exists i < \kappa)(\exists \delta \in S^2_0)(\exists K \in \omega \setminus \{0\})(\exists \beta > \delta)[\Theta_{d_i}(\beta, \delta, A, K)]\}$ has cardinality $\leq \aleph_2 \cdot \kappa < 2^{\aleph_0}$. Therefore there is some infinite $A \subseteq \omega$ not in this set. By Lemma 34 there is some ultra-metric d on ω_2 and a club $E \subseteq \omega_2$ so that for all $\delta \in S(E)$ and integer $K \geq 1$ there is $\beta > \delta$ with $\Phi_d(\beta, \delta, A, K)$. If for some $i < \kappa$ and integer $K \geq 1$ there is a K-bi-Lipschitz embedding φ of (ω_2, d) into (ω_2, d_i) then from Lemma 33 there is some club $E' \subseteq \omega_2$ so that for all $\delta \in S(E)$, $\beta > \delta$ and infinite $A \subseteq \omega$, $\varphi(\beta) > \delta$ and $\Phi_d(\beta, \delta, A, K) \Rightarrow \Theta_{d_i}(\beta, \delta, A, K)$. Let $E'' = E \cap E'$. Choose $\delta \in S(E'')$. Now for some $\beta > \delta$ it holds $\Phi_d(\beta, \delta, A)$, hence $\Theta_{d_i}(\beta, \delta, A, K)$ contrary to the choice of A. $\square$

Let us take a second look at the proof above. What is shown is that subsets of $\mathbb{N}$ — "reals" — can be coded into a metric space X of regular cardinality λ and retrieved from a larger space Y in which X is bi-Lipschitz embedded. The retrieval is done modulo some nonstationary subset of the cardinal λ.

For a metric space X of cardinality λ, let us define $S(X)$ as the set of all $A \subseteq \mathbb{N}$ so that: for every enumeration of X of ordertype λ, and for every club $E \subseteq \lambda$ there is some $\delta \in S^\lambda_0$ so that $c_\delta \subseteq E$ and for every $K > 1$ in $\mathbb{N}$ there is some $\beta > \delta$ so that $\Phi(\beta, \delta, A, K)$. The set of those As is preserved under bi-Lipschitz embeddings.

Theorem 35. *For every regular $\lambda > \aleph_1$ there is an order preserving map $F : (\mathcal{M}_\lambda, \leq_{BL} \to ([\mathcal{P}(\mathbb{N})]^{\leq \lambda}, \subseteq)$, where $\leq_{BL}$ is the quasi-ordering of bi-Lipschitz embeddability and $[\mathcal{P}(\mathbb{N})]^{\leq}$ is the family of subsets of $\mathcal{P}(\mathbb{N})$ whose cardinality is at most λ. The range of F restricted to ultra-metric spaces is cofinal in $(\mathcal{P}(\mathbb{N}), \subseteq)$.*

The picture at large is as follows: the set of all distances in a metric space is an invariant which is preserved under isometric embeddings which forbids the existence of universal metric space with respect to isometries in cardinalities below the continuum. With respect to almost isometric embeddings there is no such invariant for countable spaces, or for separable spaces of regular cardinality below the continuum. However, for general spaces of regular cardinality $\aleph_2$ or higher such invariants are again available, as proved above, and prohibit the existence of universal metric spaces.

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