PCF WITHOUT CHOICE
SH835

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ABSTRACT. We mainly investigate models of set theory with restricted choice, e.g., ZF + DC + the family of countable subsets of λ is well ordered for every λ (really local version for a given λ). We think that in this frame much of pcf theory, (and combinatorial set theory in general) can be generalized. We prove here, in particular, that there is a class of regular cardinals, every large enough successor of singular is not measurable and we can prove cardinal inequalities.

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Anotated Content

§ 0. Introduction

§ 0(A). Background, aims and results.

The thesis of [She97] was that pcf theory without full choice exists. Two theorems supporting this thesis were proved. The first ([She97, 4.6,pg.117], we shall not mention ZF) is:

Theorem 0.1. [DC] *If $\mathscr{H}(\mu)$ is well ordered, μ strong limit singular of uncountable cofinality then μ^+ is regular not measurable (and 2^μ is an $\aleph$, i.e. $\mathscr{P}(\mu)$ can be well ordered and no $\lambda \in (\mu, 2^\mu]$ is measurable).*

Note that before this Apter and Magidor [AM95] had proved the consistency of "$\mathscr{H}(\mu)$ well ordered, $\mu = \beth_\omega, (\forall \kappa < \mu)\mathrm{DC}_\kappa$ and μ^+ is measurable" so 0.1 says that this consistency result cannot be fully lifted to uncountable cofinalities. Generally without full choice, a successor cardinal being not measurable is worthwhile information.
A second theorem ([She97, §5]) was

Theorem 0.2. *Assume*

(a) $\mathrm{DC} + \mathrm{AC}_\kappa + \kappa$ *regular uncountable*
(b) $\langle \mu_i : i < \kappa \rangle$ *is increasing continuous with limit $\mu, \mu > \kappa, \mathscr{H}(\mu)$ is well ordered, μ strong limit, (we need just a somewhat weaker version, the so-called $i < \kappa \Rightarrow Tw_{\mathscr{D}_\kappa}(\mu_i) < \mu$).*

Then , we cannot have two regular cardinals θ such that for some stationary $S \subseteq \kappa$, the sequence $\langle \mathrm{cf}(\mu_i^+) : i \in S \rangle$ is constantly θ.

A dream was to prove that there is a class of regular cardinals from a restricted version of choice (see more [She97] and a little more in [**?**]).

Our original aim here is to improve those theorems. As for 0.1 we replace "$\mathscr{H}(\mu)$ well ordered" by "$[\mu]^{\aleph_0}$ is well ordered" and then by weaker statements.

We know (assuming full choice) that if, e.g., $\neg\exists 0^\#$ or there is no inner model with a measurable cardinal then though $\langle 2^\kappa : \kappa \text{ regular} \rangle$ is quite arbitrary, the size of $[\lambda]^\kappa, \lambda >> \kappa$ is strictly controlled (by Easton forcing [Eas70], and Jensen and Dodd [DJ82] respectively). It seemed that the situation here is parallel in some sense; under the restricted choice we assume, we cannot say much on the cardinality of $\mathscr{P}(\kappa)$ but can say something on the cardinality of $[\lambda]^\kappa$ for $\lambda >> \kappa$.

In the proofs we fulfill a promise from [She00, §5] about using $J[f, D]$ from Definition 0.12 instead of the nice filters used in [She97] and, to some extent, in early versions of this work which require going through inner models to prove their existence. This work is continued in Larson-Shelah [LS09] and will be continued in [She16]. On a different line with weak choice (say $\mathrm{DC}_{\aleph_0} + \mathrm{AC}_\mu, \mu$ fixed): see [She12], [She14] and [S$^+$]. The present work fits the thesies of [She94] which in particular says: it is better to look at $\langle \lambda^{\aleph_0} : \lambda \text{ a cardinality} \rangle$ then at $\langle 2^\lambda : \lambda$ a cardinal$\rangle$. Here instead well ordering $\mathscr{P}(\lambda)$ we well order $[\lambda]^{\aleph_0}$, this is enough for much.

A simply stated conclusion is (see 3.6)

Conclusion 0.3. *[DC] Assume* $[\lambda]^{\aleph_0}$ *is well ordered for every* λ.
1) If 2^{2^κ} *is well ordered* *then* *for every* $\lambda, [\lambda]^\kappa$ *is well ordered.*
2) For any set Y*, there is a derived set* Y_* *so called* $\text{Fil}^4_{\aleph_1}(Y)$ *of power near* $\mathscr{P}(\mathscr{P}(Y))$ *such that* $\Vdash_{\text{Levy}(\aleph_0,Y)}$ *"for every* $\lambda, {}^Y\lambda$ *is well ordered".*

Thesis 0.4. 1) If $\mathbf{V} \models$ "ZF + DC" and "every $[\lambda]^{\aleph_0}$ is well orderable" then $\mathbf{V}$ looks like the result of starting with a model of ZFC and using $\aleph_1$-complete forcing notions like Easton forcing, Levy collapses, and more generally, iterating of κ-complete forcing for $\kappa > \aleph_0$.
2) This approach is dual to investigating $\mathbf{L}[\mathbb{R}]$ - here we assume ω-sequences are understood (or weaker versions) and we try to understand $\mathbf{V}$ (over this), there over the reals everything is understood.

Also though our original motivation was to look at consequences of Ax_4, this was shadowed here by the try to use weaker relatives; see more in [She16].

Explanation 0.5. How do we analyze $[\mu]^\kappa$ or equivalently ${}^\kappa\mu$ here? We use $\aleph_1$-complete filters on κ and a well ordering of $[\alpha]^{\aleph_0}$ for appropriate α or less. We will consider $f : \kappa \to \mu$; now for every $\aleph_1$-complete filter D on κ, the ordinal $\text{rk}_D(f)$ gives us some information on α, but if $A, \kappa \backslash A \in D^+$ and $f{\restriction}A = 0_A$, then $\alpha = 0$ but we have no information on $f{\restriction}(\kappa\backslash A)$, then $\alpha = 0$ but we have no information on $f{\restriction}(\kappa\backslash A)$. Trying to correct this we consider the ideal $J[f,D] = \{A \subseteq \kappa : A = \emptyset \mod D$ or $A \in D^+$ but $\text{rk}_{D+(A)}(f) > \alpha\}$, this is an $\aleph_1$-complete ideal and so we may consider the pair $\bar{D} = (D_1, D_2) = (D, \text{dual}(J[f,D]))$. Now α and the pair $\bar{D}$ gives more information on f; they determine f modulo D_2. This is not enough so we use an algebra $\mathscr{B}$ on μ with no infinite decreasing sequence of sub-algebras built using the assumption "$[\mu]^{\aleph_0}$ is well ordered". So there is $Z \in D_2$ such that $A = c\ell_{\mathscr{B}}(\text{Rang}(f{\restriction}Z))$ is $\subseteq$-minimal.

Now the triple (D_1, D_2, Z) and the ordinal α almost determines f, we need one more piece of information with domain $\kappa : h(i) = \text{otp}(\alpha \cap Z)$, hence an ordinal $< \text{hrtg}(\text{Rang}(f))$. So we need a bound on it which depends on the choice of $\mathscr{B}$, usually it is $\text{hrtg}([\kappa]^{\aleph_0})$, natural by the construction of $\mathscr{B}$.

So $f{\restriction}Z$ is uniquely determined by the ordinal $\text{rk}_D(f)$ and the quadruple (D_1, D_2, Z, h), which belongs to a set defined from κ, independently of μ.

Lastly, considering all such filters D (recalling we are assuming DC) we can find countably many quadruple (D^n_1, D^n_2, Z^n, h^n) which together are enough as $\bigcup\limits_n Z^n = \kappa$.

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§ 0(B). **Preliminaries.**

Convention 0.6. We assume just $\mathbf{V} \models$ ZF if not said otherwise.

Notation 0.7. Let
1) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j$ denote ordinals.
2) $\kappa, \lambda, \mu, \chi$ denotes cardinals, infinite if not said otherwise.
3) n, m, k, ℓ denotes natural numbers.
4) D denotes a filter (on some set), I, J denote ideals on some set.

Definition 0.8. 1) $\text{hrtg}(A) = \text{Min}\{\alpha$: there is no function from A onto $\alpha\}$.
2) $\text{wlor}(A) = \text{Min}\{\alpha$: there is no one-to-one function from α into A or $\alpha = 0 \wedge A = \emptyset\}$ so $\text{wlor}(A) \leq \text{hrtg}(A)$.

Definition 0.9. 1) For D an $\aleph_1$-complete filter on Y and $f \in {}^Y\text{Ord}$ and $\alpha \in \text{Ord} \cup \{\infty\}$ we define when $\text{rk}_D(f) = \alpha$, by induction on α:

- ⊛ For $\alpha < \infty$, $\text{rk}_D(f) = \alpha$ iff $\beta < \alpha \Rightarrow \text{rk}_D(f) \neq \beta$ and for every $g \in {}^Y\text{Ord}$ satisfying $g <_D f$ there is $\beta < \alpha$ such that $\text{rk}_D(g) = \beta$.

2) We can replace D by the dual ideal. If $f \in {}^Z\text{Ord}$ and $Z \in D$ then we let $\text{rk}_D(f) = \text{rk}_{D+Z}(f \cup 0_{Y\setminus Z})$.

Galvin-Hajnal [GH75] use the rank for the club filter on ω_1. This was continued in [She80] where varying D was extensively used.

Claim 0.10. [DC] *In Definition 0.9,* $\text{rk}_D(f)$ *is always an ordinal and if* $\alpha \leq \text{rk}_D(f)$ *then for some* $g \in \prod_{y\in Y}(f(y)+1)$ *we have* $\alpha = \text{rk}_D(g)$, *(if* $\alpha < \text{rk}_D(f)$ *we can add* $g <_D f$*; if* $\text{rk}_D(f) < \infty$ *then* DC *is not necessary; if* $\text{rk}_D(f) = \alpha$ *this is trivial, as we can choose* $g = f$*).*

Claim 0.11. *1)* [DC] *If* D *is an* $\aleph_1$*-complete filter on* Y *and* $f \in {}^Y\text{Ord}$ *and* $Y = \cup\{Y_n : n < \omega\}$ *then* $\text{rk}_D(f) = \text{Min}\{\text{rk}_{D+Y_n}(f) : n < \omega$ *and* $Y_n \in D^+\}$, *(*[She80]*).*
2) [$\text{DC} + \text{AC}_{\alpha^*}$*] If* D *is a* κ*-complete filter on* Y, κ *a cardinal* $> \aleph_0$ *and* $f \in {}^Y\text{Ord}$ *and* $Y = \cup\{Y_\alpha : \alpha < \alpha^*\}, \alpha^* < \kappa$ *then* $\text{rk}_D(f) = \text{Min}\{\text{rk}_{D+Y_\alpha}(f) : \alpha < \alpha^*$ *and* $Y_\alpha \in D^+\}$.

Proof. 1) By [She80], in fact, $\text{AC}_{\aleph_0}$ suffice.
2) By [She80], in fact, DC is not necessary. $\square_{0.11}$

Definition 0.12. For Y, D, f as in 0.9 let $J[f,D] =: \{Z \subseteq Y : Y\setminus Z \in D$ or $Y\setminus Z \in D^+$ and $\text{rk}(f)_{D+(Y\setminus Z)} > \text{rk}_D(f)\}$.

Claim 0.13. [$\text{DC}+\text{AC}_{<\kappa}$] Assume D is a κ-complete filter on $Y, \kappa > \aleph_0$.
1) If $f \in {}^Y\text{Ord}$ then $J[f,D]$ is a κ-complete ideal on Y.
2) If $f_1, f_2 \in {}^Y\text{Ord}$ and $J = J[f_1,D] = J[f_2,D]$ then $\text{rk}_D(f_1) < \text{rk}_D(f_2) \Rightarrow f_1 < f_2 \mod J$ and $\text{rk}_D(f_1) = \text{rk}_D(f_2) \Rightarrow f_1 = f_2 \mod J$.

Proof. Straightforward or see [She00, §5] and the reference there to [She97] (and [She80]). $\square_{0.13}$

Definition 0.14. 1) Here $Y \leq_{\text{qu}} Z$ or $|Y| \leq_{\text{qu}} |Z|$ or $|Y| \leq_{\text{qu}} Z$ or $Y \leq_{\text{qu}} |Z|$ means that $Y = \emptyset$ or there is a function from Z (equivalently from a subset of Z) onto Y.
2) $\text{reg}(\alpha) = \text{Min}\{\partial : \partial \geq \alpha$ is a regular cardinal$\}$.

Definition 0.15. For a set Y, cardinal κ and ordinal γ we define $\mathscr{H}_{<\kappa,\gamma}(Y)$ by induction on γ: if $\gamma = 0, \mathscr{H}_{<\kappa,\gamma}(Y) = Y$, if $\gamma = \beta+1$ then $\mathscr{H}_{<\kappa,\gamma}(Y) = \mathscr{H}_{<\kappa,\beta}(Y) \cup \{u : u \subseteq \mathscr{H}_{<\kappa,\beta}(Y)$ and $|u| < \kappa\}$ and if γ is a limit ordinal then $\mathscr{H}_{<\kappa,\gamma}(Y) = \cup\{\mathscr{H}_{<\kappa,\beta}(Y) : \beta < \gamma\}$.

Observation 0.16. *1) If λ is the disjoint union of $\langle W_z : z \in Z\rangle$ and $z \in Z \Rightarrow |W_z| < \lambda$ and* $\text{wlor}(Z) \leq \lambda$ *<u>then</u> $\lambda = \sup\{\text{otp}(W_z) : z \in Z\}$ hence $\text{cf}(\lambda) < \text{hrtg}(Z)$.*
2) If $\lambda = \cup\{W_z : z \in Z\}$ and $\text{wlor}(\mathscr{P}(Z)) \leq \lambda$ <u>then</u> $\sup\{\text{otp}(W_z) : z \in Z\} = \lambda$.
3) If $\lambda = \cup\{W_z : z \in Z\}$ and $|Z| < \lambda$ <u>then</u> $\lambda = \sup\{\text{otp}(W_z) : z \in Z\}$.
4) If $Z \subseteq \text{Ord}, \bar{W} = \langle W_\alpha : \alpha \in Z\rangle, W_\alpha \subseteq \text{Ord}$ and $\lambda \geq \aleph_0, |Z|, |W_\alpha|$ for $\alpha \in Z$ <u>then</u> $\cup\{W_\alpha : \alpha \in Z\}$ has cardinality $\leq \lambda$.

Proof. 1) Let $Z_1 = \{z \in Z : W_z \neq \emptyset\}$, so the mapping $z \mapsto \text{Min}(W_z)$ exemplifies that Z_1 is well ordered hence by the definition of $\text{wlor}(Z_1)$ the power $|Z_1|$ is an aleph $< \text{wlor}(Z_1) \leq \text{wlor}(Z)$ and by assumption $\text{wlor}(Z) \leq \lambda$. Now if the desirable conclusion fails then $\gamma^* = \sup(\{\text{otp}(W_z) : z \in Z_1\} \cup \{|Z_1|\})$ is an ordinal $< \lambda$, so we can find a sequence $\langle u_\gamma : \gamma < \gamma^*\rangle$ such that $\text{otp}(u_\gamma) \leq \gamma^*, u_\gamma \subseteq \lambda$ and $\lambda = \cup\{u_\gamma : \gamma < \gamma^*\}$, so $\gamma^* < \lambda \leq |\gamma^* \times \gamma^*|$, easy contradiction.
2) For $x \subseteq Z$ let $W^*_x = \{\alpha < \lambda : (\forall z \in Z)(\alpha \in W_z \equiv z \in x)\}$ hence λ is the disjoint union of $\{W^*_x : x \in \mathscr{P}(Z)\backslash\{\emptyset\}\}$. So the result follows by part (1).
3) So let $<_*$ be a well ordering of Z and let $W'_z = \{\alpha \in W_z$: if $y <_* z$ then $\alpha \notin W_y\}$, so $\langle W'_z : z \in Z\rangle$ is a well defined sequence of pairwise disjoint sets with union equal to $\cup\{W_z : z \in Z\} = \lambda$ and $\text{otp}(W'_z) \leq \text{otp}(W_z)$. Hence if $|W_z| = \lambda$ for some $z \in Z$ the desirable conclusion is obvious, otherwise the result follows by part (1).
4) Should be clear. $\square_{0.16}$

Definition 0.17. 1) We say that $c\ell$ is a very weak closure operation on λ of character (μ, κ) <u>when</u>:

- (a) $c\ell$ is a function from $\mathscr{P}(\lambda)$ to $\mathscr{P}(\lambda)$
- (b) $u \in [\lambda]^{\leq\kappa} \Rightarrow |c\ell(u)| \leq \mu$
- (c) $u \subseteq \lambda \Rightarrow u \cup \{0\} \subseteq c\ell(u)$, the 0 for technical reasons.

1A) We say that $c\ell$ is a weak closure[1] operation on λ of character (μ, κ) <u>when</u> (a),(b),(c) above and:

- (d) $u \subseteq v \subseteq \lambda \Rightarrow u \subseteq c\ell(u) \subseteq c\ell(v)$
- (e) $c\ell(u) = \cup\{c\ell(v) : v \subseteq u, |v| \leq \kappa\}$.

1B) Let "... character $(< \mu, \kappa)$ or $(\mu, < \kappa)$, or $(< \mu, < \kappa)$" have the obvious meaning but if μ is an ordinal not a cardinal, then "$< \mu$" means of order type $< \mu$; similarly for "$< \kappa$". Let "... character (μ, Y)" means "character $(< \mu^+, < \text{hrtg}(Y))$"
1C) We omit the weak <u>when</u> in addition:

- (f) $c\ell(u) = c\ell(c\ell(u))$ for $u \subseteq \lambda$.

2) We say λ is f-inaccessible when $\delta \in \lambda \cap \text{Dom}(f) \Rightarrow f(\delta) < \lambda$.
3) We say $c\ell : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is well founded <u>when</u> for no sequence $\langle \mathscr{U}_n : n < \omega\rangle$ of subsets of λ do we have $c\ell(\mathscr{U}_{n+1}) \subset \mathscr{U}_n$ for $n < \omega$.

[1]so by actually only $c\ell{\restriction}[\lambda]^{\leq\kappa}$ count

4) For $c\ell$ a partial function from $\mathscr{P}(\alpha)$ to $\mathscr{P}(\alpha)$ (for simplicity assume $\alpha = \cup\{u : u \in \mathrm{Dom}(c\ell)\}$) let $c\ell^1_{\varepsilon,<\kappa}$ be the function from $\mathscr{P}(\alpha)$ to $\mathscr{P}(\alpha)$ defined by induction on the ordinal ε as follows:

- (a) $c\ell^1_{0,<\kappa}(u) = u$
- (b) $c\ell^1_{\varepsilon+1,<\kappa}(u) = \{0\} \cup c\ell^1_{\varepsilon,<\kappa}(u) \cup \bigcup\{c\ell(v) : v \subseteq c\ell^1_{\varepsilon,<\kappa}(u)$ and $v \in \mathrm{Dom}(c\ell), |v| < \kappa\}$
- (c) for limit ε let $c\ell^1_{\varepsilon,<\kappa}(u) = \cup\{c\ell^1_{\zeta,<\kappa}(u) : \zeta < \varepsilon\}$.

4A) Instead "$< \kappa$" we may use "$\leq \kappa$".
5) For any function $F : [\lambda]^{\aleph_0} \to \lambda$ and countable $u \subseteq \lambda$ we define $c\ell^2_\varepsilon(u, F)$ by induction on $\varepsilon \leq \omega_1$

- (a) $c\ell^2_0(u, F) = u \cup \{0\}$
- (b) $c\ell^2_{\varepsilon+1}(u, F) = c\ell^2_\varepsilon(u, F) \cup \{F(c\ell^2_\varepsilon(u, F))\}$
- (c) $c\ell^2_\varepsilon(u, F) = \cup\{c\ell^2_\zeta(u, F) : \zeta < \varepsilon\}$ when $\varepsilon \leq \omega_1$ is a limit ordinal.

6) For countable u and F as in part (5) let $c\ell^3_F(u) = c\ell^3(u, F) := c\ell^2_{\omega_1}(u, F)$ and for any $u \subseteq \lambda$ let $c\ell^4_F(u) := u \cup \bigcup\{c\ell^3_F(v) : v \in \mathrm{Dom}(F)\}$.
7) For a cardinal ∂ we say that $c\ell : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is ∂-well founded <u>when</u> for no $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ of subsets of λ do we have $\varepsilon < \zeta < \partial \Rightarrow c\ell(\mathscr{U}_\zeta) \not\supseteq \mathscr{U}_\varepsilon$.
8) If $F : [\lambda]^{\leq\kappa} \to \lambda$ and $u \subseteq \lambda$ <u>then</u> we let $c\ell_F(u) = c\ell^1_F(u)$ be the minimal subset v of λ such that $w \in [v]^{\leq\kappa} \Rightarrow F(w) \in v$ and $u \subseteq v$ (exists).

Observation 0.18. *For $F : [\lambda]^{\aleph_0} \to \lambda$, the operation $u \mapsto c\ell^3_F(u)$ is a very weak closure operation of character $(\aleph_1, \aleph_0)$.*

Remark 0.19. So for any very weak closure operation, $\aleph_0$-well founded is a stronger property than well founded, but if $u \subseteq \lambda \Rightarrow c\ell(c\ell(u)) = c\ell(u)$ which is reasonable, they are equivalent.

Observation 0.20. *$[\alpha]^\partial$ is well ordered iff ${}^\partial\alpha$ is well ordered when $\alpha \geq \partial$.*

Proof. Use a pairing function on α for showing $|{}^\partial\alpha| \leq [\alpha]^\partial$, so $\Rightarrow$ holds. If ${}^\partial\alpha$ is well ordered by $<_*$ map $u \in [\alpha]^\partial$ to the $<_*$-first $f \in {}^\partial\alpha$ satisfying $\mathrm{Rang}(f) = u$. $\square_{0.20}$

§ 1. Representing ${}^\kappa\lambda$

Here we give a simple case to illustrate what we do (see later on improvements in the hypothesis and the conclusion). Specifically, if Y is uncountable and $[\lambda]^{\aleph_0}$ is well ordered, then the set ${}^Y\lambda$ can be analyzed modulo countable union over few (i.e., their number depends on Y but not on λ) well ordered sets.

Definition 1.1. 1)

- (a) $\mathrm{Fil}_{\aleph_1}(Y) = \mathrm{Fil}^1_{\aleph_1}(Y) = \{D : D$ is an $\aleph_1$-complete filter on $Y\}$, so Y is defined from D as $\cup\{X : X \in D\}$
- (b) $\mathrm{Fil}^2_{\aleph_1}(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y, $(\emptyset \notin D_2$, of course$)\}$; in this context $Z \in \bar{D}$ means $Z \in D_2$
- (c) $\mathrm{Fil}^3_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h) : (D_1, D_2) \in \mathrm{Fil}^2_{\aleph_1}(Y)$ and $h : Y \to \alpha$ for some $\alpha < \mu\}$, if we omit μ we mean $\mu = \mathrm{hrtg}(Y) \cup \omega$
- (d) $\mathrm{Fil}^4_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h, Z) : (D_1, D_2, h) \in \mathrm{Fil}^3_{\aleph_1}(Y, \mu), Z \in D_2\}$; omitting μ means as above.

2) For $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y, \mu)$ let $Y = Y^{[\mathfrak{y}]} = Y[\mathfrak{y}]$ and $\mathfrak{y} = (D^{\mathfrak{y}}_1, D^{\mathfrak{y}}_2, h^{\mathfrak{y}}, Z^{\mathfrak{y}}) = (D_1[\mathfrak{y}], D_2[\mathfrak{y}], h[\mathfrak{y}], Z[\mathfrak{y}])$; similarly for the others and let $D^{\mathfrak{y}} = D[\mathfrak{y}]$ be $D^{\mathfrak{y}}_1 + Z^{\mathfrak{y}}$.
3) We can replace $\aleph_1$ by any $\kappa > \aleph_1$ (the results can be generalized easily assuming DC + $\mathrm{AC}_{<\kappa}$, used in §2).

Theorem 1.2. *[DC] Assume $[\lambda]^{\aleph_0}$ is well ordered.*
Then we can find a sequence $\langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\rangle$ satisfying

- (α) $\mathscr{F}_{\mathfrak{y}} \subseteq {}^{Z[\mathfrak{y}]}\lambda$
- (β) *$\mathscr{F}_{\mathfrak{y}}$ is a well ordered set by $f_1 <_{\mathfrak{y}} f_2 \Leftrightarrow \mathrm{rk}_{D[\mathfrak{y}]}(f_1) < \mathrm{rk}_{D[\mathfrak{y}]}(f_2)$ so $f \mapsto \mathrm{rk}_{D[\mathfrak{y}]}(f)$ is a one-to-one mapping from $\mathscr{F}_{\mathfrak{y}}$ into the ordinals*
- (γ) *if $f \in {}^Y\lambda$ then we can find a sequence $\langle \mathfrak{y}_n : n < \omega\rangle$ with $\mathfrak{y}_n \in \mathrm{Fil}^4_{\aleph_1}(Y)$ such that $n < \omega \Rightarrow f \restriction Z^{\mathfrak{y}_n} \in \mathscr{F}_{\mathfrak{y}_n}$ and $\cup\{Z^{\mathfrak{y}_n} : n < \omega\} = Y$.*

An immediate consequence of 1.2 is

Conclusion 1.3. *1) [DC + ${}^\omega\alpha$ is well-orderable for every ordinal α].*
For any set Y and cardinal λ there is a sequence $\langle \mathscr{F}_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\rangle$ such that

- (a) ${}^Y\lambda = \cup\{\mathscr{F}_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\}$
- (b) *$\mathscr{F}_{\bar{\mathfrak{x}}}$ is well orderable for each $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$*
- $(b)^+$ *moreover, uniformly, i.e., there is a sequence $\langle <_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y)\rangle$ such that $<_{\bar{\mathfrak{x}}}$ is a well order of $\mathscr{F}_{\bar{\mathfrak{x}}}$*
- (c) *there is a function F with domain $\mathscr{P}({}^Y\lambda)\backslash\{\emptyset\}$ such that: if $S \subseteq {}^Y\lambda$ is non-empty then $F(S)$ is a non-empty subset of S of power $\leq_{\mathrm{qu}} {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y)))$ recalling Definition 0.14. In fact, some ordinal $\alpha(*)$ and $\bar{u}$ we have:*
  - (α) *$\bar{u} = \langle \mathscr{U}_\alpha : \alpha < \alpha(*)\rangle$ is a partition of ${}^Y\lambda$*
  - (β) *if $S \subseteq {}^Y\lambda$ then $F(S) = \mathscr{U}_{f(S)} \cap S$ where $f(S) = \mathrm{Min}\{\alpha : \mathscr{U}_\alpha \cap S \neq \emptyset\}$*
  - (γ) *if $\alpha < \alpha(*)$ then $|\mathscr{U}_\alpha| < \mathrm{hrtg}({}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y)))$.*

2) [DC] For any Y, λ above, if $[\alpha(*)]^{\aleph_0}$ is well ordered where $\alpha(*) = \cup\{\mathrm{rk}_D(f)+1 : f \in {}^Y\lambda$ and $D \in \mathrm{Fil}^1_{\aleph_1}(Y)\}$ then ${}^Y\lambda$ satisfies the conclusion of part (1).

Remark 1.4. So clause (c) of 1.3(1) is a weak form of choice.

Proof. Proof of 1.3 1) Let $\langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\rangle$ be as in 1.2.

For each $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ (so $\bar{\mathfrak{x}} = \langle \mathfrak{x}_n : n < \omega\rangle$) let

$$\mathscr{F}'_{\bar{\mathfrak{x}}} = \{f : \ f \text{ is a function from } Y \text{ to } \lambda \text{ such that } n < \omega \Rightarrow f \restriction Z^{\mathfrak{x}_n} \in \mathscr{F}_{\mathfrak{x}_n} \text{ and } Y = \cup\{Z^{\mathfrak{x}_n} : n < \omega\}\}.$$

Now

$(*)_1$ ${}^Y\lambda = \cup\{\mathscr{F}'_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\}$.

[Why? By clause (γ) of 1.2.]

Let $\alpha(*) = \cup\{\mathrm{rk}_D(f)+1 : f \in {}^Y\lambda$ and $D \in \mathrm{Fil}^1_{\aleph_1}(Y)\}$. For $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ we define the function $G_{\bar{\mathfrak{x}}} : \mathscr{F}'_{\bar{\mathfrak{x}}} \to {}^\omega\alpha(*)$ by $G_{\bar{\mathfrak{x}}}(f) = \langle \mathrm{rk}_{D_1[\mathfrak{x}_n]}(f) : n < \omega\rangle$.

Next

$(*)_2$ (α) $\bar{G} = \langle G_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\rangle$ exists
(β) $G_{\bar{\mathfrak{x}}}$ is a function from $\mathscr{F}'_{\bar{\mathfrak{x}}}$ to ${}^\omega\alpha(*)$
(γ) $G_{\bar{\mathfrak{x}}}$ is one to one.

[Should be clear, e.g. for $(*)_2(\gamma)$ read the definition of $\mathscr{F}'_{\bar{\mathfrak{x}}'}$ and clause (β) of Theorem 1.2.]

Let $<_*$ be a well ordering of ${}^\omega\alpha(*)$ and for $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ let $<_{\bar{\mathfrak{x}}}$ be the following two place relation on $\mathscr{F}'_{\bar{\mathfrak{x}}}$:

$(*)_3$ $f_1 <_{\bar{\mathfrak{x}}} f_2$ iff $G_{\bar{\mathfrak{x}}}(f_1) <_* G_{\bar{\mathfrak{x}}}(f_2)$.

Obviously

$(*)_4$ (α) $\langle <_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\rangle$ exists
(β) $<_{\bar{\mathfrak{x}}}$ is a well ordering of $\mathscr{F}'_{\bar{\mathfrak{x}}}$.

By $(*)_1 + (*)_4$ we have proved clauses (a),(b),(b)$^+$ of the conclusion. Now clause (c) follows: for non-empty $S \subseteq {}^Y\lambda$, let $f(S)$ be $\min\{\mathrm{otp}(\{g : g <_{\bar{\mathfrak{y}}} f\}, <_{\bar{\mathfrak{y}}}) : \bar{\mathfrak{y}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ and $f \in \mathscr{F}'_{\bar{\mathfrak{y}}} \cap S\}$. Also for any ordinal γ let $\mathscr{U}^1_\gamma := \{f$: for some $\bar{\mathfrak{y}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ we have $\gamma = \mathrm{otp}(\{g : g <_{\bar{\mathfrak{y}}} f\}, <_{\bar{\mathfrak{y}}})\}$ and $\mathscr{U}_\gamma = \mathscr{U}^1_\gamma \setminus \cup \bigcup\{\mathscr{U}^1_\beta : \beta < \gamma\}$.

Lastly, we let $F(S) = \mathscr{U}_{f(S)} \cap S$. Now check.

2) Similarly. $\square_{1.3}$

Proof. Proof of Theorem 1.2 First

$\circledast_1$ there are a cardinal μ and a sequence $\bar{u} = \langle u_\alpha : \alpha < \mu\rangle$ listing $[\lambda]^{\aleph_0}$.

[Why? By the assumption.]

Second, we can deduce

$\circledast_2$ there are $\mu_1 \le \mu$ and a sequence $\bar{u} = \langle u_\alpha : \alpha < \mu_1\rangle$ such that:
(a) $u_\alpha \in [\lambda]^{\aleph_0}$
(b) if $u \in [\lambda]^{\le\aleph_0}$ then for some finite $w \subseteq \mu_1, u \subseteq \cup\{u_\beta : \beta \in w\}$
(c) u_α is not included in $u_{\alpha_0} \cup \ldots \cup u_{\alpha_{n-1}}$ when $n < \omega, \alpha_0, \ldots, \alpha_{n-1} < \alpha$.

[Why? Let $\bar{u}^0$ be of the form $\langle u_\alpha : \alpha < \alpha^* \rangle$ such that $(a) + (b)$ holds and $\ell g(\bar{u}^0)$ is minimal; it is well defined and $\ell g(\bar{u}^0) \le \mu$ by $\circledast_1$. Let $W = \{\alpha < \ell g(\bar{u}^0) : u^0_\alpha \nsubseteq \cup\{u^0_\beta : \beta \in w\}$ when $w \subseteq \alpha$ is finite$\}$. Let $\mu_1 = |W|$ and let $f : \mu_1 \to W$ be one-to-one onto, let $u_\alpha = u^0_{f(\alpha)}$ so $\langle u_\alpha : \alpha < \mu_1 \rangle$ satisfies $(a) + (b)$ and $\mu_1 = |W| \le \ell g(\bar{u}^0)$. So by the choice of $\bar{u}^0$ we have $\ell g(\bar{u}^0) = \mu_1$. So we can choose f such that it is increasing hence $\bar{u}$ is as required.]

- $\circledast_3$ we can define $\mathbf{n} : [\lambda]^{\le \aleph_0} \to \omega$ and partial functions $F_\ell : [\lambda]^{\le \aleph_0} \to \mu_1$ for $\ell < \omega$ (so $\langle F_\ell : \ell < \omega \rangle$ exists) as follows:
  - (a) u infinite $\Rightarrow F_0(u) =$ Min$\{\alpha$: for some finite $w \subseteq \alpha, u \subseteq u_\alpha \cup \bigcup\{u_\beta : \beta \in w\}$ mod finite$\}$
  - (b) u finite $\Rightarrow F_0(u)$ undefined
  - (c) $F_{\ell+1}(u) := F_0(u \backslash (u_{F_0(u)} \cup \ldots \cup u_{F_\ell(u)}))$ for $\ell < \omega$ when $F_\ell(u)$ is defined
  - (d) $\mathbf{n}(u) :=$ Min$\{\ell : F_\ell(u)$ undefined$\}$.

Then

- $\circledast_4$ (a) $F_{\ell+1}(u) < F_\ell(u) < \mu_1$ when they are well defined
  - (b) $\mathbf{n}(u)$ is a well defined natural number and $u \backslash \cup \{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\}$ is finite and $k < \mathbf{n}(u) \Rightarrow (u \backslash \cup \{u_{F_\ell(u)} : \ell < k\}) \cap u_{F_k(u)}$ is infinite
  - (c) if $u_1, u_2 \in [\lambda]^{\aleph_0}, u_1 \subseteq u_2$ and $u_2 \backslash u_1$ is finite then $F_\ell(u_1) = F_\ell(u_2)$ for $\ell < \mathbf{n}(u_1)$ and $\mathbf{n}(u_1) = \mathbf{n}(u_2)$
- $\circledast_5$ define $F_* : [\lambda]^{\aleph_0} \to \lambda$ by $F_*(u) =$ Min$(\cup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\} \cup \{0\} \backslash u)$ if well defined, zero otherwise
  [Note: the reader may wonder: if you add $\{0\}$ then Min$(-) = 0$ in all cases. However, if $0 \in u$ then by "$\backslash u$", zero does not belong to the set from which we choose a minimal ordinal.]
- $\circledast_6$ if $u \in [\lambda]^{\aleph_0}$ then
  - (α) $c\ell^3(u, F_*) = c\ell^3_{F_*}(u)$ is $F'(u) := u \cup \bigcup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\} \cup \{0\}$
  - (β) $c\ell^3_{F_*}(u) = c\ell^2_{\varepsilon(u)}(F)$ for some $\varepsilon(u) < \omega_1$
  - (γ) there is $\bar{F} = \langle F'_\varepsilon : \varepsilon < \omega_1 \rangle$ such that: for every $u \in [\lambda]^{\aleph_0}, c\ell^3_{F_*}(u) = \{F'_\varepsilon(u) : \varepsilon < \varepsilon(u)\}$ and $F'_\varepsilon(u) = 0$ if $\varepsilon \in [\varepsilon(u), \omega_1)$
  - (δ) in fact $F'_\varepsilon(u)$ is the ε-th member of $c\ell^3_{F_*}(u)$ if $\varepsilon < \varepsilon(u)$.

[Why? Define w^ε_u by induction on ε by $w^0_u = u, w^{\varepsilon+1}_u = w^\varepsilon_u \cup \{F_*(w^\varepsilon_u)\}$ and for limit ordinal ε we let $w^\varepsilon_u = \cup\{w^\zeta_u : \zeta < \varepsilon\}$. We can prove by induction on ε that $w^\varepsilon_u \subseteq F'(u)$ which is countable. The partial function g with domain $F'(u) \backslash u$ to Ord, $g(\alpha) =$ Min$\{\varepsilon : \alpha \in w^{\varepsilon+1}_u\}$ is one to one onto an ordinal call it $\varepsilon(*)$, so $w^{\varepsilon(*)}_u \subseteq F'(u)$ and if they are not equal that $F_*(w^{\varepsilon(*)}_u) \in F'(u) \backslash w^{\varepsilon(*)}_u$ hence $w^{\varepsilon(*)}_u \subsetneqq w^{\varepsilon(*)+1}_u$ contradicting the choice of $\varepsilon(*)$. So clause (α) holds. In fact, $c\ell^3(u, F_*) = w^{\varepsilon(*)}_u$ and clause (β) holds. CLauses $(\gamma), (\delta)$ should be clear.]

- $\circledast_7$ there is no sequence $\langle \mathscr{U}_n : n < \omega \rangle$ such that:
  - (a) $\mathscr{U}_{n+1} \subseteq \mathscr{U}_n \subset \lambda$
  - (b) $\mathscr{U}_n$ is closed under F_*, i.e. $u \in [\mathscr{U}_n]^{\aleph_0} \Rightarrow F_*(u) \in \mathscr{U}_n$
  - (c) $\mathscr{U}_{n+1} \ne \mathscr{U}_n$.

[Why? Assume toward contradiction that $\langle \mathscr{U}_n : n < \omega\rangle$ satisfies clauses (a),(b),(c). Let $\alpha_n = \text{Min}(\mathscr{U}_n \backslash \mathscr{U}_{n+1})$ for $n < \omega$ hence the sequence $\bar\alpha = \langle \alpha_n : n < \omega\rangle$ is well defined with no repetitions and let $\beta_{m,\ell} := F_\ell(\{\alpha_n : n \ge m\})$ for $m < \omega$ and $\ell < \mathbf{n}_m := \mathbf{n}(\{\alpha_n : n \in [m,\omega)\})$. As $\bar\alpha$ is with no repetition, $\mathbf{n}_m > 0$ and by $\circledast_4(c)$ clearly $\mathbf{n}_m = \mathbf{n}_0$ for $m < \omega$ and $\beta_{m,\ell} = \beta_{m,0}$ for $m < \omega, \ell < \mathbf{n}_0$. So letting $v_m = \cup\{u_{F_\ell(\{\alpha_n : n \in [m,\omega)\})} : \ell < \mathbf{n}_m\}$, it does not depend on m so $v_m = v_0$, and by the choice of F_*, as $\{\alpha_n : n \in [m,\omega)\} \subseteq \mathscr{U}_m$ and $\mathscr{U}_m$ is closed under F_* clearly $v_m \subseteq \mathscr{U}_m$. Together $v_0 = v_m \subseteq \mathscr{U}_m$ so $v_0 \subseteq \cap\{\mathscr{U}_m : m < \omega\}$. Also, by the definition of the F_ℓ's, $\{\alpha_n : n < \omega\}\backslash v_0$ is finite so for some $k < \omega, \{\alpha_m : n \in [k,\omega)\} \subseteq v_0$ but $v_0 \subseteq \mathscr{U}_{k+1}$ contradicting the choice of α_k.]

Moreover, recalling Definition 0.17(6):

$\circledast'_7$ there is no sequence $\langle \mathscr{U}_n : n < \omega\rangle$ such that
- (a) $\mathscr{U}_{n+1} \subseteq \mathscr{U}_n \subseteq \lambda$
- (b) $c\ell^4_{F_*}(\mathscr{U}_{n+1})\backslash \mathscr{U}_n \ne \emptyset$.

[Why? As above but letting $\alpha_n = \text{Min}(\mathscr{U}_n \backslash c\ell^3_{F_*}(\mathscr{U}_n))$.]

Now we define for $(D_1,D_2,h,Z) \in \text{Fil}^4_{\aleph_1}(Y)$ and ordinal α the following, recalling Definition 0.17(6) for clauses (e),(f):

$\circledast_8$ $\mathscr{F}_{(D_1 D_2,h,Z),\alpha} =: \{f :$
- (a) f is a function from Z to λ
- (b) $\text{rk}_{D_1+Z}(f \cup 0_{(Y\backslash Z)}) = \alpha$
- (c) $D_2 = \{Y\backslash X : X \subseteq Y$ satisfies $X = \emptyset \mod D_1$ or $X \in D_1^+$ and $\text{rk}_{D_1+X}(f \cup 0_{(Y\backslash Z)}) > \alpha$ that is $\text{rk}_{D_1+X}(f) > \alpha\}$
- (d) $Z \in D_2$, really follows
- (e) if $Z' \subseteq Z \wedge Z' \in D_2$ then $c\ell^3_{F_*}(\text{Rang}(f \restriction Z')) = c\ell^3_{F_*}(\text{Rang}(f))$
- (f) $y \in Z \Rightarrow f(y) =$ the $h(y)$-th member of $c\ell^3_{F_*}(\text{Rang}(f))\}$.

So we have:

$\circledast_9$ $\mathscr{F}_{(D_1,D_2,h,Z),\alpha}$ has at most one member; call it $f_{(D_1,D_2,h,Z),\alpha}$ (when defined; pedantically we should write $f_{(D_1,D_2,h,Z),c\ell,\alpha}$)

$\circledast_{10}$ $\mathscr{F}_{(D_1,D_2,h,Z)} =: \cup\{\mathscr{F}_{(D_1,D_2,h,Z),\alpha} : \alpha$ an ordinal$\}$ is a well ordered set.

[Why? Define $<_{(D_1,D_2,h,Z)}$ by the α's, i.e. $f^1 < f^2$ <u>iff</u> there are $\alpha_1 < \alpha_2$ such that $f^\ell = f_{(D_1,D_2,h,2),\alpha_\ell}$ for $\ell = 1,2$.]

$\circledast_{11}$ if $f : Y \to \lambda$ and $Z \subseteq Y$ then the set $\text{Rang}(f \restriction Z)$ has cardinality $< \text{hrtg}(Z)$.

[Why? By the definition of hrtg$(-)$ this should be clear.]

$\circledast_{12}$ if $f : Z \to \lambda$ and $Z \subseteq Y$ <u>then</u> $c\ell^4_{F_*}(\text{Rang}(f)) \subseteq \lambda$ has cardinality $< \text{hrtg}([Z]^{\aleph_0})$ or is finite.

Why? If Rang(f) is countable more holds by 0.18. Otherwise, by $\circledast_6(\beta)$ recallng Definition 0.17(6) we have $c\ell^4_{F_*}(\text{Rang}(f)) = \text{Rang}(f) \cup \{F'_\varepsilon(u) : u \in [\text{Rang}(f)]^{\aleph_0}$ and $\varepsilon < \omega_1\}$.

Let $\alpha(*)$ be minimal such that $\text{Rang}(f) \cap \alpha(*)$ has order type ω_1. Let $h_1,h_2 : \omega_1 \to \omega_1$ be such that $h_\ell(\varepsilon) < \max\{\varepsilon,1\}$ and for every $\varepsilon_1,\varepsilon_2 < \omega_1$ there is $\zeta \in [\varepsilon_1+\varepsilon_2+1,\omega_1)$ such that $h_\ell(\zeta) = \varepsilon_\ell$ for $\ell = 1,2$. Define $F : [Z]^{\aleph_0} \to \lambda$ as follows: if

$u \in [\mathrm{Rang}(f)]^{\aleph_0}$, let $\varepsilon_\ell(u) = h_\ell(\mathrm{otp}(u \cap \alpha(*))$ for $\ell = 1, 2$ and $F(u) = F'_{\varepsilon_2(u)}(\{\alpha \in u$: if $\alpha < \alpha(*)$ then $\mathrm{otp}(u \cap \alpha) < \varepsilon_1(u)\})$.

Now

- $\bullet_1$ if $u \in [\mathrm{Rang}(f)]^{\aleph_0}$ then $F(u)$ is $F_\varepsilon(v)$ for some $v \in [Z]^{\aleph_0}$ and $\varepsilon < \omega_1$.

[Why? As $F(u) \in \mathrm{Rang}(F'_{\varepsilon_2(u)} \restriction [\mathrm{Rang}(f)]^{\aleph_0})$]

- $\bullet_2$ $\{F(u) : u \in [\mathrm{Rang}(f)]^{\aleph_0}\} \subseteq c\ell^4_{F_*}(\mathrm{Rang}(f))$.

[Why? By $\bullet_1$ recalling $\circledast_6$.]

- $\bullet_3$ if $u \in [\mathrm{Rang}(f)]^{\aleph_0}$ and $\varepsilon < \omega_1$ then $F'_\varepsilon(u)$ is $F(u)$ for some $v \in [\mathrm{Rang}(f)]^{\aleph_0}$.

[Why? Let $\varepsilon_1 = \mathrm{otp}(u \cap \alpha(*)), \varepsilon_2 = \varepsilon$; now let $\zeta < \omega_1$ be such that $h_\ell(\zeta) = \varepsilon_\ell$ for $\ell = 1, 2$. Let $v = u \cup \{\alpha : \alpha \in \mathrm{Rang}(f) \cap \alpha(*)$ and $\alpha \geq \sup(u \cap \alpha(*)) + 1$ and $\mathrm{otp}(\mathrm{Rang}(f) \cap \alpha \backslash (\sup(u \cap \alpha(*) + 1)) < (\zeta - \varepsilon_1))\}$.]

So $F(u) = F'_\varepsilon(u)$. By $\bullet_2 + \bullet_3$ we can conclude:

- $\bullet_4$ in $\bullet_2$ we have equality.

Together $c\ell^4_{F_*}(\mathrm{Rang}(f)) = \{F(u) : u \in [\mathrm{Rang}(f)]^{\aleph_0}\} \cup \mathrm{Rang}(f)$ so it is the union of two sets; by the definition of $\mathrm{hrtg}(-)$ the first is of cardinality $< \mathrm{hrtg}([Z]^{\aleph_0})$ and the second is of cardinality $< \mathrm{hrtg}[Z]$, so we are easily done proving $\circledast_{12}$

$\circledast_{13}$ if $f : Y \rightarrow \lambda$ then for some sequence $\langle (\mathfrak{y}_n, \alpha_n) : n < \omega \rangle$ we have $\mathfrak{y}_n \in \mathrm{Fil}^4_{\aleph_1}(Y)$ and $\alpha_n \in \mathrm{Ord}$ for $n < \omega$ and $f = \cup\{f_{\mathfrak{y}_n, \alpha_n} : n < \omega\}$.

[Why? Let

$$\mathscr{I}^0_f = \{Z \subseteq Y : \text{ for some } \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y) \text{ satisfying } Z^{\mathfrak{y}} = Z \text{ and ordinal } \alpha, f_{\mathfrak{y},\alpha} \text{ is well defined and equal to } f \restriction Z\}$$

$$\mathscr{I}_f = \{Z \subseteq Y : Z \text{ is included in a countable union of members of } \mathscr{I}^0_f\}.$$

So recalling we are assuming DC it is enough to show that $Y \in \mathscr{I}_f$.

Toward contradiction assume not. Let $D_1 = \{Y \backslash Z : Z \in \mathscr{I}_f\}$, clearly it belongs to $\mathrm{Fil}_{\aleph_1}(Y)$, noting that $\emptyset \in \mathscr{I}_f$. So $\alpha(*) := \mathrm{rk}_{D_1}(f)$ is well defined (by 0.10) recalling that only $\mathrm{DC} = \mathrm{DC}_{\aleph_0}$ is needed.

Let

$$D_2 = \{X \subseteq Y : X \in D_1 \text{ or } \mathrm{rk}_{D_1 + (Y \backslash X)}(f) > \alpha(*)\}.$$

By 0.12 + 0.13 clearly D_2 is an $\aleph_1$-complete filter on Y extending D_1.

Now we try to choose $Z_n \in D_2$ for $n < \omega$ such that $Z_{n+1} \subseteq Z_n$ and $c\ell^4_{F_*}(\mathrm{Rang}(f \restriction Z_{n+1}))$ does not include $\mathrm{Rang}(f \restriction Z_n)$.

For $n = 0, Z_0 = Y$ is O.K.

By $\circledast'_7$ we cannot have such ω-sequence $\langle Z_n : n < \omega \rangle$; so by DC for some (unique) $n = n(*), Z_n$ is chosen but not Z_{n+1}.

Let $h : Z_n \rightarrow \mathrm{hrtg}([Y]^{\aleph_0}) \cup \omega_1$ be:

$$h(y) = \mathrm{otp}(f(y) \cap c\ell^4_{F_*}(\mathrm{Rang}(f \restriction Z_n))).$$

Now h is well defined by $\circledast_{12}$. Easily

$$f \restriction Z_n \in \mathscr{F}_{(D_1+Z_n,D_2,h,Z_n),\alpha(*)}$$

hence $Z_n \in \mathscr{I}^0_f \subseteq \mathscr{I}_f$, contradiction to $Z_n \in D_2, D_1 \subseteq D_2$.

So we are done proving $\circledast_{13}$.]

Now clause (β) of the conclusion holds by the definition of $\mathscr{F}_{\mathfrak{y}}$, clause (α) holds by $\circledast_{10}$ recalling $\circledast_8, \circledast_9$ and clause (γ) holds by $\circledast_{12}$. $\square_{1.2}$

Remark 1.5. We can improve 1.2 in some way by weakening the demands on $\bar{u}$.

We may replace the assumption "$[\lambda]^{\aleph_0}$ is well ordered" by:

- $(*)$ there is $\langle u_\alpha : \alpha < \alpha^* \rangle$, a sequence of members of $[\lambda]^{\aleph_0}$ such that $(\forall u \in [\lambda]^{\aleph_0})(\exists \alpha)(u \cap u_\alpha$ infinite).

[Why? We define $F_\varepsilon : [\lambda]^{\aleph_0} \to \alpha^*$ by induction on $\varepsilon < \omega_1$ by $F_\varepsilon(v) :=$ Min$\{\alpha < \alpha^* : (v \backslash v \cup \{F_*(v) : \zeta < \varepsilon\}) \cap u_\alpha$ infinite$\}$ if well defined and let $F : [\lambda]^{\aleph_0} \to [\lambda]^{\aleph_0}$ be defined by $F(v) = \cup\{F_\varepsilon(v) : \varepsilon < \omega_1, F_\varepsilon(v)$ well defined$\}$.

Lastly, let $F_*(u) =$ min$(F(u)\backslash u)$.]

Observation 1.6. *1) The power of* $\mathrm{Fil}^4_{\aleph_1}(Y,\mu)$ *is smaller or equal to the power of the set* $(\mathscr{P}(\mathscr{P}(Y)))^2 \times \mathscr{P}(Y) \times \mu^{|Y|}$*; if* $\aleph_0 \le |Y|$ *this is equal to the power of* $\mathscr{P}(\mathscr{P}(Y)) \times {}^Y\mu$.
2) The power of $\mathrm{Fil}^4_{\aleph_1}(Y)$ *is smaller or equal to the power of the set* $(\mathscr{P}(\mathscr{P}(Y)))^2 \times \mathscr{P}(Y) \times \cup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}$.
3) In part (2), if $\aleph_0 \le |Y|$ *this is equal to* $|\mathscr{P}(\mathscr{P}(Y))| \times \cup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}$*; also* $\alpha < \mathrm{hrtg}([Y]^{\aleph_0}) \Rightarrow |\mathscr{P}(\mathscr{P}(Y)) \times {}^Y\alpha| = |\mathscr{P}(\mathscr{P}(Y))|$ *and* $|\mathrm{Fil}^4_{\aleph_1}(Y)| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y \times Y))$.

Remark 1.7. 1) As we are assuming DC, the case $\aleph_0 \not\le |Y|$ means that Y is finite, so degenerated. Also if $|Y| = \aleph_0$ then $\mathrm{Fil}^1_{\aleph_1}(Y) = \{\{Z \subseteq Y : Z \supseteq X\} : X \subseteq Y\}$ hence $|\mathrm{Fil}^1_{\aleph_1}(Y)| = \mathscr{P}(Y)|$ hence $\mathrm{FIL}^4_{\aleph_1}(Y,\mu)$ has the same power as $\mathscr{P}(Y) \times {}^\omega\mu$ again this is a dull case.

Proof. 1) Reading the definition of $\mathrm{Fil}^4_{\aleph_1}(Y,\mu)$ clearly its power is $\le$ the power of $\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y) \times \mu^{|Y|}$. If $\aleph_0 \le |Y|$ then $|\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y)| \le |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y))| = 2^{|\mathscr{P}(Y))} \times 2^{|\mathscr{P}(Y)|} \le 2^{|\mathscr{P}(Y)|+|\mathscr{P}(Y)|} = 2^{|\mathscr{P}(Y)|} = |\mathscr{P}(\mathscr{P}(Y))| \le |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y) \times \mu^{|Y|}|$ as $\mathscr{P}(Y) + \mathscr{P}(Y) = 2^{|Y|} \times 2 = 2^{|Y|+1} = 2^{|Y|}$; so the second conclusion follows.
2) Read the definitions.
3) If $\alpha < \mathrm{hrtg}([Y]^{\aleph_0})$ then let f be a function from $[Y]^{\aleph_0}$ onto α and for $\beta < \alpha$ let $A_{f,\beta} = \{u \in [Y]^{\aleph_0} : f(u) < \beta\}$. So $\beta \mapsto A_{f,\beta}$ is a one-to-one function from α onto $\{A_{f,\gamma} : \gamma < \alpha\} \subseteq \mathscr{P}(\mathscr{P}(Y))$ so $|{}^Y\alpha| \le \mathscr{P}(\mathscr{P}(Y))$ and $\mathscr{P}(\mathscr{P}(Y)) \times |{}^Y\alpha| \le \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y)) \le 2^{|\mathscr{P}(Y)|+|\mathscr{P})Y)|} = 2^{|\mathscr{P}(Y)|}$. Better, for f a function from $[Y]^{\aleph_0}$ onto $\alpha < \mathscr{P}(Y)$ let $A_f = \{(y_1,y_2) : f(y_1) < f(y_2)\} \subseteq Y \times Y$. Define $F : \mathscr{P}(Y \times Y) \to \mathrm{hrtg}(Y)$ by $F(A) = \alpha$ if $A = A_f$ and f, α are as above, and $F(A) = 0$ otherwise.

So $|\mathscr{P}(\mathscr{P}(Y)) \cup \bigcup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y \times Y))) = |\mathscr{P}(\mathscr{P}(Y \times Y))|$. By the proof above we easily get $|\mathrm{Fil}^4_{\aleph_1}(Y)| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y \times Y))$. $\square_{1.6}$

Claim 1.8. *[DC] Assume*

(*a*) $\mathfrak{a}$ *is a countable set of limit ordinals*

(*b*) $<_*$ *is a well ordering of* $\Pi\mathfrak{a}$

(*c*) $\theta \in \mathfrak{a} \Rightarrow \mathrm{cf}(\theta) \geq \kappa$ *where* $\kappa = \mathrm{hrtg}(\mathscr{P}(\omega))$ *or just* $\Pi\mathfrak{a}/[\mathfrak{a}]^{<\aleph_0}$ *is* $< \kappa$*-directed.*

Then we can define $(\bar{J}, \bar{\mathfrak{b}}, \bar{\mathbf{f}})$ *such that*

(α) (i) $\bar{J} = \langle J_i : i \leq i(*)\rangle$ *where* $i(*) < \mathrm{hrtg}(\mathscr{P}(\omega))$

(ii) J_i *is an ideal on* $\mathfrak{a}$ *(though not necessarily a proper ideal)*

(iii) J_i *is increasing continuous with* $i, J_0 = \{\emptyset\}, J_{i(*)} = \mathscr{P}(\mathfrak{a})$

(iv) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_i : i < i(*)\rangle, \mathfrak{b}_i \subseteq \mathfrak{a}$ *and* $J_{i+1} = J_i + \mathfrak{b}_i$

(v) *so* J_i *is the ideal on* $\mathfrak{a}$ *generated by* $\{\mathfrak{b}_j : j < i\}$

(β) (i) $\bar{\mathbf{f}} = \langle \bar{f}^i : i < i(*)\rangle$

(ii) $\bar{f}^i = \langle f^i_\alpha : \alpha < \alpha_i\rangle$

(iii) $f^i_\alpha \in \prod \mathfrak{a}$ *is* $<_{J_i}$*-increasing with* $\alpha < \alpha_i$

(iv) $\{f^i_\alpha : \alpha < \alpha_i\}$ *is cofinal in* $(\prod \mathfrak{a}, <_{J_i + (\mathfrak{a} \backslash \mathfrak{b}_i)})$

(γ) (i) $\mathrm{cf}(\prod \mathfrak{a}) \leq \sum\limits_{i<i(*)} \alpha_i$

(ii) *for every* $f \in \Pi\mathfrak{a}$ *for some* n *and finite set* $\{(i_\ell, \gamma_\ell) : \ell < n\}$ *such that* $i_\ell < i(*), \gamma_\ell < \alpha_{i_\ell}$ *we have* $f < \max_{\ell<n} f^{i_\ell}_{\gamma_\ell}$*, i.e.,* $(\forall \theta \in \mathfrak{a})(\exists \ell < n)[f(\theta) < f^{i_\ell}_{\gamma_\ell}(\theta)]$.

Remark 1.9. Note that there is no harm in having more than one occurence of $\theta \in \mathfrak{a}$. See more in [She16], e.g. on uncountable $\mathfrak{a}$.

Proof. Note that:

$\circledast_1$ clause (γ) follows from $(\alpha) + (\beta)$.

[Why? Easily $(\gamma)(ii) \Rightarrow (\gamma)(i)$. Now let $g \in \Pi\mathfrak{a}$ and let $I_g = \{\mathfrak{b} \subseteq \mathfrak{a}$: we can find $n < \omega$ and $i_\ell < i(*)$ and $\beta_\ell < \alpha_{i_\ell}$ for $\ell < n$ such that $\theta \in \mathfrak{b} \Rightarrow (\exists \ell < n)(g(\theta) < f^{i_\ell}_{\beta_\ell}(\theta))\}$.

Easily I_g is an ideal on $\mathfrak{a}$ though not necessarily a proper ideal. Note that if $\mathfrak{a} \in I_g$ we are done. So assume $\mathfrak{a} \notin I_g$. Note that $I_g \subseteq J_{i(*)}$ hence $j_g = \min\{i \leq i(*)$: some $\mathfrak{c} \in \mathscr{P}(\mathfrak{a}) \backslash I_g$ belongs to $J_i\}$ is well defined (as $\mathfrak{a} \in \mathscr{P}(\mathfrak{a}) \backslash I_g \wedge \mathfrak{a} \in J_{i(*)}$). As $J_0 = \{\emptyset\}$ and clearly if $\emptyset \in \mathscr{T}_g$ we have $j_a > 0$. As $\langle J_i : i \leq i(*)\rangle$ is $\subseteq$-increasing continuous, necessarily j_g is a successor ordinal say $j_g = i_g + 1$ and let $i(g) = i_g$ and choose $\mathfrak{c} \in J_{i(g)} \backslash I_g$, clearly $J_{i(g)} \subseteq I_g$ so $\mathfrak{c}$ belongs to $J_{j_g} \backslash J_{i_g}$. By clause $(\beta)(iv)$ there is $\alpha < \alpha_{i(g)}$ such that $g < f^i_\alpha \mod (J_{i(g)} + (\mathfrak{a} \backslash \mathfrak{b}_{i(g)}))$.

Now let $\mathfrak{d} = \{\theta \in \mathfrak{a} : g(\theta) < f^i_\alpha(\theta)\}$ so by the choice of α we have $\mathfrak{d} = \mathfrak{a} \mod (J_{i(g)} + (\mathfrak{a} \backslash \mathfrak{b}_{(g)})$ which means that $\mathfrak{b}_{i(g)} \subseteq \mathfrak{d} \mod J_{i(g)}$ so as $J_{i(g)+1} = J_{i(g)} + \mathfrak{b}_{i,g}$ and $\mathfrak{c} \in J_{i(g)+1} \backslash J_{i(g)}$ clearly $\mathfrak{c} \subseteq \mathfrak{b}_{i(g)} \mod J_{i(g)}$.

But by the definition of the ideal $J_{i(g)}$ and of $\mathfrak{d}$ necessarily $\mathfrak{d} \in J_{i(g)}$ and recall $J_{i(g)} \subseteq J_{i(g)}$, contradicting the conclusion of the last sentence.]

Since (γ) follows from $(\alpha) + (\beta)$, it suffices to prove these parts. By induction on $i < \kappa$ we try to choose $(\bar{J}^i, \bar{\mathfrak{b}}^i, \bar{\mathbf{f}}^i)$ where $\bar{J}^i = \langle J_j : j \leq i\rangle, \bar{\mathfrak{b}}^i = \langle \mathfrak{b}^i_j : j < i\rangle, \bar{\mathbf{f}}^i = \langle \bar{f}^j : j < i\rangle$ which satisfies the relevant parts of the conclusion and do it uniformly from $(\mathfrak{a}, <_*)$. Once we arrive at i such that $J_i = \mathscr{P}(\mathfrak{a})$ we are done.

For $i = 0$ recalling $J_0 = \{\emptyset\}$ there is no problem.

For i limit recalling that $J_i = \cup\{J_j : j < i\}$ there is no problem and note that if $j < i \Rightarrow \mathfrak{a} \notin J_j$ then $\mathfrak{a} \notin J_i$.

So assume that $(\bar{J}^i, \mathfrak{b}^i, \bar{\mathbf{f}}^i)$ is well defined and $\mathfrak{a} \notin J_i$ and we shall define for $i+1$.

We try to choose $\bar{g}^{i,\varepsilon} = \langle g^{i,\varepsilon}_\alpha : \alpha < \delta_{i,\varepsilon}\rangle$ and $\mathfrak{b}_{i,\varepsilon}$ by induction on $\varepsilon < \omega_1$ and for each ε we try to choose $g^{i,\varepsilon}_\alpha \in \Pi\mathfrak{a}$ by induction on α (in fact $\alpha < \text{hrtg}(\Pi\mathfrak{a})$ suffice, we shall get stuck earlier) such that:

$\circledast^2_{i,\varepsilon}$ (a) if $\beta < \alpha$ then $g^{i,\varepsilon}_\beta <_{J_i} g^{i,\varepsilon}_\alpha$

(b) if $\zeta < \varepsilon$ and $\alpha < \delta_{i,\zeta}$ then $g^{i,\zeta}_\alpha \le g^{i,\varepsilon}_\alpha$

(c) if $\text{cf}(\alpha) = \aleph_1$ then $g^{i,\varepsilon}_\alpha$ is defined by
$$\theta \in \mathfrak{a} \Rightarrow g^{i,\varepsilon}_\alpha(\theta) = \text{ Min}\{\bigcup_{\beta \in C} g^{i,\varepsilon}_\beta(\theta) : C \text{ is a club of } \alpha\}$$

(d) if α is a limit ordinal and $\text{cf}(\alpha) \ne \aleph_1, \alpha \ne 0$ then $g^{i,\varepsilon}_\alpha$ is the $<_*$-first $g \in \Pi\mathfrak{a}$ satisfying clauses (a) + (b)

(e) if we have $\langle g^{i,\varepsilon}_\beta : \beta < \alpha\rangle$, $\text{cf}(\alpha) > \aleph_1$, moreover $\text{cf}(\alpha) \ge \min\{\text{cf}(\theta) : \theta \in \mathfrak{a}\}$ and there is no g as required in clause (d) then $\delta_{i,\varepsilon} = \alpha$

(f) if $\alpha = 0$ or α is a successor, then $g^{i,\varepsilon}_\alpha$ is the $<_*$-first $g \in \Pi\mathfrak{a}$ such that:
- $\bullet_1$ $\zeta < \varepsilon \wedge \alpha < \delta_{i,\zeta} \Rightarrow g^{i,\zeta}_\alpha \le g$
- $\bullet_2$ $\beta < \alpha \Rightarrow g^{i,\varepsilon}_\beta < g^{i,\varepsilon}_\alpha \mod J_i$
- $\bullet_3$ $\varepsilon = \zeta + 1 \Rightarrow (\forall \beta < \delta_{i,\zeta})[\neg(g \le_{J_i} g^{i,\zeta}_\beta)]$, follows if $\alpha > 0$

(g) J_i is the ideal on $\mathscr{P}(\mathfrak{a})$ generated by $\{\mathfrak{b}_j : j < i\}$

(h) $\mathfrak{b}_{i,\varepsilon} \in (J_i)^+$ so $\mathfrak{b}_{i,\varepsilon} \subseteq \mathfrak{a}$

(i) $\bar{g}^{i,\varepsilon}$ is increasing and cofinal in $(\Pi(\mathfrak{a}), <_{J_i + (\mathfrak{a} \backslash \mathfrak{b}_{i,\varepsilon})})$

(j) $\mathfrak{b}_{i,\varepsilon}$ is such that under clauses $(h) + (i)$ the set $\{\text{otp}(\mathfrak{a} \cap \theta) : \theta \in \mathfrak{b}_{i,\varepsilon}\}$ is $<_*$-minimal

(k) $\mathfrak{b}_{i,\zeta} \subseteq \mathfrak{b}_{i,\varepsilon} \mod J_i$ (follows by "if $\zeta < i$ then $g^{i,\varepsilon}_0$ is a $<_{J_i + \mathfrak{b}_{i,\zeta}}$-upper bound of $\bar{g}^{i,\zeta}$".

Clearly in stage ε we first choose $g^{i,\varepsilon}_\alpha$ by induction on α. As $\beta < \alpha \Rightarrow g^{i,\varepsilon}_\beta \ne g^{i,\varepsilon}_\alpha$ we are stuck in some $\delta_{i,\varepsilon}$ and then choose $\mathfrak{b}_{i,\varepsilon}$.

We now give details on some points:

$(*)_0$ if $\alpha = 0$ then we can choose $g^{2,\varepsilon}_0$.

[Why? Trivial.]

$(*)_1$ Clause (c) is O.K., that is: if we arrive to $(\varepsilon, \alpha), \text{cf}(\alpha) = \aleph_1$ then we can define $g^{i,\varepsilon}_\alpha$.

[Why? We already have $\langle g^{i,\varepsilon}_\alpha : \alpha < \delta\rangle$ and $\langle g^{i,\zeta}_\alpha : \alpha < \delta_{i,\zeta}, \zeta < \varepsilon\rangle$, and we define $g^{i,\varepsilon}_\delta$ as there. Now $g^{i,\varepsilon}_\delta(\theta)$ is well defined as the "Min" is taken on a non-empty set of ordinals as we are assuming $\text{cf}(\delta) = \aleph_1$. The value is $< \theta$ because for some club C of δ, $\text{otp}(C) = \omega_1$, so $g^{i,\varepsilon}_\delta(\theta) \le \cup\{g^{i,\varepsilon}_\beta(\theta) : \beta \in C\}$ but this set is $\subseteq \theta$ while $\text{cf}(\theta) > \aleph_1$ by clause (c) of the assumption. By $\text{AC}_{\aleph_0}$ we can find a sequence $\langle C_\theta : \theta \in \mathfrak{a}\rangle$ such that: C_θ is a club of δ of order type ω_1 satisfying $g^{i,\varepsilon}_\delta(\theta) = \cup\{g^{i,\varepsilon}_\alpha(\theta) : \alpha \in C_\theta\}$

hence for every club C of δ included in C_θ we have $g^{i,\varepsilon}_\delta(\theta) = \cup\{g^{i,\varepsilon}_\alpha(\theta) : \alpha \in C_\theta\}$. Now $\theta \in \mathfrak{a} \Rightarrow g^{i,\varepsilon}_\delta(\theta) = \bigcup\limits_{\alpha \in C} g^{i,\varepsilon}_\alpha(\theta)$ when $C := \cap\{C_\sigma : \sigma \in \mathfrak{a}\}$, because C too is a club of δ recalling $\mathfrak{a}$ is countable. So if $\alpha < \delta$ then for some β we have $\alpha < \beta \in C$ hence the set $\mathfrak{c} := \{\theta \in \mathfrak{a} : g^{i,\varepsilon}_\alpha(\theta) \geq g^{i,\varepsilon}_\beta(\theta)\}$ belongs to J_i and $\theta \in \mathfrak{a} \backslash \mathfrak{c} \Rightarrow g^{i,\varepsilon}_\alpha(\theta) < g^{i,\varepsilon}_\beta(\theta) \leq g^{i,\varepsilon}_\delta(\theta)$, so indeed $g^{i,\varepsilon}_\alpha <_{J_i} g^{i,\varepsilon}_\delta$.

Lastly, why $\zeta < \varepsilon \Rightarrow g^{i,\zeta}_\delta \leq g^{i,\varepsilon}_\delta$? As we can find a club C of δ which is as above for both $g^{i,\zeta}_\delta$ and $g^{i,\varepsilon}_\delta$ and recall that clause (b) of $\circledast_{i,\varepsilon}$ holds for every $\beta \in C$. Together $g^{i,\varepsilon}_\delta$ is as required.]

$(*)_2$ $\mathrm{cf}(\delta_{i,\varepsilon}) > \aleph_1$ and even $\mathrm{cf}(\delta_{i,\varepsilon}) \geq \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$.

[Why? We have to prove that arriving to $\alpha > 0$, if $\mathrm{cf}(\alpha) < \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$ then we can choose $g^{i,\varepsilon}_\alpha$ as required. The cases $\mathrm{cf}(\alpha) = \aleph_1, \alpha = 0$ are covered by $(*)_1, (*)_0$ respectively, otherwise let $u \subseteq \alpha$ be unbounded of order type $\mathrm{cf}(\alpha)$, and define a function g from $\mathfrak{a}$ to the ordinals by $g(\theta) = \sup(\{g^{i,\varepsilon}_\beta(\theta) : \beta \in u\} \cup \{g^{i,\zeta}_\alpha(\theta) : \zeta < \varepsilon\})$. This is a subset of θ of cardinality $< |\mathfrak{a}| + \mathrm{cf}(\alpha)$ which is $< \theta = \mathrm{cf}(\theta)$ hence $g \in \Pi\mathfrak{a}$, easily is as required, i.e. satisfies clauses (a) + (b) and the $<_*$-first such g is $g^{i,\varepsilon}_\alpha$.]

Note that clause (e) of $\circledast_{i,\varepsilon}$ follows.

$(*)_3$ if $\zeta < \varepsilon$ then $\delta_{i,\varepsilon} \leq \delta_{i,\zeta}$.

[Why? Otherwise $g^{i,\varepsilon}_{\delta_{i,\zeta}}$ contradict clause (e) of $\circledast_{i,\zeta}$.]

$(*)_4$ if $g^{i,\varepsilon} = \langle g^{i,\varepsilon}_\alpha : \alpha < \delta_{i,\varepsilon}\rangle$ is well defined and $\mathrm{cf}(\delta_{i,\varepsilon}) ge \kappa$ then $\mathfrak{b}_{i,\varepsilon}$ is well defined.

[Why? Clearly it suffices to prove that there is $\mathfrak{b}$ as required on $\mathfrak{b}_{i,\varepsilon}$ (in clauses (b),(i)). So toward contradiction assume that for every $\mathfrak{b} \in J^+_i, \bar{g}^{i,\varepsilon}$ is not $<_{J_i}$-cofinal in $\Pi\mathfrak{a}$ hence there is $h \in \Pi\mathfrak{a}$ such that $\alpha < \delta_{i,\varepsilon} \Rightarrow h \nleq_{J_i} g^{i,\varepsilon}_\alpha$ and let h_b be the $<_*$-minimal such h. Let h_* be the function with domain $\mathfrak{a}$ such that $h(\theta) = \cup\{h_\mathfrak{b}(\theta) + 1 : b \in J^+_i\}$.

As $\mathrm{hrtg}(J^+_i) \leq \mathrm{hrtg}(\mathscr{P}(\mathfrak{a})) < \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$, clearly $h_* \in \Pi\mathfrak{a}$. Now for $\alpha < \delta_{i,\varepsilon}$ let $\mathfrak{d}_{i,\varepsilon,\alpha} = \{\theta \in \mathfrak{a} : g^{i,\varepsilon}_\alpha(\theta) \leq h_*(\theta)\}$. So $\langle \mathfrak{d}_{i,\varepsilon,\alpha}/J_i : \alpha < \delta_{i,\varepsilon}\rangle$ is $\leq$-increasing in the Boolean Algebra $\mathscr{P}(\mathfrak{a})/J_i$, so for some $\beta_{i,\varepsilon} < \delta_{i,\varepsilon}$ we have $\alpha \in (\beta_{i,\varepsilon}, \delta_{i,\varepsilon}) \Rightarrow \mathfrak{d}_{i,\varepsilon,\alpha} = \mathfrak{d}_{i,\varepsilon,\beta_{i,\varepsilon}} \mod J_i$. This implies $\mathfrak{d}_{i,\varepsilon}$ can serve as $\mathfrak{b}_{i,\varepsilon}$.]

To finish consider the following two cases.

<u>Case 1</u>: We succeed to carry the induction, i.e. choose $\bar{g}^{i,\varepsilon}$ for every $\varepsilon < \kappa$.

So $\langle \mathfrak{b}_{i,\varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of subsets of $\mathfrak{a}$, pairwise distinct (by $\circledast^2_{\kappa,0}$ clauses (g) + (b)), but $\kappa \geq \mathrm{hrtg}(\mathscr{P}(\omega))$ and $\mathfrak{a}$ is countable; contradiction.

<u>Case 2</u>: We are stuck in $\varepsilon < \kappa$.

For $\varepsilon = 0$ there is no problem to define $g^{i,\varepsilon}_\alpha$ by induction on α till we are stuck, say in α, necessarily α is of large enough cofinality $\geq \kappa$ by $(*)_2$, and so $\bar{g}^{i,\varepsilon}$ is well defined. We then prove $\mathfrak{b}_{i,\varepsilon}$ exists by $(*)_4$ again using $<_*$.

For ε limit we can also choose $\bar{g}^\varepsilon$.

For $\varepsilon = \zeta + 1$, if $\mathfrak{a} \in J_\varepsilon$ then we are done; otherwise $g^{i,\varepsilon}_0$ as required can be chosen by $(*)_0$, and then we can prove that $\bar{g}^{i,\varepsilon}, \mathfrak{b}_{i,\varepsilon}$ exists as above. $\square_{1.8}$

Remark 1.10. From 1.8 we can deduce bounds on $\mathrm{hrtg}({}^Y(\aleph_\delta))$ when $\delta < \aleph_1$ and more like the one on $\aleph_\omega^{\aleph_0}$ (better the bound on $\mathrm{pp}(\aleph_\omega)$).

§ 2. No decreasing sequence of subalgebras

In this section we concentrate on weaker axioms. We consider Theorem 1.2 under weaker assumptions than "$[\lambda]^{\aleph_0}$ is well orderable". We are also interested in replacing ω by ∂ in "no decreasing ω-sequence of $c\ell$-closed sets", but the reader may consider $\partial = \aleph_0$ only. Note that for the full version, Ax^4_α, i.e., $[\alpha]^\partial$ is well orderable, the case of $\partial = \aleph_0$ is implied by the $\partial > \aleph_0$ version and suffices for the results. But for other versions, the axioms for different ∂'s seem incomparable.

Note that if we add many Cohens (not well ordering them) then Ax^4_λ fails below even for $\partial = \aleph_0$, whereas the other axioms are not affected. But forcing by $\aleph_1$-complete forcing notions preserve Ax_4.

Hypothesis 2.1. DC_∂ and let $\partial(*) = \partial + \aleph_1$. Actually we use only DC in 2.5(1) and DC_∂ in 2.5(3) and the later claims. We fix a regular cardinal ∂.

Definition 2.2. Below we should, e.g. write $\mathrm{Ax}^{\ell,\partial}$ instead of Ax^ℓ and assume $\alpha > \mu > \kappa \geq \partial$. If $\kappa = \partial$ we may omit it.
1) $\mathrm{Ax}^1_{\alpha,\mu,\kappa}$ means that there is a weak closure operation on λ of character (μ,κ), see Definition 0.17(1A), such that there is no $\subseteq$-decreasing ∂-sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ of subsets of α with $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$. We may here and below replace κ by $< \kappa$; similarly for μ; let $< |Y|^+$ means $|Y|$.
2) Let $\mathrm{Ax}^0_{\alpha,<\mu,\kappa}$ mean there is a function $c\ell : [\alpha]^{\leq\kappa} \to [\alpha]^{<\mu}$ such that $u \cup \{0\} \subseteq c\ell(u)$ and there is no $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ of members of $[\alpha]^{\leq\kappa}$ such that $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$.
2A) Writing Y instead of κ means $c\ell : [\alpha]^{<\mathrm{hrtg}(Y)} \to [\alpha]^{<\mu}$. Let $c\ell_{[\varepsilon]} : \mathscr{P}(\alpha) \to \mathscr{P}(\alpha)$ be $c\ell^1_{\varepsilon,<\mathrm{reg}(\kappa^+)}$ as defined in 0.17(4) recalling $\mathrm{reg}(\gamma) = \mathrm{Min}\{\chi : \chi$ a regular cardinal $\geq \gamma\}$.
3) Ax^2_α means that there is $\mathscr{A} \subseteq [\alpha]^\partial$ which is well orderable and for every $u \in [\alpha]^\partial$ for some $v \in \mathscr{A}, u \cap v$ has power $= \partial$.
4) Ax^3_α means that $\mathrm{cf}([\alpha]^{\leq\partial}, \subseteq)$ is below some cardinal, i.e., some cofinal $\mathscr{A} \subseteq [\alpha]^\partial$ (under $\subseteq$) is well orderable.
5) Ax^4_α means that $[\alpha]^{\leq\partial}$ is well orderable.
6) Above omitting α (or writing ∞) means "for every α", omitting μ we mean "$< \mathrm{hrtg}(\mathscr{P}(\partial))$".
7) Lastly, let $\mathrm{Ax}_\ell = \mathrm{Ax}^\ell$ for $\ell = 1,2,3$.

So easily (or we have shown in the proof of 1.2):

Claim 2.3. *1) Ax^4_α implies Ax^3_α, Ax^3_α implies Ax^2_α, Ax^2_α implies Ax^1_α and Ax^1_α implies Ax^0_α . Similarly for $\mathrm{Ax}^\ell_{\alpha,<\mu,\kappa}$.*
2) In Definition 2.2(2), the last demand, if $c\ell$ has monotonicity, then only $c\ell \restriction [\alpha]^{\leq\partial}$ is relevant, in fact, an equivalent demand is that if $\langle \beta_\varepsilon : \varepsilon < \partial\rangle \in {}^\partial\alpha$ then for some $\varepsilon, \beta_\varepsilon \in c\ell\{\beta_\zeta : \zeta \in (\varepsilon,\partial)\}$.
3) If $\mathrm{Ax}^0_{\alpha,<\mu_1,<\theta}$ and $\theta \leq \mathrm{hrtg}(Y)$ and [2] $\mu_2 = \sup\{\mathrm{hrtg}(\mu_1 \times [\beta]^\theta) : \beta < \mathrm{hrtg}(Y)\}$ then $Ax^0_{\alpha,<\mu_2,<\mathrm{hrtg}(Y)}$.

Proof. 1) Clearly $\mathrm{Ax}^2_{\alpha,<\mu,\kappa} \Rightarrow \mathrm{Ax}^1_{\alpha,<\mu,\kappa}$ holds similarly to the proof of 1.5; the other implications hold by inspection.
2) First assume that we have a $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ such that $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$. Let $\beta_\varepsilon = \min(\mathscr{U}_\varepsilon \backslash c\ell(\mathscr{U}_{\varepsilon+1}))$ for $\varepsilon < \partial$ so clearly

[2]Can do somewhat better; we can replace $[\alpha]^{<\mu_1}$ by $\{v \subseteq \alpha : \mathrm{otp}(v) \subseteq \mu_1\}$

$\bar\beta = \langle \beta_\varepsilon : \varepsilon < \partial\rangle$ exists; so by monotonicity $c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1,\partial)\} \subseteq c\ell(\mathscr{U}_{\varepsilon+1})$ hence $\beta_\varepsilon \notin c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1,\partial)\}$.

Second, assume that $\bar\beta = \langle \beta_\varepsilon : \varepsilon < \partial\rangle \in {}^\partial\alpha$ satisfies $\beta_\varepsilon \notin c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1,\partial)\}$ for $\varepsilon < \partial$. Now letting $\mathscr{U}'_\varepsilon = \{\beta_\zeta : \zeta < \partial$ satisfies $\varepsilon \le \zeta\}$ for $\varepsilon < \partial$ clearly $\langle \mathscr{U}'_\varepsilon : \varepsilon < \partial\rangle$ exists, is $\subseteq$-decreasing and $\varepsilon < \partial \Rightarrow \beta_\varepsilon \notin c\ell(\mathscr{U}'_{\varepsilon+1}) \wedge \beta_\varepsilon \in \mathscr{U}'_\varepsilon$. So we have shown the equivalence.

3) Let $c\ell(-)$ witness $\mathrm{Ax}^0_{\alpha,<\mu_1,<\theta}$. We define the function $c\ell'$ with domain $[\alpha]^{<\mathrm{hrtg}(Y)}$ by $c\ell'(u) = \cup\{c\ell(v) : v \subseteq u$ has cardinality $< \theta\}$.

Now

$(*)_0$ $c\ell'$ is a function from $[\alpha]^{<\mathrm{hrtg}(Y)}$ into $[\alpha]^{<mu_2}$.

For this it is enough to note:

$(*)_1$ if $u \in [\alpha]^{<\mathrm{hrtg}(Y)}$ then $c\ell'(u)$ has cardinality $< \mu_2 := \sup\{\mathrm{hrtg}(\mu_1 \times [\beta]^\theta : \beta < \mathrm{hrtg}(Y)\}$.

[Why? Let $C_u = \{(v,\varepsilon) : v \subseteq u$ has cardinality $< \theta$ and $\varepsilon <$ otp$(c\ell(v))$ which is $< \mu_1\}$. Clearly $|c\ell'(u)| < \mathrm{hrtg}(C_u)$ and $|C_u| = |\mu_1 \times [\mathrm{otp}(u)]^{<\theta}|$, so $(*)_1$ holds. Note that if $\alpha_* < \mu_1^+$ we can replace the demand $v \in [u]^{<\theta} \Rightarrow |c\ell(v)| < \mu_1$ by $v \in [u]^{<\theta} \Rightarrow$ otp$(c\ell(v)) < \alpha_*$.]

$(*)_2$ If $\langle u_\varepsilon : \varepsilon < \partial\rangle$ is $\subseteq$-decreasing where $u_\varepsilon \subseteq \alpha$ then $u_\varepsilon \subseteq c\ell'(u_{\varepsilon+1})$ for some $\varepsilon < \partial$.

[Why? If not we can choose a sequence $\langle \beta_\varepsilon : \varepsilon < \partial\rangle$ by letting $\varepsilon < \partial \Rightarrow \beta_\varepsilon =$ $\min(u_\varepsilon \backslash c\ell'(u_{\varepsilon+1}))$. Let $u'_\varepsilon = \{\beta_\zeta : \zeta \in [\varepsilon,\partial)\}$. As $\langle u'_\varepsilon : \varepsilon < \partial\rangle$ is $\subseteq$-decreasing by the choice of $c\ell(-)$ for some $\varepsilon, \beta_\varepsilon \in c\ell\{\beta_\zeta : \zeta \in (\varepsilon+1,\partial)\}$, but this set is $\subseteq c\ell'(u_{\varepsilon+1})$ by the definition of $c\ell'(-)$, so we are done.] $\square_{2.3}$

Claim 2.4. *Assume $c\ell$ witness $\mathrm{Ax}^0_{\alpha,<\mu,\kappa}$ so $\partial \le \kappa < \mu$ and so $c\ell : [\alpha]^{\le\kappa} \to [\alpha]^{<\mu}$ and recall $c\ell^1_{\varepsilon,\le\kappa} : \mathscr{P}(\alpha) \to \mathscr{P}(\alpha)$ is from 2.2(2A), 0.17(4).*

1) $c\ell^1_{1,\le\kappa}$ is a weak closure operation, it has character (μ_κ,κ) whenever $\partial \le \kappa \le \alpha$ and $\mu_\kappa = \mathrm{hrtg}(\mu \times \mathscr{P}(\kappa))$, see Definition 0.17.

2) $c\ell^1_{\mathrm{reg}(\kappa^+),\le\kappa}$ is a closure operation and it has character $(<\mu'_\kappa,\kappa)$ when $\partial \le \kappa \le \alpha$ and $\mu'_\kappa = \mathrm{hrtg}(\mathscr{H}_{<\partial^+}(\mu \times \kappa))$.

Proof. 1) By its definition $c\ell^1_{1,\le\kappa}$ is a weak closure operation.

Assume $u \subseteq \alpha, |u| \le \kappa$; non-empty for simplicity. Clearly $\mu \times [|u|]^{<\partial}$ has the same power as $\mu \times [u]^{<\partial}$. Define [3] the function G with domain $\mu \times [u]^{<\partial}$ as follows: if $\alpha < \mu$ and $v \in [u]^{\le\partial}$ then $G((\alpha,v))$ is the α-th member of $c\ell(v)$ if $\alpha <$ otp$(c\ell(v))$ and $G((\alpha,v)) =$ min(u) otherwise.

So G is a function from $\mu \times [u]^{\le\partial}$ onto $c\ell^1_{1,\le\kappa}(u)$. This proves that $c\ell^1_{1,\le\kappa}$ has character $(<\mu_\kappa,\kappa)$ as $\mu_\kappa = \mathrm{hrtg}(\mu \times \mathscr{P}(\kappa))$.

2) If $\langle u_\varepsilon : \varepsilon \le$ reg$(\kappa^+)\rangle$ is an increasing continuous sequence of sets <u>then</u> $[u_{\partial^+}]^{\le\partial} =$ $\cup\{[u_\varepsilon]^{\le\partial} : \varepsilon <$ reg$(\kappa^+)\}$ as reg(κ^+) is regular (even of cofinality $> \partial$ suffice) by its definition, note reg$(\partial^+) = \partial^+$ when AC_∂ holds when DC_∂ holds.

Second, let $u \subseteq \alpha, |u| \le \kappa$ and let $u_\varepsilon = c\ell^1_{\varepsilon,\kappa}(u)$ for $\varepsilon \le \partial^+$; it is enough to show that $|u_{\partial^+}| < \mu'_\kappa$. The proof is similar to earlier one. $\square_{2.4}$

[3]clearly we can replace $< \mu$ by $< \gamma$ for $\gamma \in (\mu,\mu^+)$

Definition/Claim 2.5. Let $c\ell$ exemplify $\mathrm{Ax}^0_{\lambda,<\mu,Y}$ and Y be an uncountable set such that $\partial(*) \le_{\mathrm{qu}} Y$.
1) Let $\mathscr{F}_{\mathfrak{y}}, \mathscr{F}_{\mathfrak{y},\alpha}$ be as in the proof of Theorem 1.2 for $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ and ordinal α (they depend on λ and $c\ell$ but note that $c\ell$ determines λ; so if we derive $c\ell$ by Ax^4_λ then they depend indirectly on the well ordering of $[\lambda]^\partial$) so we may write $\mathscr{F}_{\mathfrak{y},\alpha} = \mathscr{F}_{\mathfrak{y}}(\alpha, c\ell)$, etc.

That is, fully

$(*)_1$ for $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ and ordinal α let $\mathscr{F}_{\mathfrak{y},\alpha}$ be the set of f such that:
- (a) f is a function from $Z^{\mathfrak{y}}$ to λ
- (b) $\mathrm{rk}_{D[\mathfrak{y}]}(f) = \alpha$ recalling that this means $\mathrm{rk}_{D^{\mathfrak{y}}_1 + Z^{\mathfrak{y}}}(f \cup 0_{Y\backslash Z^{\mathfrak{y}}}) = \alpha$ by Definition 0.9(2)
- (c) $D^{\mathfrak{y}}_2 = D^{\mathfrak{y}}_1 \cup \{Y\backslash A : A \in J[f, D^{\mathfrak{y}}_1]\}$, see Definition 0.12
- (d) $Z^{\mathfrak{y}} \in D^{\mathfrak{y}}_2$
- (e) if $Z \in D^{\mathfrak{y}}_2$ and $Z \subseteq Z^{\mathfrak{y}}$ then $c\ell(\{f(y) : y \in Z\}) \supseteq \{f(y) : y \in Z^{\mathfrak{y}}\}$
- (f) $h^{\mathfrak{y}}$ is a function with domain $Z^{\mathfrak{y}}$ such that $y \in Z^{\mathfrak{d}} \Rightarrow h^{\mathfrak{h}}(y) = \mathrm{otp}(f(y) \cap \{c\ell(\{f(z) : z \in Z^{\mathfrak{y}}\})$

$(*)_2$ $\mathscr{F}_{\mathfrak{y}} = \cup\{\mathscr{F}_{\mathfrak{y},\alpha} : \alpha$ an ordinal$\}$.

2) Notice that $\mathscr{F}_{\mathfrak{y},\alpha}$ is a singleton or the empty set. Let $\Xi_{\mathfrak{y}} = \Xi_{\mathfrak{y}}(c\ell) = \Xi_{\mathfrak{y}}(\lambda, c\ell) = \{\alpha : \mathscr{F}_{\mathfrak{y},\alpha} \ne \emptyset\}$ and $f_{\mathfrak{y},\alpha}$ is the function $f \in \mathscr{F}_{\mathfrak{y},\alpha}$ when $\alpha \in \Xi_{\mathfrak{y}}$; it is well defined.
3) If $D \in \mathrm{Fil}_{\partial(*)}(Y)$, $\mathrm{rk}_D(f) = \alpha$ and $f \in {}^Y\lambda$ then $\alpha \in \Xi_D(\lambda, c\ell)$ and $f \restriction Z^{\mathfrak{y}} = f_{\mathfrak{y},\alpha}$ for some $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$; moreover, $(D^{\mathfrak{y}}_1, D^{\mathfrak{y}}_2) = (D, \mathrm{dual}(J(J[f,D]))$ where $\Xi_D(\lambda, c\ell) := \cup\{\Xi_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y)$ and $D^{\mathfrak{y}}_1 = D\}$.
4) If $D \in \mathrm{Fil}_{\partial(*)}(Y), f \in {}^Y\lambda, Z \in D^+$ and $\mathrm{rk}_{D+Z}(f) \ge \alpha$ then for some $g \in \prod_{y\in Y}(f(y)+1) \subseteq {}^Y(\lambda+1)$ we have $\mathrm{rk}_D(g) = \alpha$ hence $\alpha \in \Xi_D(\lambda, c\ell)$.
5) So we should write $\mathscr{F}_{\mathfrak{y}}[c\ell], \Xi_{\mathfrak{y}}[\lambda, c\ell], f_{\mathfrak{y},\alpha}[c\ell]$.

Proof. As in the proof of 1.2 recalling "$c\ell$ exemplifies $\mathrm{Ax}^0_{\lambda,<\mu,\mathrm{hrtg}(Y)}$" holds, this replaces the use of F_* there; and see the proof of 2.11 below in part (3), for this we need:

⊞ if $D \in \mathrm{Fil}^1_\partial(Y)$ and $f \in {}^\kappa\partial$, then for some $Z \in D$ we have:
- if $Y \subseteq Z$ belongs to D then $c\ell(\mathrm{Rang}(f{\restriction}Y) = c\ell(\mathrm{Rang}(f{\restriction}Z))$.

[Why ⊞ holds? By Definition 2.2(2) using the axiom DC_∂.] $\square_{2.5}$

Claim 2.6. *We have ξ_2 is an ordinal and $\mathrm{Ax}^0_{\xi_2,<\mu_2,Y}$ holds when,(note that μ_2 is not much larger than μ_1):*

- (a) $Ax^0_{\xi_1,<\mu_1,Y}$ *so* $\partial < \mathrm{hrtg}(Y)$
- (b) $c\ell$ *witnesses clause (a)*
- (c) $D \in \mathrm{Fil}_{\partial(*)}(Y)$
- (d) $\xi_2 = \{\alpha : f_{\mathfrak{y},\alpha}[c\ell]$ *is well defined for some* $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ *which satisfies* $D^{\mathfrak{y}}_1 = D$ *and necessarily* $\mathrm{Rang}(f_{\mathfrak{y},\alpha}[c\ell]) \subseteq \xi_1\}$
- (e) μ_2 *is defined as* $\mu_{2,3}$ *where:*
  - (α) *let* $\mu_{2,0} = \mathrm{hrtg}(Y)$

(β) $\mu_{2,1} = \sup_{\beta < \mu_{2,0}} \mathrm{hrtg}(\beta \times \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1))$

(γ) $\mu_{2,2} = \sup_{\alpha < \mu_{2,1}} \mathrm{hrtg}(\mu_1 \times [\alpha]^{\le \partial})$

(δ) $\mu_{2,3} = \sup\{\mathrm{hrtg}({}^Y\beta \times \mathrm{Fil}_{\partial(*)}(Y)) : \beta < \mu_{2,2}\}$
(this is an overkill).

Proof.

$\oplus_1$ ξ_2 is an ordinal.

[Why? To prove that ξ_2 is an ordinal we have to assume $\alpha < \beta \in \xi_2$ and prove $\alpha \in \xi_2$. As $\beta \in \xi_2$ clearly $\beta \in \Xi_{\mathfrak{y}}[c\ell]$ for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)$ for which $D^{\mathfrak{y}}_1 = D$ so there is $f \in {}^Y(\xi_1)$ such that $f \restriction Z^{\mathfrak{y}} \in \mathscr{F}_{\mathfrak{y},\beta}$. So $\mathrm{rk}_{D+Z[\mathfrak{y}]}(f) = \beta$ hence by 0.9 there is $g \in {}^Y\lambda$ such that $g \le f$, i.e., $(\forall y \in Y)(g(y) \le f(y))$ and $\mathrm{rk}_{D+Z[\mathfrak{y}]}(g) = \alpha$. By 2.5(4) there is $\mathfrak{z} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)$ such that $D^{\mathfrak{z}}_1 = D + Z[\mathfrak{y}]$ and $g \restriction Z^{\mathfrak{z}} \in \mathscr{F}_{\mathfrak{z},\alpha}$ so we are done proving ξ_2 is an ordinal.]

We define the function $c\ell'$ with domain $[\xi_2]^{<\mathrm{hrtg}(Y)}$ as follows:

$\oplus_2$ $c\ell'(u) = \{0\} \cup \{\alpha$: there is $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)$ such that $f_{\mathfrak{y},\alpha}[c\ell]$ is well defined [4] and $\mathrm{Rang}(f_{\mathfrak{y},\alpha}[c\ell]) \subseteq c\ell(\mathbf{v}[u])\}$.

where

$\oplus_3$ $\mathbf{v}[u] := \cup\{c\ell(v) : v \subseteq \xi_1$ is of cardinality $\le \partial$ and is $\subseteq \mathbf{w}(v)\}$.

where

$\oplus_4$ for $v \subseteq \xi_1$ we let $\mathbf{w}(v) = \cup\{\mathrm{Rang}(f_{\mathfrak{z},\beta}[c\ell]) : \mathfrak{z} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)$ and $\beta \in u$ and $f_{\mathfrak{z},\beta}[c\ell]$ is well defined$\}$.

Note that

$\oplus_5$ $c\ell'(u) = \{0\} \cup \{\mathrm{rk}_D(f) : D \in \mathrm{Fil}_{\partial(*)}(Y), Z \in D^+$ and $f \in {}^Y\mathbf{v}(u)\}$.

Note that (by 2.5(1)):

$\boxtimes_1$ for each $u \subseteq \xi_1$ and $\mathfrak{x} \in \mathrm{Fil}^4_{\partial(*)}(Y, \mu_1)$ the set $\{\alpha < \xi_2 : f_{\mathfrak{x},\alpha}[c\ell]$ is a well defined function into $u\}$ has cardinality $< \mathrm{wlor}(T_{D^{\mathfrak{y}}_2}(u))$, that is, $\langle f_{\mathfrak{x},\alpha}[c\ell] : \alpha \in \Xi_{\mathfrak{x}} \cap \xi_2 \rangle$ is a sequence of functions from $Z^{\mathfrak{x}}$ to $u \subseteq \xi_1$, any two are equal only on a set $= \emptyset \mod D^{\mathfrak{x}}_2$ (with choice it has cardinality $\le {}^{|Y|}|u|$)), call this bound $\mu'_{|u,\mathfrak{x}|}$.

Note

$\boxtimes_2$ if $u_1 \subseteq u_2 \subseteq \xi_2$ <u>then</u>

(α) $\mathbf{w}(u_1) \subseteq \mathbf{w}(u_2)$ and $\mathbf{v}(u_1) \subseteq \mathbf{v}(u_2) \subseteq \xi_1$

(β) $c\ell'(u_1) \subseteq c\ell'(u_2)$

(γ) $u \subseteq \mathbf{v}(u)$ and $\mathbf{w}[u] \subseteq \mathbf{v}[u]$

(δ) $u_1 \subseteq c\ell'(u_1)$.

---

[4] We could have used $\{t \in Y : f_{\eta,\alpha}[c\ell](t) \in c\ell(\mathbf{v}(u))\} \neq \emptyset \mod D^{\mathfrak{y}}_2$; also we could have added u to $c\ell'(u)$ but not necessarily by $\boxplus_2$.

[Why? E.g. for clause (δ); assume $\alpha \in u$ and let f be a unique function from Y into $\{\alpha\}$. Hence for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ we have $f_{\mathfrak{y},\alpha}$ is well defined. Now $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq \mathbf{w}(u)$ by the choice of $\mathbf{w}(u)$ in $\oplus_4$ and so $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq \mathbf{v}(u)$ by clause (γ) of $\boxplus_2$ hence $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq c\ell(\mathbf{v},u)$ by the assumption on $c\ell$, see by 2.6(a),(b) and 2.2(2). So we have $f_{\mathfrak{y},\beta}$ well defined and $\mathrm{Rang}(f_{\mathfrak{y},\alpha}) \subseteq c\ell(\mathbf{v}(u))$ so by the definition of $c\ell'(u)$ in $\oplus_2$ we have $\alpha \in c\ell'(u)$ so we are done.]

- $\boxtimes_3$ if $u \subseteq \xi_2, |u| < \mathrm{hrtg}(Y)$ then $\mathbf{w}(u) = \{f_{\mathfrak{y},\alpha}(z) : \alpha \in u, \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1), f_{\mathfrak{y},\alpha}$ is well defined and $z \in Z^{\mathfrak{y}}\}$ is a subset of ξ_1 of cardinality $< \mathrm{hrtg}(|u| \times \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)) \le \sup\{\mathrm{hrtg}(\beta) \times \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)) : \beta < \mathrm{hrtg}(Y)\}$ which was named $\mu_{2,1}$ in $2.6(e)(\beta)$
- $\boxtimes_4$ if $u \subseteq \xi_1$ and $|u| < \mu_{2,1}$ then $\cup\{c\ell(v) : v \in [u]^{\le\partial}\}$ is a subset of μ_1 of cardinality $< \mathrm{hrtg}(\mu_1 \times [u]^{\le\partial}) \le \sup_{\alpha<\mu_{2,1}} \mathrm{hrtg}(\mu_1 \times [\alpha]^{\le\partial})$ which we call $\mu_{2,2}$ in $2.6(e)(\gamma)$
- $\boxtimes_5$ if $u \subseteq \xi_2$ and $|u| < \mathrm{hrtg}(Y)$ then $\mathbf{v}(u)$ has cardinality $< \mu_{2,2}$.

[Why? By $\oplus_3$ and $\boxtimes_3$ and $\boxtimes_4$.]

- $\boxtimes_6$ if $u \subseteq \xi_2$ and $|u| < \mathrm{hrtg}(Y)$ then $c\ell'(u) \subseteq \xi_2$ and has cardinality $< \mu_{2,3}$ is defined in $2.6(e)(\delta)$ which we call μ_2.

[Why? Without loss of generality $\mathbf{v}(u) \ne \emptyset$. By $\oplus_5$ we have $|c\ell'(u)| < \mathrm{hrtg}({}^Y\mathbf{v}(u)) \times \mathrm{Fil}_{\partial(*)}(Y))$ and by $\boxplus_5$ the latter is $\le \sup\{\mathrm{hrtg}({}^Y\beta \times \mathrm{Fil}_{\partial(*)}(Y)) : \beta < \mu_{2,2}\} = \mu_{2,3}$ recalling clause $(e)(\delta)$ of the claim, so we are done.]

- $\boxtimes_7$ $c\ell'$ is a very weak closure operation on λ and has character $(< \mu_2, \mathrm{hrtg}(Y))$.

[Why? In Definition 0.17(1), clause (a) holds by the Definition of $c\ell'$, clause (b) holds by $\boxplus_6$ and as for clause (c), $0 \in c\ell'(u)$ by the definition of $c\ell'$ and $u \subseteq c\ell'(u)$ by clause (δ) of $\boxtimes_2$.]

Now it is enough to prove

- $\boxtimes_8$ $c\ell'$ witnesses $\mathrm{Ax}^0_{\xi_2,<\mu_2,Y}$.

Recalling $\boxtimes_7$, toward contradiction assume $\bar{\mathscr{U}} = \langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ is $\subseteq$-decreasing, $\mathscr{U}_\varepsilon \in [\xi_1]^{<\mathrm{hrtg}(Y)}$ and $\varepsilon < \partial \Rightarrow \mathscr{U}_\varepsilon \nsubseteq c\ell(\mathscr{U}_{\varepsilon+1})$. We define $\bar\gamma = \langle \gamma_\varepsilon : \varepsilon < \partial\rangle$ by

$$\gamma_\varepsilon = \mathrm{Min}(\mathscr{U}_\varepsilon \backslash c\ell(\mathscr{U}_{\varepsilon+1})).$$

As AC_∂ follows from DC_∂, we can choose $\langle \mathfrak{y}_\varepsilon : \varepsilon < \partial\rangle$ such that $f_{\mathfrak{y}_\varepsilon,\gamma_\varepsilon}[c\ell]$ is well defined for $\varepsilon < \partial$.

Let for $\varepsilon < \partial$

$$u_\varepsilon = \{\gamma_\zeta : \zeta \in [\varepsilon,\partial)\}.$$

So

- $(*)_1$ $u_\varepsilon \in [\xi_1]^{\le\partial} \subseteq [\xi_1]^{<\mathrm{hrtg}(Y)}$.

[Why? By clause (a) of the assumption of 2.6.]

- $(*)_2$ u_ε is $\subseteq$-decreasing with ε.

[Why? By the definition.]

$(*)_3$ $\gamma_\varepsilon \in u_\varepsilon \backslash c\ell(u_{\varepsilon+1})$ for $\varepsilon < \partial$.

[Why? $\gamma_\varepsilon \in u_\varepsilon$ by the definition of u_ε.]

Now if $\zeta \in [\varepsilon, \gamma)$ then $f_{\mathfrak{y}_\zeta,\gamma_\zeta}[c\ell]$ is well defined and $\gamma_\zeta \in \mathscr{U}_\zeta \backslash c\ell(\mathscr{U}_{\zeta+1})$ (see the choice of γ_ε) but $\langle \mathscr{U}_\xi : \xi < \partial\rangle$ is $\subseteq$-decreasing hence $\gamma_\zeta \in \mathscr{U}_\zeta$, by the definition of $\mathbf{w}[u_\varepsilon]$, $\mathrm{Rang}(f_{\mathfrak{y}_\zeta,\gamma_\zeta}) \in \mathbf{w}(\mathscr{U}_\varepsilon)$, hence $\mathrm{Rang}(f_{\mathfrak{y}_\zeta,\gamma_\zeta}) \in \mathbf{v}(\mathscr{U}_\varepsilon) \subseteq c\ell(\mathbf{v}(\mathscr{U}_\varepsilon))$. As this holds for every $\zeta \in [\varepsilon, \gamma)$ we can deduce $u_v arp = \{\gamma_\zeta : \zeta \in [\varepsilon, \partial)\} \subseteq c\ell'(\mathbf{v}(\mathscr{U}_\varepsilon))$.

Lastly, $\gamma_\varepsilon \notin \mathbf{v}(\mathscr{U}_{\varepsilon+1})$ by the choice of β_ε. So $\langle u_\varepsilon : \varepsilon < \partial\rangle$ contradict the assumption on $(\xi_1, c\ell)$. From the above the conclusion should be clear. $\square_{2.6}$

Claim 2.7. *Assume $\kappa < \kappa = \mathrm{cf}(\lambda) < \lambda$ hence κ is regular $\geq \partial$ of course, and D is the club filter on κ and $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ is increasing continuous with limit λ.*

Then $\lambda^+ \leq \{\mathrm{rk}_{D_\kappa}(f) : f \in \prod\limits_{i<\kappa^+} \lambda_i^+\}$.

Proof. For each $\alpha < \lambda^+$ there is a one to one [5] function g from α into $|\alpha| \leq \lambda$ and we let $f \in \prod\limits_{i<\kappa} \lambda_i$ be

$$f(i) = \mathrm{otp}(\{\beta < \alpha : g(\beta) < \lambda_i\}.$$

Let

$$\begin{aligned}\mathscr{F}_\alpha = \{f : \quad & f \text{ is a function with domain } \kappa \text{ satisfying } i < \kappa \Rightarrow f(i) < \lambda_i^+ \\ & \text{such that for some one to one function } g \text{ from } \alpha \text{ into } \lambda \\ & \text{for each } i < \kappa \text{ we have } f(i) = \mathrm{otp}(\{\beta < \alpha : g(\beta) < \lambda_i\})\}.\end{aligned}$$

Now

$(*)_1$ (α) $\mathscr{F}_\alpha \neq \emptyset$ for $\alpha < \lambda^+$
(β) $\langle \mathscr{F}_\alpha : \alpha < \lambda^+\rangle$ exists as it is well defined

[Why? For clause (α) let $g : \alpha \to \lambda$ be one to one and so the f defined above belongs to $\mathscr{F}_\alpha$. For clause (β) see the definition of $\mathscr{F}_\alpha$ (for $\alpha < \lambda^+$).]

$(*)_2$ (α) if $f \in \mathscr{F}_\beta, \alpha < \beta < \lambda^+$ <u>then</u> for some $f' \in \mathscr{F}_\alpha$ we have $f' <_{J^{\mathrm{bd}}_\kappa} f$
(β) $\langle \min\{\mathrm{rk}_D(f) : f \in \mathscr{F}_\alpha\} : \alpha < \lambda^+\rangle$ is strictly increasing hence $\min\{\mathrm{rk}_D(f) : f \in \mathscr{F}_\alpha\} \geq \alpha$.

[Why? For clause (α), let g witness "$f \in \mathscr{F}_\beta$" and define the function $f' \in \prod\limits_{i<\kappa} \lambda_i^+$ by $f'(i) = \mathrm{otp}\{\gamma < \alpha : g(\gamma) < \lambda_i\}$. So $g{\restriction}\alpha$ witness $f' \in \mathscr{F}_\alpha$, and letting $i(*) = \min\{i : g(\alpha) < \lambda_i\}$ we have $i \in [i(*), \kappa) \Rightarrow f'(i) < f(i)$ hence $f' <_{J^{\mathrm{bd}}_\kappa} f$ as promised. For clause (β) it follows.]

Note that

$(*)_3$ if $f \in \mathscr{F}_\alpha$ then, for part (2), for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\lambda, c\ell)$ and $\beta \geq \alpha$ we have $f{\restriction}Z[\mathfrak{y}] \in \mathscr{F}_{\mathfrak{y},\beta}$.

[Why? By $(*)_1 + (*)_2$.]

So we have proved 2.7. $\square_{2.7}$

[5] but, of course, possibly there is no such sequence $\langle f_\alpha : \alpha < \lambda^+\rangle$

Conclusion 2.8. *1) Assume*

(a) $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$
(b) $\lambda > \mathrm{cf}(\lambda) = \kappa$ *(not really needed in part (1)).*

Then for some $\mathscr{F}_* \subseteq {}^\kappa\lambda =: \{f : f$ *a partial function from* κ *to* $\lambda\}$ *we have*

(α) *every* $f \in {}^\kappa\lambda$ *is a countable union of members of* $\mathscr{F}_*$
(β) $\mathscr{F}_*$ *is the union of* $|\mathrm{Fil}^4_{\partial(*)}(\kappa, <\mu)|$ *well ordered sets:* $\{\mathscr{F}^*_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa,\mu)\}$
(γ) *moreover there is a function giving for each* $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa)$ *a well ordering of* $\mathscr{F}^*_{\mathfrak{y}}$.

2) Assume in addition that $\mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\kappa,<\mu)) < \lambda, \mathrm{cf}(\lambda^+)$ *and* $\mathrm{hrtg}({}^\kappa\mu) < \lambda$ *then for some* $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa)$ *we have* $|\mathscr{F}^*_{\mathfrak{y}}| > \lambda$.
3) If in part (2) we omit the assumption on $\mathrm{cf}(\lambda^+)$ *still* $\lambda^+ = \sup\{\mathrm{otp}(\Xi_{\mathfrak{y}} \cap \lambda^+) : \mathfrak{y} \in \mathrm{Fil}^4_{\partial()*}(\kappa,\mu)\}$.

Proof. 1) By the proof of 1.2.
2) Assume that this fails; so for every $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa,<\mu)$, the set $S_{\mathfrak{y}} = \Xi_{\mathfrak{y}} \cap \lambda^+$ has order type $< \lambda^+$. But we are assuming $\mathrm{cf}(\lambda^+) \ge \mathrm{hrtg}(\mathrm{Fil}^4_{\partial()*}(\kappa,\mu)$, so there is $\gamma < \lambda^+$ such that $\gamma > \mathrm{otp}(S_{\mathfrak{y}})$ for every relevant $\mathfrak{y}$, without loss of generality $\gamma > \lambda$ and let g be a one-to-one function from γ onto λ.

We choose $f \in {}^\kappa\lambda$ by

$$f(i) = \mathrm{Min}(\lambda \backslash \{f_{\mathfrak{y},\alpha}(i) : \begin{array}{l} \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa,\mu) \\ f_{\mathfrak{y},\alpha}(i) \text{ is well defined, i.e.} \\ i \in Z[\mathfrak{y}] \text{ and } \alpha \in \Xi_{\mathfrak{y}} \text{ and} \\ g(\mathrm{otp}(\alpha \cap \Xi_{\mathfrak{y}})) < \mu_i\}). \end{array}$$

Now $f(i)$ is well defined as the minimum is taken over a non-empty set of ordinals, this holds as we substruct from λ a set which has cardinality $\le \mu_i$ which is $< \lambda$. But f contradicts part (1). Note that in fact $f \in \prod_i \mu_i^+$.

3) Same proof as in part (2). $\square_{2.8}$

Conclusion 2.9. *Assume* $Ax^0_{\lambda,<\mu,\kappa}$ *so* $\lambda > \mu$.
λ^+ *is not measurable (even in cases it is regular*[6]*) when*

$\boxtimes$ (a) $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$
(b) $\lambda > \mathrm{hrtg}((\mathrm{Fil}^4_{\partial(*)}(\kappa,\mu))$.

Proof. Naturally we fix a witness $c\ell$ for $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$. Let $\mathscr{F}_{\mathfrak{y}}, \Xi_{\mathfrak{y}}, f_{\mathfrak{y},\alpha}, \mathscr{F}^\lambda_{\mathfrak{y},\alpha}$ be defined as in 2.5 so by claims 2.5, 2.7 we have $\cup\{\Xi_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa)\} \supseteq \lambda^+$; moreover, $\alpha \in \lambda^+ \cap \Xi_{\mathfrak{y}} \Rightarrow f_{\eta,\alpha} \in {}^\kappa\lambda$.

Let $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa,\mu)$ be such that $|\mathscr{F}_{\mathfrak{y}}| > \lambda$, we can find such $\mathfrak{y}$ by 2.8, as without loss of generality we can assume λ^+ is regular (or even measurable, toward contradiction). Let $Z = Z[\mathfrak{y}]$. So $\Xi_{\mathfrak{y}}$ is a set of ordinals of cardinality $> \lambda$. For $\zeta <\ \mathrm{otp}(\Xi_{\mathfrak{y}})$ let α_ζ be the ζ-th member of $\Xi_{\mathfrak{y}}$, so $f_{\mathfrak{y},\alpha_\zeta}$ is well defined. Toward

[6] the regular holds many times by 2.13

contradiction let D be a (non-principal) ultrafilter on λ^+ which is λ^+-complete. For $i \in Z$ let $\gamma_i < \lambda$ be the unique ordinal γ such that $\{\zeta < \lambda^+ : f_{\mathfrak{y},\alpha_\zeta}(i) = \gamma\} \in D$. As $|Z| \le \kappa < \lambda^+$ and D is κ^+-complete clearly $\{\zeta : \bigwedge_{i \in Z} f_{\mathfrak{y},\alpha_\zeta}(i) = \gamma_i\} \in D$, so as D is a non-principal ultrafilter, for some $\zeta_1 < \zeta_2$, $f_{\mathfrak{y},\alpha_{\zeta_1}} = f_{\mathfrak{y},\alpha_{\zeta_2}}$, contradiction. So there is no such D. $\square_{2.9}$

Remark 2.10. Similarly if D is κ^+-complete and weakly λ^+-saturated and $\mathrm{Ax}^0_{\lambda^+,<\mu}$, see [She16].

Claim 2.11. *If* $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$, *<u>then</u> we can find* $\bar{C}$ *such that:*

- (a) $\bar{C} = \langle C_\delta : \delta \in S\rangle$
- (b) $S = \{\delta < \lambda : \delta$ *is a limit ordinal of cofinality* $\ge \partial(*)\}$
- (c) C_δ *is an unbounded subset of* δ, *even a club*
- (d) *if* $\delta \in S$, $\mathrm{cf}(\delta) \le \kappa$ *then* $|C_\delta| < \mu$
- (e) *if* $\delta \in S$, $\mathrm{cf}(\delta) > \kappa$ *then* $|C_\delta| < \mathrm{hrtg}(\mu \times [\mathrm{cf}(\delta)]^\kappa)$.

Remark 2.12. 1) Recall that if we have Ax^4_λ (see 2.2(5)) then trivially there is $\langle C_\delta : \delta < \lambda, \mathrm{cf}(\delta) \le \partial\rangle, C_\delta$ a club of δ of order type $\mathrm{cf}(\delta)$ as if $<_*$ well order $[\lambda]^{\le \partial}$ we let $C_\delta :=$ be the $<_*$-minimal C which is a closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$.
2) $\mathrm{Ax}^0_{\lambda,<\xi,\kappa}$ suffices if $\kappa < \xi < \lambda$.

Proof. The "even a club" is not serious as we can replace C_δ by its closure in δ.

Let $c\ell$ witness $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$. For each $\delta \in S$ with $\mathrm{cf}(\delta) \in [\partial(*), \kappa]$ we let

$$C_\delta = \cap\{\delta \cap c\ell(C) : C \text{ a club of } \delta \text{ of order type } \mathrm{cf}(\delta)\}.$$

Now $\bar{C}' = \langle C_\delta : \delta \in S \text{ and } \mathrm{cf}(\delta) \in [\partial(*), \kappa]\rangle$ is well defined and exist. Clearly C_δ is a subset of δ.

For any club C of δ of order type $\mathrm{cf}(\delta) \in [\partial(*), \kappa]$ clearly $\delta \cap c\ell(C) \subseteq c\ell(C)$ which has cardinality $< \mu$.

The main point is to show that C_δ is unbounded in δ, otherwise we can choose by induction on $\varepsilon < \partial$, a club $C_{\delta,\varepsilon}$ of δ of order type $\mathrm{cf}(\delta)$, decreasing with ε such that $C_{\delta,\varepsilon} \nsubseteq c\ell(C_{\delta,\varepsilon+1})$, we use DC_∂. But this contradicts the choice of $c\ell$ recalling Definition 2.2(1).

If $\delta < \lambda$ and $\mathrm{cf}(\delta) > \kappa$ we let

$$\begin{aligned} C^*_\delta = \cap\{\cup\{\delta \cap c\ell(u) :\ & u \subseteq C \text{ has cardinality } \le \partial\} : \\ & C \text{ is a club of } \delta \text{ of order type } \mathrm{cf}(\delta)\}. \end{aligned}$$

A problem is a bound of $|C^*_\delta|$. Clearly for C a club of δ of order type $\mathrm{cf}(\delta)$ the order-type of the set $\cup\{\delta \cap c\ell(v) : v \subseteq C \text{ has cardinality} \le \partial\}$ is $< \mathrm{hrtg}(\mu \times [\mathrm{cf}(\delta)]^\kappa)$. As for "$C^*_\delta$ is a club" it is proved as above. $\square_{2.11}$

The following lemma gives the existence of a class of regular successor cardinals.

Lemma 2.13. *1) Assume*

- (a) δ *is a limit ordinal* $< \lambda_*$ *with* $\mathrm{cf}(\delta) = \partial$
- (b) λ^*_i *is a cardinal for* $i < \delta$ *increasing with* i

(c) $\lambda_* = \Sigma\{\lambda^*_i : i < \delta\}$

(d) $\lambda^*_{i+1} \geq \mathrm{hrtg}(\mu \times {}^\kappa(\lambda^*_i))$ *for* $i < \delta$ *and* $(\alpha) \vee (\beta)$ *hold where:*

  (α) Ax^4_λ *or*

  (β) $\lambda^*_{i+1} \geq \mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\lambda^*_i, \mu))$ *and* $\mathrm{hrtg}([\lambda^*_i]^{\leq\kappa}) \leq \lambda^*_{i+1}$

(e) $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$ *and* $\mu < \lambda^*_0$

(f) $\lambda = \lambda^+_*$

Then λ *is a regular cardinal.*
2) Assume $\mathrm{Ax}^4_\lambda, \lambda = \lambda^+_*, \lambda_*$ *singular and* $\chi < \lambda_* \Rightarrow \mathrm{hrtg}({}^\partial\chi) \leq \lambda_*$ *then* λ *is regular.*

Remark 2.14. This says that the successor of many strong limit singulars is regular.

Question 2.15. 1) Is $\mathrm{hrtg}(\mathscr{P}(\mathscr{P}(\lambda^*_i))) \geq \mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\lambda^*_i))$?
2) Is $|c\ell(f \restriction B)| \leq \mathrm{hrtg}([B]^{<\aleph_0})$ for the natural $c\ell$ and f, B as in the proof of 2.13?

Proof. 1) We can replace δ by $\mathrm{cf}(\delta)$ so without loss of generality δ is a regular cardinal so $\delta = \partial$.

So

$(*)_1$ (a) fix $c\ell : [\lambda]^{\leq\kappa} \to \mathscr{P}(\lambda)$ a witness to $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$

  (b) let $\langle C_\xi[c\ell] : \xi < \lambda, \mathrm{cf}(\xi) \geq \partial\rangle$ be as in the proof of 2.11, so $\xi < \lambda \wedge \partial \leq \mathrm{cf}(\xi) < \lambda \Rightarrow |C_\xi[c\ell]| < \lambda$.

[Why the last inequality? If $\delta < \lambda^+$, then there is i such that $\lambda^*_i > \mu + \mathrm{cf}(\partial)$ hence $\mathrm{otp}(C_\delta) < \mathrm{hrtg}(\mu \times [\mathrm{cf}(\delta)]^\kappa) \leq \mathrm{hrtg}([\lambda^*_i]^\kappa) < \lambda^*_{i+1}$.]

First, we shall use just $\lambda > \lambda_* \wedge (\forall\delta < \lambda)(\mathrm{cf}(\delta) < \lambda_*)$, a weakening of the assumption that $\lambda = \lambda^+_*$.

Now

$\boxtimes_1$ for every $i < \delta$ and $A \subseteq \lambda$ of cardinality $\leq \lambda^*_i$, we can find $B \subseteq \lambda$ of cardinality $\leq \lambda_*$ satisfying $(\forall\alpha \in A)[\alpha$ is limit $\wedge\ \mathrm{cf}(\alpha) \leq \lambda^*_i \Rightarrow \alpha = \sup(\alpha \cap B)]$.

The proof of this will take some time. By 2.11 (and 0.16) the only problem is for $Y := \{\alpha : \alpha \in A, \alpha > \sup(A \cap \alpha), \alpha$ a limit ordinal of cofinality $< \partial + \aleph_1\}$; so $|Y| \leq \lambda^*_i$. Note: if we assume Ax^4_λ this would be immediate.

We define D as the family of sets $A \subseteq Y$ such that:

$\circledast^1_A$ for some set $C \subseteq \lambda$ of $\leq \partial$ ordinals, the set $B_C =: \cup\{\mathrm{Rang}(f_{\mathfrak{x},\zeta}) : \mathfrak{x} \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i, \mu)$ and $\zeta \in C$ or for some $\xi \in C$, we have $\lambda^*_i \geq \mathrm{cf}(\xi) > \partial$ and $\zeta \in C_\xi[c\ell]\}$ satisfies $\alpha \in Y \backslash A \Rightarrow \alpha = \sup(\alpha \cap B_C)$.

Clearly

$\circledast_2$ (a) $Y \in D$

  (b) D is upward closed

  (c) D is closed under intersection of $\leq \partial$ hence of $< \partial(*)$ sets.

[Why? For clause (a) use $C = \emptyset$, for clause (b), note that if C witness a set $A \subseteq Y$ belongs to D then it is a witness for any $A' \subseteq Y$ such that $A \subseteq A'$. Lastly, for clause (c) if $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial^+$, as we have AC_∂, there is a sequence $\langle C_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ such that C_ε witnesses $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial^+$, then $C := \cup\{C_\varepsilon : \varepsilon < \varepsilon(*)\}$ witnesses $A := \cap\{A_\varepsilon : \varepsilon < \varepsilon(*)\} \in D$ and, again by AC_∂, we have $|C| \le \partial$.]

- $\circledast_3$ if $\emptyset \in D$ then we are done.

[Why? For $a = \emptyset \in D$ let $C \subseteq \lambda$ be as promised in $\circledast_1$ and then B_C is as required; its cardinality $\le \lambda^*_{i+1}$ by 2.11.]

So assume $\emptyset \notin D$, so D is an ∂^+-complete filter on Y. As $1 \le |Y| \le \lambda^*_i$, let g be a one to one function from $|Y| \le \lambda^*_i$ onto Y and let

- $\circledast_4$ (a) $D_1 := \{B \subseteq \lambda^*_i : \{g(\alpha) : \alpha \in B \cap |Y|\} \in D\}$
  - (b) $\zeta := \mathrm{rk}_{D_1}(g)$
  - (c) $D_2 := \{B \subseteq \lambda^*_i : B \notin D_1$ and $\mathrm{rk}_{D_1+(\lambda^*_i \setminus B)}(g) > \zeta\} \cup D_1$.

So D_2 is an ∂^+-complete filter on λ^*_i extending D_1.

Let $B_* \in D_2$ be such that $(\forall B')[B' \in D_2 \wedge B' \subseteq B_* \Rightarrow c\ell(\mathrm{Rang}(g \restriction B')) \supseteq (\mathrm{Rang}(g \restriction B_*)]$. Let $\mathscr{U} = \cap\{c\ell(\mathrm{Rang}(g \restriction B') : B' \in D_2\}$, so $\mathrm{Rang}(g \restriction B_*) \subseteq \mathscr{U}$, even equal.

Let h be the function with domain B_* defined by $\alpha \in B_* \Rightarrow h(\alpha) = \mathrm{otp}(g(\alpha) \cap \mathscr{U})$.

So $\mathfrak{x} := (D_1, D_2, B_*, h) \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i, \mu)$ and for some ζ we have $g \restriction B_* = f_{\mathfrak{x},\zeta}[c\ell]$.

It suffices to consider the following two subcases.

Subcase 1a: $\mathrm{cf}(\zeta) > \partial$.

So recalling $(*)_1(b)$, $C_\zeta[c\ell]$ is well defined and let $C := \{\zeta\}$ hence $B_C = \cup\{\mathrm{Rang}(f_{\mathfrak{x},\varepsilon}[c\ell] : \varepsilon \in C_\zeta[c\ell]\}$ so C exemplifies that the set $X := \{\alpha \in Y : \alpha > \sup(\alpha \cap B_C)\}$ belongs to D hence $X_* = \{\alpha < |Y| : g(\alpha) \in X\}$ belongs to D_1.

Now define g', a function from λ^*_i to Ord by $g'(\alpha) = \sup(g(\alpha) \cap B_C)+1$ if $\alpha \in X_*$ and $g'(\alpha) = 0$ otherwise. Clearly $g' < g$ mod D_1 hence $\mathrm{rk}_{D_1}(g') < \zeta$, hence there is $g'', g' <_{D_1} g'' <_{D_1} g$ such that $\xi := \mathrm{rk}_{D_1}(g'') \in C_\zeta[c\ell]$.

Now for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i)$ we have $D^{\mathfrak{y}} = D_2$ and $g'' = f_{\mathfrak{y},\xi}$ mod $D^{\mathfrak{y}}_2$.

So $B =: \{\varepsilon < |Y| : g''(\varepsilon) = f_{\mathfrak{y},\xi}(\varepsilon)\} \in D^{\mathfrak{y}}_2$ hence $B \in D^+_2$. So $B \cap B_* \cap X_* \in D^+_2$ but if $\varepsilon \in B \cap B_* \cap A_*$ then $f_{\mathfrak{y},\xi}(\varepsilon) \in B_C$ and $f_{\mathfrak{y},\xi}(\varepsilon) \in \sup((B_C \cap g(\varepsilon)), g(\varepsilon))$.

This gives contradiction.

Subcase 1b: $\mathrm{cf}(\zeta) \le \partial$.

We choose a $C \subseteq \zeta$ of order type $\le \partial$ unbounded in ζ and proceed as in subcase 1a.

As we have covered both subcases, we have proved $\boxtimes_1$.

Recall we are assuming $\delta = \partial$; now:

- $\boxtimes_2$ for every $A \subseteq \lambda$ of cardinality $\le \lambda_*$ there is $B \subseteq \lambda$ of cardinality $\le \lambda_*$ such that:
  - $\oplus$ $A \subseteq B, [\alpha + 1 \in A \Rightarrow \alpha \in B]$ and $[\alpha \in A \wedge \aleph_0 \le \mathrm{cf}(\alpha) < \lambda_* \Rightarrow \alpha = \sup(B \cap \alpha)]$.

[Why? Choose a $\subseteq$-increasing sequence $\langle A_j : j < \delta\rangle$ such that $A = \cup\{A_i : i < \delta\}$ and $j < \delta \Rightarrow |A_j| \leq \lambda^*_j$, possible as $|A| \leq \lambda_*$. For each $j < \delta$ there exists B_j such that the conclusion of $\boxplus_1$ holds with (A_j, B_j, λ^*_j) here standing for (A, B, λ_i) there, so $|B_j| \leq \lambda_*$. So as AC_δ holds (as $\delta \leq \partial$) there is a sequence $\langle \bar{B}_j : j < \delta\rangle$, each $\bar{B}_j$ as above.

Lastly, let $B = \cup\{B_j : j < \delta\}$, it is as required.]

- $\boxtimes_3$ for every $A \subseteq \lambda$ of cardinality $\leq \lambda_*$ we can find $B \subseteq \lambda$ of cardinality $\leq \lambda_*$ such that $A \subseteq B, [\alpha + 1 \in B \Rightarrow \alpha \in B]$ and $[\alpha \in B$ is a limit ordinal $\wedge\ \mathrm{cf}(\alpha) < \lambda_* \Rightarrow \alpha = \sup(B \cap \alpha)]$.

[Why? We choose B_i by induction on $i < \omega \leq \partial$ such that $|B_i| \leq \lambda_*$ by $B_0 = A, B_{2i+1} = \{\alpha : \alpha \in B_{2i}$ or $\alpha + 1 \in B_{2i+1}\}$ and B_{2i+2} is chosen as B was chosen in $\boxtimes_2$ for i with B_{2i+1}, B_{2i+2} here in the role of A, B there. There is such $\langle B_i : i < \omega\rangle$ as $\mathrm{DC} = \mathrm{DC}_{\aleph_0}$ holds. So easily $B = \cup\{B_i : i < \omega\}$ is as required.]

Now return to our main case $\lambda = \lambda^+_*$

- $\boxtimes_4$ λ^+_* is regular.

[Why? Otherwise $\mathrm{cf}(\lambda^+_*) < \lambda^+_*$ hence $\mathrm{cf}(\lambda^+_*) \leq \lambda_*$, but λ_* is singular so $\mathrm{cf}(\lambda^+_*) < \lambda_*$ hence there is a set A of cardinality $\mathrm{cf}(\lambda^+_*) < \lambda_*$ such that $A \subseteq \lambda^+_* = \sup(A)$. Now choose B as in $\boxtimes_3$. So $|B| \leq \lambda_*, B$ is an unbounded subset of $\lambda^+_*, \alpha+1 \in B \Rightarrow \alpha \in B$ and if $\alpha \in B$ is a limit ordinal then $\mathrm{cf}(\alpha) \leq |\alpha| \leq \lambda_*$, but $\mathrm{cf}(\alpha)$ is regular so $\mathrm{cf}(\alpha) < \lambda_*$ hence $\alpha = \sup(B \cap \alpha)$. But this trivially implies that $B = \lambda^+_*$, but $|B| \leq \lambda_*$, contradiction.]

2) Similar, just easier. $\square_{2.13}$

Remark 2.16. Of course, if we assume Ax^4_λ <u>then</u> the proof of 2.13 is much simpler: if $<_*$ is a well ordering of $[\lambda]^{\leq\partial}$ for $\delta < \lambda$ of cofinality $\leq \partial$ let $C_\delta =$ the $<_*$-first closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$, see 3.3.

Claim 2.17. *Assume*

- (a) $\langle \lambda_i : i < \kappa\rangle$ *is an increasing continuous sequence of cardinals* $> \kappa$
- (b) $\lambda = \lambda_\kappa = \Sigma\{\lambda_i : i < \kappa\}$
- (c) $\kappa = \mathrm{cf}(\kappa) > \partial$
- (d) $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$
- (e) $\mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\kappa, \mu)) < \lambda$ *and* $\kappa, \mu < \lambda_0$
- (f) $S := \{i < \kappa : \lambda^+_i$ *is a regular cardinal*$\}$ *is a stationary subset of* κ
- (g) *let* $D := D_\kappa + S$ *where* D_κ *is the club filter on* κ
- (h) $\gamma(*) = \mathrm{rk}_D(\langle \lambda^+_i : i < \kappa\rangle)$.

<u>*Then*</u> $\gamma(*)$ *has cofinality* $> \lambda$, *so* $(\lambda, \gamma(*)] \cap \mathrm{Reg} \neq \emptyset$.

Proof. Recall 2.5 which we shall use. Toward contradiction assume that $\mathrm{cf}(\gamma(*)) \leq \lambda_\kappa$, but λ_κ is singular hence for some $i(*) < \kappa$, $\mathrm{cf}(\gamma(*)) \leq \lambda_{i(*)}$. Let $c\ell$ witness $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$.

Let B be an unbounded subset of $\gamma(*)$ of order type $\mathrm{cf}(\gamma(*)) \leq \lambda_{i(*)}$. By renaming without loss of generality $i(*) = 0$.

For $\alpha < \gamma(*)$ let

$$\begin{aligned}\mathscr{U}_\alpha = \cup\{\mathrm{Rang}(f_{\mathfrak{y},\alpha}) : \quad & f_{\mathfrak{y},\alpha}[c\ell] \text{ is well defined } \in \Pi\{\lambda_i^+ : i \in Z^{\mathfrak{y}}\} \\ & \text{and } \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa) \text{ and } D_1^{\mathfrak{y}} = D\}.\end{aligned}$$

Clearly $\mathscr{U}_\alpha$ is well defined by 2.5; moreover, $\langle \mathscr{U}_\alpha : \alpha < \gamma(*)\rangle$ exists and $|\mathscr{U}_\alpha| \le \mathrm{hrtg}(\kappa \times \mathrm{Fil}^4_{\partial(*)}(\kappa,\mu)) = \mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\kappa,\mu))$, even $<$ recalling 0.16(4). Let $\mathscr{U} = \cup\{\mathscr{U}_\alpha : \alpha \in B\}$ so $|\mathscr{U}| \le \mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\kappa,\mu)) + |B|$.

We define $f \in \prod\limits_{i<\kappa} \lambda_i^+$ by

- (α) $f(i)$ is: $\sup(\mathscr{U} \cap \lambda_i^+) + 1$ if $\mathrm{cf}(\lambda_i^+) > |\mathscr{U}|$ and zero otherwise.

So

- (β) $f \in \prod\limits_{i<\kappa} \lambda_i^+$.

Clearly

- (γ) $\{i < \kappa : f(i) = 0\} = \emptyset \bmod D$.

Let $\alpha(*) = \mathrm{rk}_D(f)$, it is $< \mathrm{rk}_D(\langle \lambda_i^+ : i < \kappa\rangle) = \gamma(*)$, so by clause (γ) there is $\beta(*) \in B$ such that $\alpha(*) < \beta(*) < \gamma(*)$ hence for some $g \in \prod\limits_{i<\kappa} \lambda_i^+$ we have $\mathrm{rk}_D(g) = \beta(*)$ and $f < g \bmod D$, so for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\kappa)$ we have $D_1^{\mathfrak{y}} = D_\kappa + S$ and $g \in \mathscr{F}_{\mathfrak{y},\beta(*)}$, hence $f(i) < g(i) < f_{\mathfrak{y},\beta(*)}(i) \in \mathscr{U} \cap \lambda_i^+$ for every $i \in Z^{\mathfrak{y}} \cap S$.

So we get easy contradiction to the choice of g. $\square_{2.17}$

Claim 2.18. *Assume $c\ell$ witness $\mathrm{Ax}^0_{\alpha,<\mu,\kappa}$ and $\mathrm{hrtg}(Y) < \mu \in [\kappa,\mu)$. The ordinals $\gamma_\ell, \ell = 0,1,2$ are nearly equal see, i.e. ⊛ below holds where:*

- ⊠ (a) $\gamma_0 = \mathrm{hrtg}({}^Y\alpha)$, *a cardinal*
  - (b) $\gamma_1 = \cup\{\mathrm{rk}_D(\gamma) : \gamma = \mathrm{rk}_D(\alpha)$ *for some* $D \in \mathrm{Fil}_{\partial(*)}(Y)\}$
  - (c) $\gamma_2 = \sup\{\mathrm{otp}(\Xi_{\mathfrak{y}}[c\ell]) + 1 : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y)\}$
- ⊛ (α) $\gamma_2 \le \gamma_1 \le \gamma_0$
  - (β) γ_0 *is the union of* $\mathrm{Fil}^4_{\partial(*)}(Y)$ *sets each of order type* $< \gamma_2$
  - (γ) γ_0 *is the disjoint union of* $< \mathrm{hrtg}(\mathscr{P}(\mathrm{Fil}^4_{\partial(*)}(Y)))$ *sets each of order type* $< \gamma_2$
  - (δ) *if* $\gamma_0 > \mathrm{hrtg}(\mathscr{P}(\mathrm{Fil}^4_{\partial(*)}(Y)))$ *and* $\gamma_0 \ge |\gamma_2|^+$ *then* $|\gamma_0| \le |\gamma_2|^{++}$ *and* $\mathrm{cf}(|\gamma_2|^+) < \mathrm{hrtg}(\mathscr{P}(\mathrm{Fil}^4_{\partial(*)}(Y)))$.

Proof. Straightforward, see 0.16. $\square_{2.18}$

§ 3. Concluding Remarks

In May 2010, David Aspero asked whether it is true that I have results along the following lines (or that it follows from such a result):

If GCH holds and λ is a singular cardinal of uncountable cofinality, then there is a well-order of $\mathscr{H}(\lambda^+)$ definable in $(\mathscr{H}(\lambda^+), \in)$ using a parameter.

The answer is yes by [She97, 4.6,pg.117] but we elaborate this below somewhat more generally. Much earlier Gitik [Git80] had proved (using suitable large cardinals) the consistency of "ZF + every infinite cardinal has cofinality $\aleph_0$, i.e. $\aleph_0$ is the only regular cardinal". This naturally raises the question what suffices to have a class of regulars. Gitik told me that in Luming 2008 Woodin has conjectured:

$\boxplus$ let V be a model of ZF + DC, suppose that κ is a singular strong limit cardinal of cofinality ω_1 and $|\mathscr{H}(\kappa)| = \kappa$. Is then $\mathscr{P}(\kappa)$ well orderable?

Now [She97] gives some information. The results here (3.1) confirm $\boxplus$.

Claim 3.1. [DC] *Assume that μ is a singular cardinal of cofinality $\kappa > \aleph_0$ (no* GCH *needed), the parameter $X \subseteq \mu$ codes in particular the tree $\mathscr{T} = {}^{\kappa >}\lambda$ and the set $\mathscr{P}(\mathscr{P}(\kappa))$ and $F : {}^{\omega}\mu \to \mu$ which satisfies "(μ, F) has no infinite decreasing ω-chain of subalgebras"; in particular, from X a well orderings of $[\lambda]^{<\kappa} \cup \mathscr{P}(\mathscr{P}(\kappa))$ are definable. Then (with this parameter) we can define a well ordering of the set of κ-branches of the tree $({}^{\kappa >}\lambda, \triangleleft)$.*

Proof. Proof of 3.1:

Let $\langle \text{ cd}_i : i < \kappa\rangle$ satisfies

$\boxplus_1$ cd_i is a one-to-one function from ${}^{i}\mu$ into μ, (definable from X uniformly (in i))

$\boxplus_2$ let $<_\kappa$ be a well ordering of $\text{Fil}^4_\kappa(\kappa)$ definable from X.

For $\eta \in {}^{\kappa}\mu$ let $f_\eta : \kappa \to \mu$ be defined by $f_\eta(i) = \text{ cd}_i(\eta{\restriction}i)$, so $\bar{f} = \langle f_\eta : \eta \in {}^{\kappa}\mu\rangle$ is well defined.

Let $\bar{\mathscr{F}} = \langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in \text{ Fil}^4_\kappa(\kappa)\rangle$ be as in Theorem 1.2 with μ, κ here standing for λ, Y there; there is such $\bar{\mathscr{F}}$ definable from X as X codes also $[\mu]^{\aleph_0}$, see §1.

So for every $\eta \in {}^{\kappa}\mu$ there is $\mathfrak{y} \in \text{ Fil}^4_\kappa(\kappa)$ such that $f{\restriction}Z_{\mathfrak{y}} \in \mathscr{F}_{\mathfrak{y}}$ and $D^{\mathfrak{y}}_1$ contains all co-bounded subsets of κ so let $\mathfrak{y}(\eta)$ be the $<_\kappa$-first such $\mathfrak{y}$. Now we define a well ordering $<_*$ of ${}^{\kappa}\mu$: for $\eta, \nu \in {}^{\kappa}\mu$ let $\eta <_* \nu$ iff $\text{rk}_{D_1[\mathfrak{y}(\eta)]}(f_\eta{\restriction}Z_{\mathfrak{y}(\eta)}) < \text{rk}_{D_1(\mathfrak{y}(\nu))}(f_\nu{\restriction}Z_{\mathfrak{y}(\nu)})$ or equality holds and $\mathfrak{y}(\eta) < \mathfrak{y}(\nu)$.

This is O.K. because

$(*)$ if $\eta \neq \nu \in {}^{\kappa}\mu$ then $f_\eta(i) \neq f_\nu(i)$ for every large enough $i < \kappa$ (i.e. $i \geq \min\{j : \eta(j) \neq \nu(j)\}$.

$\square_{3.1}$

Conclusion 3.2. *[DC] Assume μ is a singular cardinal of uncountable cofinality and $\mathscr{H}(\mu)$ is well orderable of cardinality μ and $X \subseteq \mu$ codes $\mathscr{H}(\mu)$ and a well ordering of $\mathscr{H}(\mu)$. Then we can (with this X as parameter) define a well ordering of $\mathscr{P}(\mu)$; hence of $\mathscr{H}(\mu^+)$.*

Proof. Proof of 3.2:

Let $\langle \mu_i : i < \kappa\rangle$ be an increasing sequence of cardinals $< \mu$ with limit μ. Clearly $2^{\mu_i} < \mu$.

Let $\langle \mathrm{cd}^*_i : i < \kappa\rangle$ satisfies

$\boxplus_2$ cd^*_i is a one-to-one function from $\mathscr{P}(\mu_i)$ into μ, (definable uniformly from X).

So $\mathrm{cd}_* : \mathscr{P}(\mu) \to {}^\kappa\mu$ defined by $(\mathrm{cd}_*(A))(i) =\ \mathrm{cd}^*_i(A \cap \mu_i)$ for $A \subseteq \mu, i < \kappa$, is a one-to-one function from $\mathscr{P}(\mu)$ into ${}^\kappa\mu$. Now use 3.1. $\square_{3.2}$

We return to 2.13(2)

Claim 3.3. [DC] *1) The cardinal λ^+ is regular* *<u>when</u>:*

$\boxplus$ (a) $\mathrm{Ax}^4_{\lambda^+}$, *i.e.* $[\lambda^+]^{\aleph_0}$ *is well orderable*

(b) $|\alpha|^{\aleph_0} < \lambda$ *for* $\alpha < \lambda$

(c) λ *is singular.*

2) Also there is $\bar e = \langle e_\delta : \delta < \lambda^+\rangle, e_\delta \subseteq \delta = \sup(e_\delta), |e_\delta| \le \mathrm{cf}(\delta)^{\aleph_0}$.

Remark 3.4. Compare with 2.13; we use here more choice, but cover more cardinals.

Proof. Let $<_*$ be a well ordering of the set $[\lambda^+]^{\aleph_0}$.

As earlier let $F : {}^\omega(\lambda^+) \to \lambda^+$ be such that there is no $\subset$-decreasing sequence $\langle c\ell_F(u_n) : n < \omega\rangle$ with $u_n \subseteq \lambda^+$. Let $\Omega = \{\delta \le \lambda^+ : \delta$ a limit ordinal, $\delta < \lambda^+ \wedge\ \mathrm{cf}(\delta) < \lambda\}$, so $\mathrm{otp}(\Omega) \in \{\lambda^+, \lambda^+ + 1\}$.

We define $\bar e = \langle e_\delta : \delta \in \Omega\rangle$ as follows.

<u>Case 1</u>: $\mathrm{cf}(\delta) = \aleph_0, e_\delta$ is the $<_*$-minimal member of $\{u \subseteq \delta : \delta = \sup(u)$ and $\mathrm{otp}(u) = 0\}$.

<u>Case 2</u>: $\mathrm{cf}(\delta) > \aleph_0$.

Let $e_\delta = \cap\{c\ell_F(C) : C$ a club of $\delta\}$.

So

$(*)_1$ e_δ is an unbounded subset of δ of order type $< \lambda$.

[Why? If $\mathrm{cf}(\delta) = \aleph_0$ then e_δ has order type ω which is $< \lambda$ by clause (b) of the assumption.

If $\mathrm{cf}(\delta) > \aleph_0$ then for some club C of $\delta, e_\delta = c\ell_F(C)$ has $\mathrm{otp}(e_\delta) \le |c\ell_F(C)| \le (\mathrm{cf}(\delta)^{\aleph_0} < \lambda$. The last inequality holds as $\mathrm{cf}(\delta) \le \lambda$ as $\delta < \lambda^+$, $\mathrm{cf}(\delta) \ne \lambda$ as λ is singular by clause (c) of the assumption, and lastly $((\mathrm{cf}(\delta)^{\aleph_0}) < \lambda$ by clause (b) of the assumption.]

This is enough for part (2). Now we shall define a one-to-one function f_α from α into λ by induction on $\alpha \in \Omega$ as follows: let $\mathrm{pr}_\lambda : \lambda \times \lambda \to \lambda$ be a pairing function so one to one (can add "onto λ"); if we succeed then f_{λ^+} cannot be well defined so $\lambda^+ \notin \Omega$ hence $\mathrm{cf}(\lambda^+) \ge \lambda$, but λ is singular so $\mathrm{cf}(\lambda^+) = \lambda^+$, i.e. λ^+ is not singular so we shall be done proving part (1).

The inductive definition is:

$\boxplus$ (a) <u>if $\alpha \le \lambda$</u> <u>then</u> f_α is the identity

(b) <u>if $\alpha = \beta + 1 \in [\lambda, \lambda^+)$</u> <u>then</u> for $i < \alpha$ we let $f_\alpha(i)$ be

- $1 + f_\beta(i)$ if $i < \beta$
- 0 if $i = \beta$

(c) if $\alpha \in \Omega$ so α is a limit ordinal, $e_\alpha \subseteq \alpha = \sup(e_\alpha), e_\alpha$ of cardinality $< \lambda$ and we let f_α be defined by: for $i < \alpha$ we let $f_\alpha(i) = \mathrm{pr}_\lambda(f_{\min(e_\alpha \setminus (i+1))}(i), \mathrm{otp}(e_\alpha \cap i))$.

$\square_{3.3}$

We later add:

Claim 3.5. [ZFC] *Assume $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$ and $\mu = \mu^{\aleph_0} + 2^{2^\kappa}$.*
1) From some $X \subseteq \mu$ we can define well ordering of some set $\mathscr{G} \subseteq {}^\kappa\mu$ such that ${}^\kappa\mu = \{\sup\{f_n : n < \omega\} : f_n \in \mathscr{G}$ for $n < \omega\}$.
2) If moreover $2^{2^\theta} \leq \mu$ where $\theta = \kappa^{\aleph}$ then from some $X \subseteq \mu$ we can define a well ordering of ${}^\kappa\mu$.

Proof. 1) Let $X \subseteq \mu$ code $\mathscr{P}(\mathscr{P}(\kappa)), {}^\omega\mu$ and $F : {}^\omega\mu \to \omega$ which as in 3.1. Unlike the proof of 3.1 we do not use the $\mathrm{cd}_i(i < \kappa)$ and we use the family of $\aleph_1$-complete filters on κ, the rest should be clear.
2) As $\theta = \theta^{\aleph_0}$ there is a one-to-one onto function $\mathrm{cd} : {}^\omega\theta \to \theta$ onto θ, and for $i < \omega$ let $\mathrm{cd}_i : \theta \to \theta$ be such that:

$(*)_1$ if $\mathrm{cd}(\eta) = \zeta$, then $\mathrm{cd}_0(\zeta) = \ell g(\eta)$ and $\mathrm{cd}_{1+i}(\zeta) = \eta(i)$ for $i < \ell g(\eta)$.

Let D be $\{A \subseteq \theta$: for some $u \in [\theta]^{\leq \aleph_0}$ we have $A \supseteq \{\varepsilon < \theta : u \subseteq \{\mathrm{cd}_i(\varepsilon) : i < \omega\}\}$, so

$(*)_2$ D is an $\aleph_1$-complete filter on θ.

[Why? Should be clear.]

$(*)_3$ for $f \in {}^\theta\mu$ let g, g_f be the unique function g with doman θ such that:
- if $\varepsilon < \kappa$ and $i < \mathrm{cd}_0(\varepsilon)$, then $\mathrm{cd}_{1+i}(\varepsilon) < \theta \Rightarrow \mathrm{cd}_{1+i}(g(\varepsilon)) = f(\mathrm{cd}_{1+i}(\varepsilon))$ and $\mathrm{cd}_0(g(\varepsilon)) = \mathrm{cd}_0(\varepsilon)$ and $f(\zeta) = 0$ otherwise

[Why g_f exists? Just think.]

$(*)_4$ if $f \in {}^\theta\mu, \alpha = \mathrm{rk}_D(g_f)$ and $\mathfrak{y} = \mathfrak{y}_{g_f}$ as in the proof of 3.1 for g_f, then:
(a) from $g_f \restriction Z_\mathfrak{y}$ we can define f (using some $Y \subseteq \kappa$ as a parameter)
(b) $\mathrm{Rang}(f) \subseteq \{\mathrm{cd}_{1+i}(g_f(\varepsilon)) : \varepsilon \in Z_\mathfrak{y}$ and $i < \mathrm{cd}_0(g_f(\varepsilon))\}$.

[Why? Clause (a) follows clause (b). Clause (b) holds as for every $\xi < \kappa$, the set $\{\varepsilon < \theta : \xi \in \{\mathrm{cd}_{1+i}(\varepsilon) : i < \mathrm{cd}_0(\varepsilon)\}\} \in D$.]
We continue as in the proof of 3.1. $\square_{3.5}$

Conclusion 3.6. [DC] *Assume $[\lambda]^{\aleph_0}$ is well ordered for every λ.*
1) If 2^{2^κ} is well ordered then for every $\lambda, [\lambda]^\kappa$ is well ordered.
2) For any set Y, there is a derived set Y_* so called $\mathrm{Fil}^4_{\aleph_1}(Y)$ of power near $\mathscr{P}(\mathscr{P}(Y))$ such that $\Vdash_{\mathrm{Levy}(\aleph_0, Y)}$ "for every $\lambda, {}^Y\lambda$ is well ordered".

Proof. 1) By 3.1.
2) Follows easily. $\square_{3.6}$

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