ON LONG EF-EQUIVALENCE IN NON ISOMORPHIC MODELS SH836

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Abstract. There has been a great deal of interest in constructing models which are non-isomorphic, of cardinality λ, but are equivalent under the Ehrefeuch-Fraissé game of length α, even for every $\alpha < \lambda$. So under G.C.H. particularly for λ regular we know a lot. We deal here with constructions of such pairs of models proven in ZFC, and get their existence under mild conditions.

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0. Introduction

There has been much work on constructing pairs of $\mathrm{EF}_{\alpha,\mu}$-equivalent non-isomorphic models of the same cardinality.

In Summer of 2003, Vaanenen has asked me whether we can provably in ZFC construct a pair of non-isomorphic models of cardinality $\aleph_1$ which are EF_α-equivalent even for α like ω^2. We try to shed light on the problem for general cardinals. We construct such models for $\lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}$ for every $\alpha < \lambda$ simultaneously and then for singular $\lambda = \lambda^{\aleph_0}$. In subsequent work [HS07] we shall investigate further: weaken the assumption "$\lambda = \lambda^{\aleph_0}$" (e.g., $\lambda = \mathrm{cf}(\lambda) > \beth_\omega$) and we generalize the results for trees with no λ-branches and investigate the case of models of a first order complete T (mainly strongly dependent). We thank Chanoch Havlin and the referee for detecting some inaccuracies.

Definition 0.1. (1) We say that M_1, M_2 are EF_α-equivalent if M_1, M_2 are models (with same vocabulary) such that the isomorphism player has a winning strategy in the game $\Game^\alpha_1(M_1, M_2)$ defined below.

(1A) Replacing α by $< \alpha$ means: for every $\beta < \alpha$; similarly below.

(2) We say that M_1, M_2 are $EF_{\alpha,\mu}$- equivalent <u>when</u> M_2, M_2 are models with the same vocabulary such that the isomorphism player has a winning strategy in the game $\Game^\alpha_\mu(M_1, M_2)$ defined below.

(3) For M_1, M_2, α, μ as above and partial isomorphism f from M_1 into M_2 we define the game $\Game^\alpha_\mu(f, M_1, M_2)$ between the player ISO and AIS as follows:
   - (a) the play lasts α moves
   - (b) after β moves a partial isomorphism f_β from M_1 into M_2 is chosen increasing continuous with β
   - (c) in the $\beta+1$-th move, the player AIS chooses $A_{\beta,1} \subseteq M_1, A_{\beta,2} \subseteq M_2$ such that $|A_{\beta,1}| + |A_{\beta,2}| < 1 + \mu$ and then the player ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that
   $$A_{\beta,1} \subseteq \mathrm{Dom}(f_{\beta+1}) \text{ and } A_{\beta,2} \subseteq \mathrm{Rang}(f_{\beta+1})$$
   - (d) if $\beta = 0$, ISO chooses $f_0 = f$; if β is a limit ordinal ISO chooses $f_\beta = \cup\{f_\gamma : \gamma < \beta\}$.

The ISO player loses if he had no legal move.

(4) If $f = \emptyset$ we may write $\Game^\alpha_\mu(M_1, M_2)$. If μ is 1 we may omit it. We may write $\leq \mu$ instead of μ^+. The player ISO may be restricted to choose $f_{\beta+1}$ such that$(\forall a)(a \in \mathrm{Dom}(f_{\beta+1}) \wedge a \notin \mathrm{Dom}(f_\beta) \to a \in A_{\beta,1} \vee f_{\beta+1}(a) \in A_{\beta,2})$

1. The Case of Regular $\lambda = \lambda^{\aleph_0}$

Definition 1.1. (1) We say that $\mathfrak{x}$ is a λ-parameter <u>if</u> $\mathfrak{x}$ consists of
   - (a) a cardinal λ and ordinal $\alpha^* \leq \lambda$
   - (b) a set I, and a set $S \subseteq I \times I$ (where we shall have compatibility demand)

(c) a function $\mathbf{u} : I \to \mathcal{P}(\lambda)$; we let $\mathbf{u}_s = \mathbf{u}(s)$ for $s \in I$

(d) a set J and a function $\mathbf{s} : J \to I$, we let $\mathbf{s}_t = \mathbf{s}(t)$ for $t \in J$ and for $s \in I$ we let $J_s = \{t \in J : \mathbf{s}_t = s\}$

(e) a set $T \subseteq J \times J$ such that $(t_1, t_2) \in T \Rightarrow (\mathbf{s}_{t_1}, \mathbf{s}_{t_2}) \in S$

(1A) We say $\mathfrak{x}$ is a full λ- parameter <u>if</u> in addition it consists of:

(f) a function $\mathbf{g}$ with domain J such that $\mathbf{g}_t = \mathbf{g}(t)$ is a non-decreasing function from $\mathbf{u}_{\mathbf{s}(t)}$ to some $\alpha < \alpha^*$

(g) a function $\mathbf{h}$ with domain J such that $\mathbf{h}_t = \mathbf{h}(t)$ is a non-decreasing function from $\mathbf{u}_{\mathbf{s}(t)}$ to λ

such that

(h) if $t_1, t_2 \in J$ and $\mathbf{s}_{t_1} = s = \mathbf{s}_{t_2}, \mathbf{g}_{t_1} = g = \mathbf{g}_{t_2}$ and $\mathbf{h}_{t_1} = h = \mathbf{h}_{t_2}, \alpha^{t_1} = \alpha = \alpha^{t_2}$ <u>then</u> $t_1 = t_2$ hence we write $t = t^\alpha_{s,g,h} = t^\alpha(s, g, h)$.

(2) We may write $\alpha^* = \alpha^*_{\mathfrak{x}}, \lambda = \lambda_{\mathfrak{x}}, I = I_{\mathfrak{x}}, J = J_{\mathfrak{x}}, J_s = J^{\mathfrak{x}}_s, t^\alpha(s, g, h) = t^{\alpha,\mathfrak{x}}(s, g, h)$, etc. Many times we omit $\mathfrak{x}$ when clear from the context.

Definition 1.2. Let $\mathfrak{x}$ be a λ-parameter.

(1) For $s \in I_{\mathfrak{x}}$, let $\mathbb{G}^{\mathfrak{x}}_s$ be the group[1] generated freely by $\{x_t : t \in J_s\}$.

(2) For $(s_1, s_2) \in S_{\mathfrak{x}}$ let $\mathbb{G}_{s_1,s_2} = G^{\mathfrak{x}}_{s_1,s_2}$ by the subgroup of $\mathbb{G}^{\mathfrak{x}}_{s_1} \times \mathbb{G}^{\mathfrak{x}}_{s_2}$ generated by

$$\{(x_{t_1}, x_{t_2}) : (t_1, t_2) \in T_{\mathfrak{x}} \text{ and } t_1 \in J^{\mathfrak{x}}_{s_1}, t_2 \in J^{\mathfrak{x}}_{s_2}\}$$

(3) We say $\mathfrak{x}$ is (λ, θ)-parameter if $s \in I_{\mathfrak{x}} \Rightarrow |\mathbf{u}_s| < \theta$.

Remark 1.3. (1) We may use S a set of n-tuples from I (or $(< \omega)$-tuples) then we have to change Definitions 1.2(2) accordingly.

Definition 1.4. For a λ-parameter $\mathfrak{x}$ we define a model $M = M_{\mathfrak{x}}$ as follows (where below $I = I_{\mathfrak{x}}$, etc.).

(A) its vocabulary τ consist of

(α) P_s, a unary predicate, for $s \in I_{\mathfrak{x}}$

(β) Q_{s_1,s_2}, a binary predicate for $(s_1, s_2) \in S_{\mathfrak{x}}$

(γ) $F_{s,a}$, a unary function for $s \in I_{\mathfrak{x}}, a \in \mathbb{G}^{\mathfrak{x}}_s$

(B) the universe of M is $\{(s, x) : s \in I_{\mathfrak{x}}, x \in \mathbb{G}^{\mathfrak{x}}_s\}$

(C) for $s \in I_{\mathfrak{x}}$ let $P^M_s = \{(s, x) : x \in \mathbb{G}^{\mathfrak{x}}_s\}$

(D) $Q^M_{s_1,s_2} = \{((s_1, x_1), (s_2, x_2)) : (x_1, x_2) \in \mathbb{G}^{\mathfrak{x}}_{s_1,s_2})\}$ for $(s_1, s_2) \in S_{\mathfrak{x}}$

(E) if $s \in I_{\mathfrak{x}}$ and $a \in \mathbb{G}^{\mathfrak{x}}_s$ then $F^M_{s,a}$ is the unary function from P^M_s to P^M_s defined by $F^M_{s,a}(y) = ay$, multiplication in $\mathbb{G}^{\mathfrak{x}}_s$ (for $y \in M \setminus P^M_s$ we can let $F^M_{s,a}(y)$ be y or undefined).

Remark 1.5. We can expand $M_{\mathfrak{x}}$ by the following linear order: let $<_{\mathfrak{x}}$ linearly order I and for each $s \in I_{\mathfrak{x}}$ let $<^*_s$ be a linear order of $\mathbb{G}^{\mathfrak{x}}_s$ such that $(G^{\mathfrak{x}}_s, <^{\mathfrak{x}}_s)$

[1] we also could use abelian groups satisfying $\forall x(x + x = 0)$, in this case $\mathbb{G}_s$ is the family of finite subsets of J_2 with the symmetric difference operation also we could use the free abelian group.

is an ordered group, exists as ??$F^{\mathfrak{r}}_s$ is free and let $<_{M_{\mathfrak{r}}} = \{((s_1,\lambda_1)),(s_2,x_2) : (s_\ell,x_\ell) \in M_{\mathfrak{r}}$ for $\ell = 1,2$ and $s_1 <_{\mathfrak{r}} s_2$ or $s_1 = s_2 \wedge x_1 <^{\mathfrak{r}}_s x_2$

Definition 1.6. (1) For $\mathfrak{r}$ a λ-parameter and for $I' \subseteq I_{\mathfrak{r}}$ let $M^{\mathfrak{r}}_{I'} = M_{\mathfrak{r}} \restriction \cup \{P^{M_{\mathfrak{r}}}_s : s \in I'\}$ and let $I_\gamma = I^{\mathfrak{r}}_\gamma = \{s \in I_{\mathfrak{r}} : \sup(\mathbf{u}_s) < \gamma\}$.

(2) Assume $\mathfrak{r}$ is a full λ- parameter and $\beta < \lambda$; for $\alpha < \alpha^*_{\mathfrak{r}}$ we let $\mathcal{G}^{\mathfrak{r}}_{\alpha,\beta}$ be the set of $g : \beta \rightarrow \alpha$ which are non-decreasing; then for $g \in \mathcal{G}^{\mathfrak{r}}_{\alpha,\beta}$
  (a) we define $h = h_g : \beta \rightarrow \lambda$ as follows: $h(\gamma) = \text{Min}\{\beta' \leq \beta$: if $\beta' < \beta$ then $g(\beta') > g(\gamma)\}$
  (b) we let $I_g = I^{\mathfrak{r}}_g = \{s \in I : \mathbf{u}_s \subseteq \beta$ and $t^\alpha_{s,g\restriction \mathbf{u}_s,h_g\restriction \mathbf{u}_s}$ is well defined$\}$
  (c) we define $\bar{c}^\alpha_g = \langle c^\alpha_{g,s} : s \in I^{\mathfrak{r}}_g\rangle$ by $c^\alpha_{g,s} = x^\alpha_{t_{g,s}}$ where $t^\alpha_{g,s} = t^{\alpha,\mathfrak{r}}_{s,g\restriction \mathbf{u}_s,h_g\restriction \mathbf{u}_s}$.

(3) Let $\mathcal{G}^{\mathfrak{r}}_\alpha = \cup\{\mathcal{G}^{\mathfrak{r}}_{\alpha,\beta} : \beta < \lambda\}$ and $\mathcal{G}_{\mathfrak{r}} = \cup\{\mathcal{G}^{\mathfrak{r}}_\alpha : \alpha < \alpha^*\}$.

Definition 1.7. Let $\mathfrak{r}$ be a λ-parameter.

(1) Let $\mathbf{C}_{\mathfrak{r}} = \cup\{\mathbf{C}^{\mathfrak{r}}_{I'} : I' \subseteq I_{\mathfrak{r}}\}$ where for $I' \subseteq I_{\mathfrak{r}}$ we let $\mathbf{C}^{\mathfrak{r}}_{I'} = \{\bar{c} : \bar{c} = \langle c_s : s \in I'\rangle$ satisfies $c_s \in \mathbb{G}^{\mathfrak{r}}_s$ when $s \in I'$ and $(c_{s_1},c_{s_2}) \in \mathbb{G}_{s_1,s_2}$ when $(s_1,s_2) \in S_{\mathfrak{r}}$ and $s_1,s_2 \in I'\}$.

(2) For $\bar{c} \in \mathbf{C}^{\mathfrak{r}}_{I'}, I' \subseteq I_{\mathfrak{r}}$, let $f^{\mathfrak{r}}_{\bar{c}}$ be the partial function from $M_{\mathfrak{r}}$ into itself defined by $f^{\mathfrak{r}}_{\bar{c}}((s,y)) = (s,yc_s)$ for $(s,y) \in P^{M_{\mathfrak{r}}}_s, s \in I'$.

(3) $M_{\mathfrak{r}}$ is P_s-rigid when for every automorphism f of $M_{\mathfrak{r}}$, $f \restriction P^{M_{\mathfrak{r}}}_s$ is the identity.

Observation 1.8. *1) Let $\mathfrak{r}$ be a full λ-parameter. If $g : \gamma_2 \rightarrow \alpha$ where $\alpha < \alpha^*_{\mathfrak{r}}, \gamma_2 < \lambda$ and the function g is non-decreasing, $\gamma_1 < \gamma_2$ and $(\forall\gamma < \gamma_1)(g(\gamma) < g(\gamma_1))$ then $I_{g\restriction\gamma_1} \subseteq I_g$ and $h_{g\restriction\gamma_1} \subseteq h_g$ and $\bar{c}^\alpha_{g\restriction\gamma_1} = \bar{c}^\alpha_g \restriction I_{g\restriction\gamma_1}$.*
2) If $g \in \mathcal{G}^\alpha_{\mathfrak{r}}$ in Definition 1.6(3), then $\bar{c}^\alpha_g \in \mathbf{C}^{\mathfrak{r}}_{I^{\mathfrak{r}}_g}$.

Claim 1.9. *Assume $\mathfrak{r}$ is a full λ-parameter.*
1) For $I' \subseteq I_{\mathfrak{r}}$ and $\bar{c} \in \mathbf{C}^{\mathfrak{r}}_{I'}$, $f^{\mathfrak{r}}_{\bar{c}}$ is an automorphism of $M^{\mathfrak{r}}_{I'}$ which is the identity iff $s \in I' \Rightarrow c_s = e_{\mathbb{G}_s}$.
2) In (1) for $s \in I', f^{\mathfrak{r}}_{\bar{c}} \restriction P^{M_{\mathfrak{r}}}_s$ is not the identity iff $c_s \neq e_{\mathbb{G}_s}$.
3) If f is an automorphism of $M^{\mathfrak{r}}_{I_2}$ then $f \restriction M^{\mathfrak{r}}_{I_1}$ is an automorphism of $M^{\mathfrak{r}}_{I_1}$ for every $I_1 \subseteq I_2 \subseteq I_{\mathfrak{r}}$.
4) If $I' \subseteq I_{\mathfrak{r}}$ and f is an automorphism of $M^{\mathfrak{r}}_{I'}$, then $f = f^{\mathfrak{r}}_{\bar{c}}$ for some $\langle c_s : s \in I_{\mathfrak{r}}\rangle \in \mathbf{C}_{I'}$.
5) If $\bar{c}_\ell \in \mathbf{C}^{\mathfrak{r}}_{I_\ell}$ for $\ell = 1,2$ and $I_1 \subseteq I_2$ and $\bar{c}_1 = \bar{c}_2 \restriction I_1$ then $f_{\bar{c}_1} \subseteq f_{\bar{c}_2}$.
6) The cardinality of $M_{\mathfrak{r}}$ is $|J_{\mathfrak{r}}| + \aleph_0$

Proof: Straight, e.g.
4) For $s \in I'$ clearly $f((s,e_{\mathbb{G}_s})) \in P^{M_{\mathfrak{r}}}_s$ so it has the form $(s,c_s), c_s \in \mathbb{G}_s$ and let $\bar{c} = \langle c_s : s \in I'\rangle$. To check that $\bar{c} \in \mathbf{C}^{\mathfrak{r}}_{I'}$ assume $(s_1,s_2) \in S_{\mathfrak{r}}$; and we have to check that $(c_{s_1},c_{s_2}) \in \mathbb{G}_{s_1,s_2}$. This holds as $((s_1,e_{\mathbb{G}_{s_1}}),(s_2,e_{\mathbb{G}_{s_2}})) \in Q^{M_{\mathfrak{r}}}_{s_1,s_2}$ by the choice of $Q^{M_{\mathfrak{r}}}_{s_1,s_2}$ hence we have $((s_1,c_{s_1}),(s_2,c_{s_2})) = (f(s_1,e_{\mathbb{G}_{s_1}}),f(s_2,e_{\mathbb{G}_{s_2}})) \in Q^{M_{\mathfrak{r}}}_{s_1,s_2}$ hence $(c_{s_1},c_{s_2}) \in \mathbb{G}_{s_1,s_2}$. $\square_{1.9}$

Claim 1.10. *Let $\mathfrak{x}$ be a full λ-parameter $s \in I_{\mathfrak{x}}$ and $c_1, c_2 \in P^M_s, c^* \in \mathbb{G}_s$ and $F^{M_{\mathfrak{x}}}_{s,c^*}(c_1) = c_2$. A sufficient condition for "$(M_{\mathfrak{x}}, c_1), (M_{\mathfrak{x}}, c_2)$ are $\mathrm{EF}_{\alpha,\mu}$-equivalent" where $\alpha \le \alpha^*_{\mathfrak{x}}$, is the existence of $R, \bar{I}, \bar{\mathbf{c}}$ such that:*

$\circledast$ (a) *R is a partial order,*
(b) *$\bar{I} = \langle I_r : r \in R\rangle$ such that $I_r \subseteq I_{\mathfrak{x}}$ and $r_2 \le_R r_2 \Rightarrow I_{r_1} \subseteq I_{r_2}$*
(c) *R is the disjoint union of $\langle R_\beta : \beta < \alpha\rangle, R_0 \ne \emptyset$*
(d) *$\bar{\mathbf{c}} = \langle \bar{c}^r : r \in R\rangle$ where $\bar{c}^r \in \mathbf{C}_{I_r}$ and $r_1 \le r_2 = \bar{c}^{r_1} = \bar{c}^{r_2} \restriction I_{r_1}$ and $c^r_s = c^*$ so $s \in \cap\{I_r : r \in R\}$*
(e) *if $\langle r_\beta : \beta < \beta^*\rangle$ is $\le_R$-increasing, $\beta < \beta^* \Rightarrow r_\beta \in R_\beta$ and $\beta^* < \alpha$ <u>then</u> it has an $\le_R$-ub from R_{β^*}*
(f) *if $r_1 \in R_\beta, \beta + 1 < \alpha$ and $I' \subseteq I, |I'| < \mu$ <u>then</u> $(\exists r_2)(r_1 \le r_2 \in R_{\beta+1} \wedge I' \subseteq I_{r_2})$.*

Proof: Easy. Using 1.9(1),(5). $\square_{1.10}$

Claim 1.11. (1) *Let $\mathfrak{x}$ be a λ-parameter and $I' \subseteq I_{\mathfrak{x}}$. A necessary and sufficient condition for "$M^{\mathfrak{x}}_{I'}$ is P_s-rigid" is:*
$\circledast_1$ *there is no $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$ with $c_s \ne e_{\mathbb{G}_s}$.*

(2) *Let $\mathfrak{x}$ be a full λ-parameter and assume that $s(*) \in I_{\mathfrak{x}}, \alpha < \alpha^*_{\mathfrak{x}}, \alpha \ge \omega$ for notational simplicity and $t^* \in J^{\mathfrak{x}}_{s(*)}$. The models $M_1 = (M, (s, e_{\mathbb{G}_s})), M_2 = (M, (s, x_{t^*}))$ are $\mathrm{EF}_{\alpha,\lambda}$-equivalent <u>when</u>:*

$\circledast_{2,\alpha}$ (i) *λ is regular, $s \in I_{\mathfrak{x}} \Rightarrow |\mathbf{u}^{\mathfrak{x}}_s| < \lambda$*
(ii) *if $s \in I_{\mathfrak{x}}$ and $g \in \mathcal{G}_{\mathfrak{x}}$ and $\mathbf{u}^{\mathfrak{x}}_s \subseteq \mathrm{Dom}\ (g)$ then $t^{\alpha,\mathfrak{x}}_{s,g\restriction \mathbf{u}_s, h_g \restriction \mathbf{u}_s}$ is well defined*
(iii) *if $(s_1, s_2) \in S_{\mathfrak{x}}$ and $t_1 = t^\alpha_{s_1,g_1,h_1}, t_2 = t^\alpha_{s_2,g_2,h_2}$ are well defined then $(t_1, t_2) \in T_{\mathfrak{x}}$ when for some $g \in \mathcal{G}_{\mathfrak{x}}$ we have $g_{t_1} \cup g_{t_2} \subseteq g$ and $h_1 \cup h_2 \subseteq h_g$*
(iv) *$t^* = t^{\alpha,\mathfrak{x}}_{s(*),g,h_g}$ where $g : \mathbf{u}_{s(*)} \to \{0\}$ and h_g is constantly $\gamma^* = \cup\{\gamma + 1 : \gamma \in \mathbf{u}_{s(*)}\}$.*

Proof

(1) Toward contradiction assume that f is an automorphism of $M^{\mathfrak{x}}_{I'}$ such that $f \restriction P^{M_{\mathfrak{x}}}_s$ is not the identity. By 1.9(4) for some $\bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I'}$ we have $f = f_{\bar{c}}$. So $f_{\bar{c}} \restriction P^{M_{\mathfrak{x}}}_s = f \restriction P^{M_{\mathfrak{x}}}_s \ne \mathrm{id}$ hence by 1.9(1) we have $c_s \ne e_{\mathbb{G}_s}$, contradicting the assumption $\circledast_1$.
(2) We apply 1.10. For every $i < \alpha$ and non-decreasing function $g \in \mathcal{G}^{\mathfrak{x}}_\alpha$ from some ordinal $\gamma = \gamma_g$ into i we define $\bar{c}^\alpha_g = \langle c^\alpha_{g,s} : s \in I_{g_p}\rangle, c^\alpha_{g,s} = (s, x_{t^\alpha_{g,s}}), t^\alpha_{g,s} = t^\alpha_{s,g\restriction \mathbf{u}_s, h_g\restriction \mathbf{u}_s}$. Let $R_i = \{g : g$ a non-decreasing function from some $\gamma < \lambda$ to $1+i$ such that $\gamma^* \le \gamma, g \restriction \gamma^*$ is constantly zero, $\gamma^* < \gamma \Rightarrow g(\gamma^*) = 1\}$ and let $R = \cup\{R_i : i < \alpha\}$ ordered by inclusion. Let $\bar{I} = \langle I_g : g \in R\rangle$ and $\bar{\mathbf{c}} = \langle \bar{c}^\alpha_g : g \in R\rangle$. It is easy to check that $(R, \bar{I}, \bar{\mathbf{c}})$ is as required. $\square_{1.11}$

Claim 1.12. (1) *Assume $\alpha^* \le \lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}$. Then for some full $(\lambda, \aleph_1)$-parameter $\mathfrak{x}$ we have $|I| = \lambda = |J|, \alpha^*_{\mathfrak{x}} = \alpha^*$ and condition $\circledast_1$ of 1.11(1) holds and for every $s(*) \in I_{\mathfrak{x}} \backslash \{\emptyset\}$ condition $\circledast_{2,\alpha}$ of 1.11(2) holds whenever $\alpha < \alpha^*$.*

(2) *Moreover, if $s \in I_{\mathfrak{x}} \backslash \{\emptyset\}$ then for some $c_1 \ne c_2 \in P_s^{M_{\mathfrak{x}}}$ and $(M, c_1), (M, c_2)$ are $\mathrm{EF}_{\alpha,\lambda}$-equivalent for every $\alpha < \alpha^*_{\mathfrak{x}}$ but not $\mathrm{EF}_{\alpha^*_{\mathfrak{x}},\lambda}$-equivalent.*

Claim 1.12(1) clearly implies

Conclusion 1.13. (1) If $\lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}, \alpha^* \le \lambda$ then for some model M of cardinality λ we have:

(a) M has no non-trivial automorphism

(b) for every $\alpha < \lambda$ for some $c_1 \ne c_2 \in M$, the model $(M, c_1), (M, c_2)$ are EF_α-equivalent and even $\mathrm{EF}_{\alpha,\lambda}$-equivalent.

(2) We can strengthen clause (b) to: for some $c_1 \ne c_2$ for every $\alpha < \lambda$ the models $(M, c_1), (M, c_2)$ are $\mathrm{EF}_{\alpha,\lambda}$-equivalent.

Proof of 1.12: 1) Assume $\alpha_* > \omega$ for notational simplicity. We define $\mathfrak{x}$ by ($\lambda_{\mathfrak{x}} = \lambda$ and):

⊠ (a) (α) $I = \{u : u \in [\lambda]^{\le \aleph_0}\}$

(β) the function $\mathbf{u}$ is the identity on I

(γ) $S = \{(u_1, u_2) : u_1 \subseteq u_2 \in I\}$

(δ) $\alpha^*_{\mathfrak{x}} = \alpha^*$

(b) (α) J is the set of quadruple (u, α, g, h) satisfying

(i) $u \in I, \alpha < \alpha^*$

(ii) h is a non-decreasing function from u to λ

(iii) g is a non-decreasing function from u to α

(iv) if $\beta_1, \beta_2 \in u$ and $g(\beta_1) = g(\beta_2)$ then $h(\beta_1) = h(\beta_2)$

(v) $h(\beta) > \beta$

(β) let $t = (u^t, \alpha^t, g^t, h^t)$ for $t \in J$ so naturally $\mathbf{s}_t = u$, $\mathbf{g}_t = g^t, \mathbf{h}_t = h^t$

(γ) $T = \{(t_1, t_2) \in J \times J : \alpha^{t_1} = \alpha^{t_2}, u^{t_1} \subseteq u^{t_2}, h^{t_1} \subseteq h^{t_2}$ and $g^{t_1} \subseteq g^{t_2}\}$.

Now

$(*)_0$ $\mathfrak{x}$ is a full $(\lambda, \aleph_1)$-parameter
[Why? Just read Definition 1.1 and 1.2(3).]

$(*)_1$ for any $s(*) \in I \backslash \{\emptyset\}, \mathfrak{x}$ satisfies the demands for $\circledast_{2,\alpha}(i), (ii), (iii), (iv)$ from 1.11(2) for every $\alpha < \alpha^*$
[Why? just check]

$(*)_2$ if $u_1 \subseteq u_2 \in I$, we define the function $\pi_{u_1,u_2} : J_{u_2} \to J_{u_1}$ by $\pi_{u_1,u_2}(t) = (u_1, \alpha^t, g^t \restriction u_1, h^t \restriction u_1)$ for $t \in J_{u_2}$,
[Why is π_{u_1,u_2} a function from J_{u_2} into J_{u_1}? Just check]

$(*)_3$ for $u_1 \subseteq u_2$ we have

(α) $T \cap (J_{u_1} \times J_{u_2}) = \{(\pi_{u_1,u_2}(t_2), t_2) : t_2 \in J_{u_2}\}$ hence

(β) $\mathbb{G}_{u_1,u_2} = \{(\hat{\pi}_{u_1,u_2}(c_2), c_2) : c_2 \in \mathbb{G}_{u_2}\}$ where $\hat{\pi}_{u_1,u_2} \in \mathrm{Hom}(\mathbb{G}^{\mathfrak{x}}_{u_2}, \mathbb{G}^{\mathfrak{x}}_{u_1})$ is the unique homomorphism from $\mathbb{G}^{\mathfrak{x}}_{u_2}$ into $\mathbb{G}^{\mathfrak{x}}_{u_1}$ mapping x_{t_2}

to x_{t_1} whenever $\pi_{u_1,u_2}(t_2) = t_1$
[Why? Check.]

$(*)_4$ if $u_1 \cup u_2 \subseteq u_3 \in I, t_3 \in J_{u_3}$ and $t_\ell = \pi_{u_\ell,u_3}(t_3)$ for $\ell = 1,2$ then $\mathbf{g}_{t_1}, \mathbf{g}_{t_2}$ are compatible functions as well as $\mathbf{h}_{t_1}, \mathbf{h}_{t_2}$ and $\alpha^{t_1} = \alpha^{t_2}$ moreover $\mathbf{g}_{t_1} \cup \mathbf{g}_{t_2}$ is non-decreasing, $\mathbf{h}_{t_1} \cup \mathbf{h}_{t_2}$ is non-decreasing
[Why? just check]

$(*)_5$ clause $\circledast_1$ of 1.11(1) holds for $I' = I$, $s(*) \in I \setminus \{\emptyset\}$

[Why? Assume $\bar{c} \in C^{\mathfrak{x}}_I$ is such that $c_{s(*)} \neq e_{\mathbb{G}_{s(*)}}$. For each $u \in I, c_u$ is a word in the generators $\{x_t : t \in J_u\}$ of $\mathbb{G}_u$ and let $\mathbf{n}(u)$ be the length of this word and $\mathbf{m}(u)$ the number of generators appearing in it.
Now by $(*)_3$ we have $u_1 \subseteq u_2 \Rightarrow \mathbf{n}(u_1) \leq \mathbf{n}(u_2) \wedge \mathbf{m}(u_1) \leq \mathbf{m}(u_2)$. As $(I, \subseteq)$ is $\aleph_1$-directed, for some $u_* \in I$ we have $u_* \subseteq u \in I \Rightarrow \mathbf{n}(u) = n_* \wedge \mathbf{m}(u) = m_*$ and let $c_u = (\ldots, x^{i(\ell)}_{t(u,\ell)}, \ldots)_{\ell < n_*}$ where $i(\ell) \in \{1, -1\}$ and $t(u,\ell) \in J^{\mathfrak{x}}_u$ and $t(u,\ell) = t(u,\ell+1) \Rightarrow i(\ell) = i(\ell+1)$. Clearly $u_* \subseteq u_1 \subseteq u_2 \in I$ & $\ell < n_* \Rightarrow \pi_{u_1,u_2}(t(u_2,\ell)) = t(u_1,\ell)) \wedge \alpha^{t(u_2,\ell)} = \alpha^{t(u_*,\ell)}$. By our assumption toward contradiction necessarily $n_* > 0$.

As $\{u : u_* \subseteq u \in I\}$ is directed, by $(*)_4$ above, for each $\ell < n_*$ any two of the functions $\{g^{t(u,\ell)} : u_* \subseteq u \in I\}$ are compatible so $g_\ell =: \cup\{g^{t(u,\ell)} : u \in I\}$ is a non-decreasing function from $\lambda = \cup\{u : u \in I\}$ to α^* and $h_\ell =: \cup\{h^{t(u,\ell)} : u_* \subseteq u \in I\}$ is similarly a non-decreasing function from λ to λ. It also follows that for some α^*_ℓ we have $\alpha^*_\ell =: \alpha^{t(u,\ell)}$ whenever $u_* \subseteq u \in I$ in fact $\alpha^*_\ell = \alpha^{t(u_*,\ell)}$ is O.K. For each $i \in$ Rang$(g_\ell) \subseteq \alpha^*_\ell$ choose $\beta_{\ell,i} < \lambda$ such that $g_\ell(\beta_{\ell,i}) = i$ and let $E = \{\delta < \lambda : \delta$ a limit ordinal $> \sup(u_*)$ such that $i < \alpha^*_\ell$ & $\ell < n_*$ & $i \in \text{Rang}(g_\ell) \Rightarrow \beta_{\ell,i} < \delta$ and $\beta < \delta$ & $\ell < n \Rightarrow h_\ell(\beta) < \delta\}$, it is a club of λ. Choose u such that $u_* \subseteq u$ and $\text{Min}(u \backslash u_*) = \delta^* \in E$.

Now what can $\mathbf{g}_\ell($ Min $(u \backslash u_*))$ be?
It has to be i for some $i < \alpha^*_\ell < \alpha^*$ hence $i \in$ Rang(g_ℓ) so for some $u_1, u_* \subseteq u_1 \subseteq \delta^*$ and $\beta_{\ell,i} \in u_1$ so $h_\ell(\beta_{\ell,i}) < \delta^*$ hence considering $u \cup u_1$ and recalling clause $(\alpha)(vi)$ of (b) from definition of $\mathfrak{x}$ in the beginning of the proof we have $h_\ell(\beta_{\ell,i}) < h_\ell(\delta^*)$ hence by (clause $(b)(\alpha)(v)$) we have $i = g_\ell(\beta_{\ell,i}) < g_\ell(\delta^*)$, contradiction.]
2) A minor change is needed in the choice of $T^{\mathfrak{x}}$

$$T^{\mathfrak{x}} = \{(t_1,t_2) : \begin{array}{l} (t_1,t_2) \in J \times J \text{ and } u^{t_1} \subseteq u^{t_2}, h^{t_1} \subseteq h^{t_2}, g^{t_1} \subseteq g^{t_2}, \\ \gamma^{t_1} \leq \gamma^{t_2} \text{ and if } \text{Rang}(g^{t_1}) \nsubseteq \{0\} \text{ then } \alpha^{t_1} = \alpha^{t_2}\}. \end{array}$$

$\square_{1.12}$

2. The singular case

We deal here with singular $\lambda = \lambda^{\aleph_0}$ and our aim is the parallel of 1.13 constructing a pair of EF_α-equivalent for every $\alpha < \lambda$ non-isomorphic models of cardinality λ. But it is natural to try to construct a stronger example: This is done here:

- $\circledast$ for each $\gamma < \kappa = \mathrm{cf}(\lambda)$, in the following game the ISO player wins.

Definition 2.1. (1) For models M_1, M_2, λ and partial isomorphism f from M_1 to M_2 and $\gamma < \mathrm{cf}(\lambda)$ we define a game $\partial^*_{\gamma,\lambda}(f, M_1, M_2)$. A play lasts γ moves, in the $\beta < \gamma$ move a partial isomorphism f_β was formed increasing with β, extending f, satisfying $|\mathrm{Dom}(f_\beta)| < \lambda$. In the β-th move if $\beta = 0$, the player ISO choose $f_0 = f$, if β is a limit ordinal the ISO player chooses $f_\beta = \cup\{f_\epsilon : \epsilon < \beta\}$. In the $\beta + 1 < \gamma$ move the player AIS chooses $\alpha_\beta < \lambda$ and then they play a sub-game $\partial^{\alpha_\beta}_1(f_\beta, M_1, M_2)$ from 0.1(3) producing an increasing sequence of partial isomorphisms $\langle f^\beta_i : i < \alpha_\beta\rangle$ and let their union be $f_{\beta+1}$. ISO wins if he always has a legal move.

(2) If ISO wins the game (i.e. has a winning strategy) then we say M_1, M_2 are $\mathrm{EF}^*_{\gamma,\lambda}$-equivalent, we omit λ if clear from the context. If $f = \emptyset$ we may write $\partial^*_{\gamma,\lambda}(M_1, M_2)$

Remark: For $(M, c_1), (M, c_2)$ to be $\mathrm{EF}^*_{<\alpha,\lambda}$-equivalent not $\mathrm{EF}^*_{\alpha,\lambda}$- equivalent not just EF^*_α-equivalent not $\mathrm{EF}^*_{\alpha+1}$-equivalent we may need a minor change.

Hypothesis 2.2. $j_* \le \kappa = \mathrm{cf}(\lambda) < \lambda, \kappa > \aleph_0, \bar\mu = \langle \mu_i : i < \kappa\rangle$ is increasing continuous with limit $\lambda, \mu_0 = 0, \mu_1 = \kappa(= \mathrm{cf}(\lambda)), \mu_{i+1}$ is regular $> \mu_i^+$ and let $\mu_\kappa = \lambda$ and for $\alpha < \lambda$ let $\mathbf{i}(\alpha) = \mathrm{Min}\{i : \mu_i \le \alpha < \mu_{i+1}\}$.

Definition 2.3. Under the Hypothesis 2.2 we define a λ-parameter $\mathfrak{x} = \mathfrak{x}_{j_*,\bar\mu}$ as follows:

(a) (α) I is the set of $u \in [\lambda \setminus \kappa]^{\le\aleph_0}$
  (β) $\mathbf{u} : I \to \mathcal{P}(\lambda \setminus \kappa)$ is the identity,
  (γ) $S = \{(u_1, u_2) : u_1 \subseteq u_2 \in I\}$
  (δ) $\alpha^*_{\mathfrak{x}} = j_*$

(b) J is the set of tuples $t = (u, j, g, h) = (u^t, j^t, g^t, h^t)$ such that
  (α) $u \in I$
  (β) $j < j_*$
  (γ) (i) g is a non-decreasing function from $u_g = u \cup v_g$ to λ where $v_g = \{\mathbf{i}(\alpha) : \alpha \in u \text{ and } g(\alpha) = \mu^+_{\mathbf{i}(\alpha)}\}$
    (ii) $\alpha \in u \Rightarrow g(\alpha) \in [\mu_{\mathbf{i}(\alpha)}, \mu^+_{\mathbf{i}(\alpha)}]$
    (iii) if $i \in v_g$ then $g(i) < j^t(< \kappa = \mu_1)$
    (iv) v_g is an initial segment of $\{\mathbf{i}(\alpha) : \alpha \in u\}$
  (δ) (i) h is a non-decreasing function with domain $u_g \cup v_g$

(ii) $\alpha \in u \Rightarrow h(\alpha) \in [\mu_{\mathbf{i}(\alpha)}, \mu_{\mathbf{i}(\alpha)+1}]$ and if $i \in v_g$ then $h(i) < \kappa$

(iii) if $\beta_1 < \beta_2$ are from $u_g \cup v_g$ and $\mathbf{i}(\beta_1) = \mathbf{i}(\beta_2)$ then $g(\beta_1) = g(\beta_2) \Leftrightarrow h(\beta_1) = h(\beta_2)$

(iv) $\alpha < h(\alpha)$ for $\alpha \in u_g \cup v_g$ and $g(\alpha) = \mu^+_{\mathbf{i}(\alpha)} \Leftrightarrow h(\alpha) = \mu_{\mathbf{i}(\alpha)+1}$ for $\alpha \in u$

(c) T is the set of pairs $(t_1, t_2) \in J \times J$ satisfying

(i) $u^{t_1} \subseteq u^{t_2} \in I$ and

(ii) $g^{t_1} \subseteq g^{t_2}, h^{t_1} \subseteq h^{t_2}, j^{t_1} = j^{t_2}$

Observation 2.4. $\mathfrak{x}_\lambda = \mathfrak{x}_{j_*,\bar{\mu}}$ *is a full* λ*-parameter.*

Proof: Read the Definition 1.1(1)+1.1(1A)

Claim 2.5. *Assume* $s \in I_{\mathfrak{x}}, c_1 = (s, e_{\mathbb{G}_s}), c_2 = (s, x_t), t \in J_s$, *and for simplicity* $\mathrm{Rang}(g^t \restriction [\mu_{1+i}, \mu_{1+i+1})) \subseteq \{\mu_{1+i}\}, \mathrm{Rang}(g^t \restriction \kappa) = \{0\}$ *and* $\omega < j^t < j_*$. *Then* $(M_{\mathfrak{x}}, c_1), (M_{\mathfrak{x}}, c_2)$ *are* $\mathrm{EF}^*_{\lambda, j^t}$*-equivalent.*

Proof: So t, j^t are fixed. For $i_* < \kappa, j < j_*$ let

(a) $B_{i_*} = \{\bar{\beta} : \bar{\beta} = \langle \beta_i : i < \kappa \rangle$ and $\mu_i \le \beta_i \le \mu_{i+1}$ and $\beta_0 = i_*$ and $(\beta_{1+i} = \mu_{1+i+1} \equiv\ 1 + i < i_*)\}$

(b) for $\bar{\beta} \in B_{i_*}$ let $A_{\bar{\beta}} = \cup\{[\mu_i, \beta_i) : i < \kappa\}$ which by our conventions is equal to $i_* \cup \bigcup\{[\mu_j, \mu_{j+1}) : 1 \le j < i_*\} \cup \bigcup\{[\mu_i, \beta_i) : i \in [i_*, \kappa)\}$

(c) for $\bar{\beta} \in B_{i_*}$ let $\mathcal{G}_{j,i_*,\bar{\beta}} = \{g : g$ is a function from $A_{\bar{\beta}}$ to λ, non-decreasing and the function $g \restriction \kappa$ is into j and the function $g \restriction [\mu_{1+i}, \mu_{1+i+1})$ is into $[\mu_i, \mu^+_i]$ and $1 \le i < i_* \Leftrightarrow (\exists \alpha)(\mu_i \le \alpha < \mu_{i+1} \wedge g(\alpha) = \mu^+_i)\}$

(d) for $g \in \mathcal{G}_{j,i_*\bar{\beta}}, \bar{\beta} \in B_{i_*}$ we define $h_g : A_{\bar{\beta}} \to \lambda$ as follows: if $\gamma \in A_{\bar{\beta}}$ then $h(\gamma) = \mathrm{Min}\{\beta' \le \beta_{\mathbf{i}(\gamma)}$: if $i(\gamma) > 0 \wedge g(\gamma) = \mu^+_{\mathbf{i}(\gamma)}$ then $\beta' = \mu_{\mathbf{i}(\gamma)+1}$, otherwise $\beta' \in [\mu_{\mathbf{i}(\gamma)}, \beta_{\mathbf{i}(\gamma)}]$ and $\beta' \ne \beta_{\mathbf{i}(\gamma)} \Rightarrow g(\gamma) < g(\beta')\}$

(e) $\mathcal{G}_{j,i_*} = \cup\{\mathcal{G}_{j,i_*,\bar{\beta}} : \bar{\beta} \in B_{i_*}\}$ and $\mathcal{G}_j = \cup\{\mathcal{G}_{j,i_*} : i_* < \kappa\}$

Let $R = \mathcal{G}_{j^t}$ and for $g \in R$ let $i_*(g)$ be the unique $i_* < \kappa$ such that $g \in \mathcal{G}_{j^t,i_*}$ and $\bar{\beta}_g$ the unique $\bar{\beta} \in B_{i_*}$ such that $g \in \mathcal{G}_{j^t,i_*(g),\bar{\beta}}$ and $\bar{\beta} = \langle \beta_i(g) : i < \kappa \rangle$

On R we define a partial order $g_1 \le g_2 \Leftrightarrow g_1 \subseteq g_2 \wedge h_{g_1} \subseteq h_{g_2}$

For $g \in R$ we define $I_g, \bar{c}_g$ as follows

$\circledast$ (a) $I_g = \{u \in I : u \subseteq \mathrm{Dom}(g) \setminus \kappa\}$

(b) $\bar{c}_g = \langle c_{g,s} : s \in I_g \rangle$

(c) $c_{g,s} = x_{t_g(s)}$ where $t_g(s) = (s, j, g \restriction u_{g,s}, h_g \restriction u_{g,s})$ where $u_{g,s} = u \cup \{\mathbf{i}(\alpha) : \alpha \in u$ and $g(\alpha) = \mu^+_{\mathbf{i}(\alpha)}\}$

Let $g_* \in \mathcal{G}_1$ be chosen such that for $i > 0, \beta_i(g_*) = \sup(\{g^t(\alpha) : \alpha \in u^t \cap [\mu_i, \mu_{i+1})\} \cup \{\mu_i\})$ and $\beta_0(g_*) = \cup\{\mathbf{i}(\alpha) + 1 : \alpha \in u^t$ and $g^t(\alpha) = \mu^+_{\mathbf{i}(\alpha)}\} \cup \{1\}$. Let $\bar{c}_* = \bar{c}_{g_*}$ and $f_* = f^{\mathfrak{x}}_{\bar{c}_*}$ is the partial automorphism of M_γ with domain $\cup\{P^{M_{\mathfrak{x}}}_u : u \in I_{g_*}\}$ from Definition 1.7. We prove that the player ISO wins in the game $\eth^*_{\lambda,j}(f_*, M_1, M_1)$, as $f_*(c_1) = c_2 (\in P^{M_{\mathfrak{x}}}_{u^t})$ this is enough. Recall that a play last j moves; now the player ISO commit himself to choose in the $\beta < j$ move on the side a function $g_\beta \in \mathcal{G}_{1+\beta}$, increasing with β, $g_0 = g_*$

and his actual move f_β is $f^{\mathfrak{r}}_{\bar{c}_\beta}$ where $\bar{c}_\beta = \bar{c}_{g_\beta}$. For the β-th move if $\beta = 0$ or β limit let $g_\beta = \cup\{g_\epsilon : \epsilon < \beta\} \cup g_* \in \mathcal{G}_{1+\beta}$. In the $(\beta+1)$-th move let the AIS player choose $\alpha_\beta < \lambda$. Now the player ISO, on the side, first choose $i_\beta < \kappa$ such that $i_*(g_\beta) < i_\beta$, and $\mu_{i_\beta} > \alpha_\beta$, second he chooses $g^+_\beta \in \mathcal{G}_{1+\beta+1,i_\beta}$ satisfying:

⊛ (a) g^+_β extends g_β,
(b) $\mathrm{Dom}(g^+_\beta) \cap \kappa = i_\beta$
(c) $g^+_\beta \restriction (i_\beta \setminus \mathrm{Dom}(g_\beta))$ is constantly $1 + \beta$
(d) if $0 < i \ \in \mathrm{Dom}(g_\beta) \cap \kappa$ then $g^+_\beta \restriction [\mu_i, \mu_{i+1}) = g_\beta \restriction [\mu_i, \mu_{i+1})$
(e) if $i \notin (\mathrm{Dom}(g_\beta) \cap \kappa)$ and $i \in \mathrm{Dom}(g^+_\beta) \cap \kappa$ then $\mathrm{Dom}(g^+_\beta \restriction [\mu_i, \mu_{i+1})) = [\mu_i, \mu_{i+1})$ and $\varepsilon \in [\mu_i, \mu_{i+1}) \setminus \mathrm{Dom}(g_\beta) \Rightarrow g^+_\beta(\varepsilon) = \mu^+_i$
(f) if $i < \kappa, i \notin \mathrm{Dom}(g^+_\beta)$ then $g^+_\beta \restriction [\mu_i, \mu_{i+1}) = g_\beta \restriction [\mu_i, \mu_{i+1})$

Now ISO and AIS has to play the sub-game $\eth^{\alpha_\beta}_1(f_\beta, M_1, M_2)$. The player ISO has to play $f_{\beta,\alpha}$ in the α-th move for $\alpha \le \alpha_\beta$ and on the side he chooses $g_{\beta,\alpha} \in \mathcal{G}_{1+\beta+1}$ with large enough domain and range, to make it a legal move, increasing with α, and $g_{\beta,0} = g^+_\beta$ and $g_{\beta,\alpha} \restriction \mu_{i_\beta} = g^+_\beta \restriction \mu_{i_\beta}$. Now obviously $\{g : g \in \mathcal{G}_{1+\beta+1}, g^+_\beta \subseteq g\}$ is closed under increasing union of length $< \mu_{i_\beta}$, it is enough to show that he can make the $(\alpha+1)$-th move which is trivial so we are done. $\square_{2.5}$

Claim 2.6. *$M_{\mathfrak{r}}$ is P_s-rigid for $s \in I^*$.*

Proof: We imitate the proof of 1.12.

$(*)_0$ $\mathfrak{r}$ is a full $(\lambda, \aleph_1)$-parameter
$(*)_1$ if $u_1 \subseteq u_2 \in I$, we define the function $\pi_{u_1,u_2} : J_{u_2} \to J_{u_1}$ by $F_{u_1,u_2}(t) = (u_1, j^t, g^t \restriction u_1, h^t \restriction u_1)$ for $t \in J_{u_2}$,
$(*)_2$ if $u_1 \subseteq u_2 \subseteq u_3$ are from I then $\pi_{u_1,u_3} = \pi_{u_1,u_2} \circ \pi_{u_2,u_3}$ that is $\pi_{u_1,u_2}(t) = \pi_{u_1,u_2}(\pi_{u_2,u_3}(t))$
$(*)_3$ for $u_1 \subseteq u_2$ we have
(α) $T \cap (J_{u_1} \times J_{u_2}) = \{(\pi_{u_1,u_2}(t_2), t_2) : t_2 \in J_{u_2}\}$
(β) $\mathbb{G}_{u_1,u_2} = \{(\hat{\pi}_{u_1,u_2}(c_2), c_2) : c_2 \in \mathbb{G}_{u_2}\}$ where $\hat{\pi}_{u_1,u_2} \in \mathrm{Hom}(\mathbb{G}^{\mathfrak{r}}_{u_2}, \mathbb{G}^{\mathfrak{r}}_{u_1})$ is the unique homomorphism from $\mathbb{G}^{\mathfrak{r}}_{u_2}$ into $\mathbb{G}^{\mathfrak{r}}_{u_1}$ mapping x_{t_2} to x_{t_1} whenever $\pi_{u_1,u_2}(t_2) = t_1$
[Why? Check.]
$(*)_4$ if $u_1 \cup u_2 \subseteq u_3 \in I, t_3 \in J_{u_3}$ and $t_\ell = \pi_{u_\ell,u_3}(t_3)$ for $\ell = 1, 2$ then, recalling Definition 1.1(1A)(h), g^{t_1}, g^{t_2} are compatible functions as well as h^{t_1}, h^{t_2} and $j^{t_1} = j^{t_2}$ moreover $g^{t_1} \cup g^{t_2}$ is non-decreasing, $h^{t_1} \cup h^{t_2}$ is non-decreasing
[Why? just check]
$(*)_5$ clause $\circledast_1$ of 1.11(1) holds for $I' = I (= I_{\mathfrak{r}})$

Why? Assume $\bar{c} \in C^{\mathfrak{r}}_I$ is such that $c_{s(*)} \neq e_{\mathbb{G}_{s(*)}}$ for some $s(*) \in I$. For each $u \in I, c_u$ is a word in the generators $\{x_t : t \in J_u\}$ of $\mathbb{G}_u$ and let $\mathbf{n}(u)$ be the length of this word and $\mathbf{m}(u)$ the number of generators appearing in it.

Now by clause (β) of $(*)_3$ we have $u_1 \subseteq u_2 \Rightarrow \mathbf{n}(u_1) \le \mathbf{n}(u_2) \wedge \mathbf{m}(u_1) \le \mathbf{m}(u_2)$. As $(I, \subseteq)$ is $\aleph_1$-directed, for some $u_* \in I, n_* < \omega$ and $m_* < \omega$ we have $u_* \subseteq u \in I \Rightarrow \mathbf{n}(u) = n_* \wedge \mathbf{m}(u) = m_*$ and let $c_u = (\dots, x^{k(u,\ell)}_{t(u,\ell)}, \dots)_{\ell < n_*}$ where $k(u,\ell) \in \{1,-1\}$ and $t(u,\ell) \in J^{\mathfrak{x}}_u$ and $t(u,\ell) = t(u,\ell+1) \Rightarrow k(u,\ell) = k(u,\ell+1)$. Clearly $u_* \subseteq u_1 \subseteq u_2 \in I$ & $\ell < n_* \Rightarrow \pi_{u_1,u_2}(t(u_2,\ell)) = t(u_1,\ell) \wedge ?k(u_1,\ell) = k(u_2,\ell) = k(u_*,\ell)$ hence $j^{t(u_2,\ell)} = j^{t(u_*,\ell)} \wedge j^{t(u_2,\ell)} = j^{t(u_*,\ell)}$. By our assumption toward contradiction necessarily $n_* > 0$ and let $k(\ell) = k(u_*,\ell)$.

As $\{u : u_* \subseteq u \in I\}$ is directed, by $(*)_4$ above, for each $\ell < n_*$ any two of the functions $\{g^{t(u,\ell)} : u_* \subseteq u \in I\}$ are compatible so $g_\ell =: \cup\{g^{t(u,\ell)} : u \in I\}$ is a non-decreasing function from $Y_{i_\ell(*)}$ to λ where $Y_{i_\ell(*)} = (\lambda \setminus \kappa) \cup i_\ell(*)$ for some $i_\ell(*) \le \kappa$ and $h_\ell =: \cup\{h^{t(u,\ell)} : u_* \subseteq u \in I\}$ is similarly a non-decreasing function from $Y_{i_\ell(*)}$ to λ. Also g_ℓ maps $[\mu_i, \mu_{i+1})$ into $[\mu_i, \mu_i^+]$ for $i < \kappa$ and maps κ to κ.

Case 1: $i_\ell(*) = \kappa$.

It also follows that for some j^*_ℓ we have $j^*_\ell =: j^{t(u,\ell)}$ whenever $u_* \subseteq u \in I$ in fact $j^*_\ell = j^{t(u_*,\ell)}$ is O.K. and $j^*_\ell < j_* \le \kappa$. For each $i \in \text{Rang}(g_\ell \restriction \kappa)$ choose $\beta_{\ell,i} < \kappa$ such that $g_\ell(\beta_{\ell,i}) = i$ and let $E = \{\delta < \kappa : \delta$ a limit ordinal $> \sup(u_* \cap \kappa)$ such that $i < j^*_\ell$ & $\ell < n_*$ & $i \in \text{Rang}(g_\ell) \Rightarrow \beta_{\ell,i} < \delta$ and $\beta < \delta$ & $\ell < n \Rightarrow h_\ell(\beta) < \delta\}$, it is a club of κ. Choose u such that $u_* \subseteq u$ and $\text{Min}(u \cap \kappa \backslash u_*) = \delta^* \in E$.

Now what can $g^{t(u,\ell)}(\text{ Min } (u \backslash u_*))$ be?

It has to be i for some $i < j^*_\ell < j^*$ hence $i \in \text{Rang}(g_\ell)$ so for some $u_1, u_* \subseteq u_1 \subseteq \delta^*$ and $\beta_{\ell,i} \in u_1$ so $h_\ell(\beta_{\ell,i}) < \delta^*$ hence considering $u \cup u_1$ and recalling clause $(\delta)(iv)$ of (b) from definition 2.3 of $\mathfrak{x}$ we have $h_\ell(\beta_{\ell,i}) < h_\ell(\delta^*)$ hence by (clause $(b)(\alpha)(iii)$) we have $i = g_\ell(\beta_{\ell,i}) < g_\ell(\delta^*)$, contradiction.

Case 2: $i_\ell(*) \neq \kappa$ so $i_\ell(*) < \kappa$.

Clearly if $i \in (i_\ell(*), \kappa)$ and $\alpha \in [\mu_i, \mu_{i+1})$ then $g_\ell(\alpha) \neq \mu_i^+$ (see clause (b)(γ)(iii) of Definition 2.3) hence $g_\ell \restriction [\mu_i, \mu_{i+1})$ is a non-decreasing function from $[\mu_i, \mu_{i+1})$ to μ_i^+, but μ_{i+1} is regular $> \mu_i^+$ (see Hypothesis 2.2) hence $g_\ell \restriction [\mu_i, \mu_{i+1})$ is eventually constant say $\gamma_i \in [\mu_i, \mu_{i+1})$ and $g_\ell \restriction [\gamma_i, \mu_{i+1})$ is constantly $\epsilon_i \in [\mu_i, \mu_i^+)$. So also $h_\ell \restriction [\gamma_i, \mu_{i+1}^+)$ is constant and its value is $< \mu_{i+1}$, and we get contradiction as in case 1.

$\square_{2.6}$

Conclusion 2.7. If $\lambda = \lambda^{\aleph_0} > \text{cf}(\lambda) > \aleph_0$ <u>then</u> for every $\alpha < \text{cf}(\lambda)$ there are non-isomorphic models M_1, M_2 of cardinality λ which are $EF^*_{\alpha,\lambda}$-equivalent.

Proof: By 2.5+2.6 as the cardinality of $M_{\mathfrak{x}}$ is λ. $\square_{2.7}$

Remark 2.8. By minor changes, for some $t \in P^M_u, u = \emptyset$ letting $c_1 = e_{\mathbb{G}_u}, c_2 = x_t$ we have: $(M_{\mathfrak{x}}, c_1), (M_{\mathfrak{x}}, c_2)$ are non-isomorphism but $EF^*_{\lambda,j}$-equivalent for every $j < \kappa = \mathrm{cf}(\lambda)$. This is similar to the parallel remark in the end of §1.

Private Appendix

3. For every λ large enough

Naturally we would like to prove this for all are at least in some sense for most λ. Naturally, for me at least we do it by using the RGCH (the revised G.C.H., see [She00] or [She06, §1]). Specifically, this holds for every $\lambda \geq \beth_\omega$, moreover we phrase a weaker condition which conceivably?? is provable in every $\lambda \geq 2^{\aleph_0}$. So instead "every countable u and function g from $u \ldots$" we shall try to use "for density means?? So this leads to the following.

Conclusion 3.1. Like 1.12 (hence also 1.13) assuming just $\lambda = \mathrm{cf}(\lambda) > \beth_\omega$ or at least

- $\circledast_\lambda$ there is $\mathcal{P} \subseteq [\lambda]^{\aleph_0}$ of cardinality λ such that $(\forall A \in [\lambda]^\lambda)(\exists u \in \mathcal{P})(u \subseteq A)$.

Proof: We define $\mathfrak{y} = \mathfrak{y}_\lambda$ as in the proof of 1.12 see $\boxtimes$ there except that $[\lambda]^{<\aleph_0} \subseteq I \subseteq [\lambda]^{\leq\aleph_0}, |I| = \lambda, J \subseteq \{(u,\alpha,g,h) : u \in I, (u,\alpha,g,h)$ as in clause (b)(α) of $\boxtimes\}, |J| = \lambda$ and the pair (I,J) is quite large E.g. let $\mathfrak{B}$ be an elementary submodel of $(\mathcal{H}(\chi) \in), \lambda = \beth_2(\lambda)^+, \lambda + 1 \subseteq \mathfrak{B}, ||\mathfrak{B}|| \mathfrak{x}_\lambda \in \mathfrak{B}$ and $\mathfrak{x} = \mathfrak{x}_\lambda \restriction \mathfrak{B}$. We first have to note that the proof of "ISO wins $\Game^\alpha_\lambda((M_\mathfrak{y}, b), (M_\mathfrak{y}, c))$ for appropriate $u \in I, b \neq c \in P_u^{M_\mathfrak{y}}$" is not changed (in fact the results follows as $M_{\mathfrak{y}'_\lambda} \subseteq M_{\mathfrak{x}_\lambda}$, and moreover

$$M_{\mathfrak{y}'_\lambda} = M_{\mathfrak{x}_\lambda} \restriction (\cup\{P_u^{M_{\mathfrak{x}_\lambda}} : u \in I\}).$$

Also for simplicity we use the abelian group satisfying $x + x = 0$ version. Second, as for "$M_\mathfrak{y}$ is P_u-rigid for $u \in I_\mathfrak{y}$" again if this fail for $u \in I_\mathfrak{y}$ then we can find $\alpha < \alpha^*$ and $\bar{z}$ such that

$(*)_0$
- (a) $\bar{z} = \langle z_v : v \in I\rangle$
- (b) z_v a finite subset of $J_v^\mathfrak{y}$ such that $t \in z^v \Rightarrow \alpha^t = \alpha$
- (c) if $v \subseteq w \in I$ then $\pi^\mathfrak{y}_{v,w}$ maps z_w onto a subset of $J_v^\mathfrak{y}$ which includes z_v where $\pi^\mathfrak{y}_{v,w}$ is as in $(*)_2$ of the proof of 1.12
- (d) $z_{u_*} \neq \emptyset$
- (e) $f \in \mathrm{Aut}(M), f = f_{\bar{c}}, \bar{c} = \langle c_v : v \in I\rangle = \mathbf{C}^\mathfrak{y}_{I_\mathfrak{y}}, c_u \neq e_{\mathbb{G}_u}$, see Definition 1.7.

$(*)_1$ for each $v \in I$ we let $z_v^+ = \cup\{\mathrm{Rang}(\pi_{v,w}) : v \subseteq w \in I\}$

$(*)_2$ if $\circledast_\lambda$ from the conclusion holds then $|z_v^+| < \lambda$ for $v \in I_\mathfrak{y}$.

[Why? as in the proof of 1.11]

Now for every $\beta_1 < \beta_2 < \alpha$ let

$$B_{\beta_1,\beta_2} =: \{\gamma : \begin{array}{l} \text{for some } v \in I \text{ and } t \in z_v^+ \text{ and} \\ \gamma_1 < \gamma_2 \text{ from } u^t \text{ we have } \gamma_1 < \gamma = h^t(\beta_1) < \gamma_2 \\ \text{and } g^t(\gamma_1) = \beta_1, g^t(\gamma_2) = \beta_2\} \end{array}$$

$$B_* = \cup\{B_{\beta_1,\beta_2} : \beta_1 < \beta_2 < \alpha\}$$

$\boxtimes$ $|B_*| < \lambda$

[why? otherwise we can find $\gamma_\varepsilon \in B_*$ for $\varepsilon < \lambda$, pairwise distinct. So for $\varepsilon < \lambda$ there are $v_\varepsilon \in I, t_\varepsilon \in z^+_{v_\varepsilon}$ and be $\gamma_{1,\varepsilon}, \gamma_{2,\varepsilon} \in v_\varepsilon$ such that $h^{t_\varepsilon}(\gamma_{1,\varepsilon}) = \varepsilon$ and $\gamma_{1,\varepsilon} < \gamma_\varepsilon < \gamma_{2,\varepsilon}$. As λ is regular without loss of generality $(h^{t_\varepsilon}(\gamma_{1,\varepsilon}), h^{t_\varepsilon}(\gamma_{2,\varepsilon})) = (\beta^*_1, \beta^*_2)$ and $h^{t_\varepsilon}(\gamma_{1,\varepsilon}) = \gamma_\varepsilon$.
Let $(w_\varepsilon, t'_\varepsilon)$ be such that $v_\varepsilon \subseteq w_\varepsilon \in I, t'_\varepsilon \in z_{w_\varepsilon}$ and $\pi_{v_\varepsilon,w_\varepsilon}(t'_\varepsilon) = t_\varepsilon$. By the assumption $\circledast_\lambda$ we know that for some $\Lambda \subseteq \lambda, |\Lambda| = \aleph_0$ and $w = \cup\{w_\varepsilon : \varepsilon \in \Lambda\} \in I$. Now for each $\varepsilon \in \Lambda$ there is $s_\varepsilon \in z^+_v$ such that $\pi_{w_\varepsilon,w}(s_\varepsilon) = t'_\varepsilon$. But $\varepsilon \neq \zeta \in \Lambda \in s_\varepsilon \neq s_\zeta$, so we get a contradiction.]

So we can find $\gamma_* < \lambda$ such that

$\boxtimes_2$ if $\gamma_1 \in [\gamma_*, \lambda)$ then for no γ, γ_2 and $u \in I, t \in z^+_u$ do we have $\gamma_1, \gamma_2 \in u, \gamma_1 \leq h^t(\gamma_1) < \gamma_2$

We can find $u_1 \in I$ such that $\gamma_* \in u_1 \wedge u_* \subseteq u_1$ hence $z_{u_1} \neq \emptyset$ and let $s \in z_{u_1}, \gamma = h^t(\gamma_*)$ and let $u_2 \in I$ be such that $u_1 \cup \{\gamma + 1\} \subseteq u_2 \in I$, so there is $t \in Z_{u_2}$ such that $\pi_{u_1,u_2}(t) = s$ hence $h^t(\gamma_*) = h^s(\gamma_*) = \gamma < \gamma + 1 \in u_2$ so $(u_2, \gamma_*, \gamma + 1)$ witness then $\gamma \in B_{h^t(\gamma_*),h^t(\gamma+1)} \subseteq B_*$, contradiction. $\square_{3.1}$

Conclusion 3.2. Like 2.7 assuming only $\mathrm{cf}(\lambda) > \aleph_0$ and $\lambda > \beth_\omega \wedge \mathrm{cf}(\lambda) > \aleph_0$ or just

$\circledast'_\lambda$: there is $\mathcal{P} \subseteq [\lambda]^{\aleph_0}$ of cardinality λ such that

(a) if for every $A \subseteq \lambda$ of cardinality λ there is $u \subseteq A, u \in \mathcal{P}$
(b) for every $A \subseteq$ cf (λ) of cardinality λ there is $u \subseteq A, u \in \mathcal{P}$

TO BE FILLED :λ singular.

4. Havning trees instead "$\alpha < \lambda$"

When $\lambda < \lambda^{<\lambda}$, it is not so clear what does it mean "using EF games with trees with λ nodes, λ levels no λ-branch". We suggest here a replacement and generalize §1.

Definition 4.1. Assume that M_1, M_2 are τ-models, f a partial isomorphism from M_1 to M_2, N is a τ-model, g a partial unary function from N to N, $\tau^+ = \tau_N \cup \{F\}, F$ a unary function symbol ($\notin \tau$) and λ, μ are cardinals α an ordinal and T is a universal theory in $\mathbb{L}(\tau^+)$. We define a game $\Game^\alpha_{\lambda,\mu,\alpha}(M_1, M_2, N, T, f, g)$.
A play last up to λ moves in the α-th move a pair (f_α, g_α) is chosen such that

$\circledast$ (a) f_α is a partial isomorphism from M_1 onto M_2
(b) f_α is increasing continuous with α
(c) $f_0 = f$ and $|\text{Dom }(f_{\alpha_{\beta+1}})\backslash\text{Dom }(f_\beta)| < 1 + \mu$
(d) g_α is a partial function from N to N_1 increasing continuous with α
(e) $g_0 = g, |\text{Dom }(g_{\beta+1})\backslash\text{Dom }(g_\beta)| < 1 + \mu$
(f) (N, g_α) satisfies T as far as it is meaningful

$\circledast_2$ in the α-th move (every player can make choices only compatible with $\circledast_1$)
(a) first ISO chooses $u_\alpha \subseteq N$ of cardinality $< 1 + \mu$
(b) second AIS chooses $g_{\alpha+1}$ with $\text{Dom}(g_{\alpha+1}) = \text{Dom }(g_\alpha) \cup u_\alpha$
(c) third AIS chooses $A^1_\alpha \subseteq M_1, A^2_\alpha \subseteq M_\alpha$ such that $|A^1_\alpha| + |A^2_\alpha| < 1 + \mu$
(d) fourth ISO chooses $f_{\alpha+1}$ such that $A^1_\alpha \subseteq \text{Dom }(f_{\alpha+1}), A^2_\alpha \subseteq \text{Dom }(f_{\alpha=1})$.

A player loses the play when he has no legal move.

Definition 4.2. (1) In 4.1 if $g = \emptyset$ we may omit it, if $f = \emptyset = g$ we may omit then.
(2) We say that M_1, M_2 are $\text{EF}_{\lambda,\mu,\alpha,N,T}$-equivalent if the player ISO wins the game $\Game_{\lambda,\mu}(M_1, M_2; N, T)$.

Claim 4.3. *There are non-isomorphic models M_1, M_2 of cardinal λ which are $\text{EF}_{\lambda,\mu,N,T}$-equivalent when*

$\boxtimes$ (a) $\lambda = \lambda^{\aleph_0}$
(b) *N is a model of cardinality λ*
(c) *T is a universal first order theory in the vocabulary $\tau^T = \tau_N$ such that N has no expansion to a model of T.*

Proof: As in §1. Saharon fill.

5. On $\aleph_0$-independent theories

Our aim is to prove

⊠ if $T \subseteq T_1$ are complete first order theorem T with the $\aleph_0$-independence property, $\lambda = \mathrm{cf}(\lambda) > |T|$ then
(a) there are $M_1, M_1 \in PC(T_1, T)$ of cardinality λ which are $EF_{\alpha,\lambda}$-equivalent for every $\alpha < \lambda$ but not isomorphism.
(b) the singular.
(c) Karp complexity.

Program:

We use $EM(I, \Phi), I \in K_\lambda^{\rm orgr}$ = class of ordered graphs of cardinality λ. From a nice λ-parameter $\mathbf{p}$, we drive a model $N \in K_\lambda^{\rm orgr}$ as follows: for each $G_s^{\mathbf{p}}$ we attached $N_s^{\mathbf{p}}$ and the action of $x \in \mathcal{G}_s^{\mathbf{p}}$ and define the graph of $N^{\mathbf{p}} \cup \{N_s^p : s \in S\}$ such that the partial automorphism of $M^{\mathbf{p}}$ i.e. $\bar{e} = \langle c_s : s \in \text{set}\rangle$ induce a partial automorphism of the ordered graph. So the problem will be to make $M_1 \ncong M_2$. Better: from one λ-parameter $\mathbf{p}$ we define two ordered graphs $N_{s,1}^{\mathbf{p}} N_{s,2}^{\mathbf{p}}$ and partial automorphism of each+ partial isomorphism from one to the other- those are the really interesting objects.

Remark: Note that $\mathbf{J} \in K^{oi}$ we can use $P^{\mathbf{J}}$ only in particular defining $EM(\mathbf{J}, \Phi)$

Definition 5.1. 1)K^{oi}_λ is the class of structures $\mathbf{J}$ of the form $(A, Q, P <, F_n)_{n<\omega} = (|\mathbf{J}|, P^{\mathbf{J}}, Q^{\mathbf{J}}, <^{\mathbf{J}}, F^{\mathbf{J}}_n)$, where $\mathbf{J}$ has cardinality λ, $<^{\mathbf{J}}$ a linear order on $Q^{\mathbf{J}}, P^{\mathbf{J}} = |\mathbf{J}| \setminus Q^{\mathbf{J}}, F^{\mathbf{J}} \restriction Q^{\mathbf{J}} =$ the identity and $a \in A \setminus Q^{\mathbf{I}} \Rightarrow F_n(a) \in Q^{\mathbf{J}}$ and $a \neq b \in P^M \Rightarrow \bigvee_{n<\omega} F_n(a) \neq F_n(b)$. Let $F^{\mathbf{J}}_\omega =$ be the identity on $|\mathbf{J}|$. where (from [She09], where T being $\aleph_0$- independent follows from T having the independence property and implies T is not superstable or just not strongly dependent, see below)
2) For a linear order I and $\mathfrak{S} \subseteq {}^\omega I$, we let $\mathbf{J} = \mathbf{J}_{I,\mathfrak{S}}$ be the derived member of K^{oi} that is $|\mathbf{J}| = I \cup \mathfrak{S}, (Q^{|\mathbf{J}|}, <^{\mathbf{J}}) = I, F^{\mathbf{J}}_n(\eta) = \eta(n)$ for $n < \omega, F^{\mathbf{J}}_n(t) = t$ for $t \in I_i$; note that every $\mathbf{J} \in K^{oi} = \cup\{K^{oi}_\lambda : \lambda$ a cardinal$\}$ is isomorphic to some $\mathbf{J}_{I,\mathfrak{S}}$

Definition 5.2. (1) A (complete f.o.) T is $\aleph_0$-independent ($\equiv$ not strongly dependent) if there is a sequence $\bar\varphi = \langle \varphi_n(x, \bar y_s) : n < \omega\rangle$ (or finite $\bar x$, as usual) of (f.o.) formulas such that T is consist with Γ_λ for some ($\equiv$ every $\lambda \geq \aleph_0$)

$$\Gamma_\lambda = \{\varphi_n(x_\eta, \bar y^n_\alpha)^{\text{if } (\alpha=\eta(n))} : \eta \in {}^\omega\lambda, \alpha < \lambda, n < \omega\}$$

(2) T is strongly stable if it is stable and strongly dependent.

Claim 5.3. *If T is f.o. complete $T_1 \supseteq T$ is complete, w.l.o.g. with Skolem function and T is not strongly dependent (from* [She09]*) then we can find Φ, $\bar\varphi = \langle \varphi_n(x, \bar y_n) : n < \omega\rangle, \bar y_n \trianglelefteq \bar y_{n+1}$*

(a) *Φ is proper for K^{oi} and $\tau(T_1) \subseteq \tau(\Phi)$ and $|\tau(\Phi)| = |T_1|$*
(b) *In $M_1 = EM(\mathbf{J}, \Phi), \mathbf{J} = \mathbf{J}_{I,\mathfrak{S}}$ we have $\langle \bar a_t : t \in I\rangle$ and $\langle a_\eta : \eta \in \mathfrak{S}\rangle$ such that*
 (α) *M_1 is the Skolem full of $\{\bar a_t : t \in I, n < n\} \cup \{a_\eta : \eta \in \mathfrak{S}\}$*
 (β) *$\bar a_t \in {}^\omega M_1$*
 (γ) *$M_1 \models \varphi_n[a_\eta, \bar a_{n,t}]$ iff $\eta(n) = t$ (pedantically we should write $\varphi_n(a_\eta, \bar a_t \restriction \lg(\bar y_n))$]*
(c) *M_1 is a model of T_1*

Proof: Let I be an infinite linear order. We can find $M_1 \models T_1$ and sequence $\langle \bar a_q : q \in I\rangle, \bar a_\alpha \in {}^\omega(M_1)$ such that for every $\eta \in {}^\omega I, \{\varphi_n(x, \bar a_q)^{\text{if}(\eta(n)=q)} : q \in I, n < \omega\}$.
Now w.l.o.g. $\langle \bar a_q : q \in I\rangle$ is an indiscernible sequence in M_1. W.l.o.g. M_1 is λ^+-saturated, we then expand M_1 to M^+_1 by function $F^{M^+_1}_n$ $(n < \omega)$, (of finite arity) such that $F_n(\bar a_{q_0}, \bar a_{q_1}, \dots \bar a_{q_{n-1}})$ or more exactly $F_n(\bar a_{q_0} \restriction \lg \bar y_0, \bar a_{q_1} \restriction \lg(\bar y_1), \dots, \bar a_{q_{n-1}} \restriction \mathbf{lg}(\bar y_{n-1}))$ realizes in M_1 the type $\{\varphi_\ell(x, \bar a_q)^{\text{if}(\eta(\ell)=q)} : q \in I, \ell < n\}$. W.l.o.g. $\langle \bar a_q : q \in I\rangle$ is an indexed sequence in M_1. Let D be a non-principal ultrafilter on ω and in $M^+_2 = (M^+_1)^\omega/D$, we let $\bar a_q = \langle \bar a_q : n < \omega\rangle/D$, and

$\bar a_\eta = \langle F_n(\bar a_{\eta(0)}, \bar a_{\eta(1)}, \dots, \bar a_{\eta(n-1)}) : n < \omega\rangle / D$ for $\eta \in {}^\omega I$. Now has the right vocabulary and from the quantifier free types realized by $\langle \bar a_q : q \in I\rangle^\frown \langle \bar a_\eta : \eta \in {}^\omega I\rangle$ in M_2^+ we can read Φ. $\square_{6.3}$

As in [Shear, III].

Claim 5.4. *Assume $\mathbf{J}_1, \mathbf{J}_2 \in K^{oi}$, and $\Phi, \bar\varphi, T_1, T$ as in 6.3. A sufficient condition for $EM_{\tau(T)}(\mathbf{J}_1, \Phi) \ncong EM_{\tau(T)}(\mathbf{J}_2, \Phi)$ is*

- (*) *if f is a function from $\mathbf{J}_1$ (i.e. its universe) into $\mathcal{M}_{|T_1|,\aleph_0}(\mathbf{J}_2)$ (i.e. the free algebra generated by $\{x_t : t \in \mathbf{J}_1\}$ the vocabulary $\tau_{|T_1|,\aleph_0} = \{F^n_\alpha : n < \omega$ and $\alpha < |T_1|\}, F^n_\alpha$ has arity n, see* [Shear, III 1]*) we can find $t \in P^{\mathbf{J}_1}, n < \omega$, and $s_1, s_2 \in Q^{\mathbf{J}_1}$ such that:*
  - (α) $F_n^{\mathbf{J}_1}(t) = s_1 \neq s_2$
  - (β) *$f(s_\ell) = \sigma(r^\ell_0, \dots, r^\ell_{k-1})$ so $k < \omega, r^\ell_t \in \mathbf{J}_2$ for $i < k$ so σ is a $\tau_{|T_1|,\aleph_0}$-term not dependent on ℓ*
  - (γ) *$f(t) = \sigma^*(r_0, \dots, r_{m-1}), \sigma^*$ is a $\tau_{|T_1|,\aleph_0}$-term and $r_0, \dots, r_{m-1} \in \mathbf{J}_2$*
  - (δ) *the sequences*

$$\langle r^1_i : i < k\rangle^\frown \langle r_i : i < m\rangle$$

$$\langle r^2_i : i < k\rangle^\frown \langle r_i : i < m\rangle$$

realize the same quantifier free type in $\mathbf{J}_2$ (note: we should close by the $F_n^{\mathbf{J}_2}$, so type mean the truth value of the inequalities $F_{n_1}(r') \neq F_{n_2}(r')$ (including F_ω) and the order between those terms)

Proof: As in [Shear, III].

Remark: We could have replaced Q by the disjoint union of $\langle Q^{\mathbf{J}}_n : n < \omega\rangle, <^{\mathbf{J}}$ linearly order each $Q^{\mathbf{J}}_n$ (and $<^{\mathbf{J}} = \cup\{< \restriction Q^{\mathbf{J}_1}_n : n < \omega\}$ and use Q_n to index parameters for $\varphi_n(x, \bar y_n)$. Does not matter. If you like just to get the main point for [S$^+$], i.e. to show that $\aleph_0$-independent is a relevant dividing line note the following claim.

Claim 5.5. *Assume $(\Phi, \bar\varphi, T, T_1)$ is an in 6.3 and $\lambda = \lambda^{<\lambda}$. Then for some λ-complete λ^+. c.c. forcing notion $\mathbb{Q}$ we have: $\Vdash_{\mathbb{Q}}$ "there are $\mathbf{J}_1, \mathbf{J}_2 \in K^{oi}$ of cardinality λ such that $EM_{\tau(T)}(\mathbf{J}_1, \Phi), EM_{\tau(T)}(\mathbf{J}_2, \Phi)$ are $EF_{\alpha,\lambda}$ equivalent for every $\alpha < \lambda$ but are not isomorphic".*

Remark 5.6. It should be clear that we can improve it allowing $\alpha < \lambda^+$ and replacing forcing and e.g. $2^\lambda = \lambda^+ + \lambda = \lambda^{<\lambda}$, but anyhow we shall get better result

Proof: We define $\mathbb{Q}$ as follows

- $\circledast_1$ $p \in \mathbb{Q}$ iff p consist of the following objects satisfying the following conditions
  - (a) $u = u^p \in [\lambda^+]^{<\lambda}$ such that $\alpha + i \in u \wedge i < \lambda \Rightarrow \alpha \in u$

(b) $<^p$ a linear order of u such that

$$\alpha, \beta \in u \wedge \alpha + \lambda \leq \beta \Rightarrow \alpha <^p \beta$$

$$\alpha < \beta \in u \wedge \alpha \in u \wedge \lambda | \alpha \Rightarrow \alpha <^p \beta$$

(c) for $\ell = 1, 2$ $\quad \mathfrak{S}^p_\ell$ is a subset of $\{\eta \in {}^\omega u : \eta(n) + \lambda \leq \eta(n+1)$ for $n < \omega\}$ such that $\eta \neq \nu \in \mathfrak{S}^p_\ell \Rightarrow \mathrm{Rang}(\eta) \cap \mathrm{Rang}(\nu)$ is finite; note that in particular $\eta \in \mathfrak{S}^p_\ell$ is without repetitions

(d) Λ^p a set of $< \lambda$ increasing sequence of ordinals from $\{\alpha \in u^p : \lambda | \alpha\}$ hence of length $< \lambda$

(e) $\bar{f}^p = \langle f^p_\rho : \rho \in \Lambda^p \rangle$
such that

(f) f^p_ρ is a partial automorphism of the linear order $(u^p, <^p)$ and we let $f^{1,p}_\rho = f^p_\rho, f^{2,p}_\rho = (f^p_\rho)^{-1}$

(g) if $\eta \in \mathfrak{S}^p_\ell, \rho \in \Lambda^p, \ell \in \{1, 2\}$ then $\mathrm{Rang}(\eta)$ is included in $\mathrm{Dom}(f^{\ell,p}_\rho)$ or is almost disjoint to it (i.e. except finitely many "errors").

(h) if $\rho \triangleleft \varrho \in \Lambda^p$ then $\rho \in \Lambda^p$ and $f^p_\rho \subseteq f^p_\varrho$

(i) if $\rho \in \Lambda^p$ has limit length then

$$f^p_\rho = \cup\{f^p_{\rho \restriction i} : i < \lg(\rho)\}$$

(j) if $\rho \in \Lambda^p$ has length $i + 1$ then $\mathrm{Dom}(f^{\ell,p}_\rho) \subseteq \rho(i)$ for $\ell = 1, 2$

(k) if $\rho \in \Lambda$ and $\eta \in {}^\omega(\mathrm{Dom}(f^p_\rho))$ then $\eta \in \mathfrak{S}^p_1 \Leftrightarrow \langle f^p_\rho(\eta(n)) : n < \omega\rangle \in \mathfrak{S}^p_2$

(ℓ) if $\rho_n \in \Lambda^p$ for $n < \omega$ and $\rho_n \triangleleft \rho_{n+1}$ and $\lambda > \aleph_0$ then $\cup\{\rho_n : n < \omega\} \in \Lambda$

$\circledast_2$ We define the order on $\mathbb{Q}$ as follows: $p \leq q$ iff ($p, q \in \mathbb{Q}$ and)

(a) $u^p \subseteq u^\varphi$

(b) $\leq^p = \leq^q \restriction u^p$

(c) $\mathfrak{S}^p_\ell \subseteq \mathfrak{S}^q_\ell$ for $\ell = 1, 2$

(d) $\Lambda^p \subseteq \Lambda^q$

(e) if $\rho \in \Lambda^p$ then $f^p_\rho \subseteq f^q_\rho$

(f) if $\eta \in \mathfrak{S}^q_\ell \setminus \mathfrak{S}^p_\ell$ then $\mathrm{Rang}(\eta) \cap u^p$ is finite

(g) if $\rho \in \Lambda^p$ and $f^p_\rho \neq f^q_\rho$ then $u^p \subseteq \mathrm{Dom}(f^{\ell,q}_\rho)$ for $\ell = 1, 2$

(h) if $\rho \in \Lambda^p$ and $\ell \in \{1, 2\}, \alpha \in u^p \setminus \mathrm{Dom}(f^{\ell,p}_\rho)$ and $\alpha \in \mathrm{Dom}(f^{\ell,q}_\rho)$ then $f^{\ell,p}_\rho(\alpha) \notin u^p$

(i) if $n < \omega$ and $\rho_k \in \Lambda^p, \ell_k \in \{1, 2\}$ for $k < n$ and $\alpha_k \in u^q$ for $k \leq \gamma, f^{\ell_k,q}_\rho(\alpha_k) = \alpha_{k+1}$ for $k < n$, and for no $k, \ell_k \neq \ell_{k+1} \wedge (\exists \rho)[\rho \trianglelefteq \rho_k \wedge \rho \trianglelefteq \rho_{k+1} \wedge \alpha_k \in \mathrm{Dom}(f^{\ell_k,p}_\rho))]$ and $\alpha_0 = \alpha_n$ then $\alpha_0 \in \mathrm{Dom}(f^{\ell_0,p}_{\rho_0})$.

Having defined the forcing notion $\mathbb{Q}$ we start to investigate it.

$\circledast_3$ $\mathbb{Q}$ is a partial order of cardinality λ^+

$\circledast_4$ (i) if $\bar{p} = \langle p_i : i < \delta\rangle$ is $\leq^{\mathbb{Q}}$-increasing , δ a limit ordinal $< \lambda$ of uncountable cofinality then $p_\delta := \cup\{p_i : i < \delta\}$ defined naturally is an upper bound of $\bar{p}$

[Why? think]

(ii) if $\delta < \lambda^+$ is a limit ordinal of cofinality $\aleph_0$ and the sequence $\bar p = \langle p_i : i < \delta\rangle$ is increasing (in $\mathbb{Q}$), then it has an upper bound. [We define $q \in \mathbb{Q}$ as follows: $u^q = \cup\{u^{p_i} : i < \delta\}, <^q = \cup\{<^{p_i} : i < \delta\}, \Lambda^q = \cup\{\Lambda^{p_i} : i < \delta\} \cup \{\rho : \rho$ is an increasing sequence of ordinals from u^q of length a limit ordinal of cofinality $\aleph_0$ such that $\varepsilon < \lg(\rho) \Rightarrow \rho\restriction\varepsilon \in \cup\{\Lambda^{p_i} : i < \delta\}\}$. Lastly $\mathfrak{S}^q_\ell$ is the closure of $\cup\mathfrak{S}^{p_i}_\ell : i < \delta\}$ under clause (g) of $\circledast_1$, where by clauses (f)-(i) of $\circledast_2$ this works MORE DETAILS.]

$\circledast_5$ $\mathbb{Q}$ satisfies the λ^+-c.c.

[Why? use $\triangle$-system lemma and check]

$\circledast_6$ if $\alpha < \lambda^+$ then $\mathcal{I}^1_\alpha := \{p \in \mathbb{Q} : \alpha \in u^p\}$ is dense and open

[Why? Easy]

$\circledast_7$ if $\varrho \in \Lambda^* := \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by λ of length $< \lambda\}$ then $\mathcal{I}^2_\varrho = \{p \in \mathbb{Q} : \varrho \in \Lambda^p\}$ is dense open

[Why? let $p \in \mathbb{Q}$ by $\circledast_6$+$\circledast_4$ there is $q \geq p$ such that $\mathrm{Rang}(\varrho) \subseteq u^q_1$. If $\varrho \in \Lambda^q$ we are done otherwise define q' as follows: $u^{q'} = u^q, <^{q'} = <^q$ $, \mathfrak{S}^{q'}_\ell = \mathfrak{S}^q_\ell, \Lambda^{q'} = \Lambda^q \cup \{\varrho\restriction\varepsilon : \varepsilon \leq \lg(\varrho)\}$ and if $i \leq \lg(\varrho)$, $\varrho\restriction i \notin \Lambda^q$ then we let $f^{q'}_{\varrho\restriction i} = \cup\{f^q_\rho : \rho \in \Lambda^q$ and $\rho \triangleleft \varrho\restriction i\}]$

$\circledast_8$ For ϱ as in $\circledast_7$ and $\alpha < \lambda^+$ and $\ell \in \{1,2\}$

$$\mathcal{I}^3_{\varrho,\alpha,\ell} = \Big\{p \in \mathbb{Q} : \alpha \in \mathrm{Dom}(f^{\ell,p}_\varrho) \text{ so } \varrho \in \Lambda^p, \alpha \in u^p\Big\} \text{ is dense open}$$

[Why? for any $p \in \mathbb{Q}$ there is $p^1 \geq p$ such that $\varrho \in \Lambda^{p_1}, \alpha \in u^{p_1}$, now use disjoint amalgamation]

$\circledast_9$ define $\underset{\sim}{\mathbf{J}}_\ell \in K^{oi}_\lambda$ a $\mathbb{Q}$-name as follows:

$$Q^{\underset{\sim}{\mathbf{J}}_\ell} = \lambda^+$$

$$\mathfrak{S}^{\underset{\sim}{\mathbf{J}}_\ell} = \cup\{\mathfrak{S}^p_\ell : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$$

$$<^{\underset{\sim}{\mathbf{J}}_\ell} = \cup\{<^p : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$$

$F^{\underset{\sim}{\mathbf{J}}_\ell}_{\underset{\sim}{n}}$ is a unary function, the identity on λ^+ and

$$\eta \in \mathfrak{S}^{\underset{\sim}{\mathbf{J}}_\ell} \Rightarrow F^{\underset{\sim}{\mathbf{J}}_n}_{\underset{\sim}{n}}(\eta) = \eta(n)$$

$\circledast_{10}$ $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{J}}_\ell \in K^{oi}_{\lambda^+}$ for $\ell = 1, 2$

[Why? think]

$\circledast_{11}$ $\Vdash_{\mathbb{Q}}$" $EM_{\tau(T)}(\underset{\sim}{\mathbf{J}}_1, \Phi), EM_{\tau(T)}(\underset{\sim}{\mathbf{J}}_2, \Phi)$ are EF_{λ,λ^+}-equivalent (i.e. games of length $< \lambda$, and the player INC chooses sets of cardinality $< \lambda^+$).

[Why? recall $\Lambda^* = \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by λ of length $< \lambda\}$ (is the same in $\mathbf{V}$ and $\mathbf{V}^{\mathbb{Q}}$). For $\rho \in \Lambda^*$ let $\underset{\sim}{f}_\rho = \cup\{f^p_\rho : \rho \in \underset{\sim}{G}, \rho \in \Lambda^p\}$. Easily $\Vdash_{\mathbb{Q}}$" $\underset{\sim}{f}_\rho$ an isomorphism from $\underset{\sim}{\mathbf{J}}_1\restriction\sup\mathrm{Rang}(\rho)$ onto $\underset{\sim}{\mathbf{J}}_2\restriction\sup\mathrm{Rang}(\rho)$ where for any $\delta < \lambda^+$ (divisible by λ),

$\mathbf{J}_\ell\restriction\delta = ((\delta \cup (P^{\mathbf{J}_\ell} \cap {}^\omega\delta), Q^M \cap \delta, P^M\restriction\delta, F^{\mathbf{J}_\ell}_n\restriction(\delta \cup (P^{\mathbf{J}_\ell} \cap {}^\omega\delta))).$

Also $\rho \triangleleft \varrho \Rightarrow \Vdash_{\mathbb{Q}} \underset{\sim}{f}_\rho \subseteq \underset{\sim}{f}_\varrho$. So $\langle f_\rho : \rho \in \Lambda^* \rangle$ exemplify the equivalence]

Remark: Note that $\lambda | \delta \wedge \delta < \lambda^+ \wedge \delta \in \mathrm{Dom}(f_\rho) \Rightarrow \{f_\rho(\alpha) : \alpha < \delta\} = \delta$

So to finish we need just $\circledast_{13}$ but first

$\circledast_{12}$ for $p \in \mathbb{Q}$ let $\mathbf{J}^p_\ell \in K^{oi}$ has universe $u^p \cup \mathfrak{S}^p_\ell, <^{\mathbf{J}_\ell} = <^p, Q^{\mathbf{J}^p_\ell} = u^p, F^{\mathbf{J}^p_\ell}_n(\eta) = \eta(n)$. We do not distinguish

$\circledast_{13}$ $\Vdash_{\mathbb{Q}}$ "$M_1 = EM_{\tau(T)}(\underset{\sim}{\mathbf{J}}_1, \Phi), \underset{\sim}{M}_2 = EM_{\tau(T)}(\underset{\sim}{\mathbf{J}}_2, \Phi)$ are not isomorphic"

Why? let $\underset{\sim}{M}^+_\ell = EM(\underset{\sim}{\mathbf{J}}_1, \Phi)$, and assume toward contradiction that $p \in \mathbb{Q}$, and $p \Vdash_{\mathbb{Q}}$ "g is an isomorphism from M_1 onto M_2". For each $\delta \in S^{\lambda^+}_\lambda := \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ we can find $p_\delta \in \mathbb{Q}$ and g_δ such that:

$\boxdot$ (a) $p \leq p_\delta, \delta \in u^{p_\delta}$
(b) $p_\delta \Vdash$" $\underset{\sim}{g}_\delta$ is $\underset{\sim}{g} \restriction EM(\mathbf{J}^{p_\delta}, \Phi)$"
(c) g_δ is an isomorphism from $EM_{\tau(T)}(\mathbf{J}^p_1, \Phi)$ onto $EM_{\tau(T)}(\mathbf{J}^p_2, \Phi)$.

We can find stationary $S \subseteq S^{\lambda^+}_\lambda$ and p^* such that

$\boxdot_2$ (a) $p_\delta \restriction \delta$, naturally defined is p^* for $\delta \in S$.
(b) for $\delta_1, \delta_2 \in S$, $u^{p_{\delta_1}}, u^{p_{\delta_2}}$ has the same order type and the order preserving mapping π_{δ_1,δ_2} from $u^{p_{\delta_2}}$ onto $u^{p_{\delta_1}}$ induce an isomorphism from p_{δ_2} onto p_{δ_1}.

Now choose $\eta^* = \langle \delta^*_n : n < \omega \rangle$ such that

$\boxtimes_3$ (c) $\delta^*_n < \delta^*_{n+1}$
(d) $\delta^*_n = \sup(S \cap \delta^*_n)$

We define $q \in \mathbb{Q}$ as follows

$\boxdot_4$ (e) $u^q = \cup \{p_{\delta^*_n} : n < \omega\}$
(f) $<^q = \{(\alpha, \beta) : \alpha <^{p_{\delta^*}}_n \beta$ for some n or for some $m < m, \alpha \in u^{p_{\delta^*_m}} \setminus \delta^*_m, \beta \in u^{p_{\delta^*_n}} \setminus \delta^*_n$
(g) $\mathfrak{S}^q_1 = \cup\{\mathfrak{S}^{p_{\delta^*_n}}_1 : n < \omega\} \cup \{\eta^*\}$
(h) $\mathfrak{S}^q_2 = \cup\{\mathfrak{S}^{p_{\delta^*_n}}_2 : n < \omega\}$
(i) $\Lambda^q = \cup\{\Lambda^{p_{\delta^*_n}} : n < u\}$
(j) $f^q_\rho = f^{p_{\delta^*_n}}_\rho$ if $\rho \in \Lambda^{p_{\delta^*_n}}$

Now q forces contradiction. $\square_{5.5}$

6.

Our aim is

Theorem 6.1. *Let $T \subseteq T_1$ be complete f.o., T is $\aleph_0$-independent or unstable. Some non-isomorphic $M_1, M_2 \in PC(T_1, T)$ of cardinality λ are $EF_{\alpha,\lambda}$-equivalent when $\lambda = \lambda^{\aleph_0} = \mathrm{cf}(\lambda) > |T_1| + \aleph_1$*

Proof: If T is $\aleph_0$-independent. We can find Φ as in 5.3(for T, T_1). If T is not $\aleph_0$-independent but is unstable we can find Φ satisfies the conclusion of 5.3 except that for some $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_\varphi)$ which linearly order some infinite set of m-types is some model of $T, m = \lg(\bar{x}) = \lg(\bar{y})$ we replace clause (c) there by

(c)' $M \models \varphi[\bar{a}_\eta, \bar{a}_\nu]$ iff $\eta <^{\mathbf{J}}_{\ell x} \nu$ which mean $\eta, \nu \in \mathbf{J}$, and $I^{\mathbf{J}} \models \eta < \nu$ or $\eta \in P^{\mathbf{J}}, \nu \in Q^{\mathbf{J}}$ or for some $n, m < n \to F^{\mathbf{J}}_m(\eta) = F^{\mathbf{J}}_m(\nu)$ and $I^{\mathbf{J}} \models$ "$F^{\mathbf{J}}_n(\eta) < F^{\mathbf{J}}_n(\nu)$.

(e)$\langle \bar{a}_\eta : \eta \in \mathbf{J} \rangle$ an indiscernible sequence in M_1.

Now use Definition 6.2 and claims 6.3,6.5 below.

Definition 6.2. (1) We say $\mathbf{y}$ is an ordered full λ-parameter <u>if</u>

- (a) $\mathbf{y} = (\mathfrak{x}, <, s, t) = (\mathfrak{x}_{\mathbf{y}}, <_{\mathbf{y}}, s_{\mathbf{y}}, t_{\mathbf{y}})$
- (b) $\mathfrak{x}$ is a full λ-parameter, see Definition 1.1(1A), so $M_{\mathbf{y}} =: M_{\mathfrak{x}}$ is from Definition 1.4
- (c) $s \in I_{\mathfrak{x}}, t \in J^{\mathfrak{x}}_s$
- (d) $<_{\mathbf{y}}$ is a linear order of $J_{\mathfrak{x}}$ such that
- (e) $J^{\mathfrak{x}}_s$ is a convex subset of $J_{\mathfrak{x}}$ for each $s \in I_{\mathfrak{x}}$
- (f) may add: in J_s there is a first element (hence in $\mathbb{G}_s$, every element has an immediate successor and an immediate predecessor).

(1A) We let $I_{\mathbf{y}} = I_{\mathfrak{x}}$ etc., and $s_1 <_{\mathbf{y}} s_2$ where $s_1, s_2 \in I_{\mathbf{y}}$ mean $\mathbf{s}_{t_1} = s_1 \wedge \mathbf{s}_{t_2} = s_2 \Rightarrow t_1 <_{\mathbf{y}} t_2$. We use $\leq_{\mathbf{y}}$ also for the following linear order on each $\mathbb{G}_s$ and on $M_{\mathbf{y}}$

- (a) for $s \in I_{\mathfrak{x}}, (\mathbb{G}_s, \leq_{\mathbf{y}})$ is an ordered abelian group, $\mathbb{G}_s = \mathbb{G}^{\mathbf{y}}_s$ is the abelian group generated freely by $\{x_t : \mathbf{s}_t = s\}$ and for $n < \omega, t_0 <_{\mathbf{y}} t_1 <_{\mathbf{y}} \ldots <_{\mathbf{y}} t_{n-1} \in J_s$ and $a_0, a_1, \ldots a_{n-1} \in \mathbb{Z} \setminus \{0\}$ we have $0_{\mathbb{G}_s} <_{\mathbf{y}} \sum\limits_{i=1}^{n} a_i x_{t_i}$ iff $a_{n-1} > 0$ so $n > 0$.
- (c) for $s_1 <_{\mathbf{y}} s_2$ all member of $\{s_1\} \times \mathbb{G}_{s_1}$ are $<_{\mathbf{y}}$ below those of $\{s_2\} \times \mathbb{G}_{s_2}$

(3) Let $\mathfrak{S}_{\mathbf{y}} = \{\eta : \eta \text{ an } \omega\text{-sequence from } (M_{\mathbf{y}}, <_{\mathbf{y}})\}$.

(4) We define a graph $H_{\mathbf{y}}$ on $\{1,2\} \times \mathfrak{S}_{\mathbf{y}}$: it consist of the pairs $\{(1, \eta_1), (2, \eta_2)\}$ such that $\eta_1, \eta_2 \in \mathfrak{S}_{\mathbf{y}}$ and for some $\alpha < \lambda, \bar{c} \in \mathbf{C}^{\mathfrak{x}}_{I_2}$ we have $f^{\mathfrak{x}}_{\bar{c}}$ maps η_1 to η_2 so necessarily $n < \omega \Rightarrow \eta_\ell(n) \in \mathrm{Dom}(f^{\mathfrak{x}}_{\bar{c}})$

(5) $E_{\mathbf{y}}$ is the equivalence relation on $\mathfrak{S}_{\mathbf{y}}$ which is being $H_{\mathbf{y}}$-connected.

(6) We say $(\mathfrak{S}_1, \mathfrak{S}_2)$ is a $\mathbf{y}$ -candidate when

(a) $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq \mathfrak{S}_{\mathbf{y}}$
(b) if $\{(1,\eta_1),(2,\eta_2)\} \in H$ then $\eta_1 \in \mathfrak{S}_1 \Leftrightarrow \eta_2 \in \mathfrak{S}_2$ (hence $(\{1\} \times \mathfrak{S}_1) \cup (\{2\} \times \mathfrak{S}_2)$ is closed under E-equivalence.

(7) For $\mathfrak{S} \subseteq \mathfrak{S}_{\mathbf{y}}$ let $\mathbf{J}_{\mathbf{y},\mathfrak{S}} = J_{I,\mathfrak{S}}$ where I is the linear order $(|M_{\mathbf{y}}|, <_{\mathbf{y}})$, clearly $\mathbf{J}_{\mathbf{y},\mathfrak{S}} \in K^{oi}_\lambda$

Claim 6.3. (1) *Assume* $\mathbf{y}$ *is an ordered full* λ*-parameters satisfying* $\circledast_{2,\alpha}$ *from 1.11(2) and* $(\mathfrak{S}_1,\mathfrak{S}_2)$ *is a* $\mathbf{y}$*-candidate and* $\Phi, \bar\varphi, T_1, T$ *are as in 6.3. Then* $EM_{\tau(T)}(\mathbf{J}_{\mathbf{y},\mathfrak{S}_1},\Phi), EM_{\tau(T)}(\mathbf{J}_{\mathbf{y},\mathfrak{S}_2},\Phi)$ *are* $EF_{\alpha,\lambda}$*-equivalent for every* $\alpha < \alpha^*_{\mathbf{y}}$

Proof: Recall that for any $\bar c \in \mathbf{C}_{\mathfrak{x}}, f^{\mathfrak{x}}_{\bar c}$ is a partial automorphism of $M_{\mathfrak{x}}$ (in fact an automorphism of $M^{\mathfrak{x}}_{I[\bar c]}$ where $\bar c \in \mathbf{C}^{\mathfrak{x}}_{I[\bar c]}$, so $I[\bar c] \subseteq I$ is uniquely determined by $\bar c$). Let $f^{\mathfrak{x}}_{\bar c}$ be the partial mapping from $J_{y,\mathfrak{S}_1}$ to $\mathbf{J}_{\mathbf{y},\mathfrak{S}_2}$ defined by $x \in M^{\mathfrak{x}}_{I[\bar c]} \Rightarrow f^{\mathfrak{x}}_{\bar c}(x) = f^{\mathfrak{x}}_{\bar c}(x)$ and
$\eta \in \mathfrak{S}_1 \Rightarrow f^{\mathfrak{x},*}_{\bar c}(\eta) = \langle f^{\mathfrak{x}}_{\bar c}(\eta(n)) : n < \omega\rangle$. It is easy to check that $\mathrm{Rang}(f^{\mathfrak{x},*}_{\bar c}) \subseteq \mathbf{J}_{y,\mathfrak{S}_2}$.

Now for each $\alpha < \lambda$ we can prove that $\{f^{\mathfrak{x},*}_{\bar c} : \bar c \in \mathbf{C}_{\mathfrak{x}}\}$ exemplifies that M_1, M_2 are $EF_{\alpha,\lambda}$- equivalent exactly as in the proof of 1.10. $\square_{6.3}$

Discussion 6.4. Now we need two steps
Step A: Characterize E (or a less fine E)?? effectively.

Step B: Construct $(\mathfrak{S}_1,\mathfrak{S}_2)$ such that the criterion from 5.4 unto holds for $\mathbf{J}_{\mathbf{y},\mathfrak{S}_1}, \mathbf{J}_{\mathbf{y},\mathfrak{S}_2}$

Claim 6.5. *Assume* $\lambda = \lambda^{\aleph_0} = \mathrm{cf}(\lambda) > \aleph_1 + |T_1$ *(we may concentrate on the case* $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$*). Let* $\mathfrak{x} = \mathfrak{x}_\lambda$ *be the full* λ*-candidate constructed in the proof of 1.12 (hence* $\circledast_{4\alpha}$ *for* $\alpha < \lambda$ *holds by its proof).* *Then we can find a* $\mathbf{y}$*-candidate* $(\mathfrak{S}_1,\mathfrak{S}_{2?})$ *such that letting* $M_\ell = M^+_\ell{\restriction}\tau(T)$ *where* $M^+_\ell = EM(J_{\mathbf{y},\mathfrak{S}_\ell},\Phi)$ *the models* M_1, M_2 *are* $EF_{\alpha,\lambda}$*-equivalent for every* $\alpha < \lambda$ *but are not isomorphic.*

Proof: By renaming $|M_{\mathbf{y}}| = \lambda$ let $S \subseteq \{\delta < \aleph_0 : \mathrm{cf}(\delta) = \aleph_0\}$ be stationary and we use the appropriate black box (see [Shear, IV]), $\langle (N_\alpha,\eta_\alpha) : \alpha < \alpha^*\rangle, \zeta : \alpha^* \to S$ non-decreasing, and $\dot\zeta(\alpha_1) = \delta = \dot\zeta(\alpha_2) \wedge \alpha_1 \neq \alpha_2 \Rightarrow \sup(N_{\alpha_1} \cap N_\alpha \cap \lambda) < \delta$ etc. [Maybe: for the sets $N_{\alpha_1} \cap \lambda, N_{\alpha_2} \cap \lambda$ interlacing is simple]
We choose $\nu_\alpha \in {}^\omega(|N_\alpha| \cap \lambda)$ as used in the later part of the proof (for some $\alpha \in S$) and let $\mathfrak{S}_\ell = \{(\ell,\nu)$: for some α, in the graph H, $(1,\nu_\alpha),(\ell,\nu)$ are connected (i.e. finite path)$\}$. The $EF_{\alpha,\lambda}$- equivalence holds by 6.3. To prove the models are not isomorphic assume f is an isomorphism from M_1 onto M_2. [Probably into is enough, not crucial for the main result.]?
For every $\alpha < \lambda$ let $s_\alpha = s(\alpha) = \{\alpha\} \in I_{\mathfrak{x}}$, and $t_\alpha = t(\alpha) \in J_s$. Let $f((s_\alpha, 0_{\mathbb{G}_{s(\alpha)}})) = \sigma_\alpha(a_{r(\alpha,0)},\ldots,a_{r(\alpha,n(\alpha)-1)})$ where $r(\alpha,\ell) \in J_{\mathbf{y}} \cup \mathfrak{S}_2$. By earlier remark w.l.o.g. $r(\alpha,\ell) \in \mathfrak{S}_2$. Let $S_1 = \{\delta < \lambda : \mathrm{cf}(\delta) > \aleph_0\}$ and

assuming for simplicity $(\forall\beta < \lambda)(|\beta|^{\aleph_0} < \lambda)$ for the time being, there is a stationary $S_2 \subseteq S_1$ such that

(a) $\delta \in S_2 \Rightarrow \sigma_\delta = \sigma_*$ so $\delta \in S_2 \Rightarrow n(\delta) = n(*)$.
(b) for each $n < n(*), k < \omega$ one of the following occurs
  (α) for $\delta \in S, r(\delta,n)(k) \in J_{\mathbf{y}}$, so in fact
  (β) $r(\delta,n)(k) = \sum\limits_{\ell<\ell(2)} a_{\delta,k,n,\ell} t_{\delta,k,n,\ell}$ where $t_{\delta,k,n,0} <_{\mathbf{y}} \ldots <_{\mathbf{y}} t_{\delta,k,n,\ell,\alpha}$
  (γ) $t_{\delta,k,n,\ell} \in J_{s,\delta,k,n}$ and
  (δ) $s_{\delta,k,0} <_{\mathbf{y}} \ldots <_{\mathbf{y}} s_{\delta,k,\ell(n)-1} \in I_{\mathbf{y}}$
  (ϵ) $s_{\delta,k,n} \cap \delta = u^*_{k,n}$ `kak? mqur lo mxuq` [[so $\langle (g^{t_{\delta,k,n,\ell}}, h^{t_{\delta,k,n,\ell}}) : \delta \in S_2\rangle$ is like a $\triangle$-system.]]
(c) (α) $s_{\delta,k,n} \subseteq \mathrm{Min}(S_2 \setminus (\delta+1))$ moreover if $t \in \{t_{\delta,k,n,\ell} : k,n,\ell\}$ then $\mathrm{Rang}(h^t) \cup \mathrm{Rang}(g^t) \subseteq \mathrm{Min}(S_2 \setminus (\delta+1)$

Now we choose $\beta < \alpha^*$ (the α^* of the B.B) such that N_β guess this situation, in particular

(*) (a) N_β is closed under f
  (b) $S_2 \cap N_\beta$ is P^{N_β}, for a fine predicate P relation of N_β and the function $\delta \mapsto \langle s_{\delta,k,n}, t_{\delta,k,n,\ell} : k,n,\ell\rangle$ is F^{N_β}, for some fixed function symbol F is P^{N_β}, for a fine predicate P.

Now we can choose $\nu_\beta \in {}^\omega(S_2 \cap N_\beta)$ increasing with limit $\dot\zeta(\beta) \in S$. Note: each $\nu_\beta(n)$ has $<_{J_{\mathbf{y}}}$-successor which we call $\rho_\beta(n)$ (see clause (f) of Definition 6.2(1)). The type of $f(a_{\nu_\beta})$ "mark" the $q_{\nu_\beta(n)}$. The rest should be straight. FILL

The $(\exists\mu)(\mu < \lambda = \mathrm{cf}(\lambda) \le \mu^{\aleph_0} \wedge \lambda > 2^{\aleph_0}$: Should be similar somewhat more complicated case.
λ singluar case have not thought.
The unstable case

Question: The case

(a) set theory $\aleph_1 = \mathrm{cf}(\lambda) < \mathrm{cf}(\mu) < \mu < \lambda < \lambda^{\aleph_0} \le 2^\mu, -$
(b) model theory: T = the theory of the rational order, T_1- make it home, see Droste ...

Question: Karp complexly?? [for Chris ??] for $\mathbb{L}_{\infty,\kappa}$, for simplicity $(2^{\aleph_0})^+ < \kappa = \mathrm{cf}(\kappa), (\forall\alpha < \kappa)(|\alpha|^{\aleph_0} < \kappa$.
first case: depth $\gamma < \kappa$.
second case: arbitrary γ.

Discussion 6.6. Given κ, γ we use the linear order $I = \{(\alpha,\eta) : \alpha < \kappa, \eta \in d??(\gamma)\}$, ordered but $(\alpha_1,\eta_1) \le_I (\alpha_2,\eta_1)$ iff $\alpha_1 < \alpha_2 \vee (\alpha_1 = \alpha_2 \wedge \lg\eta_1 < \lg\eta_2), \wedge(\alpha_1 = \alpha_2 \wedge \lg\eta_1 = \lg\eta_2 \wedge \eta_1 <_{\ell x} \eta_2$ (or simpler)
In the depth we use $\bar{\mathbf{a}}_\eta = \langle a_{\alpha(\eta)} : \alpha < \kappa\rangle$. All as in [LS03]. But we have to do a specific work here: for every pretender to an $\bar{\mathbf{a}}_\eta$ there is

$\langle \sigma(\ldots, a_{(\alpha_{\epsilon,\ell},\eta_{\epsilon,\ell})}, \ldots)_{\ell<n_*} : \epsilon < \kappa\rangle, n_* > 1$ if possible we give witness to its being a "composite"; similarly for a pair of $(\bar{a}', \bar{a}'')$ of pretenders.

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