SACCHARINITY

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Abstract. We present a method to iterate finitely splitting lim-sup tree forcings along non-wellfounded linear orders. As an application, we introduce a new method to force (weak) measurability of all definable sets with respect to a certain (non-ccc) ideal.

Introduction

Non-wellfounded iterations. We introduce a method to iterate lim-sup finitely splitting tree forcings along linear, non-wellfounded orders.

There is quite some literature about non-wellfounded iteration. E.g., Jech and Groszek [4] investigated the wellfounded but non-linear iteration of Sacks forcings. Building on this, Kanovei [7] and Groszek [5] develop non-wellfounded iterations of Sacks forcing. In spirit, their construction is close to the construction of this paper, but the implementation is quite different. Zapletal gives an illfounded iteration construction in the framework of "idealized forcing" [15], it seems that his results give some of the properties of our construction (e.g., ω^ω-bounding) for a more general class of forcings, cf. his Theorem 5.4.12.[1] Regarding finite support, Brendle [1] developed finite-support non-wellfounded iteration constructions, based on the second author's method of iterations along smooth templates [13]. Brendle's paper also contains the important observation by Hjorth (answering a question of Hechler) that it is impossible to have an illfounded iteration of forcings that all add dominating reals.

Measurability. As an application of our method, we introduce a new way to force measurability of definable sets.

In the seminal paper [14] Solovay proved that in the Levy model (after collapsing an inaccessible) every definable set is measurable and has the Baire property.

In [12] the second author showed that the inaccessible is necessary for measurability, but the Baire property of every definable set can be obtained by a forcing P without the use of an inaccessible (i.e., in ZFC). This forcing P is constructed by amalgamation of universally meager forcings Q. So every Q adds a co-meager set of generics and has many automorphisms, and the forcing P has similar properties to the Levy collapse. The property of Q that implies that Q can be amalgamated is called "sweetness" (a strong version of ccc). One can ask about other ccc ideals than Lebesgue-null and meager (or their defining forcings, random and Cohen), and classify such ideals (respectively forcings) according to whether they behave like Cohen or like randoms see, e.g., Sweet & Sour [10].

For (non-ccc) ideals corresponding to tree forcings Q, forcing measurability can be much simpler, see Section 6 about the Cohen model. In this model, all definable set are Q-measurable (e.g., Marczewski measurable for Q = Sacks forcing). The proof is a simpler

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[1] note that Zapletal's basic construction can be applied to countable orders only, for longer orders additional work is required, see Section 5.5 there.

version of Solovay's: Cohen forcing is homogeneous and adds subtrees $S \in Q \cap V[G]$ to all $T \in Q \cap V$ such that all branches of S are Cohen reals.

In this paper, we introduce a new construction that gives a variant of measurability (weak measurability, as defined in 3.3) for all definable sets: Instead of iterating basic forcings Q that have many automorphisms and add a measure 1 set of generics, we use a Q that adds only a null set of generics (a single one in our case, and this real remains the only generic over V even in the final limit). So Q has to be very non-homogeneous. Instead of having many automorphisms in Q, we assume that the skeleton of the iteration has many automorphisms (so in particular a non-wellfounded iteration has to be used).

We use the word Saccharinity for this concept: a construction that achieves the same effect as (an amalgamation of) sweet forcings, but using entirely different means.

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Annotated contents.

1. FINITELY SPLITTING LIM-SUP TREE-FORCINGS

We will define a class of finitely splitting tree forcings with "lim-sup norm". The simplest example is Sacks forcing. Such forcings (and generalizations) have been investigated by many authors, e.g. in [9] under the name $\mathbb{Q}_0^{\text{tree}}$ (see Definition 1.3.5 there).

1.1. **Basics.** Let us first introduce some notation:

Definition 1.1. Let $T \subseteq \omega^{<\omega}$ be a tree (i.e., T is closed under initial segments), let $s, t \in \omega^{<\omega}$, $A \subseteq T$.

- We write sequences as $\langle a_1, \ldots, a_n\rangle$ or as $(a_1, \ldots, a_n)$. In particular, $\langle\rangle$ denotes the empty sequence.
- $s \leq t$ means that s is a restriction of t (or equivalently that $s \subseteq t$).
- t is immediate successor of s if $t \geq s$ and $\text{length}(t) = \text{length}(s) + 1$.
- $\text{succ}_T(t)$ is the set of immediate successors of t in a tree T. If the tree T is clear from the context we will also write $\text{succ}(t)$.

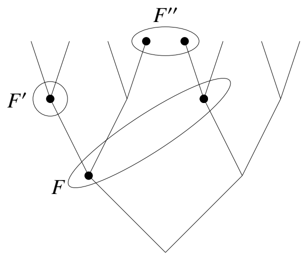

FIGURE 1. F' is stronger than F, F'' is purely stronger than F.

- Nodes s and t are compatible ($s \parallel t$), if they are comparable, i.e., if $s \leq t$ or $t \leq s$. Otherwise, s and t are incompatible ($s \perp t$).
- The order in forcing notions is usually chosen such that $q < p$ means that q is stronger than p. We try to stick to Goldstern's alphabetic convention [3, 1.2]: Whenever two conditions are comparable the notation is chosen so that the variable used for the stronger condition comes "lexicographically" later.
- Two forcing conditions p and p' are compatible ($p \parallel p'$), if there is a q stronger than both p and p'. Otherwise, p and p' are incompatible ($p \perp p'$).
- $T^{[t]} := \{s \in T : s \parallel t\}$. (So $T^{[t]}$ is a tree.) If T is clear, we might also just write $[t]$.
- $T \restriction n := \{t \in T : \text{length}(t) < n\}$.
- $A \subseteq T$ is a chain if $s \parallel t$ for all $s, t \in A$.
- $b \subseteq T$ is a branch if it is a maximal chain.
  If there exists a $t \in b$ with length n then this t is unique and denoted by $b \restriction n$.
- $A \subseteq T$ is an antichain if $s \perp t$ for all $s \neq t \in A$. Unless noted otherwise, we will assume that antichains are nonempty.
- $A \subseteq T$ is a front if it is an antichain and every branch b meets A (i.e., $|b \cap A| = 1$).
- $t \leq A$ stands for: "$t \leq s$ for some $s \in A$".
- $T^A_{\text{cldn}} := \{t \in \omega^{<\omega} : t \leq A\}$.
  (We will use this downwards-closure only for finite sets A. Then T^A_{cldn} is a finite tree.)
- If A and A' are antichains, then A' is stronger than A if for each $t \in A'$ there is a $s \in A$ such that $s \leq t$ (cf. Figure 1.1).
- If A and A' are antichains then A' is purely stronger than A if it is stronger and for each $s \in A$ there is a $t \in A'$ such that $s \leq t$ (cf. Figure 1.1).
- $\lim(T)$ are the maximal branches of T. We use this notation only for T that are "pruned", i.e., have no finite maximal branches; then $\lim(T) \subseteq \omega^\omega$ is the closed set corresponding to T.

We are only interested in finitely splitting trees (i.e., $\text{succ}(t)$ is finite for all $t \in T$). Then all fronts are finite. Note that being a front is stronger than being a maximal antichain. For example, $\{0^n 1 : n \in \omega\}$ is a maximal antichain in $2^{<\omega}$, but not a front.

Assumption. Assume $T_{\max}$ and μ satisfy the following:

- $T_{\max}$ is a finitely splitting tree.
- μ assigns a non-negative real to every subset of $\text{succ}_{T_{\max}}(t)$ for every $t \in T_{\max}$.
- μ is monotone, i.e., if $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
- The measure of singletons is smaller than 1, i.e., $\mu(\{s\}) < 1$.
- For all branches b in $T_{\max}$, $\limsup_{n\to\infty}(\mu(\text{succ}(b \restriction n))) = \infty$.

Note that such a $T_{\max}$ has to be perfect.

Definition 1.2. (The tree forcing Q.)

- If T is a subtree of $T_{\max}$ and $t \in T$, then $\mu_T(t)$ is defined as the measure of the T-successors of t, i.e., $\mu_T(t) := \mu(\mathrm{succ}_T(t))$.
- Q consists of all subtrees T of $T_{\max}$ (ordered by inclusion) such that along every branch b of T
$$\limsup(\mu_T(b \restriction n)) = \infty.$$

So $T_{\max}$ itself is the weakest element of Q.

For example, Sacks forcing can be defined in this way: Set $T_{\max} := 2^{<\omega}$, and for $t \in T_{\max}$ and $A \subseteq \mathrm{succ}(t)$ set
$$\mu(A) := \begin{cases} \mathrm{length}(t) & \text{if } |A|= 2, \\ 0 & \text{otherwise.} \end{cases}$$
Then a subtree T of $2^{<\omega}$ is in Q iff T is a Sacks tree, i.e., iff along every branch there are infinitely many splitting nodes.[2]

Definition 1.3. A (finite or infinite) subtree T of $T_{\max}$ is n-dense if there is a front F in T such that $\mu_T(t) > n$ for every $t \in F$.

Lemma 1.4. (1) *A subtree T of T_{max} is in Q iff T is n-dense for every $n \in \mathbb{N}$.*
(2) *"$T \in Q$" and "$T \leq_Q S$" are Borel statements, and "$S \perp T$" is $\underset{\sim}{\Pi}^1_1$ (in the real parameters T_{max} and μ).*

Proof. (1) $\rightarrow$: If $D_n := \{s \in T : \mu_T(s) > n\}$ meets every branch, then
$$F_n := \{s \in D_n : (\forall s' \lneqq s)\, s' \notin D_n\}$$
is a front.
$\leftarrow$: If b is a branch, then b meets every F_n, i.e., $\mu_T(b \restriction m) > n$ for some m. Since $\mu_T(b \restriction m)$ is finite, $\limsup(\mu_T(b \restriction n))$ has to be infinite.

(2) Since $T_{\max}$ is finitely splitting, "F is a front" is equivalent to "F is a finite maximal antichain". □

A finite antichain A can be seen as an approximation to a tree: "A approximates T" means that A is a front in T. If A' is purely stronger than A, then A' gives more information about the tree T that is approximated (i.e., every tree approximated by A' is also approximated by A). And, informally, a stronger antichain approximates smaller (i.e., stronger) trees.

We will usually identify a finite antichain F and the corresponding finite tree T^F_{cldn}.

Definition 1.5.
- A finite antichain F is n-dense if T^F_{cldn} is n-dense.
- $\bar{F} = (F_n)_{n \in \omega}$ is a front-sequence, if F_{n+1} is n-dense and purely stronger than F_n.
- A front-sequence $\bar{F}$ and a tree $T \in Q$ correspond to each other if F_n is a front in T for all n.

Facts 1.6.
- *If F is n-dense and F' is purely stronger than F, then F' is n-dense as well. (This is not true if F' is just stronger than F.)*
- *If $T \in Q$ then there is a front-sequence corresponding to T.*
- *If $\bar{F}$ is a front-sequence then there exists exactly one $T \in Q$ corresponding to $\bar{F}$, which we call* $\lim(\bar{F})$. *It is the tree*
$$\lim(\bar{F}) := \{t \in T_{max} : (\exists i \in \omega)\, t \leq F_i\},$$
or equivalently
$$\lim(\bar{F}) := \{t \in T_{max} : (\forall i \in \omega)\, (\exists s \in F_i)\, t \parallel s\}.$$

Lemma 1.7. *Assume that Q is a finitely splitting lim-sup tree-forcing.*

[2] This example is "atomic" in the following sense: For a node $s \in T$ there is an $A \subset \mathrm{succ}(s)$ such that $\mu(A)$ is large but $\mu(B) < 1$ for every $B \subsetneq A$. In this paper, we will be interested in "finer" norms. In particular we will require the Ramsey property defined in 5.4.

(1) *If $T \in Q$ and $t \in T$ then $T^{[t]} \in Q$. (Sometimes this fact is formulated as "Q is strongly arboreal".)*
(2) *The finite union of elements of Q is in Q.*[3]
(3) *The generic filter on Q is determined by a real $\underaccent{\tilde}{\eta}$ defined by $\Vdash_Q \{\underaccent{\tilde}{\eta}\} = \bigcap_{T \in \underaccent{\tilde}{G}_Q} \lim(T)$; or equivalently: $\underaccent{\tilde}{\eta}$ is the union of the stems of the trees in G_Q.*
It is forced that $\underaccent{\tilde}{\eta} \notin V$ and that $T \in G_Q$ iff $\underaccent{\tilde}{\eta} \in \lim(T)$.
For every $T \in Q$ and $t \in T$, $t \prec \underaccent{\tilde}{\eta}$ is compatible with T. (In other words: $T \not\Vdash t \not\prec \underaccent{\tilde}{\eta}$.)
(4) *(Fusion) If $(T_i)_{i \in \omega}$ is a decreasing sequence in Q and $\bar{F}$ is a front-sequence such that F_i is a front in T_i for all i, then $\lim(\bar{F}) \leq_Q T_i$.*
(5) *(Pure decision) If $D \subseteq Q$ is dense, $T \in Q$ and F is a front of T, then there is an $S \leq T$ such that F is a front of S and for every $t \in F$, $S^{[t]} \in D$.*
(6) *Q is proper*[4] *and ω^ω-bounding.*

Sketch of proof. (1) and (2) and (4) are clear. (1) and (2) imply (5).

(3): Let G be Q-generic over V, and define $X := \bigcap_{T \in G} \lim(T)$. Since $\lim(T_{\max})$ is compact, it satisfies the finite intersection property. So X is nonempty. For every $T \in G$ and $n \in \omega$ there is exactly one $t \in T$ of length n such that $T^{[t]} \in G$. So X has at most one element.

If $r \in V$, then the set of trees $S \in Q^V$ such that $r \notin \lim(S)$ is dense: If r is a branch of $T \in Q$ then pick an m such that $\mu_T(r \restriction m) > 2$. Since singletons have measure less than 1, $r \restriction m$ has at least two immediate successors in T, and one of them (we call it t) is not an initial segment of r. So $S := T^{[t]}$ forces that $\underaccent{\tilde}{\eta} \neq r$.

Assume towards a contradiction that $\underaccent{\tilde}{\eta} \in \lim(T)$ for some $T \in Q^V \setminus G$. Then this is forced by some $S \in G$. In particular S can not be a subtree of T. So pick an $s \in S \setminus T$. Then $S^{[s]} \leq S$ forces that $\underaccent{\tilde}{\eta} \notin \lim(T)$, a contradiction.

If $T \in Q$ and $t \in T$ then $T^{[t]}$ forces that $t \prec \underaccent{\tilde}{\eta}$.

(4) and (5) imply that Q is ω^ω-bounding and satisfies a version of Axiom A (with fronts as indices instead of natural numbers).[5] So we get properness. (We will prove a more general case in 2.24.) □

So a front can be seen as a finite set of (pairwise incompatible) possibilities for initial segments of the generic real $\underaccent{\tilde}{\eta}$. In the next section we will generalize this to finite sequences of generic reals instead of a single one.

1.2. **Some additional facts needed later.**

Lemma 1.8. *If $S \in Q$ and the forcing R adds a new real $\underaccent{\tilde}{r} \in 2^\omega$, then R forces that there is a $\underaccent{\tilde}{T} \leq_Q S$ such that $\lim(\underaccent{\tilde}{T}) \cap V = \emptyset$, and moreover $\lim(\underaccent{\tilde}{T}) \cap V$ remains empty in every extension of the universe.*

Proof. Assume S corresponds to the front-sequence $\bar{F}$. Without loss of generality we can assume that along every branch in S there is exactly one split between F_{n-1} and F_n and this split has measure $> n$.

[3] Q is generally not closed under countable unions.

[4] there even are generic conditions for arbitrary countable transitive ZFC models M, similarly to Suslin proper. Sometimes this is called "totally proper".

[5] In the formulation of fusion and pure decision we could use the classical Axiom A version as well: Define F^T_n to be the minimal n-dense front, i.e.,

$$F^T_n := \{t \in T : \mu_T(t) > n \,\&\, (\forall s \subsetneq t)\, \mu_T(s) \leq n\},$$

and define $T \leq_n S$ by $T \leq S$ and $F^T_n = F^S_n$. It should be clear how to formulate fusion and pure decision for this notion, and that this proves Axiom A for Q. But in 1.7, we do not use this notion, instead we (implicitly) use the following one: $T \leq_A S$ means that $T \leq S$ and A that is a front in both T and S. The reason is that this is the notion that will be generalized for the non-wellfounded iteration.

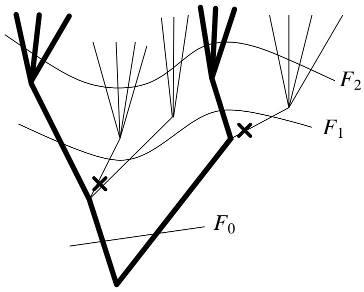

FIGURE 2. An example for S and its subtree $\underset{\sim}{T}$ (bold) when $\underset{\sim}{r}(0) = 0$.

We define an R-name of a sequence of finite antichains $(\underset{\sim}{F}'_n)$ the following way (cf. Figure 1.2): If n is even, we "take all splits", i.e., $\underset{\sim}{F}'_n$ is the set of nodes in F_n that are compatible with $\underset{\sim}{F}'_{n-1}$. If n is odd, then we add no splittings at all: for every $s \in \underset{\sim}{F}'_{n-1}$ we put exactly one successor $t \in F_n$ of s into $\underset{\sim}{F}'_n$, namely the one continuing the $\underset{\sim}{r}(\frac{n-1}{2})$-th successor of the (unique) splitting node over s. It is clear that the sequence $(\underset{\sim}{F}'_n)$ defines a subtree $\underset{\sim}{T}$ of S that is in Q.

Assume V' is an arbitrary extension of V containing an R-generic filter G over V. If $\eta \in \lim(\underset{\sim}{T}[G]) \cap V$, then $\underset{\sim}{r}[G]$ can be decoded in V using S and η. This is a contradiction to $\Vdash_R \underset{\sim}{r} \notin V$. □

We will also need the following family of definable dense subsets of Q:

Definition 1.9. Fix a recursive bijection ψ from ω to $2^{<\omega}$. Assume that $f : \omega \to \omega$ is strictly increasing and that $A \subseteq \omega$.

- For $g \in 2^\omega$, define $A^\psi_g := \{n \in \omega : \psi(n) < g\}$.
- Q^f_A is the set of all $T \in Q$ such that for all splitting nodes $t \in T$, length(t) is in the interval $[f(n), f(n+1) - 1]$ for some $n \in A$.
- $T \in Q$ has full splitting with respect to f if for all $n \in \omega$ and $t \in T$ of length $f(n+1)$ there is an $s \leq t$ of length at least $f(n)$ such that $\mu_T(s) > n$.
- D^{spl}_f is the set of all $T \in Q$ such that either $T \in Q^f_{A^\psi_g}$ for some $g \in 2^\omega$ or $T \perp_Q S$ for all $g \in 2^\omega$ and $S \in Q^f_{A^\psi_g}$.

Of course the notions Q^f_A and D^{spl}_f depend on the forcing Q (i.e., on $T_{\max}$ and μ), so maybe it would be more exact to write $Q^f_A[T_{\max}, \mu]$ etc. However, we always assume that the Q is understood. 3.3).

Lemma 1.10. *Assume that $f : \omega \to \omega$ is strictly increasing and $A, B \subseteq \omega$.*

1. *If $g \neq g'$, then $A^\psi_g \cap A^\psi_{g'}$ is finite.*
2. *$Q^f_\omega = Q$. If A is finite then $Q^f_A = \emptyset$.*
3. *$Q^f_A \cap Q^f_B = Q^f_{A\cap B}$. If $A \subseteq B$, then $Q^f_A \subseteq Q^f_B$.*
4. *If $T \leq_Q S$ and $S \in Q^f_A$ then $T \in Q^f_A$.*
5. *For every $T \in Q$ there is a strictly increasing f such that T has full splitting with respect to f.*
6. *If $T \in Q$ has full splitting with respect to f and $|A| = \aleph_0$ then there is an $S \leq_Q T$ such that $S \in Q^f_A$.*

(7) *D^{spl}_f is an (absolute definition of an) open dense subset of Q (using the parameters f, T_{max} and μ).*[6]

(8) *In any extension V' of V the following holds: If $r \in 2^\omega \setminus V$ and $S \in Q^f_{A^\psi_r}$, then $T \perp_Q S$ for all $T \in V \cap D^{spl}_f$.*

Proof. (1)–(4) and (6) are clear.

(5): Let T be an element of Q. Assume we already constructed $f(n)$. Let N be the maximum of $\mu_T(t)$ for $t \in T \restriction f(n)$. There is an $N+n+1$-dense front F in T. Let $f(n+1)$ be the maximum of $\{\mathrm{length}(t) : t \in F\}$.

(7): "T is incompatible with all $S \in Q^f_{A^\psi_g}$" is absolute, since it is equivalent to

$$(\forall g \in 2^\omega)\,(\forall S \subseteq T_{\max})\,\left[S \notin Q^f_{A^\psi_g} \ \vee\ T \perp_Q S\right],$$

which is a $\underset{\sim}{\Pi}^1_1$ statement.

(8): Let $r \in 2^\omega \setminus V$ and $T \in V \cap D^{\mathrm{spl}}_f$. If $T \in Q^f_{A^\psi_g}$ for some $g \in 2^\omega \cap V$, then $g \neq r$, so $A^\psi_g \cap A^\psi_r$ is finite and $Q^f_{A^\psi_r} \cap Q^f_{A^\psi_g}$ is empty. If on the other hand T is incompatible with all $S \in Q^f_{A^\psi_g}$ in V then this holds in V' as well. $\square$

Assume $f'(n) \geq f(n)$ for all $n \in \omega$. Define $h(n)$ by induction: $h(n+1) := f'(h(n)+1)$. If T has full splitting with respect to f, then T has full splitting with respect to h: $h(n) \leq f(h(n))$, since f is strictly increasing. $f(h(n)+1) \leq f'(h(n)+1) = h(n+1)$, and there are $h(n)$-dense splits between the levels $f(h(n))$ and $f(h(n)+1)$. So there are n-dense splits between the levels $h(n)$ and $h(n+1)$. So we get:

Lemma 1.11. *If V' is an ω^ω-bounding extension of V and $T \in Q^{V'}$, then there is a strictly increasing $h \in V$ such that (in V') T has full splitting with respect to h.*

2. A non-wellfounded Iteration

In this section we introduce a general construction to iterate lim-sup tree-forcings Q_i (as defined in the last section) along non-wellfounded linear orders I. It turns out that the limit P is proper, ω^ω-bounding and has other nice properties similar to the properties of Q_i itself. If I is wellfounded, then P is equivalent to the usual countable support iteration of (the evaluations of the definitions) Q_i.

2.1. Conditions and approximations, the nw-iteration.

Definition 2.1. Let I be a linear order. For $i \in I$ we set $I_{<i} := \{j \in I : j < i\}$ and analogously we define $I_{\leq i}$ and $I_{>i}$. We also set $I_{<\infty} := I$.

[6] $X_0 := \{A^\psi_g : g \in 2^\omega\}$ is an almost disjoint family, but not maximal. So of course $Q(X_0) := \bigcup_{A \in X_0} Q^f_A \subset Q$ is not dense. We add the incompatible conditions to get the dense set D^{spl}_f. One could ask whether $Q(X)$ is dense for a m.a.d. family X. The following holds:

(a) For every f there is a m.a.d. family X such that $Q(X)$ is not dense.

(b) (CH) For every f there is a m.a.d. family X such that $Q(X)$ is dense.

Proof: Fix f. A node $s \in T_{\max}$ has level m if $f(m) \leq \mathrm{length}(s) < f(m+1)$. $S \in Q$ has unique splitting if S has at most one splitting point of level n for all $n \in \omega$. For every $T \in Q$ there is an $S \leq_Q T$ with unique splitting.

For (a), fix a $T \in Q$ with unique splitting. Set $Y := \{A \in [\omega]^{\aleph_0} : (\forall S \leq_Q T)\, S \notin Q^f_A\}$. Y is open dense in $([\omega]^{\aleph_0}, \subseteq)$, therefore there is a m.a.d. $X \subseteq Y$.

For (b), list Q as $(T_\alpha)_{\alpha \in \omega_1}$, and build $B_\alpha \in [\omega]^{\aleph_0}$ by induction on $\alpha \in \omega_1$: Find an $S \leq_Q T_\alpha$ with unique splitting. If some $S' \leq_Q S$ is in $Q^f_{B_\beta}$ ($\beta < \alpha$) (or equivalently in $Q^f_{\bigcup_{i \in l} B_{\beta_i}}$ for some $l \in \omega$, $\beta_0, \dots, \beta_{l-1} < \alpha$), then just pick any almost disjoint B_α. Otherwise enumerate $(B_\beta)_{\beta \in \alpha}$ as $(C_n)_{n \in \omega}$, and construct B_α and $S' \leq_Q S$ inductively: At stage n, add a split of S to S' whose level is not in $\bigcup_{m \leq n} C_m$, and use some bookkeeping to guarantee that $S' \in Q$. Let B_α be the set of splitting-levels of S'.

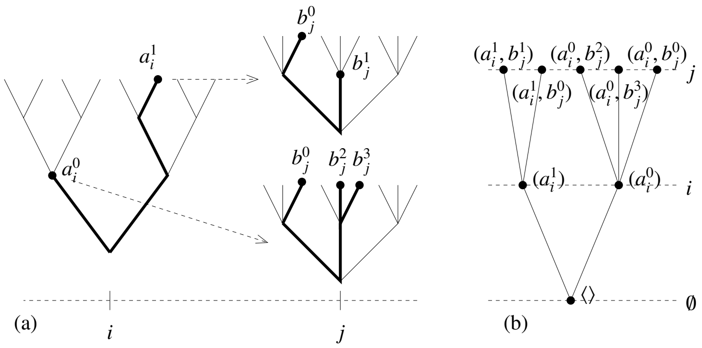

FIGURE 3. An approximation $\mathfrak{g}$: $u = \{i, j\}$, $T^i_{\max} = 2^{<\omega}$, $T^j_{\max} = 3^{<\omega}$.
$\mathrm{Pos}(\mathfrak{g}) = \mathrm{Pos}_{\leq j}(\mathfrak{g}) = \{(a_i^1, b_j^0), (a_i^1, b_j^1), (a_i^0, b_j^2), (a_i^0, b_j^3), (a_i^0, b_j^0)\}$.
(a): viewed as function: $\mathfrak{g}(i)(\langle\rangle) = \{a_i^0, a_i^1\}$, $\mathfrak{g}(j)(\langle a_i^1\rangle) = \{b_j^0, b_j^1\}$ etc.
(b): viewed as tree, the heights labeled with $\{\emptyset\} \cup u$.

For every $i \in I$ we fix a finitely splitting lim-sup tree-forcing Q_i (to be more exact, we fix a pair $T^i_{\max}, \mu^i$). In the application of this paper, each Q_i will be the same forcing Q.

Definition 2.2. (Pre-condition) We call p a pre-condition, if p is a function, the domain of p is a countable[7] subset of I, and for each $i \in \mathrm{dom}(p)$, $p(i)$ consists of the following:

- Dom^p_i, a countable subset of $\mathrm{dom}(p) \cap I_{<i}$, and
- a (definition of a) Borel function $B^p_i : (\omega^\omega)^{\mathrm{Dom}^p_i} \to Q_i$

Remark 2.3. The idea is that we calculate the condition $B^p_i \in Q_i$ using countably many generic reals $(\eta_j)_{j \in \mathrm{Dom}^p_i}$ that have already been produced at stage i. The forcing conditions p of the non-wellfounded iterations will be pre-conditions that satisfy additional properties, in particular: all B^p_i are continuous (on a certain Borel set), i.e., if we want to know B^p_i up to some finite height we only have to know $(\eta_i \restriction m)_{i \in u}$ for some finite u and $m \in \omega$. Moreover, we will assume that we will have "wellfounded continuity parts". This will be explained in the following, here just an example: Assume that $I = \omega^* = \{\dots, 3, 2, 1, 0\}$, and each $T^i_{\max} = 2^{<\omega}$. Let p be the pre-condition with $\mathrm{Dom}^p_n = \{n + 1\}$, i.e., B^p_n only depends on the generic real $\underset{\sim}{\eta}_{n+1}$, and $B^p_n(x) = [0]$ if $x(0) = 0$ and $B^p_n(x) = [1]$ otherwise. Then p is continuous, but will not be a valid condition, since it is not well founded enough.

We now define finite "approximations" to conditions of the iteration; they will have the same role for the iteration that finite antichains have for Q (see, e.g., Lemma 1.7). The following definition looks rather unpleasant, but really is quite simple, as Figure 2.1 hopefully demonstrates. (We first define approximations as functions as in (a) of the figure; sometimes it is more useful to think of them as trees as in (b), which will be described in 2.6.)

Definition 2.4. (Approximation)

- $\mathfrak{g}$ is an approximation, if $\mathfrak{g}$ is a function with finite domain $u \subseteq I$ of the following form: Let i_0 be the smallest element of u. We set $\mathrm{Pos}_{<i_0}(\mathfrak{g}) := \{\langle\rangle\}$. By induction on $i \in u$, we assume that $\mathrm{Pos}_{<i}(\mathfrak{g})$ is a set of sequences indexed by the set $\{j \in u : j < i\}$, and require the following: $\mathfrak{g}(i)$ is a function from $\mathrm{Pos}_{<i}(\mathfrak{g})$ to finite antichains in $T^i_{\max}$, and we set

$$\mathrm{Pos}_{\leq i}(\mathfrak{g}) := \{\bar{a}^\frown b : \bar{a} \in \mathrm{Pos}_{<i}(\mathfrak{g}),\ b \in \mathfrak{g}(i)(\bar{a})\}.$$

If j is the successor of i in u, we set $\mathrm{Pos}_{<j}(\mathfrak{g})$ to be $\mathrm{Pos}_{\leq i}(\mathfrak{g})$.

[7] this includes finite and empty.

- For any $i \in I \cup \{\infty\}$, we define $\mathrm{Pos}_{<i}(\mathfrak{g})$ as $\mathrm{Pos}_{\leq j}(\mathfrak{g})$, where $j = \max(\mathrm{dom}(\mathfrak{g}) \cap I_{<i})$ (or as $\{\langle\rangle\}$, if $\mathrm{dom}(\mathfrak{g}) \cap I_{<i}$ is empty). We set $\mathrm{Pos}(\mathfrak{g}) := \mathrm{Pos}_{<\infty}(\mathfrak{g})$ and call it the set of possibilities of $\mathfrak{g}$.
- If $i \notin \mathrm{dom}(\mathfrak{g})$ or $\bar{a} \notin \mathrm{Pos}_{<i}(\mathfrak{g})$ we set $\mathfrak{g}(i)(\bar{a}) := \{\langle\rangle\}$ (i.e., the front in $T^i_{\max}$ consisting only of the root. This corresponds to "no information").
- Let $\mathfrak{g}$ be an approximation, $J \subset I$, and $\bar{\eta} = (\eta_i)_{i \in J}$ a sequence of reals. Then "$\bar{\eta}$ is compatible with $\mathfrak{g}$", if there is an $\bar{a} \in \mathrm{Pos}(\mathfrak{g})$ such that $a_i \prec \eta_i$ for all $i \in \mathrm{dom}(\mathfrak{g}) \cap J$. If in addition $J \supseteq \mathrm{dom}(\mathfrak{g})$, then this $\bar{a}$ is uniquely defined and called $\bar{\eta} \restriction \mathfrak{g}$. If $J \supseteq \mathrm{dom}(\mathfrak{g}) \cap I_{<i}$, then $\bar{a} \restriction I_{<i}$ is uniquely defined, and therefore we can set $\mathfrak{g}(i)(\bar{\eta}) := \mathfrak{g}(i)(\bar{a} \restriction I_{<i})$.

  If $\bar{b} = (b_i)_{i \in J}$ is a sequence of elements of $\omega^{<\omega}$, we define $\bar{b}$ to be compatible with $\mathfrak{g}$ if there is a sequence $\bar{\eta}$ extending $\bar{b}$ and compatible with $\mathfrak{g}$. If $J \supseteq \mathrm{dom}(\mathfrak{g})$ and additionally each b_i is long enough, then such a $\bar{b}$ defines a unique $\bar{a} \in \mathrm{Pos}(\mathfrak{g})$ called $\bar{b} \restriction \mathfrak{g}$; if $J \supseteq \mathrm{dom}(\mathfrak{g}) \cap I_{<i}$ and additionally each b_i is long enough, then we can define $\mathfrak{g}(i)(\bar{b})$ as above.
- If $\mathfrak{g}$ and $\mathfrak{g}'$ are both approximations, then "$\mathfrak{g}'$ is stronger than $\mathfrak{g}$" if $\mathrm{dom}(\mathfrak{g}') \supseteq \mathrm{dom}(\mathfrak{g})$ and for all $\bar{b} \in \mathrm{Pos}(\mathfrak{g}')$ there is an $\bar{a} \in \mathrm{Pos}(\mathfrak{g})$ such that $\bar{b} \geq \bar{a}$ (i.e., $b_i \geq a_i$ for all $i \in \mathrm{dom}(\mathfrak{g})$). In this case $\bar{a}$ is $\bar{b} \restriction \mathfrak{g}$.

  Equivalently, $\mathfrak{g}'$ is stronger than $\mathfrak{g}$ iff for all $i \in \mathrm{dom}(\mathfrak{g})$ and all $\bar{b} \in \mathrm{Pos}_{<i}(\mathfrak{g}')$ there is a (unique) $\bar{a} \in \mathrm{Pos}_{<i}(\mathfrak{g})$ such that $\bar{b} \geq \bar{a}$ and the antichain $\mathfrak{g}'(i)(\bar{b})$ is stronger than $\mathfrak{g}(i)(\bar{a})$.
- $\mathfrak{g}'$ is purely stronger than $\mathfrak{g}$ if $\mathfrak{g}'$ is stronger than $\mathfrak{g}$ and for all $i \in \mathrm{dom}(\mathfrak{g})$ and $\bar{b} \in \mathrm{Pos}_{<i}(\mathfrak{g}')$ the front $\mathfrak{g}'(i)(\bar{b})$ is purely stronger than $\mathfrak{g}(i)(\bar{b} \restriction \mathfrak{g})$.
- For $u \subseteq \mathrm{dom}(\mathfrak{g})$, $\mathrm{maxlength}_u(\mathfrak{g})$ is $\max(\{\mathrm{length}(a_i) : i \in u, \bar{a} \in \mathrm{Pos}(\mathfrak{g})\})$. $\mathrm{maxlength}(\mathfrak{g})$ is $\mathrm{maxlength}_{\mathrm{dom}(\mathfrak{g})}(\mathfrak{g})$. Analogously we define $\mathrm{minlength}(\mathfrak{g})$.
- $\mathfrak{g}$ is n-dense at $i \in I$, if $i \in \mathrm{dom}(\mathfrak{g})$ and for all $\bar{a} \in \mathrm{Pos}_{<i}(\mathfrak{g})$, $\mathfrak{g}(i)(\bar{a})$ is n-dense for Q_i. (See Definition 1.5.)
- For all $\bar{a} = (a_i)_{i \in u}$ such that $a_i \in T^i_{\max}$ there is a (unique) approximation $\mathfrak{g}$ such that $\mathrm{Pos}(\mathfrak{g}) = \{\bar{a}\}$. We will call this approximation $\bar{a}$ as well.

Facts 2.5.
- *"stronger" is a partial order on the set of approximations; the same holds for "purely stronger".*
- *If $\mathfrak{h}$ is stronger than $\mathfrak{g}$, then all $\bar{\eta}$ compatible with $\mathfrak{h}$ are compatible with $\mathfrak{g}$.*

We could equivalently define approximations as trees, cf. Figure 2.1(b): Given an approximation $\mathfrak{g}$, we can define an approximation-tree with $u = \mathrm{dom}(\mathfrak{g})$ labeling the heights above the root, and the set of nodes at height $i_n \in u$ is $\mathrm{Pos}_{\leq i_n}(\mathfrak{g})$; the tree order is just extension of sequences. Every such approximation-tree corresponds to an approximation:

Fact 2.6. *Consider a finite tree where the heights above the root are labeled by the increasing sequence $u = \{i_1, \ldots, i_n\}$ in I. Assume that each node at height i_m is a sequence $(a_j)_{j=i_1,\ldots,i_m}$ and that the tree order is the extension relation. Then this tree corresponds to an approximation, iff each branch has maximal height and the successors of each node at level i_{n-1} form an antichain in $T^{i_n}_{max}$.*

In particular, if we take a subset of the (maximal) branches in the approximation-tree $\mathfrak{g}$, we get a "sub-approximation" $\mathfrak{h}$. A single branch $\bar{a}$ is a special case of such a sub-approximation.

Definition 2.7. (Approximation to p) Let p be a pre-condition.
- $\mathfrak{g}$ approximates p, or: $\mathfrak{g}$ is a p-approximation, if $\mathrm{dom}(\mathfrak{g}) \subseteq \mathrm{dom}(p)$ and $\mathfrak{g}$ is an approximation with the following property: If $i \in \mathrm{dom}(\mathfrak{g})$, $\bar{a} \in \mathrm{Pos}_{<i}(\mathfrak{g})$, and $\bar{\eta} = (\eta_j)_{j \in \mathrm{Dom}^p_i}$ is compatible with $\bar{a}$, then $\mathfrak{g}(i)(\bar{a})$ is a front in $B^p_i(\bar{\eta})$.
- $\mathfrak{g}$ is an indirect approximation to p witnessed by $\mathfrak{g}'$, if $\mathfrak{g}'$ approximates p and $\mathfrak{g}'$ is purely stronger than $\mathfrak{g}$.

Example 2.8. The following trivial example should demonstrate the difference between approximation and indirect approximation: Assume each $T^i_{\max}$ is $2^{<\omega}$, and p is a condition with $\mathrm{dom}(p) = \{i, j\}$ for some $i < j$ in I. Accordingly Dom^p_i has to be empty, and B^p_i is constant; we set it to have constant value $[1]$. We set $\mathrm{Dom}^p_j = \{i\}$ and $B^p_j(x) = [x(0)]$, i.e., if the real x starts with 0 then B^p_j is $[0]$ and otherwise it is $[1]$. We define the approximation $\mathfrak{g}$ by $\mathrm{Pos}(\mathfrak{g}) = \{(\langle\rangle, 1)\}$ and $\mathfrak{h}$ by $\mathrm{Pos}(\mathfrak{h}) = \{(1, 1)\}$. Then $\mathfrak{g}$ indirectly approximates p, witnessed by $\mathfrak{h}$.

Now we can define the forcing P, the non-wellfounded countable support limit along I:

Definition 2.9. (The nwf-iteration $P = \text{nwf-lim}_I(Q_i)$)

- $p \in P$ means:
  p is a pre-condition, and for all finite $u \subseteq \mathrm{dom}(p)$, $i \in u$ and $n \in \omega$ there is a p-approximation $\mathfrak{g}$ such that $\mathrm{dom}(\mathfrak{g}) \supseteq u$, $\mathfrak{g}$ is n-dense for i, and $\mathrm{minlength}_u(\mathfrak{g}) > n$.
- For $p, q \in P$, $q \leq p$ means:
  for all p-approximations $\mathfrak{g}$ there is a q-approximation $\mathfrak{h}$ which is stronger than $\mathfrak{g}$ (so in particular, $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$).
- $q \leq_{\mathfrak{g}} p$ if $q \leq p$ and $\mathfrak{g}$ indirectly approximates p and q.

Remark. The definition of $q \leq_P p$ is *not* equivalent to "for all i and $\bar\eta$, $B^q_i(\bar\eta)$ is a subtree of $B^p_i(\bar\eta)$." (Informally speaking, we are only interested in "the generic $\bar\eta$, not in "all $\bar\eta$".) We will see in Lemma 2.23(6) that $q \leq_P p$ is equivalent to: for each $i \in I$ it is forced by $q \restriction P_{<i}$ that $B^q_i(\bar\eta)$ is a subtree of $B^p_i(\bar\eta)$, where $\bar\eta$ is the generic sequence up to i.

Facts 2.10.
- *$\leq$ is transitive, and for a fixed approximation $\mathfrak{g}$ the relation $\leq_{\mathfrak{g}}$ is transitive as well.*
- *If $\mathfrak{h}$ is purely stronger than $\mathfrak{g}$ then $\leq_{\mathfrak{h}}$ implies $\leq_{\mathfrak{g}}$.*
- *For every $p \in P$, the approximations of p are directed: If $\mathfrak{g}$ and $\mathfrak{g}'$ both (indirectly) approximate p, then there is a $\mathfrak{h}$ approximating p that is (purely) stronger than both $\mathfrak{g}$ and $\mathfrak{g}'$. In fact,* every *p-approximation $\mathfrak{h}$ has this property if* $\mathrm{dom}(\mathfrak{h}) \supseteq \mathrm{dom}(\mathfrak{g}) \cup \mathrm{dom}(\mathfrak{g}')$ *and if* $\mathrm{minlength}_{\mathrm{dom}(\mathfrak{g})\cup\mathrm{dom}(\mathfrak{g}')}(\mathfrak{h})$ *is large enough.*

So in particular for every $p \in P$ there is an approximating sequence:

Definition 2.11. An approximating sequence for $p \in P$ is a sequence $(\mathfrak{g}_n)_{n\in\omega}$ of approximations of p such that $\mathfrak{g}_{n+1}$ is purely stronger than $\mathfrak{g}_n$, and $\mathfrak{g}_{n+1}$ is n-dense for each $i \in \mathrm{dom}(\mathfrak{g}_n)$, and $\mathrm{dom}(p) = \bigcup_{n\in\omega} \mathrm{dom}(\mathfrak{g}_n)$.

An approximating sequence contains all relevant information about p. In particular, $\mathfrak{g}$ is an indirect approximation to p iff there is an n such that $\mathfrak{g}_n$ is purely stronger than $\mathfrak{g}$. So if p and q both have the approximating sequence $(\mathfrak{g}_n)_{n\in\omega}$, then $p =^* q$ (i.e., $p \leq q$ and $q \leq p$), furthermore $\mathfrak{g}$ indirectly approximates p iff it indirectly approximates q.

Approximating sequences provide an equivalent definition for P:

Definition 2.12. (Alternative definition of the nwf-iteration P) Define the p.o. P' as follows: $\bar{\mathfrak{g}} \in P'$ iff $\bar{\mathfrak{g}}$ is a sequence of approximations $(\mathfrak{g}_n)_{n\in\omega}$ such that $\mathfrak{g}_{n+1}$ is purely stronger than $\mathfrak{g}_n$ and n-dense for every $i \in \mathrm{dom}(\mathfrak{g}_n)$. We define $\bar{\mathfrak{h}} < \bar{\mathfrak{g}}$ as: For every n there is an m such that $\mathfrak{h}_m$ is stronger than $\mathfrak{g}_n$.

Lemma 2.13. *There is a dense embedding*[8] $\phi : P' \to P$.

Proof. Given a sequence $\bar{\mathfrak{g}} \in P'$, define $p = \phi(\bar{\mathfrak{g}})$ the following way: $\mathrm{dom}(p) = \bigcup \mathrm{dom}(\mathfrak{g}_n)$. For $i \in \mathrm{dom}(p)$, set $\mathrm{Dom}^p_i := \mathrm{dom}(p) \cap I_{<i}$. Define $T = B^p_i(\bar\eta)$ as follows: If $\bar\eta$ is compatible with all $\mathfrak{g}_n$, then let T be $\{t \in T^i_{\max} : (\exists n \in \omega)\, t \leq \mathfrak{g}_n(i)(\bar\eta)\}$. Otherwise, let n be maximal such that $\bar\eta$ is compatible with $\mathfrak{g}_n$, and let T be $\{t \in T^i_{\max} : (\exists s \in \mathfrak{g}_n(i)(\bar\eta))\, t \parallel s\}$. Clearly,

[8] ϕ is even an isomorphism modulo $=^*$, where $p =^* q$ if $q \leq p$ and $q \leq p$.

B_i^p is a Borel function, $B_i^p(\bar\eta) \in Q_i$ and each $\mathfrak{g}_n$ approximates p. Therefore $(\mathfrak{g}_n)_{n\in\omega}$ is an approximating sequence for $p \in P$. It is clear that ϕ preserves the order.

Let ψ map $p \in P$ to any approximating sequence for p. $\psi : P \to P'$ preserves order as well and $\phi(\psi(p)) =^* p$. Therefore ϕ is a dense embedding. $\square$

Notes 2.14.

(1) If $\mathfrak{g}$ indirectly approximates p, then there is a $q =^* p$ such that $\mathfrak{g}$ approximates q. (Just let q correspond to an approximating sequence of p starting with $\mathfrak{g}_0 = \mathfrak{g}$.)
(2) It doesn't matter whether the $\mathfrak{g}_n$ in an approximating sequence are approximations to p or just indirect approximations.
(3) It doesn't matter whether $\mathfrak{g}_{n+1}$ proves n-density for every $i \in \mathrm{dom}(\mathfrak{g}_n)$ or for just some i_n, provided that the sequence $(i_n)_{n\in\omega}$ covers $\bigcup \mathrm{dom}(\mathfrak{g}_n)$ infinitely often.
(4) In Definition 2.2 of pre-condition, instead of requiring B_i^p to be a function into Q_i, we could have defined B_i^p to be a function to subtrees of $T^i_{\max}$. The additional "n-dense" requirements on a condition guarantee $B_i^p(\bar\eta) \in Q_i$ anyway (for generic sequences $\bar\eta$).
(5) Every approximation $\mathfrak{g}$ can be interpreted as a condition in P, by
$$B_i^{\mathfrak{g}}(\bar\eta) := \{t : t \parallel \mathfrak{g}(i)(\bar\eta)\} \text{ for } i \in \mathrm{dom}(\mathfrak{g}).$$
(Where we set $\mathfrak{g}(i)(\bar\eta) := \{\langle\rangle\}$ if $\bar\eta$ is incompatible with $\mathfrak{g}$.) Then $\mathfrak{g}$ approximates itself.
(6) For any approximation $\mathfrak{g}$ and $u \subseteq I$ finite we can assume $u \subseteq \mathrm{dom}\,\mathfrak{g}$: Just set $\mathfrak{g}(i)$ to be the constant function with value $\{\langle\rangle\}$ for $i \notin \mathrm{dom}\,\mathfrak{g}$. (Recall that $\{\langle\rangle\}$ is the "trivial front" corresponding to "no information".)
(7) If $\mathfrak{g}$ and $\mathfrak{h}$ are approximations, we can assume without loss of generality that $\mathrm{dom}(\mathfrak{g}) = \mathrm{dom}(\mathfrak{h})$.
(8) For any $U \subseteq I$ countable and $p \in P$ we can assume without loss of generality that $\mathrm{dom}(p) \supseteq U$. This is clear if p is interpreted as a sequence of Borel-functions: just set B_i^p to be (the constant function with value) $T^i_{\max}$ for $i \notin \mathrm{dom}(p)$. If p is interpreted as sequence $(\mathfrak{g}_n)_{n\in\omega}$ of approximations, we have to set $\mathfrak{g}_n(i)$ to be (the constant function with value) $T^i_{\max} \cap \omega^{k(n)}$ for some sufficiently large $k(n)$. (Using $\{\langle\rangle\}$ does not work here, since it does not satisfy n-density.)
(9) So if $q \leq p$ we can assume $\mathrm{dom}(q) = \mathrm{dom}(p)$, and if p is interpreted as sequence $(\mathfrak{g}_n)_{n\in\omega}$ and q as $(\mathfrak{g}_n)_{n\in\omega}$ then we can assume $\mathrm{dom}(\mathfrak{g}) = \mathrm{dom}(\mathfrak{h})$.

2.2. **Fusion and pure decision.** We have seen: Every $p \in P$ corresponds to a purely increasing sequence $(\mathfrak{g}_n)$ of approximations such that $\bigcup \mathrm{dom}(\mathfrak{g}_n) = \mathrm{dom}(p)$ and $\mathfrak{g}_{n+1}$ is n-dense for $\mathrm{dom}(\mathfrak{g}_n)$. The approximating sequences immediately prove a version of fusion:

Lemma 2.15. *(Fusion) Assume that $(p_n)_{n\in\omega}$ is a sequence of conditions, $(\mathfrak{g}_n)_{n\in\omega}$ a sequence of approximations, and $i_n \in \mathrm{dom}(\mathfrak{g}_n)$ such that:*

- $p_{n+1} \leq_{\mathfrak{g}_n} p_n$,
- $\mathfrak{g}_{n+1}$ *is purely stronger than* $\mathfrak{g}_n$ *and n-dense for i_n,*
- $(i_n)_{n\in\omega}$ *covers* $\bigcup \mathrm{dom}(p_n)$ *infinitely often.*

Then there is a condition p_ω such that $p_\omega \leq_{\mathfrak{g}_n} p_n$ for all n.

Proof. We already know that the sequence $(\mathfrak{g}_n)_{n\in\omega}$ of approximations defines a condition p_ω such that each $\mathfrak{g}_n$ approximates p_ω. If $\mathfrak{h}$ approximates p_n, then some $\mathfrak{g}_m$ is stronger than $\mathfrak{h}$. Then $\mathfrak{g}_m$ approximates p_ω, so $p_\omega \leq p_n$. $\square$

Definition 2.16. $\mathfrak{h}$ is sub-approximation of $\mathfrak{g}$ if $\mathrm{Pos}(\mathfrak{h}) \subseteq \mathrm{Pos}(\mathfrak{g})$. (So in particular $\mathrm{dom}(\mathfrak{g}) = \mathrm{dom}(\mathfrak{h})$.)

Obviously any sub-approximation of $\mathfrak{g}$ is stronger than $\mathfrak{g}$. In the interpretation of approximations as trees, a sub-approximation is just a nonempty subset of the (maximal) branches, see Fact 2.6.

Lemma 2.17. *(Sub-approximation) Assume that $\mathfrak{g}$ indirectly approximates p and that $\mathfrak{h}$ is a sub-approximation of $\mathfrak{g}$. Then there is a weakest condition stronger than p and approximated by $\mathfrak{h}$, which we call $p\restriction\mathfrak{h}$.*

Proof. Without loss of generality, we can think of p as an approximation-sequence $(\mathfrak{g}_n)_{n\in\omega}$ with $\mathfrak{g} = \mathfrak{g}_0$. We define approximations $\mathfrak{h}_n$ as follows: $\mathfrak{h}_n$ consists of those nodes in the approximation-tree $\mathfrak{g}_n$ that are compatible with an element of $\mathfrak{h}$. Then $p\restriction\mathfrak{h}$ is the sequence $(\mathfrak{h}_n)_{n\in\omega}$. □

A special case of a sub-approximation is a singleton:

Definition 2.18. Assume that $\mathfrak{g}$ (indirectly) approximates p and $\bar{a} \in \mathrm{Pos}(\mathfrak{g})$. We can interpret $\bar{a}$ as an approximation, a sub-approximation of $\mathfrak{g}$. Instead of $p\restriction\bar{a}$ we also write $p^{[\bar{a}]}$.

Corollary 2.19. *(Specialization and pure decision) Assume that $\mathfrak{g}$ indirectly approximates p and that $\bar{a} \in Pos(\mathfrak{g})$.*

(1) *$p^{[\bar{a}]} \in P$, $p^{[\bar{a}]} \leq p$ and $\bar{a}$ indirectly approximates $p^{[\bar{a}]}$. If $q \leq p$ and $\bar{a}$ indirectly approximates q, then $q \leq p^{[\bar{a}]}$.*
(2) *If $q \leq_{\mathfrak{g}} p$, then $q^{[\bar{a}]} \leq p^{[\bar{a}]}$.*
(3) *If $q \leq p^{[\bar{a}]}$ then there is a $r \leq_{\mathfrak{g}} p$ such that $r^{[\bar{a}]} =^* q$.*
(4) *The set $\{p^{[\bar{a}]} : \bar{a} \in Pos(\mathfrak{g})\}$ is predense below p.*
(5) *Abusing notation, we denote with (i, a) the approximation $\mathfrak{g}$ with domain $\{i\}$ such that $\mathfrak{g}(i)(\langle\rangle) = \{a\}$. For all $i \in I$, $n \in \omega$ the following set is dense:*
$$\{p \in P : (\exists a \in \omega^n)\,(i, a) \text{ approximates } p\}.$$
(Or, in the notation introduced later: We can densely determine the generic $\underset{\sim}{\eta}_i$ up to n.)
(6) *(Pure decision) If $D \subseteq P$ is open dense, and $\mathfrak{g}$ indirectly approximates p, then there is an $r \leq_{\mathfrak{g}} p$ such that $r^{[\bar{a}]} \in D$ for all $\bar{a} \in Pos(\mathfrak{g})$.*

Proof. (1) and (2) follow easily from the definition.

(3) We set r to be q "below $\bar{a}$" and p otherwise. Let p correspond to $(\mathfrak{g}_n)_{n\in\omega}$ with $\mathfrak{g}_0 = \mathfrak{g}$, and q corresponds to $(\mathfrak{h}_n)_{n\in\omega}$ with $\mathfrak{h}_0 = \bar{a}$ such that each $\mathfrak{h}_n$ is stronger than $\mathfrak{g}_n$. According to Note 2.14(9), we can assume that $\mathrm{dom}(\mathfrak{h}_n) = \mathrm{dom}(\mathfrak{g}_n) = u_n$. We define by induction on n a sub-approximation $\mathfrak{f}_n$ of $\mathfrak{g}_n$: Let i_0 be minimal in u_n. So $\mathrm{Pos}_{<i_0}(\mathfrak{f}_n) = \{\langle\rangle\}$. By induction on $i \in u_n$, define for all $\bar{b} \in \mathrm{Pos}_{<i}(\mathfrak{f}_n)$
$$\mathfrak{f}_n(i)(\bar{b}) := \begin{cases} \mathfrak{g}_n(i)(\bar{b}) & \text{if } \bar{b} \text{ is incompatible with } \mathfrak{h}_n, \\ \mathfrak{h}_n(i)(\bar{b}) \cup \{t \in \mathfrak{g}_n(i)(\bar{b}) : t \perp \mathfrak{h}_n(i)(\bar{b})\} & \text{otherwise.} \end{cases}$$
It is clear that the possibilities of $\mathfrak{f}_n$ follow $\mathfrak{h}_n$ up to some $i \in \mathrm{dom}\,\mathfrak{g}_n$ and from then on become incompatible with $\mathfrak{h}_n$ and follow $\mathfrak{g}_n$. To be more exact: $\bar{b} \in \mathrm{Pos}(\mathfrak{f}_n)$ iff $\bar{b} \in \mathrm{Pos}(\mathfrak{g}_n)$ and for some $i \in \mathrm{dom}(\mathfrak{g}_n) \cup \{\infty\}$, $\bar{a}\restriction I_{<i}$ is in $\mathrm{Pos}(\mathfrak{h}_n)$ and either $i = \infty$ or $a_i \perp \mathfrak{h}_n(i)(\bar{a})$. From this it follows that $\mathfrak{f}_n$ is purely stronger than $\mathfrak{g}_n$, and that the $\mathfrak{f}_n$ are an approximating sequence (converging to some $r \leq p$).

(4) If $\mathfrak{g}$ indirectly approximates p and $q \leq p$, then there is a $\mathfrak{h}$ stronger than $\mathfrak{g}$ approximating q. Let $\bar{b} \in \mathrm{Pos}(\mathfrak{h})$ and $\bar{a} = \bar{b}\restriction\mathfrak{g} \in \mathrm{Pos}(\mathfrak{g})$. Then $q^{[\bar{b}]} \leq q, p^{[\bar{a}]}$.

(5) Let $\mathfrak{h}$ approximate p such that $\mathrm{minlength}_{\{i\}}(\mathfrak{h}) > n$. Let $\bar{a} \in \mathrm{Pos}(\mathfrak{h})$. Then (a_i) indirectly approximates $p^{[\bar{a}]} \leq p$. By 2.14(1) we can find a $q =^* p$ such that (a_i) approximates q.

(6) Let $\mathrm{Pos}(\mathfrak{g}) = \{\bar{a}_0, \dots, \bar{a}_l\}$. Pick $q_0 \leq p^{[\bar{a}_0]}$ in D, and $r_0 \leq_{\mathfrak{g}} p$ as in (3). So $r_0^{[\bar{a}_0]} \in D$. Pick $q_1 \leq r_0^{[\bar{a}_1]}$ in D and $r_1 \leq_{\mathfrak{g}} r_0$ as above, etc. Then r_l has the required property. □

Remark. Similarly, we can define conjunctions of two approximations $\mathfrak{g}, \mathfrak{g}'$. More specifically: let us call $\mathfrak{g}$ and $\mathfrak{g}'$ compatible if there is an $\mathfrak{h}$ stronger than both $\mathfrak{g}$ and $\mathfrak{g}'$. Then for every compatible pair $\mathfrak{g}, \mathfrak{g}'$ there is a weakest approximation $\mathfrak{g} \wedge \mathfrak{g}'$ stronger than $\mathfrak{g}$ and $\mathfrak{g}'$. If p and q have incompatible approximations, then they are incompatible (in Q). This can be used to define the conjunction of an approximation and a condition (if the condition p corresponds to the sequence $\mathfrak{g}_n$, let $p \wedge \mathfrak{h}$ correspond to the sequence $\mathfrak{g}_n \wedge \mathfrak{h}$; it is the weakest condition stronger than p that is approximated by $\mathfrak{h}$). Similarly one can define the conjunction of two conditions. However, all of this will not needed in this paper.

2.3. Restrictions.

We now list some trivial properties of P regarding restriction:

Definition 2.20. For $i \in I \cup \{\infty\}$ we define $P_{<i} := \{p \in P : \operatorname{dom}(p) \subseteq I_{<i}\}$. In particular, $P = P_{<\infty}$. Analogously we define $P_{\leq i}$ for $i \in I$.

Facts 2.21. *(Restriction) Assume $p, q \in P$ and $i, j \in I \cup \{\infty\}$.*

- *If $\operatorname{dom}(q) \supseteq \operatorname{dom}(p)$, $q \restriction \operatorname{dom}(p) = p$ and $\mathfrak{g}$ approximates p, then $q \leq_{\mathfrak{g}} p$.*
- *$p \restriction I_{<i} \in P_{<i}$ and $p \leq p \restriction I_{<i}$.*
- *If $p' \leq p$ then $p' \restriction I_{<i} \leq p \restriction I_{<i}$. If $p \in P_{<i}$ then $p \restriction I_{<i} = p$.*
- *Let $q \in P_{<i}$, $q \leq p \restriction I_{<i}$. Define $q \wedge p := q \cup p \restriction I_{\geq i}$. Then $q \wedge p \in P$ is the weakest condition stronger than both q and p.*
- *$p \restriction I_{<i}$ is a reduction of p (i.e., $r' \in P_{<i}$ and $r' \leq p \restriction I_{<i}$ implies $r' \parallel p$).*
- *In particular, $P_{<i} \lessdot P_{<j}$ (i.e., $P_{<i}$ is a complete subforcing of $P_{<j}$) for $i \leq j$.*
- *If $p \restriction I_{<i} \parallel q \restriction I_{<i}$ and $\operatorname{dom}(p) \cap \operatorname{dom}(q) \subseteq I_{<i}$, then $p \parallel q$.*
- *Similar facts hold for $P_{\leq i}$. E.g., if $i < j$, then $P_{\leq i} \lessdot P_{<j}$.*

Definition 2.22. Assume that $j \in I \cup \{\infty\}$ and $i < j$, and that $G_{<j}$ is a $P_{<j}$-generic filter over V.

- Since $P_{<i}$ is a complete subforcing of $P_{<j}$, the filter $G_{<j} \cap P_{<i} =: G_{<i}$ is $P_{<i}$-generic over V. We set $V_{<i} := V[G_{<i}]$. The canonical Q_i-generic filter over $V_{<i}$ is called $G(i)$. Analogously we can define $V_{\leq i}$ and $G_{\leq i}$ (which turns out to be $V_{<i}[G(i)]$ and $G_{<i} * G(i)$, respectively).
- In $V_{<j}$ or $V_{\leq i}$ we define η_i to be the union of all $t \in \omega^{<\omega}$ such that (i, t) is an approximation[9] of p for some $p \in G_{<j}$ (or $G_{\leq i}$).

Lemma 2.23. *Let $i, j, G_{<j}$ be as above, $p \in G_{<j}$, and set $\bar{\eta} = (\eta_l)_{l<j}$.*

1. *η_i is a well-defined real. In particular we can calculate $B^q_i(\bar{\eta} \restriction Dom^q_i)$ for all $q \in P$; abusing notation, we will just write $B^q_i(\bar{\eta})$.*
2. *If $\mathfrak{g}$ indirectly approximates p, then $\bar{\eta}$ is compatible with $\mathfrak{g}$.*
3. *$\{\eta_i\} = \bigcap\{\lim B^q_i(\bar{\eta}) : q \in G_{<j},\ i \in \operatorname{dom}(q)\}$.*
4. *$q \in G_{<j}$ iff $\eta_i \in \lim(B^q_i(\bar{\eta}))$ for all $i \in \operatorname{dom}(q)$.*
5. *(in V): $q \leq_P p$ iff $\operatorname{dom}(q) \supseteq \operatorname{dom}(p)$ and $q \Vdash \underset{\sim}{\eta}_i \in \lim(B^p_i(\underset{\sim}{\bar{\eta}}))$ for all $i \in \operatorname{dom}(p)$.*
6. *(in V): $q \leq_P p$ iff $\operatorname{dom}(q) \supseteq \operatorname{dom}(p)$ and $q \restriction I_{<i} \Vdash B^q_i(\underset{\sim}{\bar{\eta}}) \subseteq B^p_i(\underset{\sim}{\bar{\eta}})$ for all $i \in \operatorname{dom}(p)$.*

Proof. (1) By 2.19(5), the set of conditions q such that for some t of length n the approximation (i, t) approximates q is dense. Therefore η_i is infinite. Also, if $s \perp t$, if (i, t) is an approximation of q, and if (i, s) is an approximation of q', then q and q' are incompatible. This shows that η_i is indeed a real.

(2) According to 2.19(4), the set $\{p^{[\bar{a}]} : \bar{a} \in \operatorname{Pos}(\mathfrak{g})\}$ is predense below p. Let $\bar{a}$ be such that $p^{[\bar{a}]} \in G$. Any $q \in G$ that is stronger than $p^{[\bar{a}]}$ and decides $\underset{\sim}{\eta}_i$ up to the length of a_i forces that $\underset{\sim}{\eta}_i \supset a_i$. So $\bar{\eta}$ is compatible with $\bar{a}$ and therefore with $\mathfrak{g}$.

(3) Let $\underset{\sim}{n} \in \omega$. We have to show that $\eta_i \restriction n \in B^q_i(\bar{\eta})$. First pick an approximation $\mathfrak{g}$ of q with $\operatorname{minlength}_{\{i\}}(\mathfrak{g}) \geq n$. We already know that $\bar{\eta}$ is compatible with $\mathfrak{g}$, in particular η_i is compatible with $\mathfrak{g}(i)(\bar{\eta})$. And $\mathfrak{g}(i)(\bar{\eta})$ is a front in $B^q_i(\bar{\eta})$, since $\mathfrak{g}$ approximates q. It remains to be seen that the intersection on the right-hand side is a singleton; this is clear by genericity.

[9] as in Corollary 2.19(5)

(5) One direction follows immediately from the definition of the order in P: Assume that $q \leq p$ and that $i \in \mathrm{dom}(p)$. Assume towards a contradiction that $r \leq q$ forces that $\eta_i \notin \lim(B^p_i(\bar{\underaccent{\tilde}{\eta}}))$, more specifically that $\underaccent{\tilde}{\eta}_i \restriction M \notin B^p_i(\bar{\underaccent{\tilde}{\eta}})$ for some M (already determined by r). Pick a p-approximation $\mathfrak{g}$ that has minimal height greater than M at position i; and an r-approximation $\mathfrak{h}$ stronger than $\mathfrak{g}$. Pick $\bar{b} \in \mathrm{Pos}(\mathfrak{h})$ and let $\bar{a} \in \mathrm{Pos}(\mathfrak{g})$ be the restriction. Then $r^{[\bar{b}]}$ forces that $\underaccent{\tilde}{\eta}_i \restriction M < a_i < b_i$ for any $\bar{b} \in \mathrm{Pos}(\mathfrak{h})$, but $a_i \in \mathfrak{g}(i)(\bar{a})$ which is a front in $B^p_i(\bar{\underaccent{\tilde}{\eta}})$, a contradiction.

For the other direction, let $\mathfrak{g}$ approximate p and $\mathfrak{h}$ approximate q such that $\mathrm{dom}(\mathfrak{h}) \supseteq \mathrm{dom}(\mathfrak{g})$ and the length of $\mathfrak{h}$ is sufficiently large on $\mathrm{dom}(\mathfrak{g})$. Then $\mathfrak{h}$ must be stronger than $\mathfrak{g}$, which shows that $q \leq p$.

(4) follows from (3) and (5); (6) follows from (5) (see also the proof of Lemma 2.25). $\square$

2.4. **Properness, bounding, continuous reading.** As immediate consequence of fusion and pure decision we get:

Theorem 2.24.
(1) *P is ω^ω-bounding. For every p and P-name $\underaccent{\tilde}{\tau}$ for an ω-sequence of ordinals there is a $q \leq p$ such that q reads $\underaccent{\tilde}{\tau}$ continuously.*[10]
(2) *Assume that the cofinality*[11] *of I is $\geq \aleph_1$, that G is P-generic over V and that $r \in \mathbb{R}^{V[G]}$. Then there is an $i \in I$ such that $r \in \mathbb{R}^{V_{<i}}$.*
(3) *P is proper.*[12]
(4) *P forces that $\underaccent{\tilde}{\eta}_i$ is a Q_i-generic real over $V_{<i}$.*
(5) *If $I = I_1 + I_2$, then* $\text{nwf-lim}_I(Q_i) \cong \text{nwf-lim}_{I_1}(Q_i) * \text{nwf-lim}_{I_2}(Q_i)$*, the forcing-composition of* $\text{nwf-lim}_{I_1}(Q_i)$ *and (the* $\text{nwf-lim}_{I_1}(Q_i)$*-name for)* $\text{nwf-lim}_{I_2}(Q_i)$.
(6) *If $I = \Sigma_{\beta\in\epsilon} J_\beta$ is the concatenation of the orders J_β along the ordinal ϵ, then* $\text{nwf-lim}_I(Q_i)$ *is equivalent to the countable support limit $(P_\beta, \underaccent{\tilde}{Q}'_\beta)_{\beta\in\epsilon}$, where $\underaccent{\tilde}{Q}'_\beta$ is (the P_β-name for)* $\text{nwf-lim}_{J_\beta}(Q_i)$.
(7) *If I is well-founded, then* $\text{nwf-lim}_I(Q_i)$ *is the countable support limit of the Q_i.*

Proof. (1) Fix for every countable subset J of I an enumeration $\{j_m : m \in \omega\}$, and denote $\{j_m : m \in n\}$ by $\mathrm{first}(n, J)$.

Assume $\underaccent{\tilde}{\tau}$ is a name of a real and $p \in P$. We have to show that there is a $p_\omega \leq p$ and an $f \in \omega^\omega$ such that $p_\omega \Vdash \underaccent{\tilde}{\tau}(n) < f(n)$. Let $p_0 \leq p$, $f(0) \in \omega$ be such that $p_0 \Vdash \underaccent{\tilde}{\tau}(0) = f(0)$, and let $\mathfrak{g}_0$ approximate p_0. Assume that $\mathfrak{g}_n$ and p_n are already defined. We define $p_{n+1} \leq_{\mathfrak{g}_n} p_n$, $f(n)$ and $\mathfrak{g}_{n+1}$ the following way: Let $p_{n+1} \leq_{\mathfrak{g}_n} p_n$ be such that $p^{[\bar{a}]}_{n+1}$ decides $\underaccent{\tilde}{\tau}(n)$ for every $\bar{a} \in \mathrm{Pos}(\mathfrak{g}_n)$, see 2.19(6). Let $f(n)$ be the maximum of the possible values for $\underaccent{\tilde}{\tau}(n)$. Let $\mathfrak{g}_{n+1}$ be a p_{n+1}-approximation stronger than $\mathfrak{g}_n$ which is n-dense at every $i \in \mathrm{first}(n, \mathrm{dom}(p_1)) \cup \cdots \cup \mathrm{first}(n, \mathrm{dom}(p_n))$. Then the sequence $(p_n)_{n\in\omega}$ satisfies the conditions for fusion 2.15 so there is a $p_\omega \leq p_n$. Clearly, $p_\omega \Vdash \underaccent{\tilde}{\tau}(n) \leq f(n)$.

The same argument shows continuous reading of ω-sequences: Now we do not require $\underaccent{\tilde}{\tau}(n)$ to be a natural number, and we do not care about the maximum possible value; the rest is the same.

(2) The p_ω above completely determines $\underaccent{\tilde}{\tau}$, so if $p_\omega \in P_{<i}$, then $p_\omega \Vdash_P \underaccent{\tilde}{\tau} \in V_{<i}$.

(3) is very similar to the above: Assume that $N \prec H(\chi)$ and $p_0 \in N$. Let $\{D_m : m \in \omega\}$ enumerate the dense sets in N. Assume p_n, $\mathfrak{g}_n \in N$ are already defined. Find (in N) $p_{n+1} \leq_{\mathfrak{g}_n} p_n$ such that $p^{[\bar{a}]}_{n+1} \in D_n$ for all $\bar{a} \in \mathrm{Pos}(\mathfrak{g}_n)$, and pick $\mathfrak{g}_{n+1} \in N$ big enough. Then

[10] In more detail: Let $(\underaccent{\tilde}{\tau}(n))_{n\in\omega}$ be a sequence of P-names for ordinals and $p \in P$. Then there is a $q \leq p$ corresponding to a sequence $(\mathfrak{g}_n)_{n\in\omega}$ of approximations, and there are functions f_n from $\mathrm{Pos}(\mathfrak{g}_n)$ into the ordinals such that $q^{[\bar{a}]}$ forces $\underaccent{\tilde}{\tau}(n) = f_n(\bar{a})$ for all $\bar{a} \in \mathrm{Pos}(\mathfrak{g}_n)$. If each $\underaccent{\tilde}{\tau}(n)$ is a natural number then this defines (in V) a continuous function F from $(\omega^\omega)^{\mathrm{dom}(q)}$ into ω^ω such that q forces that $F(\bar{\underaccent{\tilde}{\eta}} \restriction \mathrm{dom}(q)) = \bar{\underaccent{\tilde}{\tau}}$.

[11] We always mean the "upwards cofinality", i.e., the minimal size of an upwards cofinal subset. $A \subset I$ is upwards cofinal if for every $i \in I$ there is an $a \in A$ such that $a \geq i$.

[12] P even is non-elementary-proper (nep), i.e., there are generic conditions for all (non-transitive, non-elementary, but ord-transitive) countable ZFC models; cf. [11] or [8].

we can (in V) fuse this sequence into a $p_\omega \in P$. Note that $\mathrm{dom}(p_\omega) \subseteq N \cap I$. If G is P-generic over V and $p_\omega \in G$, then $p_n \in G$ and $\{p_n^{[\bar a]} : \bar a \in \mathrm{Pos}(\mathfrak{g}_n)\}$ is predense below p_n, so some $p_n^{[\bar a]} \in G$, and $p_n^{[\bar a]} \in D_n \cap N$.

(4) is a special case of (5): Set $I_1 := I_{<i}$ and $I_2 := \{i\}$. So $\underset{\sim}{\eta}_i$ is $V_{<i}$-generic in $V_{\leq i}$ and therefore in $V_{<\infty}$ as well.

(5) Set $P := \text{nwf-lim}_I(Q_i)$, $P_1 := \text{nwf-lim}_{I_1}(Q_i)$, and $\underset{\sim}{P}_2$ (the P_1-name of) $\text{nwf-lim}_{I_2}(Q_i)$.

There is a natural map $\phi : p \mapsto (p_1, \underset{\sim}{p}_2)$ from P to $P_1 * \underset{\sim}{P}_2$: $p_1 := p \restriction I_1$, and $\underset{\sim}{p}_2$ is defined by $\mathrm{dom}(\underset{\sim}{p}_2) := \mathrm{dom}(p) \setminus I_1$ and $B_i^{\underset{\sim}{p}_2}(\bar\eta) := B_i^p((\underset{\sim}{\eta}_i)_{(i\in I_1)}{}^\frown\bar\eta)$.

It is clear that ϕ preserves $\leq$. We claim that it is dense and preserves $\perp$. Assume $\phi(p) = (p_1, \underset{\sim}{p}_2)$, $\phi(q) = (q_1, \underset{\sim}{q}_2)$, and $(r_1, \underset{\sim}{r}_2) \leq (p_1, \underset{\sim}{p}_2), (q_1, \underset{\sim}{q}_2)$. We have to find a $r' \leq_P p, q$ such that $\phi(r') \leq (r_1, \underset{\sim}{r}_2)$.

r_1 forces that $\underset{\sim}{p}_2$, $\underset{\sim}{q}_2$ and $\underset{\sim}{r}_2$ correspond to approximating sequences $(\underset{\sim}{\mathfrak{g}}_n^p)$, $(\underset{\sim}{\mathfrak{g}}_n^q)$ and $(\underset{\sim}{\mathfrak{g}}_n^r)$. As in (1) we can find an $r_1' \leq r_1$ with an approximating sequence $(\mathfrak{h}_n)$ such that $\mathfrak{h}_n$ decides $\underset{\sim}{\mathfrak{g}}_n^i$ (for $i \in \{p, q, r\}$) in a way such that $\underset{\sim}{\mathfrak{g}}_n^r$ is stronger than both $\underset{\sim}{\mathfrak{g}}_n^p$ and $\underset{\sim}{\mathfrak{g}}_n^q$. Then we can concatenate $(\mathfrak{h}_n)$ with the $(\underset{\sim}{\mathfrak{g}}_n^r)$ to an approximating sequence to some $r' \in P$. Then $r' \leq p, q$ and $\phi(r') \leq (r_1, \underset{\sim}{r}_2)$.

(6) By induction on ϵ. The successor step follows from (5). Let $\mathrm{cf}(\epsilon) > \omega$. Then the nwf-limit as well as the cs-limit are just the unions of the smaller limits, and therefore equal by induction. If $\mathrm{cf}(\epsilon) = \omega$, then the nwf-limit as well as the cs-limit are the full inverse limits of the iteration system, and therefore again equal by induction.

(7) follows from (6). $\square$

We will also use the following fact:

Lemma 2.25. *Assume that $\underset{\sim}{S}$ is a $P_{<i}$-name for an element of Q_i, that $q \restriction I_{<i}$ reads $\underset{\sim}{S}$ continuously and that $q \Vdash \underset{\sim}{\eta}_i \in \underset{\sim}{S}$. Then $q \restriction I_{<i}$ forces that $B_i^q(\underset{\sim}{\bar\eta}) \leq_{Q_i} \underset{\sim}{S}$.*

Proof. Assume otherwise. Then there is an approximation $\mathfrak{g}$ of $p := q \restriction I_{<i}$, an $\bar a \in \mathrm{Pos}(\mathfrak{g})$ and a $t \in T^i_{\max}$ such that $p^{[\bar a]}$ forces t to be in $B_i^q(\underset{\sim}{\bar\eta})$ but not in $\underset{\sim}{S}$. Let $\bar a^+$ be $\bar a^\frown t$. Then $\bar a^+$ is a possible value of some approximation of q, and $q^{[\bar a^+]}$ forces that $\underset{\sim}{\eta}_i \notin \underset{\sim}{S}$, a contradiction. $\square$

Remark. The iteration technique defined here also works for larger classes of forcings, e.g., for the tree forcings $\mathbb{Q}_0^{\text{tree}}$ of [9] mentioned already. If we assume additional properties such as bigness and halving, we could also use lim-inf forcings. It is also possible to extend the construction to non-total orders, or to allow $T^i_{\max}, \mu^i$ to be $P_{<i}$-names.

3. The ideal $\mathbb{I}^c$

To every tree forcing such as Q defined in Section 1 (and many other tree forcings as well) there is an associated ideal $\mathbb{I}$ and a notion of measurability. We will also use $\mathbb{I}^c$, the $< 2^{\aleph_0}$-closure of $\mathbb{I}$, and the associated notion of weak measurability. The application in this paper of a nw-iteration will be: for certain Q we can force weak measurability for all definable sets.

Definition 3.1.
- The ideal $\mathbb{I}$ on the reals is defined by: $X \in \mathbb{I}$ if for all $S \in Q$ there is a $T \leq S$ such that $X \cap \lim(T) = \emptyset$.
- $\mathbb{I}^c$ is the $< 2^{\aleph_0}$-closure of $\mathbb{I}$.
- X has weak measure 1 if $\mathbb{R} \setminus X \in \mathbb{I}^c$. X has strong measure 1, if $\mathbb{R} \setminus X \in \mathbb{I}$.

Notes.
- Of course these notions depend on the forcing Q, so it might be more exact to use notation such as $\mathbb{I}_Q$ or $\mathbb{I}_{(T_{\max},\mu)}$ etc. In this paper this is not necessary, since we will always use a fixed Q.
- We use the phrase "measure 1" although the ideals $\mathbb{I}$ and $\mathbb{I}^c$ are not related to a measure (they are not even ccc).
- Of course, if CH holds, then $\mathbb{I}^c = \mathbb{I}$.
- $\mathbb{I}$ is always nontrivial (i.e., $\lim(T_{\max}) \notin \mathbb{I}$), but this is not clear for $\mathbb{I}^c$.

$F : Q \to Q$ is a witness for $X \in \mathbb{I}$ if $F(S) \leq S$ and $X \cap \lim(F(S)) = \emptyset$ for all $S \in Q$.

So every $X \in \mathbb{I}$ is contained in a set $\bigcap\{\omega^\omega \setminus \lim(F(S)) : S \in Q\}$.[13]

Lemma 3.2. *$\mathbb{I}$ is a non-trivial σ-ideal.*

Proof. This follows from fusion: Assume $X_i \in \mathbb{I}$ ($i \in \omega$) and $S = S_0 \in Q$. Pick any front $F_0 \in S_0$, so $S_0 = \bigcup_{t \in F_0} S_0^{[t]}$. For each $t \in F_0$ pick an $S_{1,t} \leq S_0^{[t]}$ such that $\lim(S_{1,t}) \cap X_1 = \emptyset$. Set $S_1 := \bigcup_{t \in F_0} S_{1,t}$. So $S_1 \in Q$, and F_0 is a front in S_1. Pick a 1-dense front F_1 in S_1 (purely) stronger than F_0. Iterate the construction. Fusion produces a $T < S$ such that $\lim(T) \cap X_i = \emptyset$ for all $i \in \mathbb{N}$. □

For example, if Q is Sacks forcing, then $\mathbb{I}$ is called Marczewski ideal. $X \in \mathbb{I}$ iff in every perfect set A there is a perfect subset A' of A such that $A' \cap X = \emptyset$. So if X is Borel (or if X has the perfect set property, e.g., X is $\underset{\sim}{\Sigma}^1_1$), then $X \in \mathbb{I}$ iff X is countable. $\mathbb{I}$ is not a ccc ideal: For $A \subseteq \omega$, set

$$X_A := \{f \in 2^\omega : (\forall n \notin A)\, f(n) = 0\}.$$

Clearly $X_A \cap X_B = X_{A \cap B}$, and $|X_A| = 2^{|A|}$. So if $\{A_i : i \in 2^{\aleph_0}\}$ is an almost disjoint family, then $\{X_{A_i}\}$ is a family of closed sets not in $\mathbb{I}$ such that $X_{A_i} \cap X_{A_j}$ is finite for $i \neq j$.

For a Borel ccc ideal I, "$X \subseteq \mathbb{R}$ is measurable" can be defined by "there is a Borel set A such that $A \Delta X \in I$". (Usually the basis of the ideal is simpler, e.g., one can use open sets instead of Borel sets for meager, or G_δ sets for Lebesgue-null.) Equivalently, X is measurable iff for every I-positive Borel set A there is an I-positive Borel set $B \subseteq A$ such that either $B \cap X \in I$ or $B \setminus X \in I$. For non-ccc ideals that do not live on the Borel sets, this second notion is usually the one used to define measurability:

Definition 3.3.
- $X \subseteq \mathbb{R}$ is measurable if for every $T \in Q$ there is an $S \leq_Q T$ such that either $\lim(S) \cap X \in \mathbb{I}$ or $\lim(S) \setminus X \in \mathbb{I}$.
- $X \subseteq \mathbb{R}$ is weakly measurable if for every $T \in Q$ there is an $S \leq_Q T$ such that either $\lim(S) \cap X \in \mathbb{I}^c$ or $\lim(S) \setminus X \in \mathbb{I}^c$.

Since $\mathbb{I}^c$ is the bigger ideal, measurability implies weak measurability.

In the rest of the paper, we will construct a specific Q and a nwf-iteration P and show that P forces all definable sets to be weakly measurable:

Theorem 3.4. *Assume CH and that Q satisfies the Ramsey property 5.4. Then there is a proper, $\aleph_2$-cc, ω^ω-bounding p.o. P forcing that every set of reals which is (first-order) definable using a parameter in $L(\mathbb{R})$ is weakly measurable.*

We will see in Lemma 5.5 that there is such a Q, and the Theorem will be proven by 4.8, 4.10 and 5.8.

Remark 3.5. It is natural to ask whether in our forcing extension every definable set is measurable (and not just weakly measurable, as stated in the theorem). This seems unlikely, but it is not clear how to prove it. It is not even clear how to prove that in our forcing model $\mathbb{I} \neq \mathbb{I}^c$ (i.e., that $\mathrm{add}(\mathbb{I}) < 2^{\aleph_0}$). (Of course, $\mathbb{I} = \mathbb{I}^c$ would trivially imply that measurable sets and weakly measurable sets are the same, so in particular that all definable sets are measurable.)

Let us first list some facts about (weak) measurability:

Lemma 3.6. *Every Borel set is measurable. The family of measurable sets is closed under complements and countable unions; the same holds for weakly measurable sets.*

Proof. Closure under complement is trivial.

Every closed set is measurable: Let $X = \lim(T')$ be closed and $T \in Q$. If there is a $t \in T \setminus T'$ then $S := T^{[t]}$ satisfies $\lim(S) \cap X = \emptyset$. Otherwise $T \subseteq T'$ and $S := T$ satisfies $\lim(S) \setminus X = \emptyset$.

[13] Note that this is not a countable intersection.

Assume that $(X_i)_{i\in\omega}$ is a sequence of weakly measurable sets and that $T \in Q$. If for some $i \in \omega$ there is an $S \leq T$ such that $\lim(S) \setminus X_i \in \mathbb{I}^c$ then the same obviously holds for $\bigcup_{i\in\omega} X_i$. So assume that for all $i \in \omega$ and $T' \leq T$ there is an $S \leq T'$ such that $\lim(S) \cap X_i \in \mathbb{I}^c$. Now repeat the proof of 3.2.

The same proof also shows that the measurable sets are closed under countable unions. $\square$

$\mathbb{I}^c$ could be trivial (i.e., $\mathrm{cov}(\mathbb{I})$ could be less than $2^{\aleph_0}$). If $\mathbb{I}^c$ is "everywhere nontrivial", then $\mathbb{I}^c$ and $\mathbb{I}$ are the same on measurable (in particular, Borel) sets:

Lemma 3.7. *Assume that* $\lim(S) \notin \mathbb{I}^c$ *for all* $S \in Q$*. Then* $\mathbb{I}^c$ *and* $\mathbb{I}$ *agree on measurable sets. I.e., if* X *is measurable and* $X \in \mathbb{I}^c$*, then* $X \in \mathbb{I}$.

Proof. For every $T \in Q$ there is an $S \leq_Q T$ such that $\lim(S)\cap X \in \mathbb{I}$: Otherwise $\lim(S)\setminus X \in \mathbb{I} \subseteq \mathbb{I}^c$, a contradiction to $X \in \mathbb{I}^c$ and $\lim(S) \notin \mathbb{I}^c$. So by the definition of $\mathbb{I}$ there is a $S' \leq_Q S \leq_Q T$ such that $\lim(S') \cap X = \emptyset$. So $X \in \mathbb{I}$. $\square$

Since any Borel set B is measurable, $B \in \mathbb{I}$ iff $(\forall S \in Q) \lim(S) \nsubseteq B$, so we get:

Fact 3.8. *For a Borel code* B*, the statement "*$B \in \mathbb{I}$*" is* $\undertilde{\Pi}^1_2$ *and therefore invariant under forcing.*

On the other hand, since $\mathbb{I}$ is not a Borel ideal (i.e., not every $X \in \mathbb{I}$ is contained in a Borel set $B \in \mathbb{I}$), there is no reason why $X \in \mathbb{I}$ should be upwards absolute between universes.

For later reference, we will reformulate the definition of $\mathbb{I}$: If $S \in Q$, $X \subseteq Q$, $T \in X$ and $T' \leq_Q S, T$, then $\lim(T') \cap (2^\omega \setminus \bigcup_{R\in X} \lim(R)) \subseteq \lim(T') \setminus \lim(T) = \emptyset$. So we get:

Lemma 3.9. *If* $X \subseteq Q$ *is predense then* $\bigcup_{T\in X} \lim(T)$ *is of strong measure 1.*

4. An order with many automorphisms

In this section we assume CH. We will construct an order I and define P to be the nwf-limit of Q along I. I is ω_2-like,[14] has a cofinal sequence j_α ($\alpha \in \omega_2$) and many automorphisms. We show that these properties imply that P forces the following:

- $2^{\aleph_0} = \aleph_2$,
- $\mathbb{I}^c$ is nontrivial (and moreover $\lim(S) \notin \mathbb{I}^c$ for all $S \in Q$),
- for every definable set X, "locally" either all or no η_{j_δ} are in X and
- $\{\eta_{j_\delta} : \delta \in \omega_2\}$ is of weak measure 1 in $\{\eta_i : i \in I\}$.

In the next section it will be shown that the set $\{\eta_i : i \in I\}$ is of weak measure 1, which will finish the proof Theorem 3.4

First note that for any I with uncountable cofinality, P makes the old reals null:

Lemma 4.1. *If* I *has cofinality* $\geq \aleph_1$ *and* $i \in I$ *then* $\Vdash_P V_{<i} \cap \lim(T_{max}) \in \mathbb{I}$.

Proof. Let G_P be P-generic over V. If $T \in V[G_P]$ then $T \in V_{<j}$ for some $i < j < \infty$ because of 2.24(2). So in $V_{\leq j}$ there is an $S < T$ such that $\lim(S) \cap V_{<i} = \emptyset$ (in $V_{\leq j}$ and $V[G_P]$ as well, according to 1.8). $\square$

Lemma 4.2. *Assume that CH holds and that* I *is* ω_2*-like. Then*

(1) *P has the $\aleph_2$-cc (and therefore preserves all cofinalities).*
(2) *$P_{<i} \Vdash CH$ for each $i \in I$ and $P \Vdash 2^{\aleph_0} = \aleph_2$.*

Proof. (1) If $|I_{<i}| \leq 2^{\aleph_0}$ then $|P_{<i}| \leq 2^{\aleph_0}$: There are at most $|I_{<i}|^{\aleph_0} \leq 2^{\aleph_0}$ may countable subsets of $|I_{<i}|$. For each $p \in P_{<i}$ with a fixed domain and each $j \in \mathrm{dom}(p)$ there are $2^{\aleph_0}$ many possibilities for Dom^p_j and $2^{\aleph_0}$ many possibilities for the Borel definition B^p_j.

[14] I is ω_2-like if $|I_{<i}| < \aleph_2$ for all $i \in I$ and $|I| = \aleph_2$.

If CH holds, then the usual delta system lemma applies: If $A \subseteq P$ is a maximal antichain of size $\aleph_2$ then without loss of generality the domains of $p \in A$ form a delta system (i.e., there is a countable $x \subseteq I$ such that $\mathrm{dom}(p_1) \cap \mathrm{dom}(p_2) = x$ for all $p_1 \neq p_2 \in A$). Since I is ω_2-like, x cannot be cofinal. Let i be an upper bound of x. Without loss of generality $p_1 \restriction I_{<i} = p_2 \restriction I_{<i}$ for $p_1 \neq p_2 \in A$ (since there are only $\aleph_1$ many elements of $P_{<i}$). But then $p_1 \parallel p_2$ by Fact 2.21.

Proper and $\aleph_2$-cc imply preservation of all cofinalities and cardinalities.

(2) Let G be P-generic over V. Then the reals in $V[G]$ are the union of the reals in $V_{<i}$. Every real in $V_{<i}$ is read continuously from a condition $p \in G_{<i}$. There are only $|P_{<i}| = (2^{\aleph_0})^V = \aleph_1$ many conditions, and given a condition there are only $(2^{\aleph_0})^V = \aleph_1$ many possibilities to continuously read a real from the condition. So there are at most $\aleph_1$ many reals in $V_{<i}$. And $\eta_i \notin V_{<i}$, so in particular $\eta_{i_1} \neq \eta_{i_2}$ for $i_1 \neq i_2$. □

The following is well known:

Lemma 4.3. *If CH holds, then there is an $\aleph_1$ saturated*[15] *linear order $\tilde{I}$ of size $\aleph_1$, and all such orders are isomorphic.*

Proof. Induction of length ω_1: Assume at stage α we have a linear order L_α of size $\omega_1 = 2^{\aleph_0}$. List all the (ω_1 many) countable gaps and add points to fill these gaps. At limits, take the union. Then at stage ω_1 we get a saturated order.

Uniqueness is proven by the standard back and forth argument. □

Definition 4.4. Let $\mathfrak{S}$ be the set of $0 < \alpha < \omega_2$ such that $\mathrm{cf}(\alpha) \in \{1, \omega_1\}$. Note that $\mathfrak{S} \subseteq \omega_2$ is stationary.

We will now define the order I along which we iterate. (We do this assuming CH.)

Given $\tilde{I}$ as above, let I be the following order:

$$\underbrace{\tilde{I}}_{0} + \underbrace{\{j_1\} + \tilde{I}}_{1} + \cdots + \underbrace{\tilde{I}}_{\omega} + \underbrace{\{j_{\omega+1}\} + \tilde{I}}_{\omega+1} + \cdots + \underbrace{\{j_{\omega_1}\} + \tilde{I}}_{\omega_1} + \cdots$$

So at stages $\alpha \in \mathfrak{S}$, we add an order of the type $\{c\} + \tilde{I}$, in other stages we add just $\tilde{I}$.

Facts 4.5.
- *I is ω_2-like,*
- *$(j_\alpha)_{\alpha \in \mathfrak{S}}$ is an increasing (and therefore cofinal) continuous sequence in I, and*
- *every j_α has cofinality $\aleph_1$ in I.*

Continuous means that $j_\delta = \sup(j_\alpha : \alpha \in \mathfrak{S}, \alpha < \delta)$ whenever $\delta = \sup(\mathfrak{S} \cap \delta) \in \mathfrak{S}$ (which is equivalent to $\mathrm{cf}(\delta) = \omega_1$).

Note. We could just as well define j_α for α with cofinality ω_1 only, or for all $\alpha \in \omega_2$ (and require continuity for points of cofinality ω_1 only). All these versions are equivalent by simple relabeling, cf. the beginning of the proof of 4.8.

Definition 4.6. We set $Q_i = Q$ for all $i \in I$ and let P be the nwf-iteration of Q_i along I.

We will use the notation I_α, P_α, V_α and η_α for $I_{<j_\alpha}$, $P_{<j_\alpha}$, $V_{<j_\alpha}$ and η_{j_α}. We set G_{ω_2} to be (the name for) the P-generic (in previous notation, $G_{<\infty}$) and V_{ω_2} the generic extension $V[G_{\omega_2}]$ (in previous notation, $V_{<\infty}$).

[15] A linear order $\tilde{I}$ is $\aleph_1$ saturated if "there are no countable gaps", more exactly:
- I has neither a smallest or a largest element, i.e., no $(-\infty, 1)$ and no $(1, \infty)$ gaps.
- I does not have a cofinal sequence of order type ω nor a coinitial one of order type ω^*, i.e., no (ω, ∞) and no $(-\infty, \omega^*)$ gaps.
- If $A \subset I$ has order type ω and $c > a$ for all $a \in A$ ($c > A$ in short) then there is a $b < c$ such that $b > A$. I.e., there are no $(\omega, 1)$ gaps.
- Analogously for B of order type ω^* and $c < B$. I.e., no $(1, \omega^*)$ gaps.
- If A has order type ω and B has order type ω^* and $A < B$, then there is an $x \in I$ such that $A < x < B$. I.e., there are no (ω, ω^*) gaps.

Lemma 4.7. *(CH) Let $S_0 \subseteq \mathfrak{S}$ be stationary. P forces the following:*

(1) *$\{\eta_\delta : \delta \in S\} \notin \mathbb{I}^{\mathrm{c}}$ for every stationary $S \subseteq \mathfrak{S}$, and*
(2) *$\{\eta_\delta : \delta \in S_0\} \cap \lim(T_0) \notin \mathbb{I}^{\mathrm{c}}$ for every $T_0 \in Q$.*

This lemma implies that in V_{ω_2} the assumption of Lemma 3.7 is satisfied (i.e., that $\mathbb{I}^{\mathrm{c}}$ is "everywhere nontrivial"). This lemma holds for all I satisfying 4.5.

Proof. (1) Assume otherwise, i.e., there are P-names $\underset{\sim}{F}_\zeta$ $(\zeta \in \omega_1)$ for functions from Q to Q and $\underset{\sim}{S}$ for a stationary set such that $p_0 \in P$ forces

$$\underset{\sim}{F}_\zeta(T) \leq T \text{ and } (\forall \delta \in \underset{\sim}{S})\,(\exists \zeta \in \omega_1)\,(\forall T \in Q)\,\eta_\delta \notin \lim(\underset{\sim}{F}_\zeta(T)).$$

P forces that for each $\alpha \in \mathfrak{S}$ there is a $\beta \in \mathfrak{S}$ such that $\underset{\sim}{F}_\zeta(T) \in Q^{V_\beta}$ for all $T \in Q^{V_\alpha}$ and $\zeta \in \omega_1$. We need something slightly stronger: For every name $\underset{\sim}{T}$ for an element of Q^{V_α} and $\zeta \in \omega_1$ there is a maximal antichain $A \subset P$ such that for every $q \in A$ there is a P-name $\underset{\sim}{T}'_q$ such that q forces $\underset{\sim}{F}_\zeta(\underset{\sim}{T}) = \underset{\sim}{T}'_q$ and q continuously reads $\underset{\sim}{T}'_q$. So if $q \in G_{\omega_2}$ and β is bigger than $\mathrm{dom}(q)$,[16] then V_β not only contains $T'_q = F_\zeta(T)$, but also knows that T'_q will be $F_\zeta(T)$ in V_{ω_2}.

Define $f^-(\alpha)$ to be the smallest β which is bigger than $\mathrm{dom}(q)$ for every $q \in A$, where A is an antichain for some $\underset{\sim}{T}$ and $\zeta \in \omega_1$ as above. P is $\aleph_2$-cc, every $q \in A$ has countable domain, and there are only $\aleph_1$ many reals in V_α. So $f^-(\alpha) < \omega_2$, and we can define $f(\alpha)$ to be the smallest $\beta \in \mathfrak{S}$ that is larger or equal to $\max(\alpha, f^-(\alpha))$.

If $\mathrm{cf}(\alpha) = \omega_1$, then $f(\alpha)$ is the supremum of $\{f(\gamma) : \gamma \in \mathfrak{S} \cap \alpha\}$, since the reals in V_α are the union of the reals in V_γ. So f is continuous.

Then P forces the following: Since $\underset{\sim}{S}$ is stationary, there is a $\beta \in \underset{\sim}{S}$ such that $f(\beta) = \beta$. V_β can calculate every F_ζ, and $\underset{\sim}{F}''_\zeta Q$ is dense in Q. Since $\underset{\sim}{\eta}_\beta$ is a Q-generic real over V_β, there is (for every $\zeta \in \omega_1$) a $T \in Q^{V_\beta}$ such that $\underset{\sim}{\eta}_\beta \in \lim(\underset{\sim}{F}_\zeta(T))$, a contradiction.

(2): We can assume that $T_0 \in V$. Again, choose names $\underset{\sim}{F}_\zeta$ as above, and assume that $p_0 \in P$ forces that

$$\underset{\sim}{F}_\zeta(T) \leq T \text{ and } (\forall \delta \in S_0)\,(\exists \zeta \in \omega_1)\,(\forall T \in Q)\,\eta_\delta \notin \lim(\underset{\sim}{F}_\zeta(T)) \cap \lim(T_0).$$

Define f as above, so there is a $\beta > \mathrm{dom}(p)$ such that $\beta \in S_0$ and $f(\beta) = \beta$. So the same argument proves that p_0 forces that $\underset{\sim}{\eta}_\beta \notin \lim(T_0)$, a contradiction. $\square$

We also get the following:

Lemma 4.8. *(CH) For every $C \subseteq \omega_2$ club, P forces the following:*

$$\{\underset{\sim}{\eta}_i : i \in I\} \setminus \{\underset{\sim}{\eta}_\alpha : \alpha \in \mathfrak{S} \cap C\} \in \mathbb{I}^{\mathrm{c}}.$$

Again, this lemma applies to all I satisfying 4.5.

Proof. We can assume that $C = \omega_2$, since we can just relabel the sequence $\{j_\alpha : \alpha \in \mathfrak{S} \cap C\}$: Set $j'_\alpha := j_\beta$, where β is the α-th element of $C \cap \mathfrak{S}$. Then $(j'_\alpha)_{\alpha \in \mathfrak{S}}$ satisfies 4.5 as well.

Recall Definition 1.9 of $Q^f_{A^\psi_r}$ and D^{spl}_f (for $f : \omega \to \omega$ increasing and $r \in 2^\omega$). Enumerate all increasing $f : \omega \to \omega$ in V as f_ζ $(\zeta \in \omega_1)$. (CH holds in V.)

Claim: In V, we can find P_α-names $\underset{\sim}{T}^\zeta_\alpha$ $(\zeta \in \omega_1,\ \alpha < \omega_2$ successor$)$ for elements of Q such that the following is forced by P_{ω_2}:

(1) The set $\{\underset{\sim}{T}^\zeta_\alpha : \alpha < \omega_2 \text{ successor}\} \subseteq Q$ is dense for all $\zeta \in \omega_1$.
(2) $\underset{\sim}{T}^\zeta_\alpha \in D^{\mathrm{spl}}_{f_\zeta}$ (in V_α or equivalently in V_{ω_2}).[17]
(3) If $\beta < \alpha$ is a successor, then $\underset{\sim}{T}^\zeta_\alpha$ has no branch in V_β, and for all $i < j_\alpha$ there is a ζ_0 such that $\underset{\sim}{T}^\zeta_\alpha$ has no branch in $V_{<i}$ for all $\zeta \geq \zeta_0$.

[16]More formally: if $j_\beta > i$ for all $i \in \mathrm{dom}(q)$.

[17]Recall 1.9 and 1.10.

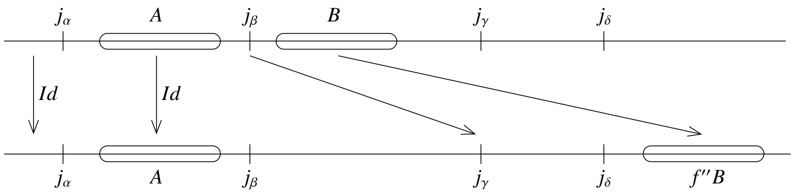

FIGURE 4. An automorphism f.

Proof of the claim: Pick for all $\alpha+1$ a function $\phi_{\alpha+1} : \omega_1 \to I_{<j_{\alpha+1}} \setminus I_{<j_\alpha}$ which is increasing and cofinal. Also pick an enumeration $(\underset{\sim}{S}_{\alpha+1})_{\alpha\in\omega_2}$ such that $\underset{\sim}{S}_\alpha$ is an P_α-name and P forces that $Q = \{\underset{\sim}{S}_{\alpha+1} : \alpha \in \omega_2\}$. (This is possible since P forces that $Q^{V_{\omega_2}} = \bigcup Q^{V_\alpha}$, cf. 2.24(2).)

To find $\underset{\sim}{T}^\zeta_\alpha$ (α successor) note that P_α forces that we can perform the following construction in V_α: First pick an $S' \leq \underset{\sim}{S}_\alpha$ such that $S' \in D^{\mathrm{spl}}_{f_\zeta}$ (cf. 1.10(7)). $\mathrm{cf}(j_\alpha) = \aleph_1$, so $S' \in V_{<i}$ for some $i < j_\alpha$. Pick some i' bigger than $\max(i, \phi_\alpha(\zeta))$ and smaller than j_α. There is a real $r \in V_\alpha \setminus V_{<i'}$ (e.g., $\eta_{i'}$). Therefore there is a $T^\zeta_\alpha < S'$ such that $\lim(T^\zeta_\alpha) \cap V_{<i'} = \emptyset$ (in V_α and V_{ω_2} as well, cf. 1.8). Let $\underset{\sim}{T}^\zeta_\alpha$ be a P_α-name for T^ζ_α.

The $\underset{\sim}{T}^\zeta_\alpha$ constructed this way satisfy the claim: (1): $\underset{\sim}{T}^\zeta_\alpha \leq \underset{\sim}{S}_\alpha$, (2): $D^{\mathrm{spl}}_{f_\zeta}$ is open dense and absolute, (3): pick ζ_0 such that $\phi_\alpha(\zeta_0) > i$. This ends the proof of the claim.

From now on assume G is P-generic over V. We work in V_{ω_2} and set $T^\zeta_\alpha := \underset{\sim}{T}^\zeta_\alpha[G]$. So if $i \in I$ then the sequence $(T^\zeta_{\alpha+1})_{j_{\alpha+1}<i,\zeta\in\omega_1}$ is in $V_{<i}$.

For all $\zeta \in \omega_1$, $X_\zeta := \bigcup_{\alpha+1<\omega_2} \lim(T^\zeta_{\alpha+1})$ is of strong measure 1 (cf. 3.9). So the set $Y := \bigcap_{\zeta\in\omega_1} X_\zeta$ is of weak measure 1. It is enough to show that

$$(\{\eta_i : i \in I\} \setminus \{\eta_\alpha : \alpha \in \mathfrak{S}\}) \cap Y = \emptyset.$$

Assume towards a contradiction that some η_i is in Y and $\eta_i \neq \eta_\alpha$ for all $\alpha \in \mathfrak{S}$.

Let $\alpha \in \mathfrak{S}$ be minimal such that $\eta_i \in V_\alpha$ (i.e., $i < j_\alpha$). So α is a successor (but not necessarily a successor of a $\beta \in \mathfrak{S}$), and $i > j_\beta$ for all $\beta \in \mathfrak{S} \cap \alpha$. So according to (3) there is a ζ_0 such that $\eta_i \notin \lim(T^\zeta_{\gamma+1})$ for all $\zeta > \zeta_0$ and all $\gamma + 1 \geq \alpha$.

So we know the following: $\eta_i \in Y$, i.e.,

$$\eta_i \in \bigcup_{\gamma+1<\omega_2} \lim(T^\zeta_{\gamma+1}) \quad \text{for all } \zeta \in \omega_1.$$

But

$$\eta_i \notin \bigcup_{\alpha\leq\gamma+1<\omega_2} \lim(T^\zeta_{\gamma+1}) \quad \text{for all } \zeta \geq \zeta_0.$$

Therefore

$$\eta_i \in \bigcup_{\gamma+1<\alpha} \lim(T^\zeta_{\gamma+1}) \quad \text{for all } \zeta \geq \zeta_0.$$

Recall that $V_{<i}$ sees the sequence $(T^\zeta_{\gamma+1})_{\gamma+1<\alpha,\zeta\in\omega_1}$. So in $V_{<i}$, some $T \in Q$ forces that for all $\zeta > \zeta_0$ there is a successor $\underset{\sim}{\beta}(\zeta) < \alpha$ such that $\underset{\sim}{\eta}_i \in \lim(T^\zeta_{\underset{\sim}{\beta}(\zeta)})$. In $V_{<i}$, T has full splitting for some $f_\zeta \in V$, $\zeta > \zeta_0$ (see 1.10(5), 1.11 and 2.24(1)).

Let r be a real in $V_{<i} \setminus \bigcup_{\gamma+1<\alpha} V_{\gamma+1}$. Pick in $V_{<i}$ a $T' \leq T$ such that $T' \in Q^{f_\zeta}_{A^\psi_r}$ (cf. 1.10(6)) and T' decides $\underset{\sim}{\beta}(\zeta)$. Then T' forces that $\underset{\sim}{\eta}_i \in \lim(T' \cap T^\zeta_{\beta(\zeta)})$, a contradiction to $T' \perp T^\zeta_{\beta(\zeta)} \in V_{<i}$ (because of (2), either $T^\zeta_{\beta(\zeta)}$ is in $Q^{f_\zeta}_{A^\psi_s}$ for some old real s, or incompatible to all $Q^{f_\zeta}_{A^\psi_s}$). □

We call f an automorphism if it is a $<$-preserving bijection from I to I.

If $f : I \to I$ is an automorphism, then f defines an automorphism of P in a natural way as well (provided of course that $f(i) = j$ implies $Q_i = Q_j$, but in our case all the Q_i are the same). Also, f defines a map on all P-names, and we have: $p \Vdash \varphi(\underset{\sim}{\tau})$ iff $f(p) \Vdash \varphi(f\underset{\sim}{\tau})$.

If $\Vdash_P \underset{\sim}{x} \in V_{<i}$, then there is a $V_{<i}$-name $\underset{\sim}{\tau}$ such that $\Vdash_P \underset{\sim}{x} = \underset{\sim}{\tau}$. If $f \restriction I_{<i}$ is the identity, then $f(\underset{\sim}{\tau}) = \underset{\sim}{\tau}$. So in this case $p \Vdash \phi(\underset{\sim}{\tau})$ iff $f(p) \Vdash \phi(\underset{\sim}{\tau})$. Also, if $f \restriction \mathrm{dom}(p) \cap I_{<i}$ is the identity then $B_i^p(\bar{\eta}) = B_{f(i)}^{f(p)}(\bar{\eta})$.

Lemma 4.9. *The following holds for I (see Figure 4): If $\alpha < \beta < \gamma < \delta$ are in $\mathfrak{S}$, and if $A \subseteq I_\beta$ and $B \subseteq I \setminus I_\beta$ are countable, then there is an automorphism f of I such that $f \restriction (I_\alpha \cup A)$ is the identity, $f(j_\beta) = j_\gamma$ and $f''B > j_\delta$.*

Proof. For every $i < j \in I$, $I_{<i}$ and $\{k : i < k < j\}$ are isomorphic and also isomorphic to $\tilde{I}$ (since they are all $\aleph_1$ saturated linear orders of size $\aleph_1$). If $A \subset I$ is countable, then there are $i < A < j$, and for all such i, j the sets $\{k : i < k < A\}$ and $\{k : A < k < j\}$ are again isomorphic to $\tilde{I}$. Also, $I_{>i}$ is isomorphic to I (since $\omega_2 \setminus \alpha$ is isomorphic to ω_2).

So assume $\alpha < \beta < \gamma \in \mathfrak{S}$, $A < i < j_\beta$ countable, $i > j_\alpha$. Then $I_{<j_\beta} \setminus I_{\leq i} \cong I_{<j_\gamma} \setminus I_{\leq i} \cong \tilde{I}$. Also, if $B \subset I$ is countable, $\delta \in \mathfrak{S}$ and $B > j_\beta$, then there is an $j_\beta < i < B$, and $I_{<i} \cong I_{<j_\delta} \cong \tilde{I}$, $I \setminus I_{\leq i} \cong I \setminus I_{\leq j_\delta} \cong I$. Now combine these automorphisms. □

Lemma 4.10. *For $\beta \in \omega_2$ set $Y_\beta := \{\underset{\sim}{\eta}_\gamma : \gamma \in \mathfrak{S}, \gamma > \beta\}$. P forces the following: If X is a set of reals defined with a parameter $\underset{\sim}{x} \in \bigcup_{i \in I} V_{<i}$, and if $T \in Q$, then there is an $S \leq T$ and a $\beta \in \omega_2$ such that either* $\lim(S) \cap X \cap Y_\beta = \emptyset$ *or* $(\lim(S) \setminus X) \cap Y_\beta = \emptyset$.

This lemma holds for all I satisfying 4.5 and 4.9.

Note that every real in V_{ω_2} is in $\bigcup_{i \in I} V_{<i}$.

We will see in the next section that (using additional assumptions) Y_β is a weak measure 1 set. Then this lemma implies that X is weakly measurable, i.e., Theorem 3.4. Because of 4.8, it will be enough to show that the set $\{\eta_i : i \in I\}$ is of weak measure 1.

Proof. Assume $\underset{\sim}{X} = \{r : \varphi(r, \underset{\sim}{x})\}$ and fix some $\underset{\sim}{T}$. Some p_0 forces that $\underset{\sim}{x}$ and $\underset{\sim}{T}$ are in V_α, so without loss of generality $\underset{\sim}{x}, \underset{\sim}{T}$ are P_α names and $\mathrm{dom}(p_0) \subset I_\alpha$. Pick a $p_1 \leq p_0$, $p_1 \in P_\alpha$ such that p_1 continuously reads $\underset{\sim}{T}$. Fix some $\beta > \alpha$. Then $p_2 := p_1 \cup \{(j_\beta, \underset{\sim}{T})\}$ is an element of $P_{\leq j_\beta}$ (since $\underset{\sim}{T}$ is read continuously).

Let $p \leq p_2$ decide $\varphi(\underset{\sim}{\eta}_\beta, \underset{\sim}{x})$. Without loss of generality $p \Vdash \varphi(\underset{\sim}{\eta}_\beta, \underset{\sim}{x})$. $p \restriction I_\beta$ forces that $\underset{\sim}{S} := B_{j_\beta}^p(\underset{\sim}{\bar{\eta}}) \leq_Q \underset{\sim}{T}$ (since $p \leq p_2$).

Assume towards a contradiction that for some $q \leq p$, $\gamma \in \mathfrak{S}$ and $\gamma > \beta$

$$q \Vdash \underset{\sim}{\eta}_\gamma \in \lim(\underset{\sim}{S}) \ \&\ \neg\varphi(\underset{\sim}{\eta}_\gamma, \underset{\sim}{x}).$$

Note that $q \restriction I_\gamma$ reads $\underset{\sim}{S}$ continuously and forces that $B_{j_\gamma}^q(\underset{\sim}{\bar{\eta}}) \leq_Q \underset{\sim}{S}$ (cf. 2.25).

Set $A := \mathrm{dom}(p) \cap I_\beta$ and $B := \mathrm{dom}(p) \cap I_{>j_\beta}$. Let j_δ be bigger than $\mathrm{dom}(q)$, and let f be an automorphism of I such that $f \restriction (I_\alpha \cup A)$ is the identity, $f(j_\beta) = j_\gamma$ and $f''B > \mathrm{dom}(q)$ (cf. 4.9 or Figure 4).

$\mathrm{dom}(f(p)) \cap \mathrm{dom}(q) \subseteq A \cup \{j_\gamma\}$. $f(p) \restriction A = p \restriction A \geq q \restriction A$, and $q \restriction I_\gamma$ forces that

$$B_{j_\gamma}^{f(p)}(\underset{\sim}{\bar{\eta}}) = B_{j_\beta}^p(\underset{\sim}{\bar{\eta}}) = \underset{\sim}{S} \geq_Q B_{j_\gamma}^q(\underset{\sim}{\bar{\eta}}).$$

So $f(p)$ and q are compatible, a contradiction to $f(p) \Vdash \varphi(\underset{\sim}{\eta}_\gamma, \underset{\sim}{x})$. □

5. A very non-homogeneous tree

For the proof of Theorem 3.4 it remains to be shown that $\{\eta_i : i \in I\}$ is of weak measure 1. For this we will need a certain Ramsey property for Q.

Definition 5.1. A subtree T of $T_{\max}$ is called (n, r)-meager if $\mu_T(t) < r$ for all $t \in T$ with length at least n.

Lemma 5.2. *If T is meager for some (n, r), then* $\lim(T) \in \mathbb{I}$.

Proof. For any $S \in Q$ there is an $s \in S$ of length at least n such that $\mu_S(s) > r$. So there is an immediate successor t of s in S such that $t \notin T$. Then $\lim(S^{[t]}) \cap \lim(T) = \emptyset$. $\square$

Definition 5.3. Let M, N be natural numbers. $N \to M$ means: If

- $r_1, \dots, r_M \in T_{\max}$ such that $\text{length}(r_i) > N$,
- $t \in T_{\max}$ such that $r_i \perp t$ for $1 \le i \le M$,
- $A \subseteq \text{succ}(t)$ such that $\mu(A) > N$,
- $f_i : A \to T_{\max}^{[r_i]}$ for $1 \le i \le M$,

then there is a $B \subseteq A$ such that

- $\mu(B) > M$ and
- $\{s \in T_{\max} : (\exists i \le M)(\exists t \in B)\, s \le f_i(t)\}$ is $(N, 1/M)$-meager.

Definition 5.4. A lim-sup tree-forcing Q is strongly non-homogeneous if μ is sub-additive[18] and for all M there is an N such that $N \to M$.

There are many similar notions of bigness, see e.g., [9, 2.2].

Lemma 5.5. *There is a forcing Q that is strongly non-homogeneous.*

Proof. First note that it is enough to show that for each M there is an N such that $N \to^- M$, where $N \to^- M$ is defined as above but with just one r and f instead of M many. To see this, just set $K_0 := M^2$ and find K_i such that $K_{i+1} \to^- K_i$. Then $K_M \to M$. (Here we use that μ is sub-additive, since we need that the union of m many (n, x)-meager trees is $(n, x \cdot m)$-meager.)

We will construct $T_{\max}$ and μ by induction. We define $s \lhd t$ by: $\text{length}(s) < \text{length}(t)$ or $\text{length}(s) = \text{length}(t)$ and s is lexicographically smaller than t.

Fix some $t \in \omega^{<\omega}$. Assume that we already decided which $s \lhd t$ will be elements of $T_{\max}$ and that we already defined the set of successors of all these s as well as the measure of their subsets. Assume that we have decided to put t into $T_{\max}$. So we have to define $\text{succ}(t)$ and the measure on it.

Let m_t be the number of nodes $s \lhd t$ already defined, including the already defined successors of s for $s \lhd t$. Set $M_t := (2m_t)^{m_t}$. Then we define $\text{succ}(t)$ to be of size ${M_t}^{m_t}$.[19] For $A \subseteq \text{succ}(t)$ we set $\mu(A) := \log_{M_t}((|A|/M_t) + 1)$.

Then $0 \le \mu(A) < m_t$, $\mu(A) = 0$ iff $A = \emptyset$, and μ is strictly monotonous and sub-additive.[20] If $A, B \subseteq \text{succ}(t)$ and $|B| \ge |A|/M_t$, then $\mu(B) > \mu(A) - 1$. If $|B| \le m_t$ then $\mu(B) < 1/m_t$. If $\mu(\text{succ}(t)) > M$, then $m_t > M$.

Now fix an arbitrary $M \in \omega$. There is an N_0 such that $\mu(A) < 1/M$ for all s with $\text{length}(s) > N_0$ and all $A \subseteq \text{succ}(s)$ with $|A| < m_s$. (Just note that m_s strictly increases with $\text{length}(s)$.) Let N be larger than $M + 1$ and N_0.

So assume that $r \perp t \in T_{\max}$, $\text{length}(r) > N \ge N_0$, $A \subseteq \text{succ}(t)$, $\mu(A) > N \ge M + 1$ (in particular $m_t > M$), and $f : A \to T_{\max}^{[r]}$.

Set $X := \{s' \ge r : s' \lhd t, \text{length}(s') \ge N\}$. Enumerate X as $\{s_0, \dots, s_{l-1}\}$ (for some $l \ge 0$). Set $A_0 := A$. Assume that A_n is already defined, and define

$$S_n := \{s' \in T_{\max} : (\exists t' \in A_n)\, s' \le f(t')\}.$$

If $n > 0$ assume that $|\text{succ}_{S_n}(s_{n-1})| \le 1$ and that $|A_n| > |A_{n-1}|/(2m_t)$.

Then we define A_{n+1} as follows: Since $s_n \in X$, $|\text{succ}(s_n)| < m_t$. By a simple pigeon-hole argument, there is an $A_{n+1} \subseteq A_n$ such that $|A_{n+1}| > |A_n|/(2m_t)$ and $|\text{succ}_{S_{n+1}}(s_n)| \le 1$. So in the end we get a $B := A_l$ with cardinality at least $|A|/(2m_t)^{m_t} = |A|/M_t$, i.e., $\mu(B) > \mu(A) - 1 \ge M$. Also, $|\text{succ}_{S_l}(s')| \le 1$ for every $s' \in X$, so $\mu_{S_l}(s') \le 1/M$ (since $\text{length}(s')$ was sufficiently large).

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[18] $\mu(A \cup B) \le \mu(A) + \mu(B)$.

[19] We can e.g., set $\text{succ}(t) := \{t^\frown k : 0 \le k < M_t^{m_t}\}$.

[20] Since the function $g(x) := \log_{M_t}(x + 1)$ is concave and satisfies $g(0) = 0$.

We claim that B is as required. We have to show that S_l is $(N, 1/M)$-meager. Pick an $s' \in S_l$ of length $\geq N$. We already dealt with the case $s' \in X$. Otherwise $s' \rhd t$ (note that $s' \neq t$ since $s' \perp t$). In this case $|\operatorname{succ}_{S_l}(s')| \leq |\operatorname{succ}_{T_{\max}}(t)| \leq m_{s'}$. So $\mu(\operatorname{succ}_{S_l}(s')) \leq 1/M$, since length$(s') > N_0$. $\square$

Lemma 5.6. *If Q is strongly non-homogeneous, then P forces the following: If $r \in \lim(T_{max}) \setminus \{\underset{\sim}{\eta}_i : i \in I\}$ then there is a $T \in V$ such that $r \in \lim(T)$ and T is $(1,1)$-meager.*

If additionally the assumptions of Lemma 4.2 hold, then there are only $\aleph_1$ many $T \in V$, and $\aleph_1 < (2^{\aleph_0})^{V_{\omega_2}}$. This implies that the set $\{\underset{\sim}{\eta}_i : i \in I\}$ is of weak measure 1: If $r \in \lim(T_{\max}) \setminus \{\eta_i : i \in I\}$, then $r \in \bigcup_{T \in V \text{ meager}} \lim(T) \in \mathbb{I}^{c}$.

Proof. Fix a P-name $\underset{\sim}{r}$ for a real and a $p \in P$ such that $p \Vdash \underset{\sim}{r} \notin \{\underset{\sim}{\eta}_i : i \in I\}$. We will show that there is a $p_\omega \leq p$ and a $(1,1)$-meager tree T such that $p_\omega \Vdash \underset{\sim}{r} \in \lim(T)$.

We will by induction construct $p_n \in P$, approximations $\mathfrak{g}_n$, $k_n \in \omega$ and $i_n \in u_n = \operatorname{dom}(\mathfrak{g}_n)$ such that

(1) $p_{n+1} \leq_{\mathfrak{g}_n} p_n$, $\mathfrak{g}_{n+1}$ is purely stronger than $\mathfrak{g}_n$.
(2) $\mathfrak{g}_n$ is n-dense at i_n.
(3) the sequence $(i_n)_{n\in\omega}$ covers $\bigcup \operatorname{dom}(p_n)$ infinitely often.
(4) $k_n \to \max(n+1, |\operatorname{Pos}(\mathfrak{g}_n)|)$.
(5) If $n > 0$, then for each $\bar{a} \in \operatorname{Pos}(\mathfrak{g}_n)$, $p_n^{[\bar{a}]}$ forces a value to $\underset{\sim}{r} \restriction k_n$, and the tree $\{r_n^{\bar{a}} : \bar{a} \in \operatorname{Pos}(\mathfrak{g}_n)\} \subseteq T_{\max} \restriction k_n$ is $(k_{n-1}, 1)$-meager.

(1)–(3) allow us to fuse the $(p_n)_{n\in\omega}$ into a $p_\omega \leq p$ (cf. 2.15), and (5) implies that the tree of all initial segments of r compatible with p_ω is meager.

We start by picking any $i_0 \in \operatorname{dom}(p)$, some p-approximation $\mathfrak{g}_0$ that is 0-dense at i_0, a k_0 satisfying (4). So assume by induction we have found p_n, $\mathfrak{g}_n$ and k_n satisfying (1,2,4).

(a) Set $p := p_n$, $\mathfrak{g} := \mathfrak{g}_n$, $M := |\operatorname{Pos}(\mathfrak{g}_n)|$ and $N := k_n$. So we have $N \to M$.
(b) Choose the position $i_{n+1} \in \operatorname{dom}(p_n)$ according to some simple bookkeeping. This takes care of (3). Set $j := i_{n+1}$.
(c) Find a $p_1 \leq_{\mathfrak{g}} p$ and $m > N$ such that $p_1 \Vdash (\underset{\sim}{\eta}_j \restriction m \neq \underset{\sim}{r} \restriction m)$ and for all $\bar{a} \in \operatorname{Pos}(\mathfrak{g})$ the condition $p_1^{[\bar{a}]}$ determines $\underset{\sim}{\eta}_j \restriction m$ and $\underset{\sim}{r} \restriction m$.
    (How to do this? First apply pure decision 2.19(6) to get a $p' \leq_{\mathfrak{g}} p$ such that for all $\bar{a} \in \operatorname{Pos}(\mathfrak{g})$ there is an $m^{\bar{a}} > N$ and $\eta^* \neq r^*$ such that $p'^{[\bar{a}]} \Vdash (r^* = \underset{\sim}{r} \restriction m^{\bar{a}}, \eta^* = \underset{\sim}{\eta}_j \restriction m^{\bar{a}})$. Then we apply pure decision again to get $p_1 \leq_{\mathfrak{g}} p'$ determining $\underset{\sim}{r}$ and $\underset{\sim}{\eta}_j$ up to $\max\{m^{\bar{a}} : \bar{a} \in \operatorname{Pos}(\mathfrak{g})\}$.)
(d) Pick a p_1-approximation $\mathfrak{h}_1$ which is $\max(n, N)$-dense at j and (purely) stronger than $\mathfrak{g}$.
(e) Pick a $k_{n+1} > m$ such that $k_{n+1} \to \max(n+2, |\operatorname{Pos}(\mathfrak{h}_1)|)$.
(f) Pick a $q \leq_{\mathfrak{h}_1} p_1$ such that $q^{[\bar{b}]}$ determines $\underset{\sim}{r} \restriction k_{n+1}$ up to k_{n+1} for all $\bar{b} \in \operatorname{Pos}(\mathfrak{h}_1)$.

So far we have taken care of (1–4): $q \leq_{\mathfrak{g}} p$, $\mathfrak{h}_1$ approximates q and witnesses N-density (at j). However, the tree of possible values for $\underset{\sim}{r}$ could be very thick in the levels between k_n and k_{n+1}. We will thin out the approximation $\mathfrak{h}_1$ so that we still have $(n+1)$-density, and the tree of possible values for $\underset{\sim}{r}$ gets sufficiently thin. We do this in two steps:

(g) Find a sub-approximation $\mathfrak{h}_2$ of $\mathfrak{h}_1$ that is still purely stronger than $\mathfrak{g}$ and has only as many splittings as $\mathfrak{g}$, apart from one additional split (for each possibility) that witnesses N-density at j (see Figure 5).
    In more detail: we construct $\mathfrak{h}_2$ the following way: Given $\bar{b} \in \operatorname{Pos}_{<i}(\mathfrak{h}_2)$, set $\bar{a} = \bar{b} \restriction \mathfrak{g}$. We have to define $\mathfrak{h}_2(i)(\bar{b})$. If $i \neq j$, pick for each $t \in \mathfrak{g}(i)(\bar{a})$ exactly one successor $s \in \mathfrak{h}_1(i)(\bar{b})$. So $\mathfrak{h}_2$ makes the branches of $\mathfrak{g}$ longer, but does not add any splittings. At j, we have the front $F := \mathfrak{g}(j)(\bar{a})$ and the purely stronger n'-dense front $F' := \mathfrak{h}_1(j)(\bar{b})$. Recall that $T := T^{F'}_{\text{cldn}} = \{s : s \leq F'\}$ is the finite tree corresponding to the front F'. We continue each $t \in F$ in T uniquely (without splits) until we reach a node t with many (i.e., n'-dense) splittings. We call t

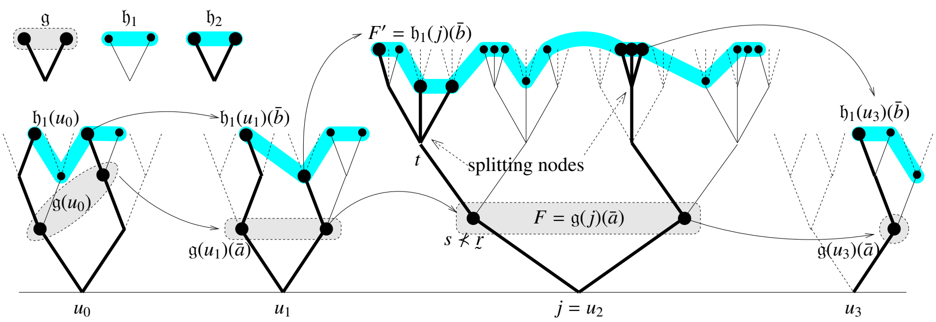

FIGURE 5. $\mathfrak{h}_2$ (bold) is a subapprox. of $\mathfrak{h}_1$ and still purely stronger than $\mathfrak{g}$. Here, we assume $\mathrm{dom}(\mathfrak{h}_1) = \{u_0, \dots, u_3\}$, $j = u_2$, $\bar b \in \mathrm{Pos}(\mathfrak{h}_2)$ and $\bar a = \bar b \restriction \mathfrak{g}$.

"splitting node". We take all the immediate successors of the splitting node and continue them uniquely in T until we reach a leaf of T, i.e., an element of F'. This process leads to a subset F'' of F'. Set $\mathfrak{h}_2(j)(\bar b) \coloneqq F''$.

(h) So we get: There are $|\mathrm{Pos}_{\leq j}(\mathfrak{g})| \leq M$ many pairs $(\bar b, t)$, where $\bar b \in \mathrm{Pos}_{<j}(\mathfrak{h}_2)$ and t is a splitting node.

Also, for $\bar b \in \mathrm{Pos}_{\leq j}(\mathfrak{h}_2)$, there are at most M continuations of $\bar b$ to some $\bar b' \in \mathrm{Pos}(\mathfrak{h}_2)$.

Such a $\bar b \in \mathrm{Pos}_{\leq j}(\mathfrak{h}_2)$ corresponds to a pair $(\bar a, t)$ as above together with a choice of an (immediate) successor of t.

(i) Now we are ready to apply the Ramsey property. First fix a $\bar b \in \mathrm{Pos}_{<j}(\mathfrak{h}_2)$ and a splitting node t. (There are at most M many such pairs.)

This pair corresponds to a unique $\bar a \in \mathrm{Pos}_{\leq j}(\mathfrak{g})$. There are at most M many continuations of $\bar a$ to some $\bar c \in \mathrm{Pos}(\mathfrak{g})$. Fix an enumeration $\bar c_1 \dots \bar c_M$ of these possible continuations. According to (c), each $\bar c_l$ forces a value to $\underaccent{\sim}{r} \restriction m$, call this value r_l.

Back to $\mathfrak{h}_2$. Set $A \coloneqq \mathrm{succ}(t)$ in the tree $T^{\mathfrak{h}_2(j)(\bar b)}_{\mathrm{cldn}}$ (or equivalently $T^{\mathfrak{h}_1(j)(\bar b)}_{\mathrm{cldn}}$). So $\mu(A) \geq n' > N$. For every $s \in A$ there is a unique $s' \geq s$ such that $\bar a \cup \{(j, s')\} \in \mathrm{Pos}_{\leq j}(\mathfrak{h}_2)$, and for every $s \in A$, $l \in M$ there is a unique $\bar d \in \mathrm{Pos}(\mathfrak{h}_2)$ continuing $\bar c_l \in \mathrm{Pos}(\mathfrak{g})$ and $\bar a \cup \{(j, s')\}$. Each such $\bar d$ decides $\underaccent{\sim}{r}$ up to k_{n+1}. We call this value $r^{s,l}$. So $r^{s,l} \restriction m = r_l$. According to (d) we know that $\mathrm{length}(t) > m$, so in particular $t \perp r_l$, according to (c).

So for every $l \in M$ we define a function $f_l : A \to T^{[r^l]}_{\max} \restriction k_{n+1}$ by mapping s to $r^{s,l}$. So we can apply the Ramsey property and get a $B \subseteq A$ such that $\mu(B) > M \geq n+1$, and the tree of possibilities for $\underaccent{\sim}{r}$ induced by $\bar a, B$ is $(k_n, 1/M)$-meager. We repeat that for all pairs $(\bar a, t)$ where $\bar a \in \mathrm{Pos}_{<j}(\mathfrak{h}_2)$ and t is a splitting node, and get a subapproximation $\mathfrak{g}_{n+1}$ of $\mathfrak{h}_2$ such that the tree of possibilities for $\underaccent{\sim}{r}$ induced by $\mathfrak{g}_{n+1}$ is $(k_n, 1)$-meager (here we again use the sub-additivity of μ).

This results in a sub-approximation $\mathfrak{g}_{n+1}$ of $\mathfrak{h}_2$ (and therefore $\mathfrak{h}_1$) which is still purely stronger than $\mathfrak{g} = \mathfrak{g}_n$. Since $\mathfrak{g}_{n+1}$ is a sub-approximation of $\mathfrak{h}_1$, $|\mathrm{Pos}(\mathfrak{g}_{n+1})| \leq |\mathrm{Pos}(\mathfrak{h}_1)|$, and therefore $k_{n+1}, \mathfrak{g}_{n+1}$ satisfy (4). $\square$

Note that we did not use the j_α or automorphisms of I, the proof works for all I. In particular, for $I = \{i\}$ we get: If G is Q-generic over V, and if $r \neq \eta$ in $V[G]$, then there is a $(1,1)$-meager T in V such that $r \in \lim(T)$. In particular, such an r cannot be Q-generic over V. So we get:

Corollary 5.7. *If Q is strongly non-homogeneous then Q forces that $\underset{\sim}{\eta}$ is the only Q-generic real over V in $V[G_Q]$.*

Remark. A similar forcing Q^{JeSh} (finitely splitting, rapidly increasing number of successors) was used in [6] to construct a complete Boolean algebra without proper atomless complete subalgebra. Q^{JeSh} can also be written as lim-sup forcing. However, the difference is that the norm in Q^{JeSh} is "binary" (as e.g., Sacks): either s has a minimum number of successors, then the norm is large, or the norm is 0. Such a norm cannot satisfy a Ramsey property as the one above. For Q^{JeSh} we can only prove Corollary 5.7 for the "single step iteration", but not Lemma 5.6 for the iteration.

We have already mentioned another corollary:

Corollary 5.8. *If Q is strongly non-homogeneous, then P forces that $\{\underset{\sim}{\eta}_i : i \in I\}$ is of weak measure 1.*

This, together with 4.8 and 4.10 proves Theorem 3.4.

Remark. There are various ways to extend the constructions in this paper. As already mentioned, we could use non-total orders I or allow Q_i to be a $P_{<i}$-name. A more difficult change would be to use lim-inf trees instead of lim-sup trees. In this case we need additional assumptions such as bigness and halving. This could allow us to apply Saccharinity to a ccc ideal $\mathbb{I}$, i.e., to force (without inaccessible or amalgamation) weak measurability of all definable sets.

6. The Cohen model

We thank the referee for providing this section.

There is a well known and much simpler way to force that every definable set is even measurable (not just weakly measurable) with respect to many tree forcings: Just add many Cohen reals.

Let $\mathbb{C}^\kappa$ be the forcing notion adding κ many Cohen reals (in a finite support product, or, equivalently, a finite support iteration). Any κ with uncountable cofinality will work. We call the forcing extension the "Cohen model". If in the ground model $\kappa^{\aleph_0} = \kappa$, then the continuum has size κ in the Cohen model.

Lemma 6.1. *In the Cohen model, every definable (e.g., projective) set is Q-measurable.*

This works for all Q as in Section 1, in particular for Sacks forcing, and also many other tree forcings, such as Silver forcing (as was shown in [2]). So in particular, in the Cohen model all definable sets are Marczewski measurable (corresponding to Q = Sacks) and have the doughnut property (corresponding to Q = Silver).

Proof. This is similar to, but simpler than, Solovay's argument that all definable sets are Lebesgue measurable in the Solovay model.

Assume that in the Cohen model the parameter p is in the union of the intermediate extensions (i.e., already added by the first α Cohen reals for some $\alpha < \kappa$) and that

$$X = \{x : \varphi(x, p)\}.$$

for some first order formula φ. Pick $T \in Q$. We can assume without loss of generality (by factoring $\mathbb{C}^\kappa$) that p and T are in V.

Work in V and consider the (countable) forcing notion T (ordered by $\leq_T$, the standard tree order). This forcing (which is obviously equivalent to a single Cohen forcing) adds a real $\underset{\sim}{c}$ that is Cohen over V in the natural topology of $\lim(T)$ (we call such a real T-Cohen, for short). In the same way as for "standard Cohen" forcing, one can see that $\underset{\sim}{c}$ determines the T-generic filter, and c^* is T-Cohen iff c^* is $\underset{\sim}{c}[G]$ for some T-generic G over V.

In particular, whenever R is some forcing notion, G_R is R-generic over V and $c^* \in V[G_R]$ is T-Cohen (over V), then we can factor the extension by first adding the T-generic c^* and then forcing with some quotient forcing to extend $V[c^*]$. If R is $\mathbb{C}^\kappa$, then the quotient forcing is again equivalent to $\mathbb{C}^\kappa$.

Let c^* be T-Cohen (i.e., T-generic) over V. In $V[c^*]$ consider the forcing notion $\mathbb{C}^\kappa$. Since this forcing is homogeneous, either $\Vdash_{\mathbb{C}^\kappa} \varphi(c^*, p)$ or $\Vdash_{\mathbb{C}^\kappa} \neg\varphi(c^*, p)$. Without loss of generality assume the former. So in V we can pick some condition $t^* \in T$ such that

$$t^* \Vdash_T \Vdash_{\mathbb{C}^\kappa} \varphi(\underset{\sim}{c}, p).$$

Let c^* in a $\mathbb{C}^\kappa$-extension V' of V be any T-Cohen real extending t^*. As described above, we can get V' by first extending V with the T-generic c^* and then some $\mathbb{C}^\kappa$-extension of $V[c^*]$. In particular, $\varphi(c^*, p)$ holds in V'. To summarize:

(1) In the Cohen model V', all T-Cohen reals c^* that extend t^* satisfy $\varphi(c^*, p)$.

Back in V, let T' be the tree $T^{[t^*]}$. So $T' \in Q^V$. Set

$$P = \{(t, n) : n \geq \text{length}(t^*), t \text{ is a subtree of } T', \text{ each maximal branch has height } n\}$$

ordered by end-extension (more exactly: (t, n) is stronger than (s, m) iff $n \geq m$ and t end-extends s). Obviously P is equivalent to Cohen forcing as well, and P adds a generic subtree S of T' (and S determines the generic filter). By density, the lim-sup condition will be satisfied, so S is in $Q^{V[S]}$. In any forcing extension V' of $V[S]$, we get:

(2) Every branch $\nu \in \lim(S)$ is T-Cohen over V and extends t^*.

To see this, fix some nowhere dense set N in V. Without loss of generality N is closed, i.e., corresponds to a nowhere dense subtree N' of T. Then (by a simple density argument) there is some (t, n) in the P-generic such that each maximal branch of t is not in N'. So any $\nu \in \lim(S)$ extends one of the maximal branches of t, and therefore is not in N.

Now we can finally fix a $\mathbb{C}^\kappa$-extension V' of V. We can use the equivalence of $\mathbb{C}^\kappa$ and $P * \mathbb{C}^\kappa$ to get in V' some $S \leq_Q T$ such that (2) holds. Then by (1) we get that each $c^* \in \lim(S)$ satisfies $\varphi(c^*, p)$, i.e., that $\lim(S) \subseteq X$. □

What is the difference between the Cohen model and the model obtained in the non-wellfounded iteration (let us call it nw-model, for short)? Note that in our nw-model, the continuum has size $\aleph_2$ (of course we can get larger continuum as well). One obvious difference is that in the nw-model $\mathbb{I}^c$ (the $< \aleph_2$-closure of $\mathbb{I}$) is non-trivial (or, in the language of cardinal characteristics, $\text{cov}(\mathbb{I}) = \aleph_2$), which is not the case in the Cohen model for $\kappa \geq \aleph_2$:

Lemma 6.2. *In the Cohen model,* $\text{cov}(\mathbb{I}) = \omega_1$.

Proof. The Cohen model is obtained by a finite support product of κ many Cohen reals. We can write κ as the strictly increasing union $\bigcup_{\alpha\in\omega_1} A_\alpha$ (each A_α of size κ). Let $\mathbb{C}_\alpha$ be the complete subforcing of $\mathbb{C}^\kappa$ consisting of the conditions that only use coordinates in A_α. Let G be $\mathbb{C}^\kappa$-generic over V, and let G_α be the induced $\mathbb{C}_\alpha$-generic filters over V. Then we get:

(1) $V[G_\alpha] \cap \omega^\omega$ is a proper subset of $V[G_{\alpha+1}] \cap \omega^\omega$.
(2) $V[G] \cap \omega^\omega = \bigcup_{\alpha\in\omega_1} V[G_\alpha] \cap \omega^\omega$.

From (1) and Lemma 1.8 we know that each $V[G_\alpha] \cap \omega^\omega$ is Q-null in the final Cohen extension; so by (2) ω^ω is the union of $\aleph_1$ many Q-null sets. □

(This argument works not only for the Cohen extension, but also for the random model and similarly for finite support iteration of Suslin ccc forcings of length $\aleph_2$; also, it works for other ideals than the ones defined by lim-sup tree forcings.)

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