POWER SET MODULO SMALL, THE SINGULAR OF UNCOUNTABLE COFINALITY F1873

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Abstract. Let μ be singular of uncountable cofinality. If $\mu > 2^{\text{cf}(\mu)}$, we prove that in $\mathbb{P} = ([\mu]^\mu, \supseteq)$ as a forcing notion we have a natural complete embedding of Levy$(\aleph_0, \mu^+)$ (so $\mathbb{P}$ collapses μ^+ to $\aleph_0$) and even Levy$(\aleph_0, \mathbf{U}_{J^{\text{bd}}_\kappa}(\mu))$, well, when the sup is attained. The "natural" means that the forcing $(\{p \in [\mu]^\mu : p$ closed$\}, \supseteq)$ is naturally embedded and is equivalent to the Levy algebra. Also if $\mathbb{P}$ fails the χ-c.c. then it collapses χ to $\aleph_0$ (and the parallel results for the case $\mu > \aleph_0$ is regular or of countable cofinality).

The 2019 version has more:

(a) instead of just collapsing χ to $\aleph_0$ we add a generic to Levy$(\aleph_0, \chi)$

(b) we intend (as promised) to deal also with singular $\mu < 2^{\text{cf}(\mu)}$.

Date: May 8, 2019.

2010 *Mathematics Subject Classification.* Primary; Secondary:

Key words and phrases. ?

This research was supported by the United States-Israel Binational Science Foundation. The author thanks Alice Leonhardt for the beautiful typing. References like [She, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

§ 0. Introduction

§ 0(A). Background.

This work tries to confirm the "pcf is effective" thesis.

We may consider the completions of the Boolean Algebras $\mathscr{P}(\mu)/\{u \subseteq \mu : |u| < \mu\} = \mathscr{P}(\mu)/[\mu]^{<\mu}$. This is equivalent to considering the partial orders $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$, viewing them as forcing notions, so actually looking at their completion $\hat{\mathbb{P}}_\mu$, which are complete Boolean Algebras. Recall that forcing notions $\mathbb{P}^1, \mathbb{P}^2$ are equivalent <u>iff</u> their completions are isomorphic Boolean Algebras. The Czech school has investigated them, in particular, (letting $\ell(\mu)$ be 0 if $\mathrm{cf}(\mu) > \aleph_0$ and 1 if $\mu > \mathrm{cf}(\mu) = \aleph_0$, (and $\aleph_{\ell(\mu)} = \mathfrak{h}$ if $\mu = \aleph_0$) consider the questions:

$\otimes_1$ (a) is $\hat{\mathbb{P}}_\mu$ isomorphic to the completion of the Levy collapse $\mathrm{Levy}(\aleph_{\ell(\mu)}, 2^\mu)$?

(b) which cardinals χ the forcing notion $\mathbb{P}_\mu$ collapse to $\aleph_{\ell(\mu)}$

(c) is $\mathbb{P}_\mu$ is (θ, χ)-nowhere dense distributive for $\theta = \aleph_{\ell(\mu)}$? this can be phrased as: for some $\mathbb{P}_\mu$-name $\underset{\sim}{f}$ of a function from $\aleph_{\ell(\mu)}$ to χ, for every $p \in \mathbb{P}_\mu$ for some $i < \theta$ the set $\{\alpha < \chi : p \nVdash \underset{\sim}{f}(i) \neq \alpha\}$ has cardinality χ.

The first, (a) is a full answer, (b) the second seems central for set theories, the last is sufficient if the density is right, to get the first. The case of collapsing seems central (it also implies clause (c) if χ large enough) so we repeat the summary from Balcar Simon [BS95] of what was known of the collapse of cardinals by $\mathbb{P}_\kappa$, i.e., $\otimes_1(b)$. Let $\lambda \to_\kappa \mu$ denote the fact that λ is collapsed to μ by $\mathbb{P}_\kappa$

$\boxtimes_1$ (i) for $\kappa = \aleph_0, 2^{\aleph_0} \to_\kappa \mathfrak{h}$, (but $\mathbb{P}_\kappa$ adds no new sequence of length $< \mathfrak{h}$ so we are done); Balcan Pelant Simon [BPS80]

(ii) for κ uncountable and regular, $\mathfrak{b}_\kappa \to_\kappa \aleph_0$, (hence $\kappa^+ \to_\kappa \aleph_0$) Balcar Simon [BS88]

(iii) for κ singular with $\mathrm{cf}(\kappa) = \aleph_0, 2^{\aleph_0} \to_\kappa \aleph_1$, Balcar Simon [BS95]

(iv) for κ singular with $\mathrm{cf}(\kappa) \neq \aleph_0, \mathfrak{b}_{\mathrm{cf}(\kappa)} \to_\kappa \aleph_0$, Balcar Simon [BS95];

under additional assumptions for singular cardinals more is known

(v) for κ singular with $\mathrm{cf}(\kappa) = \aleph_0$ and $\kappa^{\aleph_0} = 2^\kappa, \kappa^{\aleph_0} \to_\kappa \aleph_1$, Balcar Simon [BS88]

(vi) for κ singular with $\mathrm{cf}(\kappa) \neq \aleph_0$ and $2^\kappa = \kappa^+, 2^\kappa \to_\kappa \aleph_0$, [BS88].

Now [BS95] end with the following very reasonable conjecture.

Conjecture 0.1. (Balcar and Simon) in ZFC: for a singular cardinal κ with countable cofinality, $\kappa^{\aleph_0} \to \aleph_1$ and for a singular cardinal κ with an uncountable cofinality $\kappa^+ \to \aleph_0$ (here we concentrate on the case $\mathrm{cf}(\mu) > \aleph_0$, see below).

Concerning the other questions they prove

$\boxtimes_2$ (i) Balcar Franek [BF87]:
if $\mu > \mathrm{cf}(\mu) > \aleph_0, 2^{\mathrm{cf}(\mu)} = \mathrm{cf}(\mu)^+$ then $\mathbb{P}_\mu$ is $(\omega; \kappa^+)$-nowhere distributive

(ii) Balcar Simon [BS89, 5.20,pg.38]:
if $2^\mu = \mu^+$ and $2^{\mathrm{cf}(\mu)} = \mathrm{cf}(\mu)^+$ then $\mathbb{P}_\mu$ is equivalent: to Levy$(\aleph_0, \mu^+)$ if $\mathrm{cf}(\mu) > \aleph_0$ and to Levy$(\aleph_1, \mu^+)$ if $\mathrm{cf}(\mu) = \aleph_0$

(iii) Balcar Franek [BF87]:
if $2^\mu = \mu^+, \mu = \mathrm{cf}(\mu) > \aleph_0, J$ a μ-complete idea on μ and J nowhere precipitous extending $[\mu]^{<\mu}$ then $\mathscr{P}(\mu)/J$ is equivalent to Levy$(\aleph_0, \mu^+)$; also the parallel of (ii).

So under G.C.H. the picture was complete; getting clause (ii) of $\boxtimes_2$, and, in ZFC for regular cardinals $\mu > \aleph_0$ the picture is reasonble, particularly if we recall that by Baumgartner [Bau76]

$\boxtimes_3$ if $\kappa = \mathrm{cf}(\mu) < \theta = \theta^{<\theta} < \mu < \chi$ and $\mathbf{V} \models$ G.C.H. for simplicity and $\mathbb{P}$ is adding χ-Cohen subsets to θ then
- (a) forcing with $\mathbb{P}$ collapse no cardinal, change no cofinality, adds no new sets of $< \theta$ ordinals
- (b) in $\mathbf{V}^{\mathbb{P}}, ([\mu]^\mu, \supseteq)$ satisfies the μ_1^+-c.c. where $\mu_1 = (2^\mu)^{\mathbf{V}}$.

Lately Kojman Shelah [KS01] prove the conjecture 0.1 for the case when $\mu > \mathrm{cf}(\mu) = \aleph_0$; morever

$\boxtimes_4$ (i) if $\mu > \mathrm{cf}(\mu) = \aleph_0$ then Levy$(\aleph_1, \mu^{\aleph_0})$ can be completely embeddable into the completion of $\mathbb{P}_\mu$.
Moreover,
(ii) the embedding is "natural": Levy$(\aleph_1, \mu^{\aleph_0})$ is equivalent to $\mathbb{Q}_\mu = (\{A \subseteq \mu : A$ a closed subset of μ of cardinality $\mu\}, \supseteq))$.

Here we continue [KS01] in §1, [BS89] in §2 but make it self contained. Naturally we may add to the questions (answered positively for the case $\mathrm{cf}(\kappa) = \aleph_0$ by [KS01])

$\otimes_2$ (a) can we strengthen "$\mathbb{P}_\mu$ collapse χ to $\aleph_{\ell(\mu)}$" to "Levy$(\aleph_{\ell(\mu)}, \chi)$ is completely embeddable into $\mathbb{P}_\mu$ (really $\hat{\mathbb{P}}_\mu$)"
(b) can we find natural such embeddings.

We may add that by [BS95] the Baire number of $\mathscr{U}[\mu]$, the space of all uniform ultrafilters over uncountable μ is $\aleph_1$, except when $\mu > \mathrm{cf}(\mu) = \aleph_0$ and in that case it is $\aleph_2$ and under some reasonable assumptions. By [KS01] the Baire number of $\mathscr{U}[\mu]$ is always $\aleph_2$ when $\mu > \mathrm{cf}(\mu) = \aleph_0$.

§ 0(B). **The Results.**

Here we deal mainly with the case $\mu > 2^{\mathrm{cf}(\mu)}$ (in §1) and prove more on regular $\mu > \aleph_0$ (in §2).

Our original aim in this work has been to deal with $\mu > \mathrm{cf}(\mu) > \aleph_0$, proving the conjecture of Balcar and Simon above (i.e., that μ^+ is collapsed to $\aleph_0$), first of all when $2^{\mathrm{cf}(\mu)} < \mu$ answering $\circledast_2(a) + (b)$ using pcf and (replacing μ^+ by $\mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}}(\mu)$). In fact this seems, at least to me, the best we can reasonably expect. But a posteriori we have more to say.

For $\mu = \kappa = \mathrm{cf}(\mu) > \aleph_0$, though by the above we know that some cardinal $> \mu$ is collapsed, (that is $\mathfrak{b}_\kappa$) we do not know what occurs up to 2^μ or when the c.c.

fails. This leads to the following conjecture, (stronger than the Balcar Simon one mentioned above).

Of course, it naturally breaks to cases for according to μ.

Conjecture 0.2. If $\mu > \aleph_0$ and $\mathbb{P}_\mu$ does not satisfy the χ-c.c., then forcing with $\mathbb{P}_\mu$ collapse χ to $\aleph_{\ell(\mu)}$, see Definition 0.6 below.

Note that

Observation 0.3. *If conjecture 0.2 holds for $\mu > \aleph_0$ then $\mathbb{P}_\mu$ is equivalent to a* Levy *collapse iff it fails the $d(\mathbb{P}_\mu)$-c.c. where $d(\mathbb{P}_\mu)$ is the density of $\mathbb{P}_\mu$.*

Lastly, we turn to the results; by 1.17(1):

Theorem 0.4. *1) If $\mu > \kappa = \text{cf}(\mu) > \aleph_0$ and $\mu > 2^\kappa$ then $\mathbb{Q}_\mu$ (a natural complete subforcing of $\mathbb{P}_\mu$, forcing with closed sets) is equivalent to* $\text{Levy}(\aleph_0, \mathbf{U}_{J^{\text{bd}}_\kappa}(\mu))$.
2) Hence if in addition $\mathbf{U}_{J^{\text{bd}}_\kappa}(\mu) = 2^\mu$, which holds if M is strong limit then $\mathbb{P}_\mu$ is equivalent to $\text{Levy}(\aleph_0, 2^\mu)$.

By 1.18, 1.19 and 2.7 we have:

Theorem 0.5. *Conjecture 0.2 holds except possibly when $\aleph_0 < \text{cf}(\mu) < \mu < 2^{\text{cf}(\mu)}$.*

We hope in a subsequent paper to prove the Balcar Simon conjecture fully, i.e., in all cases.

Definition 0.6. For $\mu > \aleph_0$ we define $\ell(\mu) \in \{0, 1\}$ by:
$\ell(\mu) = 0$ if $\text{cf}(\mu) > \aleph_0$
$\ell(\mu) = 1$ if $\mu > \text{cf}(\mu) = \aleph_0$
and may add
$\ell(\mu) = \alpha$ when $\mu = \aleph_0, \mathfrak{h} = \aleph_\alpha$.

We thank Menachem Kojman for discussions on earlier attempts; Shimoni Garti for corrections and Bohuslav Balcar and Petr Simon for improving the presentation in the published version.

A posteriori (– 2019) we add the conclusions concerning

Conjecture 0.7. If $\mathbb{P}_\mu$ fails the χ-c.c., then forcing by $\mathbb{P}_\mu$ adds a generic for $\text{Levy}(\aleph_{\ell(\mu)}, \chi)$.

§ 1. §1 Forcing with closed set is equivalent to the Levy algebra

Definition 1.1. 1) For $f \in {}^\kappa(\mathrm{Ord}\backslash\{0\})$ and ideal I on κ let

$$\mathbf{U}_I(f) = \mathrm{Min}\{|\mathscr{P}| : \begin{array}{l}\mathscr{P} \subseteq [\sup \mathrm{Rang}(f)]^{\leq\kappa} \\ \text{such that for every } g \leq f \text{ for some } u \in \mathscr{P} \\ \text{we have } \{i < \kappa : g(i) \in u\} \in I^+\}.\end{array}$$

2) Let $\mathbf{U}_I(\lambda)$ means $\mathbf{U}_I(f)$ where f is the function with domain $\mathrm{Dom}(I)$ which is constantly λ.

Hypothesis 1.2.
 (a) μ is a singular cardinal
 (b) $\kappa = \mathrm{cf}(\mu) > \aleph_0$ (but try to mention when used).

Definition 1.3. 1) $\mathbb{P}_\mu$ is the following forcing notion

$$p \in \mathbb{P}_\mu \text{ \underline{iff} } p \in [\mu]^\mu$$

$$\mathbb{P}_\mu \models p \leq q \text{ iff } p \supseteq q.$$

2) $\mathbb{P}'_\mu$ is the forcing notion with the same set of elements and with the partial order

$$\mathbb{P}'_\mu \models p \leq q \text{ iff } |p\backslash q| < \mu.$$

3) $\mathbb{Q}_\mu = \mathbb{Q}^0_\mu$ is $\mathbb{P}_\mu \restriction \{p \in \mathbb{P}_\mu : p \text{ is closed in the order topology of } \mu\}$.

Choice/Definition 1.4. 1) Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of regular cardinals $> \kappa$ with limit μ.
2) Let $\lambda^-_i = \cup\{\lambda_j : j < i\}$.
3) For $p \in \mathbb{P}_\mu$ let $a(p) = \{i < \kappa : p \cap [\lambda^-_i, \lambda_i) \neq \emptyset\}$.
4) $\mathbb{Q}^1_\mu = \{p \in \mathbb{Q}_\mu : i < \kappa \Rightarrow |p \cap \lambda_i| < \lambda_i$ and for each $i \in a(p)$ the set $p \cap \lambda_i\backslash\lambda^-_i$ has no last element is closed in its supremum and has cardinality $> |p \cap \lambda^-_i|\}$.
5) For $p \in \mathbb{Q}^1_\mu$ let $\mathrm{ch}_p \in \prod\limits_{i \in a(p)} \lambda_i$ be $\mathrm{ch}_p(i) = \cup\{\alpha + 1 : \alpha \in p \cap [\lambda^-_i, \lambda_i)\}$ and $\mathrm{cf}_p \in \prod\limits_{i \in a(p)} \lambda_i$ be $\mathrm{cf}_p(i) = \mathrm{cf}(\mathrm{ch}_p(i))$.
6) $\mathbb{Q}^2_\mu = \{p \in \mathbb{Q}^1_\mu{:}\mathrm{cf}_p(i) > |p \cap \lambda^-_i|$ for $i \in a(p)\}$.
7) $p \in \mathbb{P}_\mu$ is $\bar\lambda'$-normal when $\bar\lambda' = \langle \lambda'_i : i \in a(p)\rangle$ and $\mathrm{otp}(p \cap [\lambda^-_i, \lambda_i)) = \lambda'_i$ for $i \in a(p)$.

Claim 1.5. *1) $\mathbb{Q}^0_\mu, \mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$ are complete sub-forcings of $\mathbb{P}_\mu$.*
2) For $\ell = 0, 1, 2$ and $p, q \in \mathbb{Q}^\ell_\mu$ we have $p \Vdash_{\mathbb{Q}^\ell_\mu}$ "$q \in \underset{\sim}{\mathbf{G}}$" \underline{iff} $|q\backslash p| < \mu$ and similarly for $\mathbb{P}_\mu$.
3) $\mathbb{Q}_\mu = \mathbb{Q}^0_\mu, \mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$ are equivalent, in fact $\mathbb{Q}^2_\mu$ a dense subset of $\mathbb{Q}^1_\mu$ which is dense in Q^0_μ.

Proof. Easy. $\square_{1.5}$

Recall

Claim 1.6. *1) $\mathbb{P}_\kappa$ can be completely embedded into $\mathbb{P}_\mu$ (naturally).*
2) $\mathbb{Q}_\mu$ can be completely embeddable into $\mathbb{P}_\mu$ (naturally).
3) $\mathbb{P}_\kappa$ is completely embeddable into $\mathbb{Q}_\mu$ (naturally).

Proof. 1) Known: just $a \in [\kappa]^\kappa$ can be mapped to $\cup\{[\lambda_i^-, \lambda_i) : i \in a\}$.
2) By [KS01, **?**].
3) Should be clear (again map $A \in [\kappa]^\kappa$ to $\cup\{[\lambda_i^-, \lambda_i) : i \in A\}$). $\square_{1.6}$

Choice/Definition 1.7. 1) $\lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$.
2) χ is, e.g., $(\beth_8(\lambda_*))^+$, $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$.
3) $\mathfrak{B}$ is an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality λ_* such that $\lambda_* + 1 \subseteq \mathfrak{B}$.

Recall

Claim 1.8. *Assume $\mu > 2^\kappa$.*
1) $\lambda_* = \sup\{{\rm pp}_{J^{\rm bd}_\kappa}(\mu') : \kappa < \mu' \le \mu, {\rm cf}(\mu') = \kappa\} = \sup\{{\rm tcf}(\prod\limits_{i<\kappa} \lambda'_i / J^{\rm bd}_\kappa) : \lambda'_i \in {\rm Reg} \cap (\kappa, \mu)$ and $\prod\limits_{i<\kappa} \lambda'_i / J^{\rm bd}_\kappa$ has true cofinality$\}$.
2) For every regular cardinal $\theta \in [\mu, \lambda_*]$, for some increasing sequence $\langle \lambda^*_i : i < \kappa\rangle$ of regulars $\in (\kappa, \mu)$ we have $\theta = {\rm tcf}(\prod\limits_{i<\kappa} \lambda^*_i, <_{J^{\rm bd}_\kappa})$.

Proof. 1) Note that $J^{\rm bd}_\kappa \restriction A \approx J^{\rm bd}_\kappa$ if $A \in (J^{\rm bd}_\kappa)^+$, we use this freely. By their definition the second and third terms are equal. Also by the definition the second is smaller or equal to the first.

By [She00, 1.1], the first, $\lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$ is $\ge$ than the second number (well it speaks on $T^2_{J^{\rm bd}_\kappa}(\mu)$, instead $\mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$ but as $2^\kappa < \mu$ they are the same).
2) As λ_* is regular by [She00, 1.1] we actually get the stronger conclusion. $\square_{1.8}$

Convention 1.9. We fix $\bar{C}^*$ as in 1.10.

Claim/Definition 1.10. *1) There is $\bar{C}^* = \langle C^*_\alpha : \alpha < \mu\rangle \in \mathfrak{B}$ such that:*

(a) *C^*_α is a subset of $[\lambda_i^-, \lambda_i)$ closed in its supremum when $\alpha \in [\lambda_i^-, \lambda_i)$*
(b) *if $i < \kappa, \gamma < \lambda_i, C$ is a closed subset of $[\lambda_i^-, \lambda_i)$ of order type $> |\gamma|^{++}$ (or γ a regular cardinal, ${\rm otp}(C) = \gamma^{++}$) then for some $\alpha \in [\lambda_i^-, \lambda_i), C^*_\alpha \subseteq C$ and ${\rm otp}(C^*_\alpha) = \gamma$.*

2) $\mathbb{Q}^3_{\mu,\bar{C}^*} = \{p \in \mathbb{Q}^2_\mu$: if $i \in a(p)$ then $p \cap [\lambda_i^-, \lambda_i) \in \{C^*_\alpha : \alpha \in [\lambda_i^-, \lambda_i)\}\}$ is a dense subset of $\mathbb{Q}^1_\mu, \mathbb{Q}^2$ hence of $\mathbb{Q}_\mu$ as we are fixing $\bar{C}^*$, we may write $\mathbb{Q}^3_\mu$.
3) For $p \in \mathbb{Q}^3_\mu$ let ${\rm cd}_p \in \prod\limits_{i \in a(p)} \lambda_i$ be such that ${\rm cd}_p(i) \in [\lambda_i^-, \lambda_i)$ is the minimal $\alpha \in (\lambda_i^-, \lambda_i)$ such that $p \cap [\lambda_i^-, \lambda_i) = C^*_\alpha$.

Proof. 1) It is enough, for any limit $\delta \in (\lambda_i^-, \lambda_i]$ and regular $\theta, \theta^+ < {\rm cf}(\delta)$, to find a family $\mathscr{P}_{\delta,\theta}$ of closed subsets of (λ_i^-, δ) of order type θ such that any club of δ contains (at least) one of them. This holds by guessing clubs, see [She94, Ch.III,§2]; in fact also singular θ is O.K.
2) By the definitions. $\square_{1.10}$

Claim 1.11. *1) If $\mu > 2^\kappa$ (or just $\lambda_* \ge 2^\kappa$) <u>then</u> $\mathbb{Q}^2_\mu$ (hence $\mathbb{Q}^1_\mu$) has a dense subset of cardinality λ_*.*
2) If $\mu > 2^\kappa$ <u>then</u> $\mathbb{Q}^3_\mu \cap \mathfrak{B}$ is a dense subset of $\mathbb{Q}^1_\mu$ and has cardinality λ_*.

Proof. 1) By part (2).
2) By 1.10(2) it suffices to deal with $\mathbb{Q}^3_\mu$. There is a set $\mathscr{P}$ exemplifying $\lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$, so without loss of generality $\mathscr{P} \in \mathfrak{B}$ hence $\mathscr{P} \subseteq \mathfrak{B}$. For every $p \in \mathbb{Q}^3_\mu$ apply the choice of $\mathscr{P}$ to the function cd_p, hence there is $u \in \mathscr{P} \subseteq [\mu]^{\le\kappa} \cap \mathfrak{B}$ such that $b := \{i \in a(p) : \mathrm{cd}_p(i) \in u\} \in [\kappa]^\kappa$. As $2^\kappa \le \lambda_*$ clearly $b \in \mathfrak{B}$, hence $\mathrm{cd}_p{\restriction}b \in \mathfrak{B}$ recall that $\langle |p \cap [\lambda^-_i, \lambda_i)| : i \in a(p)\rangle$ is increasing hence $q := \cup\{p \cap [\lambda^-_i, \lambda_i) : i \in b\} = \cup\{C^*_{\mathrm{cd}_p(i)} : i \in b\}$ belongs to $\mathbb{Q}^3_\mu \cap \mathfrak{B}$ and is above p. So $\mathfrak{B} \cap \mathbb{Q}^3_\mu$ is a dense subset of $\mathbb{Q}^3_\mu$ (hence of $\mathbb{Q}_\mu$) of cardinality $\|\mathfrak{B}\| = \lambda_*$, as required.
3) Should be clear. $\square_{1.11}$

From now on (till the end of this section)

Hypothesis 1.12. $2^\kappa < \mu$ (in addition to 1.2). Recall (Claim 1.14(1) is Balcar Simon [BS89, 1.15] and 1.14(2) is a variant).

Definition 1.13. A forcing notion $\mathbb{P}$ is (θ, λ)-nowhere distributive <u>when</u> there are maximal antichains $\bar{p}^\varepsilon = \langle p^\varepsilon_\alpha : \alpha < \alpha_\varepsilon\rangle$ of $\mathbb{P}$ for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta, \lambda \le |\{\alpha < \alpha_\varepsilon : p, p^\varepsilon_\alpha$ are compatible$\}|$.

Recall that

Fact 1.14. 1) If

(a) $\mathbb{P}$ is a forcing notion, (θ, λ)-nowhere distributive
(b) $\mathbb{P}$ has density λ
(c) $\theta > \aleph_0 \Rightarrow \mathbb{P}$ has a θ-complete dense subset

<u>then</u> $\mathbb{P}$ is equivalent to $\mathrm{Levy}(\theta, \lambda)$.
2) If $\mathbb{P}$ is a forcing notion of density λ collapsing λ to $\aleph_0$,<u>then</u> $\mathbb{P}$ is equivalent to $\mathrm{Levy}(\aleph_0, \lambda)$.
3) If $\mathbb{P}$ is a forcing notion of density λ and is nowhere (θ, λ)-distributive, <u>then</u> $\mathbb{P}$ collapses λ to θ (and may or may not collapse θ).

Claim 1.15. *Assume $\langle b_\varepsilon : \varepsilon < \kappa\rangle$ is a sequence of pairwise disjoint members of $[\kappa]^\kappa$ with union b. <u>Then</u> we can find anti-chain $\mathscr{I}$ of $\mathbb{Q}^3_\mu$ such that:*

(*) *if $q \in \mathbb{Q}^3_\mu$ and $(\forall \varepsilon < \kappa)(a(q) \cap b_\varepsilon \in [\kappa]^\kappa)$, <u>then</u> q is compatible with $\lambda_* =: \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$ of the members of $\mathscr{I}$.*

Proof. Let

$$\begin{aligned}\mathscr{I}^* = \{p \in \mathbb{Q}^3_\mu :\ & p \in \mathfrak{B} \text{ and we can find an increasing sequence} \\ & \langle i_\varepsilon : \varepsilon < \kappa\rangle \text{ such that } i_\varepsilon \in b_\varepsilon \backslash (\varepsilon + 1) \\ & a(p) \subseteq \{i_\varepsilon : \varepsilon < \kappa\} \text{ and } i_\varepsilon \in a(p) \Rightarrow p \cap [\lambda^-_{i_\varepsilon}, \lambda_{i_\varepsilon}) \\ & \text{has order type } \lambda_\varepsilon\}.\end{aligned}$$

Let $\mathscr{J}^* = \{p \in \mathbb{Q}^3_\mu$: for every $\varepsilon < \kappa$ we have $a(p) \cap b_\varepsilon \in [\kappa]^\kappa\}$.
Clearly

(a) $|\mathscr{I}^*| \le \lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$.

[Why? As $\mathscr{I}^* \subseteq \mathfrak{B}$.]

(b) if $\mathscr{I} \subseteq \mathscr{I}^*, |\mathscr{I}| < \lambda_*$ and $q \in \mathscr{J}^*$ then there is r such that $q \le r \in \mathscr{I}^*$ and r is incompatible with every $p \in \mathscr{I}$.

[Why? Let $\theta = |\mathscr{I}| + \mu$ it is $< \lambda_*$, hence we can find an increasing sequence $\langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ of regular cardinals with limit μ such that $\prod\limits_{\varepsilon<\kappa} \theta_\varepsilon / J^{\rm bd}_\kappa$ has true cofinality θ^+, this by 1.8 + the no hole lemma [She94, Ch.II,§3]. By renaming without loss of generality $\theta_\varepsilon > \lambda_\varepsilon$.

Let $u = \{\varepsilon < \kappa : a(q) \cap b_\varepsilon \in [\kappa]^\kappa\}$, so we know that u is κ. For each $\varepsilon \in u$ we know that $a(q) \cap b_\varepsilon \in [\kappa]^\kappa$, and so for some $\zeta_\varepsilon < \kappa$ we have $\theta_\varepsilon < \lambda_{{\rm otp}(a(q)\cap\zeta_\varepsilon)}$. Now choose $i(\varepsilon) \in b_\varepsilon$ such that $i(\varepsilon) > \varepsilon \wedge i(\varepsilon) > \zeta_\varepsilon \wedge (\forall \varepsilon_1 < \varepsilon)(i(\varepsilon_4) < i(\varepsilon))$. As $q \in \mathbb{Q}^3_\mu$ it follows that $(q \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}))$ has order type $\ge \lambda_{{\rm otp}(a(q)\cap\zeta_\varepsilon)} > \theta_\varepsilon$. Let $C_{q,\varepsilon} = \{\alpha : \alpha \in q, \alpha \in [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)})$ and ${\rm otp}(q \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}) \cap \alpha)$ is $< \theta_\varepsilon\}$. Now for every $p \in \mathscr{I}^*$ the set $p \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}) \subseteq \cup\{[\lambda^-_i, \lambda_i) : i \in b_\varepsilon\}$ if non-empty has cardinality $\le \lambda_\varepsilon$ which is $< \theta_\varepsilon$ hence $p \cap C_{q,\varepsilon}$ is a bounded subset of $C_{q,\varepsilon}$, call the lub $\alpha_{p,\varepsilon}$. As $\theta = |\mathscr{I}| + \mu < {\rm tcf}(\prod\limits_{\varepsilon<\kappa} \theta_\varepsilon / J^{\rm bd}_\kappa)$ clearly there is $h \in \prod\limits_{\varepsilon\in u} C_{q,\varepsilon}$ such that $p \in \mathscr{I} \Rightarrow \langle \alpha_{p,\varepsilon} : \varepsilon < \kappa \rangle <_{J^{\rm bd}_\kappa} h$ and let

$$r = \{\alpha : \text{ for some } \varepsilon \in u \text{ we have } \alpha \in C_{q,\varepsilon} \backslash h(\varepsilon) \text{ and } |C_{q,\varepsilon} \cap \alpha \backslash h(\varepsilon)| < \lambda_\varepsilon\}.$$

So r is as required in clause (b). (We can assume that $r \in \mathfrak{B}$, since by the density propositions of 1.11 we can find $r \le r' \in \mathscr{B}$ as required.) So clause (b) holds.]

As by 1.11(2) in the conclusion of the claim it is enough to deal with $q \in \mathbb{Q}^3_\mu \cap \mathfrak{B}$, there are only λ_* such q's so we can finish easily by diagonalization. $\square_{1.15}$

Claim 1.16. *The forcing notion $\mathbb{Q}^3_\mu$ is $(\mathfrak{b}_\kappa, \lambda_*)$-no-where distributive.*

Proof. Let $\langle \bar A_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ be such that: $\bar A_\alpha = \langle A_{\alpha,i} : i < \kappa \rangle, A_{\alpha,i} \in [\kappa]^\kappa, i < j \Rightarrow A_{\alpha,i} \cap A_{\alpha,j} = \emptyset$ and $(\forall B \in [\kappa]^\kappa)(\exists \alpha < \mathfrak{b}_\kappa)(\forall i < \kappa)[\kappa = |B \cap A_{\alpha,i}|]$, exists by 2.7(2) below. Hence for each $\alpha < \mathfrak{b}_\kappa, \mathscr{I}^*_\alpha \subseteq \mathbb{Q}^3_\mu \cap \mathfrak{B}$ as in 1.15 for the sequence $\bar A_\alpha$ exists. So $\langle \mathscr{I}^*_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ is a sequence of $\mathfrak{b}_\kappa$ antichains of $\mathbb{Q}^3_\mu$ and we shall show that it witnesses the conclusion.

Now

⊛ if $q \in \mathbb{Q}^3_\mu$ then for some $\alpha < \mathfrak{b}_\kappa$ the set $\{p \in \mathscr{I}^*_\alpha : p$ compatible with $q \in \mathbb{Q}^3_\mu\}$ has cardinality λ_*.

Why? By the choice of $\langle \bar A_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ there is $\alpha < \mathfrak{b}_\kappa$ such that:

(∗) $a(q) \cap A_{\alpha,i} \in [\kappa]^\kappa$ for every $i < \kappa$.

Hence q fits the demand in 1.15 with $\bar A_\alpha$ here standing for $\langle b_\varepsilon : \varepsilon < \kappa \rangle$ there. Hence it is compatible with λ_* members of $\mathscr{I}^*_\alpha$ which, of course, shows that we are done. $\square_{1.16}$

Conclusion 1.17. *1) If $2^\kappa < \mu$ (and $\aleph_0 < \kappa = {\rm cf}(\mu) < \mu$, of course) then $\mathbb{Q}_\mu$ is equivalent to* Levy$(\aleph_0, \lambda_*)$*, i.e., they have isomorphic completions (recalling $\mathbb{Q}_\mu$ is naturally completely embeddable into the completion of $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$).*
2) If $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$ then $\mathbb{Q}_\mu$ is equivalent to Levy$(\aleph_0, \mu^\kappa)$.
3) If μ is strong limit (singular of uncountable cofinality κ), then $\mathbb{P}_\mu$ is equivalent to Levy$(\aleph_0, \mu^\kappa) =$ Levy$(\aleph_0, 2^\mu)$.

Proof. 1) By 1.11(1), $\mathbb{Q}^3_\mu$ has density (and even cardinality) λ_* and by 1.16 it is $(\mathfrak{b}_\kappa, \lambda_*)$-no-where distributive hence by 1.14(3), we know that $\mathbb{Q}^3_\mu$ collapse λ_* to $\mathfrak{b}_\kappa$. But $\mathbb{P}_\kappa$ is completely embeddable into $\mathbb{Q}^2_\mu$ (see 1.6(3)) and $\mathbb{P}_\kappa$ collapses $\mathfrak{b}_\kappa$ to $\aleph_0$ (see §2) and $\mathbb{Q}^3_\mu$ is dense in $\mathbb{Q}^2_\mu$. Together forcing with $\mathbb{Q}^3_\mu$ collapse λ_* to $\aleph_0$. As $\mathbb{Q}^3_\mu$ has density λ_*, by 1.14(2) we get that $\mathbb{Q}^2_\mu$ is equivalent to Levy$(\aleph_0, \lambda_*)$.

Lastly $\mathbb{Q}_\mu, \mathbb{Q}^3_\mu$ are equivalent by 1.5(3) + 1.10(2) so we are done.

2) Recalling 1.8 by [She94] we have $\lambda_* = \mu^\kappa$ (alternatively directly as in [She97, §3]). Now apply part (1).

3) By easy cardinal arithmetic $\mu^\kappa = 2^\mu$. Enough to check the demands in 1.14(2). Now as $\mathbb{Q}_\mu$ collapse λ_* to $\aleph_0$ by part (1) and $\mathbb{Q}_\mu$ can be completely embeddable into $\mathbb{P}_\mu$ (see 1.6(2)) clearly $\mathbb{P}_\mu$ collapse λ_* to $\aleph_0$. But $|\mathbb{P}_\mu| \le |[\mu]^\mu| = 2^\mu$, so $\mathbb{P}_\mu$ has density $\le 2^\mu$.

Lastly $\lambda_* = 2^\mu$ by [She94, Ch.VIII]. So we are done. $\square_{1.17}$

Claim 1.18. *Assume that $\mathbb{P}_\mu$ does not satisfy the χ-c.c. Then forcing with $\mathbb{P}_\mu$ collapse χ to $\aleph_0$.*

Proof. By the nature of the conclusion without loss of generality χ is regular. Now we can find $\bar{X}$ such that:

$(*)_1$ (a) $\bar{X} = \langle X_\xi : \xi < \chi \rangle$
(b) $X_\xi \in \mathbb{P}_\mu$
(c) $X_\zeta \cap X_\xi \in [\mu]^{<\mu}$ for $\zeta \ne \xi < \chi$.

As $\mathbb{Q}_\mu \lessdot \mathbb{P}_\mu$, by the earlier proof (e.g., 1.17(1)) it suffices to prove that $\mathbb{P}_\mu$ collapses χ to λ_*. Let $\mathbf{P} = \{\bar{A} : \bar{A} = \langle A_\alpha : \alpha < \mu \rangle$, the A_α's are pairwise disjoint and each A_α belongs to $[\mu]^\mu\}$, such that $|\mathbf{P}| = \lambda_*$ and

$(*)_2$ for every $p \in \mathbb{P}_\mu$ there is a $\bar{A} \in \mathbf{P} \cap \mathfrak{B}$ such that $(\forall \alpha < \mu)[|A_\alpha \cap p| = \mu]$.

[Why? By induction on $\varepsilon < \kappa$ we can find $\delta_\varepsilon < \mu$ of cofinality λ^{++}_ε such that $p \cap \delta_\varepsilon$ is unbounded in δ_ε and $\delta_\varepsilon > \cup\{\delta_\zeta : \zeta < \varepsilon\}$. There is a club $C^1_\varepsilon \in \mathfrak{B}$ of δ_ε of order type λ^{++}_ε with $\min(C^1_\varepsilon) > \cup\{\delta_\zeta : \zeta < \varepsilon\}$. Let $C^2_\varepsilon = \{\delta \in C^1_\varepsilon : \mathrm{otp}(\delta \cap C^1_\varepsilon) = \mathrm{otp}\{\alpha \in C^1_\varepsilon : (\alpha, \mathrm{Min}(C^1_\varepsilon \backslash (\alpha+1)) \cap p \ne \emptyset\}\}$, it is a club of δ_ε but in general not from $\mathfrak{B}$. But by the club guessing see 1.10 there is $C^3_\varepsilon \in \mathfrak{B}$ such that: $C^3_\varepsilon \subseteq C^2_\varepsilon (\subseteq C^1_\varepsilon)$ and $\mathrm{otp}(C^3_\varepsilon) = \lambda_\varepsilon$. We can find in $\mathfrak{B}$ also $\langle \langle W_{\varepsilon,\alpha} : \alpha < \lambda_\varepsilon \rangle : \varepsilon < \kappa \rangle$ a sequence such that $\langle W_{\varepsilon,\alpha} : \alpha < \lambda_\varepsilon \rangle$ is a partition of λ_ε to λ_ε (pairwise disjoint) sets each of cardinality λ_ε.

As $\lambda_* = \mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$ and $2^\kappa < \mu$, there is $a \in [\kappa]^\kappa \subseteq \mathfrak{B}$ such that $\langle C^3_\varepsilon : \varepsilon \in a \rangle \in \mathfrak{B}$. Lastly, let us define $\bar{A} = \langle A_\alpha : \alpha < \mu \rangle$ by

$$A_\alpha = \cup\{[\beta, \min(C^3_\varepsilon \backslash (\beta+1)) : \ \varepsilon \in a \text{ satisfies } \alpha < \lambda_\varepsilon \text{ and } \beta \in C^3_\varepsilon \text{ and } \mathrm{otp}(C^3_\varepsilon \cap \beta) \in W_{\varepsilon,\alpha}\}.$$

Easily $\langle A_\alpha : \alpha < \mu \rangle \in \mathfrak{B}$ is as required in $(*)_2$.]

Now for $\bar{A} \in \mathbf{P} \cap \mathfrak{B}$ we define a $\mathbb{P}_\mu$-name $\underset{\sim}{\tau}_{\bar{A}}$ as follows: for $\mathbf{G} \subseteq \mathbb{P}_\mu$ generic over $\mathbf{V}$,

$(*)_3$ $\underset{\sim}{\tau}_{\bar{A}}[\mathbf{G}] = \xi$ iff ξ is minimal such that $\cup\{A_\alpha : \alpha \in X_\xi\} \in \mathbf{G}$.

Clearly

$(*)_4$ $\underset{\sim}{\tau}_{\bar{A}}[\mathbf{G}]$ is defined in at most one way;

$(*)_5$ for every $p \in \mathbb{P}_\mu$ for some $\bar{A} \in \mathbf{P} \cap \mathfrak{B}$ for every $\xi < \chi$ we have $p \nVdash$ "$\underset{\sim}{\tau}_{\bar{A}} \neq \xi$".

[Why? Let $\bar{A} \in \mathbf{P} \cap \mathbf{B}$ be such that $(\forall \alpha < \mu)(\mu = |p \cap A_\alpha|)$, it exists by $(*)_2$. Now we can find q satisfying $p \leq q \in \mathbb{P}_\mu$ such that $(\forall \alpha < \mu)(q \cap A_\alpha$ is a singleton) and for each $\xi < \chi$ let $q_\xi = \cup\{A_\alpha \cap q : \alpha \in X_\xi\}$. Clearly $\zeta < \xi \Rightarrow |X_\zeta \cap X_\xi| < \mu \Rightarrow \cup\{A_\alpha : \alpha \in X_\zeta\} \cap q_\xi \subseteq \cup\{A_\alpha \cap q_\xi : \alpha \in X_\zeta\} = \cup\{A_\alpha \cap q : \alpha \in X_\zeta \cap X_\xi\} \in [\mu]^{<\mu}$, hence $q_\xi \Vdash$ "$\xi = \underset{\sim}{\tau}_{\bar{A}}$".]

So

$(*)_6$ $\Vdash_{\mathbb{P}_\mu}$ "$\chi = \{\underset{\sim}{\tau}_{\bar{A}}[\mathbf{G}] : \bar{A} \in \mathbf{P} \cap \mathfrak{B}\}$".

Together clearly $\mathbb{P}_\mu$ collapses χ to $\lambda_* + |\mathbf{P} \cap \mathfrak{B}|$ which is $\leq \|\mathfrak{B}\| = \lambda_*$, so as said above we are done. $\square_{1.18}$

Lastly, concerning the singular μ_* of cofinality $\aleph_0$ so we forget the hypothesis 1.2, 1.12.

Claim 1.19. *If $\mu_* > \mathrm{cf}(\mu_*) = \aleph_0$ and $\mathbb{P}_{\mu_*}$ fails the χ-c.c., <u>then</u> $\mathbb{P}_{\mu_*}$ collapse χ to $\aleph_1$; note that in this case $\mathbb{Q}_{\mu_*}$ is equivalent to* Levy$(\aleph_1, \mu_*^{\aleph_0})$ *by* [KS01].

Proof. Let $\lambda_* = \mu_*^{\aleph_0}$.

By Kojman Shelah [KS01], $\mathbb{P}_{\mu_*}$ collapse λ_* to $\aleph_1$ hence it suffices to prove that $\mathbb{P}_\mu$ collapse χ to λ_* assuming $\chi > \lambda_*$ (otherwise the conclusion is known). Let $\langle \lambda_n : n < \omega \rangle$ be a sequence of regular uncountable cardinals with limit μ_*. Now repeat the proof of 1.18 $\square_{1.19}$

§ 2. The regular uncountable case

We prove that (for κ regular uncountable), $\mathbb{P}_\kappa$ collapse λ to $\aleph_0$ iff $\mathbb{P}_\kappa$ fail the λ-c.c. This continues Balcar Simon [BS88, 2.8] so we first re-represent what they do; the proof of 2.6 is made to help later. In the present notation they let $\lambda = \mathfrak{b}_\kappa$ (rather than $\lambda \in \mathfrak{b}^{\rm spc}_\kappa$ as below a minor point); let $\langle f_\alpha : \alpha < \mathfrak{b}_\kappa\rangle$ be a sequence exemplifying it; let $C_\alpha = \{\delta < \kappa : (\forall \beta < \delta)(f_\alpha(\beta) < \delta), \delta \text{ a limit ordinal}\}$ and let $B_\alpha = \kappa \backslash C_\alpha$, so $\langle B_\alpha : \alpha < \lambda\rangle$ is a (κ,λ)-sequence (see Definition 2.5(1)), derive a good $(\kappa, {}^{\omega >}\lambda)$-sequence from it (see 2.5(2)), define $\alpha_n(A), \beta_n(A)$ and used the $A_{\eta,\delta,i}$'s to define the $\mathbb{P}_\kappa$-names $\underset{\sim}{\beta}_n$ and prove $\Vdash_{\mathbb{P}_\kappa}$ "$\{g^*(\underset{\sim}{\beta}_n) : n < \omega\} = \mathfrak{b}_\kappa$" (see 2.6). Then comes the major point we prove the new result: if $\mathbb{P}_\kappa$ fail the χ-c.c. then it collapses χ to $\aleph_0$.

A major point is that the proof splits to two cases: when $\lambda > \kappa^+$ and when $\lambda = \kappa^+$. In the 2019 revision we strengthen the result to "adding a generic to Levy$(\aleph_0, \chi)$".

Context 2.1. κ is a fixed regular uncountable cardinal.

Definition 2.2. 1) Let $\mathfrak{b}^{\rm spc}_\kappa$ be the set of regular $\lambda > \kappa$ such that there is a $<_{J^{\rm bd}_\kappa}$-increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of members of ${}^\kappa\kappa$ with no $\le_{J^{\rm bd}_\kappa}$-upper bound in ${}^\kappa\kappa$.
2) Let $\mathfrak{b}_\kappa = \text{Min}(\mathfrak{b}^{\rm spc}_\kappa)$.

Note that in Definition 2.3(1) we do not require that the B_α's define a MAD family.

Definition 2.3. 1) We say $\bar{B}$ is a (κ,λ)-sequence <u>when</u>:

- (a) $\bar{B} = \langle B_\alpha : \alpha < \lambda\rangle$
- (b) $B_\alpha \in [\kappa]^\kappa$ and $\kappa\backslash B_\alpha \in [\kappa]^\kappa$ and $B_{\alpha+1}\backslash B_\alpha \in [\kappa]^\kappa$
- (c) for every $B \in [\kappa]^\kappa$ for some $\alpha, B \cap B_\alpha \in [\kappa]^\kappa$
- (d) $B_\alpha \subseteq^* B_\beta$ when $\alpha < \beta < \lambda$, i.e., $B_\alpha\backslash B_\beta \in [\kappa]^{<\kappa}$.

2) We say that $\bar{B}$ is a $(\kappa, {}^{\omega >}\lambda)$-sequence <u>when</u>:

- (a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega >}\lambda\rangle$
- (b) $B_\eta \in [\kappa]^\kappa$
- (c) if $\eta_1 \triangleleft \eta_2 \in {}^{\omega >}\lambda$ then $B_{\eta_2} \subseteq^* B_{\eta_1}$ which means $B_{\eta_2}\backslash B_{\eta_1} \in [\kappa]^{<\kappa}$
- (d) $B_{<>} = \kappa$
- (e) if $\eta \in {}^{\omega >}\lambda$ and $A \in [B_\eta]^\kappa$ then for some $\alpha < \lambda$ we have $A \cap B_{\eta^\frown <\alpha>} \in [\kappa]^\kappa$
- (f) if $\eta \in {}^{\omega >}\lambda$ and $\alpha < \beta < \lambda$ then $B_{\eta^\frown <\alpha>} \subseteq^* B_{\eta^\frown <\beta>}$ and $B_\eta\backslash B_{\eta^\frown <\alpha>} \in [\kappa]^\kappa$ and $B_{\eta^\frown <\beta>}\backslash B_{\eta^\frown <\alpha>} \in [\kappa]^\kappa$.

3) For a $(\kappa, {}^{\omega >}\lambda)$-sequence $\bar{B}$ and $A \in [\kappa]^\kappa$ we try to define an ordinal $\alpha_k(A,\bar{B})$ by induction on $k < \omega$. If $\eta = \langle \alpha_\ell(A,\bar{B}) : \ell < k\rangle$ is well defined (which trivially holds for $k = 0$) and there is an $\alpha < \lambda$ such that $A \subseteq^* B_{\eta^\frown <\alpha>} \wedge (\forall \beta < \alpha)(A \cap B_{\eta^\frown <\beta>} \in [\kappa]^{<\kappa})$ then we let $\alpha_k(A,\bar{B}) = \alpha$; note that α, if exists, is unique.
3A) Let $n(A,\bar{B})$ be the $n \le \omega$ such that $\alpha_\ell(A,\bar{B})$ is well defined iff $\ell < n$.
4) We say that $(\bar{B},\bar{\nu})$ is a $(\kappa, {}^{\omega >}\lambda)$-parameter <u>when</u>:

(a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda\rangle$ is a $(\kappa, {}^{\omega>}\lambda)$-sequence

(b) $\bar{\nu}$ is an S^λ_κ-ladder which means that $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa\rangle, \nu_\delta$ is an increasing sequence of ordinals of length κ with limit δ, where $S^\lambda_\kappa = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$.

5) We say $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter <u>when</u> (a)+(b) of part (4) holds and

(c) if $A \in [\kappa]^\kappa$ then for some $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ and $A' \in [A]^\kappa$ we have
- (α) $\alpha_\ell(A', \bar{B}) = \eta(\ell)$ for $\ell < n$
- (β) for κ many ordinals $\zeta < \kappa$ we have $(\forall \varepsilon < \zeta)(A' \cap B_{\eta^\frown<\nu_\delta(\zeta)>} \backslash B_{\eta^\frown<\nu_\delta(\varepsilon)>}$ belongs to $[\kappa]^\kappa$).

6) $\bar{B}$ is a good $(\kappa, {}^{\omega>}\lambda)$-sequence if clause (a) of part (4) and clause (c) of part (5) hold for some S^λ_κ-ladder (see above). We say $(\bar{B}, \bar{\nu})$ is a weakly good $(\kappa, {}^{\omega>}\lambda)$-parameter if clause (a) of part (4) and clause (c)$^-$ of part (5) which means that we ignore subclause (α) there. Similarly $\bar{B}$ is a weakly good sequence.

Observation 2.4. *1) In 2.3(5)(c)(β), the "for κ many ordinals $\zeta < \kappa$" implies "for club many ordinals $\zeta < \kappa$".*
2) In 2.3(6) it does not matter which S^λ_κ-ladder you choose.

Proof. Notice, if $\nu_1, \nu_2 \in {}^\kappa\delta$ are increasing and $\sup(\nu_1) = \sup(\nu_2) = \delta$, then $\{i < \kappa : \bigcup\limits_{j<i} \nu_1(j) = \bigcup\limits_{j<i} \nu_2(j)\}$ is a club of κ, so it doesn't matter which S^λ_κ-ladder you choose. $\square_{2.4}$

Note that for §1 (except 1.18, 1.19) we need no more than Claim 2.5 (actually the weakly good version is enough for §1 except presenting the proof that $\mathfrak{b}_\kappa$ is collapsed).

Claim 2.5. *1) Assume $\lambda = \mathfrak{b}_\kappa$ or just $\lambda \in \mathfrak{b}^{\text{spc}}_\kappa$. <u>Then</u> λ is regular $> \kappa$ and there is a $\subseteq^*$-decreasing sequence $\langle C_\alpha : \alpha < \lambda\rangle$ of clubs of κ such that for no $\alpha < \lambda \Rightarrow$ no $A \in [\kappa]^\kappa$ do we have $\alpha < \lambda \Rightarrow A \subseteq^* C_\alpha$ and for $|C_\alpha \backslash C_{\alpha+1}| = \kappa$. Hence $\langle \kappa \backslash C_\alpha : \alpha < \lambda\rangle$ is a (κ, λ)-sequence.*
2) Assume $\bar{C} = \langle C_\alpha : \alpha < \lambda\rangle$ is as above and $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa\rangle$ is an S^λ_κ-ladder, see Definition 2.3(4), clause (b) (such $\bar{\nu}$ always exists). <u>Then</u> $\bar{B} = \bar{B}_{\bar{C}}, \bar{f} = \bar{f}_{\bar{C}}$ are well defined and $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter when we define $\bar{B}$ and $\bar{f}$ as follows:

⊛ (a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda\rangle$
- (b) $\bar{f} = \langle f_\eta : \eta \in {}^{\omega>}\lambda\rangle$
- (c) $B_{<>} = \kappa, f_{<>} = \text{id}_\kappa$
- (d) $B_\eta \in [\kappa]^\kappa, f_\eta$ *is a function from B_η onto κ, non-decreasing, and not eventually constant*
- (e) *if the pair (B_η, f_η) is defined and $\alpha < \lambda$ we let*
$$B_{\eta^\frown<\alpha>} = \{\gamma \in B_\eta : f_\eta(\gamma) \in \kappa \backslash C_\alpha\}$$
- (f) *if $\eta = \rho^\frown\langle\alpha\rangle$ and B_ρ, f_ρ are defined and B_η is defined as in clause (e), <u>then</u> we let $f_\eta : B_\eta \to \kappa$ be defined by $f_\eta(i) = \text{otp}(\{j$: for some $i_1 < i_2 < i$ we have $j = f_\rho(i_1) < f_\rho(i_2)$ and $f_\rho(i_1) \in C_\alpha$ and $f_\rho(i_2) \in C_\alpha\})$ for each $i < \kappa$*

(g) *if* $\eta\hat{\ }\langle\alpha\rangle \in {}^{\omega>}\lambda$ *then* $B_{\eta\frown<\alpha>} \subseteq B_\eta$ *and* $i \in B_{\eta\frown<\alpha>} \wedge f_\eta(i) > 0 \Rightarrow f_\eta(i) > f_{\eta\frown<\alpha>}(i)$.

Proof. 1) Recall $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

By the definition of $\mathfrak{b}^{\rm spc}_\kappa$ there is an $<_{J^{\rm bd}_\kappa}$-increasing sequence $\langle f^*_\alpha : \alpha < \lambda\rangle$ of members of ${}^\kappa\kappa$ with no $<_{J^{\rm bd}_\kappa}$-upper bound from ${}^\kappa\kappa$. Let $C_\alpha := \{\delta < \kappa : \delta$ is a limit ordinal such that $(\forall\gamma < \delta)(f^*_\alpha(\gamma) < \delta)\}$.

Clearly

$(*)_1$ C_α is a club of κ.

[Why? As κ is regular uncountable]

$(*)_2$ if $\alpha < \beta < \lambda$ then $C_\beta \subseteq^* C_\alpha$; i.e., $C_\beta\backslash C_\alpha \in [\kappa]^{<\kappa}$.

[Why? As if $\alpha < \beta$ then $f^*_\alpha <_{J^{\rm bd}_\kappa} f^*_\beta$, i.e., for some $\varepsilon < \kappa, (\forall\zeta)(\varepsilon \le \zeta < \kappa \Rightarrow f^*_\alpha(\zeta) < f^*_\beta(\zeta < \kappa))$; without loss of generality $\varepsilon \in C_\alpha \cap C_\beta$ hence $C_\beta\backslash(\varepsilon+1) \subseteq C_\alpha$ as required.]

$(*)_3$ for every club C of κ for some $\zeta < \lambda$ we have $C\backslash C_\zeta \in [\kappa]^\kappa$.

[Why? Toward contradiction assume C is a counterexample. Let $f : \kappa \to \kappa$ be defined by $f(i) =$ the $(i+1)$-th member of C, clearly for every $\alpha < \lambda$ for some $j < \kappa$, we have $C\backslash j \subseteq C_\alpha$ so possibly increasing j, without loss of generality $\mathrm{otp}(c \cap J) = \mathrm{otp}(C_\alpha \cap j)$. Hence easily $i \in (j,\kappa) \Rightarrow f_\alpha(i) < \min(C_\alpha\backslash i) \le \min(C\backslash i) \le f(i)$, hence $f_\alpha < f \mod J^{\rm bd}_\kappa$. As this holds for every $\alpha < \theta, f$ contradicts the choice of $\bar f$, i.e. $\bar f$ has no $\le_{J^{\rm bd}_\kappa}$-bound in ${}^\kappa\kappa$.]

Hence

$(*)_4$ for every unbounded subset A of κ for some $\zeta < \lambda$ we have $A\backslash C_\zeta \in [\kappa]^\kappa$.

[Why? Otherwise the closure of A contradicts $(*)_3$.]

Clearly without loss of generality $\alpha < \lambda \Rightarrow (C_\alpha\backslash C_{\alpha+1}| = \kappa$ hence $\langle C_\alpha : \alpha < \lambda\rangle$ is as required.

Lastly, let $B_\alpha = \kappa\backslash C_\alpha$, it is easy to check that $\langle B_\alpha : \alpha < \lambda\rangle$ is a (κ,λ)-sequence.

2) Clearly $\bar B_{\bar C}, \bar f_{\bar C}$ are well defined and $(\bar B, \bar\nu)$ is a $(\kappa, {}^{\omega>}\lambda)$-parameter and clauses (a)-(f) of $\circledast$ holds. Why is it good? Toward contradiction assume that it is not, so choose $A \in [\kappa]^\kappa$ which exemplify the failure of clause (c) of Definition 2.3(5) and define

$$\mathscr{T}_0 = \mathscr{T}^0_A = \{\eta \in {}^{\omega>}\lambda : \quad \text{there is } A' \in [A]^\kappa \text{ such that } \langle\alpha_\ell(A',\bar B) : \ell < \ell g(\eta)\rangle \text{ is well defined and equal to } \eta\}.$$

and define

$$\mathscr{T}_1 = \mathscr{T}^1_A := \{\eta \in \mathscr{T}^0_A : \quad \text{for every } k < \ell g(\eta) \text{ there are } < \kappa \text{ ordinals } \alpha < \eta(k) \text{ such that } \eta\hat{\ }\langle\alpha\rangle \in \mathscr{T}_0\}.$$

Clearly

$(*)_1$ $\mathscr{T}_0 \supseteq \mathscr{T}_1$ are non-empty subsets of ${}^{\omega>}\lambda$ (in fact $<>\in \mathscr{T}_1 \subseteq \mathscr{T}_0$)

$(*)_2$ $\mathscr{T}_0, \mathscr{T}_1$ are closed under initial segments.

For $\eta \in \mathscr{T}_\ell$ let $\mathrm{Suc}_{\mathscr{T}_\ell}(\eta) = \{\rho \in \mathscr{T}_\ell : \ell g(\rho) = \ell g(\eta) + 1 \text{ and } \eta \triangleleft \rho\}$.

We try to choose $A_\eta \in [B_\eta]^\kappa$ for $\eta \in \mathscr{T}_1$ by induction on $\ell g(\eta)$:

$(*)_3$ (a) $A_{<>} = A$

(b) if A_ν is defined and $\nu \frown \langle\alpha\rangle \in \mathscr{T}_1$ then we let $A_{\nu\frown\langle\alpha\rangle} = A_\nu \cap B_{\nu\frown\langle\alpha\rangle} \backslash \bigcup\{B_{\nu\frown<\beta>} : \beta < \alpha \text{ and } \nu \frown \langle\beta\rangle \in \mathscr{T}_1\}$.

Now

$(*)_4$ if $\nu \in \mathscr{T}_1$ then

(a) if $B \in [A]^\kappa$ and $\langle\alpha_\ell(B, \bar{B}) : \ell < \ell g(\nu)\rangle$ is well defined and equal to ν <u>then</u> A_ν is well defined and $B \subseteq^* A_\nu$

(b) if $j \in \{0, 1\}$ and $\mathrm{suc}_{I_j}(\nu)$ has cardinality $< \kappa$ then $A_\nu \backslash \cup \{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_j}(\nu)\}$ has cardinality $< \kappa$.

[Why? First we can prove clause (a) by induction on $\ell g(\nu)$ using the definition of $\mathscr{T}_1$ and clause (c) of 2.3(2). Second, we can prove clause (b) from it.]

$(*)_5$ $|\mathscr{T}_1| \geq \kappa$.

[Why? Otherwise by $(*)_4(b)$ the set $A' := \cup\{A_\nu \backslash \cup \{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_0}(\nu)\} : \nu \in \mathscr{T}_1\}$ is a subset of κ of cardinality $< \kappa$ and by clause (d) of $\circledast$ of the present claim also $A'' = \cup\{f_\nu^{-1}\{0\} : \nu \in \mathscr{T}_1\}$ is a subset of κ of cardinality $< \kappa$. So we can choose $j \in A \backslash (A' \cup A'')$. Now we try to choose $\nu_n \in \mathscr{T}_1$ by induction on n such that $\ell g(\nu_n) = n, \nu_{n+1} \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$ and $j \in A_{\nu_n}$.

First, $\nu_0 = <>$ belongs to $\mathscr{T}_1$ by clause $\circledast(c)$. Second, assume ν_n is well defined, then $\mathrm{Suc}_{\mathscr{T}_0}(\nu_n) = \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$.

[Why? Otherwise, as $\mathscr{T}_1 \subseteq \mathscr{T}_0$ there is an α with $\nu_n\hat{\ }\langle\alpha\rangle \in \mathrm{Suc}_{\mathscr{T}_0}(\nu_n) \backslash \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$, hence by the definition of $\mathscr{T}_1$ the set $u := \{\beta < \alpha : \nu_n\hat{\ }\langle\beta\rangle \in \mathscr{T}_0\}$ has cardinality $\geq \kappa$ but then $\beta \in u \wedge |\beta \cap u| < \kappa \Rightarrow \nu_n\hat{\ }\langle\beta\rangle \in \mathscr{T}_1$ which implies that $|\mathrm{Suc}_{\mathscr{T}_1}(\nu_n)| \geq \kappa$, contradiction to the "otherwise").

Now $j \notin A'$ and $A' \supseteq A_{\nu_n} \backslash \cup \{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)\}$ but $j \in A_{\nu_n}$ hence clearly $j \in \cup\{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)\}$ so we can choose ν_{n+1} as required. As $j \in A_{\nu_n} \subseteq B_{\nu_n}$ by $(*)_3(b)$ above clearly $f_{\nu_n}(j)$ is well defined (for each $n < \omega$). For each n, as $j \notin A''$ and $f_{\nu_n}^{-1}\{0\} \subseteq A''$, so $j \notin f_{\nu_n}^{-1}\{0\}$, necessarily $f_{\nu_n}(j) \neq 0$ and so $f_{\nu_n}(j) > f_{\nu_{n+1}}(j)$ by the choice of $f_{\nu_{n+1}}$ in clauses (g) of $\circledast$. Hence $\langle f_{\nu_n}(j) : n < \omega\rangle$ is decreasing, contradiction. So $(*)_5$ holds.]

Let $n < \omega$ be maximal such that $|\mathscr{T}_1 \cap {}^{n\geq}\lambda| < \kappa$, it exists as $|\mathscr{T}_1| \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $n = 0 \Rightarrow |\mathscr{T}_1 \cap {}^{n\geq}\lambda| = 1 < \kappa$, and let $\eta \in \mathscr{T}_1 \cap {}^n\lambda$ be such that $\mathrm{Suc}_{\mathscr{T}_1}(\eta)$ has $\geq \kappa$ members; it exist as κ is regular. We can choose an increasing sequence $\langle\alpha_i : i < \kappa\rangle$ of ordinals such that α_i is the i-th member of the set $\{\alpha < \lambda : \eta\frown\langle\alpha\rangle \in \mathscr{T}_1\}$ and let $A_i \in [A]^\kappa$ be such that $\langle\alpha_\ell(A_i, \bar{B}) : \ell \leq n\rangle = \eta\frown\langle\alpha_i\rangle$ and let $\delta = \cup\{\alpha_i : i < \kappa\}$ so $\delta \in S^\lambda_\kappa$.

Let

$$A_* = \cup\{A_i : i < \kappa\} \cap B_\eta \backslash A_\bullet$$

where

$$A_\bullet = \cup\{A_{(\eta\restriction\ell)\frown<\gamma>} : \ell < \ell g(\eta) \text{ and } \gamma < \eta(\ell) \text{ and } (\eta \restriction \ell)\frown\langle\gamma\rangle \in \mathscr{T}_1\}$$

(note that the number of pairs (ℓ, γ) as mentioned above is $< \kappa$).

Clearly $\alpha_\ell(A_*, \bar{B}) = \eta(\ell)$ for $\ell < \ell g(\eta)$ hence $\alpha_\ell(A_* \cap A_i, \bar{B}) = \eta(\ell)$ for $i < \kappa, \ell < n$ so clause (α) of (c) of Definition 2.3(5) holds (recalling 2.3(3)), as well as clause (β) because $\alpha_n(A_* \cap A_i, \bar{B}) = \alpha_i$ for $i < \kappa$.

So $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter indeed, hence we are done proving 2.5(2).

For later use note that we have:

$(*)_6$ for every $A \in [\kappa]^\kappa$ there are $n, \eta, \delta, \langle A_i = A_{\delta,\eta,i} : i < \kappa\rangle, A_*$ as above.

$\square_{??}$

Claim 2.6. *1) If there is a good $(\kappa, {}^{\omega>}\lambda)$-parameter and $\lambda_1 \in \mathfrak{b}^{\rm spc}_\kappa$, then the forcing notion $\mathbb{P}_\kappa$ collapses λ_1 to $\aleph_0$.*
2) Moreover, forcing with $\mathbb{P}_\kappa$ adds a generic to Levy$(\aleph_0, \lambda_1)$.

Proof. 1) Let $(\bar{B}, \bar{\nu})$ be a good $(\kappa, {}^{\omega>}\lambda)$-parameter.

Note

$\circledast_1$ if $A_1 \subseteq A_2$ are from $[\kappa]^\kappa$ and $\alpha_\ell(A_2, \bar{B})$ is well defined then $\alpha_\ell(A_1, \bar{B})$ is well defined and equal to $\alpha_\ell(A_2, \bar{B})$, recalling Definition 2.3(3).

Let $\bar{h} = \langle h_\gamma : \gamma < \lambda_1\rangle$ exemplify $\lambda_1 \in \mathfrak{b}^{\rm spc}_\kappa$, i.e., is as in Definition 2.2 and (as in the proof of 2.5) without loss of generality $[i < j < \kappa \Rightarrow i < h_\gamma(i) < h_\gamma(j)]$. For each $\delta \in S^\lambda_\kappa, i < \kappa$ and $\eta \in {}^{\omega>}\lambda$ let $A_{\eta,\delta,i} = B_{\eta\frown<\nu_\delta(i+1)>} \backslash \cup \{B_{\eta\frown<\nu_\delta(j+1)>} : j < i\}$ for $i < \kappa$ so $\langle A_{\eta,\delta,i} : i < \kappa\rangle$ are pairwise disjoint subsets of κ (each of cardinality κ). For $A \in [\kappa]^\kappa$ and $n < \omega$ we try to define an ordinal $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h})$ as follows:

$\circledast_2$ $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$ iff for some $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ we have $\langle \alpha_\ell(A, \bar{B}) : \ell \leq n\rangle = \eta\frown\langle\delta\rangle$ so in particular is well defined and $A \subseteq^* \cup\{A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa\}$ but for every $\beta < \gamma$ we have $A \cap \cup\{A_{\eta,\delta,i} \cap h_\beta(i) : i < \kappa\} \in [\kappa]^{<\kappa}$.

Next we define a $\mathbb{P}_\kappa$-name $\underset{\sim}{\beta}_n = \underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h})$ by:

$\circledast_3$ for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}$ we let: $\underset{\sim}{\beta}_n[\mathbf{G}] = \gamma$ iff for some $A \in \mathbf{G}$ we have $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$ or there is no such A and $\gamma = 0$.

[Why is this really a well defined name? Because

- if $A_1, A_2 \in [\kappa]^\kappa$ and $\beta_n(A_1, \bar{B}, \bar{\nu}, \bar{h}) = gamma > 0$ and $A_2 \subseteq^*_\kappa A_1$ then $\beta_n(A_2, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$.]

Now

$\circledast_4$ if $A \in [\kappa]^\kappa$ and (and $\mathscr{T}^0_A, \mathscr{T}^1_A$) n, η, δ are chosen as in the proof of 2.5(2), see after $(*)_5$, then the set $u := \{\beta < \lambda_1 : A \nVdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h}) \neq \beta$"$\}$ is a κ-closed unbounded subset of λ_1.

[Why? We know that $w := \{i < \kappa : A \cap A_{\eta,\delta,i} \in [\kappa]^\kappa\}$ has cardinality κ.

First, why is u "unbounded"? For any $\gamma_1 < \lambda_1$, we define a function $h \in {}^\kappa\kappa$ as follows, $h(i)$ is the minimal $i_1 < \kappa$ such that for some $i_0, i < i_0 < i_1$ the set $A \cap A_{\eta,\delta,i_0} \cap i_1 \backslash h_{\gamma_1}(i_0)$ is not empty, clearly h is well defined because $|w| = \kappa$. So for some $\gamma_2 \in (\gamma_1, \lambda_1)$ the set $v := \{i < \kappa : h(i) < h_{\gamma_2}(i)\}$ has cardinality κ. Let C be the club $\{\delta < \kappa : \delta$ is a limit ordinal and $i < \delta \Rightarrow h(i) < \delta \wedge h_{\gamma_2}(i) < \delta\}$ and let $\langle \alpha_\varepsilon : \varepsilon < \kappa\rangle$ list $C \cup \{0\}$ increasing order κ and let $A' = \cup\{A \cap A_{\eta,\delta,i} \cap [\alpha_\varepsilon, \alpha_{\varepsilon+1}) : i < \kappa, \varepsilon < \kappa$ and $\alpha_\varepsilon \leq i < \alpha_{\varepsilon+1}\}$, now $A' \in [\kappa]^\kappa$. So $\mathbb{P}_\kappa \models$ "$A \leq A'$" and $A' \Vdash$ "$\underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h}) \in (\gamma_1, \gamma_2]$", recalling that the h_γ's are increasing.

Second, why "the set u is κ-closed" (that is the limit of any increasing sequence of length κ of members belong to it)? Easy, too.]

Let $\langle S_\varepsilon : \varepsilon < \lambda_1 \rangle$ be pairwise disjoint stationary subsets of λ_1 included in $S^{\lambda_1}_\kappa$ and define $g^* : \lambda_1 \to \lambda_1$ by $g^*(\gamma) = \varepsilon$ if $\gamma \in S_\varepsilon \vee (\gamma \in \lambda_1 \backslash \bigcup\limits_{\zeta < \lambda_1} S_\zeta \wedge \varepsilon = 0)$.

So

$\circledast_5$ for every $p \in \mathbb{P}_\kappa$ for some n, for every $\varepsilon < \lambda_1, p \nVdash$ "$g^*(\underset{\sim}{\beta}_n) \neq \varepsilon$"

so we are done.

2) Let $\langle \rho_\varepsilon : \varepsilon < \lambda_1 \rangle$ list ${}^{\omega >}(\lambda_1)$ and we define the $\mathbb{P}_\kappa$-name $\underset{\sim}{\rho}$ by:

$\oplus$ $\underset{\sim}{\rho}$ is the concatenation of $\rho_{g^*(\underset{\sim}{\beta}_0)} \hat{} \rho_{g^*(\underset{\sim}{\beta}_1)} \Psi \ldots$.

Clearly $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\rho} \in {}^\omega(\lambda_1)$" and by $\circledast_4$ above.

$\circledast_5$ if $A \in [\kappa]^\kappa$ then for some n, η, δ we have:

- (a) A forces a value to $\underset{\sim}{\beta}_0, \ldots, \underset{\sim}{\beta}_{n-1}$ hence to $g^*(\underset{\sim}{\beta}_0) \ldots, g^*(\underset{\sim}{\beta}_{n-1})$ call them $\gamma_0, \ldots, \gamma_{n-1}$
- (b) for some club C of λ_1, for every $\delta \in C \cap S^{\lambda_1}_\kappa, A \nVdash$ "$\underset{\sim}{\beta}_n \neq \delta$"
- (c) for every $\varepsilon < \lambda_1, A \nVdash_{\mathbb{P}_\kappa}$ "$g^*(\underset{\sim}{\beta}_n) \neq \varepsilon$
- (d) for every $\nu \in {}^{\omega >}(\lambda_1)$ there is A' such that $\mathbb{P}_\kappa \models$ "$A \leq A'$" and $A' \Vdash_{\mathbb{P}_\kappa}$ "$\rho_{\gamma_0} \hat{} \ldots \hat{} \rho_{\gamma_{n-1}} \hat{} \nu \triangleleft \underset{\sim}{\rho}$".

This clearly suffices. $\square_{2.6}$

Now we arrive to the main point.

Main Claim 2.7. *1) If $\mathbb{P}_\kappa$ does not satisfy the χ-c.c. <u>then</u> forcing with $\mathbb{P}_\kappa$ collapses χ to $\aleph_0$. Morever, if $\mathfrak{b}^{\rm spec}_\lambda \neq \{\kappa\}$ then $\Vdash_{\mathbb{P}_\kappa}$ "there is $\rho \in {}^\omega\chi$ generic for* Levy$(\aleph_0, \chi)$ *over* $\mathbf{V}$*".*
*2) There is $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ such that $\bar{A}_\alpha = \langle A_{\alpha,i} : i < \kappa \rangle$ is a sequence of pairwise disjoint subsets of κ (*without loss of generality *each is a partition of κ to sets each of cardinality κ) such that for every $B \in [\kappa]^\kappa$ for some $\alpha < \mathfrak{b}_\kappa$ we have $i < \kappa \Rightarrow \kappa = |A_{\alpha,i} \cap B|$; i.e., for every $i < \kappa$ not just of κ many $i < \kappa$.*
3) In part (2) we can replace $\mathfrak{b}_\kappa$ by $\lambda \in \mathfrak{b}^{\rm spc}$ (so $\lambda = \kappa^+ \Rightarrow \mathfrak{b}_\kappa = \kappa^+$).

Proof. The proof is divided to two cases:

<u>Case 1</u>: $\lambda \in \mathfrak{b}^{\rm spc}_\kappa, \lambda > \kappa^+$.

So λ is regular $> \kappa^+$ and a good $(\kappa, {}^{\omega >}\lambda)$ sequence $\bar{B}$ exists (by 2.5).

Let $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa \rangle$ be such that $\nu_\delta \in {}^\kappa\delta$ is increasing continuous with limit δ and $\bar{\nu}$ guess clubs (i.e. for every club C of λ, for stationarily many $\delta \in S^\lambda_\kappa$ we have ${\rm Rang}(\nu_\delta) \subseteq C$); exists by [She94, Ch.III,§2] because $\lambda = {\rm cf}(\lambda) > \kappa^+$. As $\bar{B}$ be a good $(\kappa, {}^{\omega >}\lambda)$-sequence, $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega >}\lambda)$-parameter.

Let $\langle h_\alpha : \alpha < \lambda \rangle$ exemplify $\lambda \in \mathfrak{b}^{\rm spc}_\kappa$; without loss of generality $i < j < \kappa \Rightarrow i < h(i) <_{J^{\rm bd}_\omega} h(j)$.

For $\eta \in {}^{\omega >}\lambda, \delta \in S^\lambda_\kappa$ and $i < \kappa$, recall that $A_{\eta,\delta,i} = B_{\eta \frown <\nu_\delta(i+1)>} \backslash \cup \{B_{\eta \frown <\nu_\delta(j+1)>} : j < i\}$ and let $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}), \underset{\sim}{\beta}_n = \underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h})$ be defined as in the proof of 2.6. For $\eta \in {}^{\omega >}\lambda, \delta \in S^\lambda_\kappa, \delta^* \in S^\lambda_\kappa$ and $i < \kappa$ and $\gamma < \lambda$ let $B^*_{\eta,\delta,\gamma} := \cup\{A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa\}$. So clearly (for each $\eta \in {}^{\omega >}\lambda, \delta \in S^\lambda_\kappa$) the sequence $\langle B^*_{\eta,\delta,\gamma} : \gamma < \lambda \rangle$ is $\subseteq^*$-increasing. Let $A^*_{\eta,\delta,\delta^*,i} := B^*_{\eta,\delta,\nu_{\delta^*}(i+1)} \backslash \cup \{B^*_{\eta,\delta,\nu_{\delta^*}(j+1)} : j < i\}$. So $\langle A^*_{\eta,\delta,\delta^*,i} : i < \kappa \rangle$ are pairwise disjoint subsets of κ. Note that (by the proof of 2.6 but not used) for each pair (η,δ) as above for some club $E_{\eta,\delta}$ of λ, for every $\delta^* \in S^\lambda_\kappa \cap E_{\eta,\delta}$ satisfying $\text{Rang}(\nu_{\delta^*}) \subseteq E_{\eta,\delta}$ we have $i<\kappa \Rightarrow A^*_{\eta,\delta,\delta^*,i}$ has[1] of cardinality κ. We shall show during the proof of (1) that $\{\langle A^*_{\eta,\delta,\delta^*,i} : i<\kappa\rangle : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \delta^* \in S^\lambda_\kappa\}$ is as required in part (2), so this will prove part (2) when $\mathfrak{b}_\kappa > \kappa^+$ and also part (3) when $\lambda > \kappa^+$.

Let $\langle X^*_\xi : \xi < \chi\rangle$ be an antichain of $\mathbb{P}_\kappa$, exist by the assumption of 2.7(1). We now, for η,δ,δ^* as above define $\mathbb{P}_\kappa$-names $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$ if an ordinal $<\chi$: for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}$ we let:

$\circledast_0$ for $\xi \in (0,\chi), n<\omega$ and $\eta \in {}^n\lambda$ and $\delta,\delta^* \in S^\lambda_\kappa$ we have:

- $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}[\mathbf{G}] = \xi$ <u>iff</u> for some $A \in \mathbf{G}$
  - (a) $\langle \alpha_\ell(A,\bar B) : \ell < n\rangle = \eta$ so in particular is well defined
  - (b) $\alpha_n(A,\bar B) = \delta \in S^\lambda_\kappa$
  - (c) $\beta_n(A,\bar B,\bar\nu,\bar h) = \delta^* \in S^\lambda_\kappa$
  - (d) $A \cap A^*_{\eta,\delta,\delta^*,i}$ has at most one member for each $i<\kappa$
  - (e) $A \subseteq \cup\{A^*_{\eta,\delta,\delta^*,i} : i \in X^*_\xi\}$.

Note that demands (a),(b),(c) are natural but actually not being used for proving just the first half of 2.7(1); with them we can define the $\mathbb{P}_\kappa$-names $\underset{\sim}{\gamma}_n$ which is $\underset{\sim}{\gamma}_{n,\delta,\delta^*}$ when defined, see below.

Now clearly

$\circledast_1$ $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$ is a $\mathbb{P}_\kappa$-name of an ordinal $<\chi$ (may have no value)

$\circledast_2$ for every $p \in \mathbb{P}_\kappa$ for some $n,\eta \in {}^n\lambda \subseteq {}^{\omega>}\lambda$ and $\delta,\delta^* \in S^\lambda_\kappa$, for every $\varepsilon < \chi$ there is q such that $p \le q \in \mathbb{P}_\kappa$ and $q \Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} = \varepsilon$" and $\eta = \langle \alpha_\ell(q,\bar B) : \ell < n\rangle$.

[Why? We start as in the proof of 2.6. First, possibly increasing p, there are $n<\omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ such that $p \cap A_{\eta,\delta,i} \in [\kappa]^\kappa$ for κ many ordinals $i<\kappa$ and $\eta = \langle \alpha_\ell(p,\bar B) : \ell < n\rangle$.
Second, the set $W_p \subseteq \lambda$ is unbounded in λ (by the proof of 2.6) where

$$W_p = \{\beta < \lambda : \text{ for some } \gamma \in (\beta,\lambda) \text{ we have } p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \text{ is from } [\kappa]^\kappa\}.$$

Third, the club C_p of λ defined by $C_p = \{\delta < \lambda : \delta \text{ is a limit ordinal suchthat } \delta = \sup(W_p \cap \delta)\}$ satisfies:

$\circledast_{2.1}$ if $\beta < \gamma < \lambda$ are from C_p then $p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa$.

Now by the choice of $\bar\nu$, i.e., its being club guessing, there is $\delta^* \in \text{acc}(C_p) \cap S^\lambda_\kappa$ such that $(\forall i<\kappa)(\nu_{\delta^*}(i) \in C_p)$. So (note that we have used $\nu_{\delta^*}(i+1), \nu_{\delta^*}(j+1)$ in the definition of $A^*_{\eta,\delta,\delta^*,i}$)

$\circledast_{2.2}$ $i<\kappa \Rightarrow p \cap A^*_{\eta,\delta,\delta^*,i} \in [\kappa]^\kappa$.

[1] and also $i<\kappa \Rightarrow A_{\eta_1,\delta^*,i} \in [\kappa]^\kappa$

This fulfills the promise needed for proving part (2) in the present case 1. Choose $\zeta_i \in p \cap A^*_{\eta,\delta,\delta^*,i}$ for $i < \kappa$. Now for every $\xi < \chi$ let $q_\xi = \{\zeta_i : i \in X^*_\xi\}$. Recall that $\langle X^*_\zeta : \zeta < \chi\rangle$ is an antichain in $\mathbb{P}_\kappa$. Clearly for $\xi < \chi$ we have $\mathbb{P}_\kappa \models$ "$p \le q_\xi$" and $q_\xi \Vdash$ "$\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} = \xi$"; so we have finished proving $\circledast_2$.]

This is enough for proving

> $\circledast_3$ forcing with $\mathbb{P}_\kappa$ collapse χ to $\aleph_0$.

[Why? By $\circledast_1 + \circledast_2$ we know that $\Vdash_{\mathbb{P}_\kappa}$ "$\chi = \{\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$ and $\delta^* \in S^\lambda_\kappa\}$", so it is forced that $|\chi| \le |\lambda|$. As we already have by 2.6 that $\Vdash_{\mathbb{P}_\kappa}$ "$|\lambda| = \aleph_0$", we are done.]

For proving the "moreover" in 2.7(1):

> $\circledast_4$ for $n < \omega$ we define the $\mathbb{P}_\kappa$-name $\underset{\sim}{\gamma}_n$ of an ordinal $< \chi$ by: for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}$ and $\xi \in (0,\chi)$ we have $\underset{\sim}{\gamma}_n[\mathbf{G}] = \xi$ <u>iff</u>
> $\{\xi\} = \{\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} : \delta, \delta^* \in S^\lambda_\kappa$ and $\eta \in {}^n\lambda\}\backslash\{0\}$.

Next

> $\circledast_5$ if $A \in \mathbb{P}_\kappa$ then for some $n, \eta, \langle \beta_\ell : \ell, n < \omega, \eta \in {}^n\lambda, \beta_\ell < \lambda, \xi_\ell < \chi$ for $\ell < n, \delta \in S^\lambda_\kappa, \delta^* \in S^\lambda_\kappa$ we have: for every $\xi < \chi$ for some $A' \in [\kappa]^\kappa$ we have:
>
> - $\bullet_1$ $A' \Vdash$ "$\underset{\sim}{\beta}_\ell = \beta_\ell$" for $\ell < n$
> - $\bullet_2$ $A' \Vdash$ "$\underset{\sim}{\gamma}_\ell = \xi_\ell$" for $\ell < n$
> - $\bullet_3$ $A' \Vdash$ "$\underset{\sim}{\beta}_n = \delta, \underset{\sim}{\gamma}_n = \xi$".

This is proved as in the proof of 2.6(2).

Now as there:

> $\circledast_6$ in $\mathbf{V}^{\mathbb{P}_\kappa}$ there is $\eta \in {}^\omega\chi$ which is a generic for Levy$(\aleph_0, \chi)$ over $\mathbf{V}$.

So we are done with proving 2.7 in Case 1, i.e. when "there is $\lambda \in \mathfrak{b}^{\rm spc}_\kappa, \lambda > \kappa^+$".

<u>Case 2</u>: $\mathfrak{b}_\kappa = \kappa^+$.

Let [2] $\lambda = \kappa^+$ and $\bar{B}$ be a good $(\kappa, {}^{\omega>}\lambda)$-sequence. Let $\langle S_\varepsilon : \varepsilon < \kappa\rangle$ be a partition of $S^{\kappa^+}_\kappa$ to (pairwise disjoint) stationary sets. For $\alpha < \kappa^+$ let $\langle u^\alpha_i : i < \kappa\rangle$ be an increasing continuous sequence of subsets of α each of cardinality $< \kappa$ with union α and without loss of generality $\alpha < \beta \Rightarrow (\forall^\infty i < \kappa)(u^\alpha_i = u^\beta_i \cap \alpha)$. Let $\bar{h} = \langle h_\beta : \beta < \kappa^+\rangle$ exemplifying $\kappa^+ \in \mathfrak{b}^{\rm spc}_\kappa$ be such that each h_β is strictly increasing, $(\forall i) h_\beta(i) > i$ and let $C_\beta = \{\delta < \kappa : \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta\}$ and let $(\bar{B}, \bar{\nu})$ be a good $(\kappa, {}^{\omega>}\lambda)$-parameter; exists by 2.5(2). Now for $\eta \in {}^{\omega>}\lambda$ and $\delta \in S^\lambda_\kappa$ and define $A_{\eta,\delta,i}(i < \kappa), B^*_{\eta,\delta,\gamma}(\gamma < \lambda)$ as in Case 1. Now for $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define the sequence $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha\rangle$ by

$$\begin{aligned} Y_{\eta,\delta,\alpha,\beta,\gamma} = \quad & \cup\{B^*_{\eta,\delta,\gamma} \cap [i, \mathrm{Min}(C_\beta \backslash (i+1)) \backslash \cup \{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma \cap u^\alpha_i\} : \\ & i \in C_\beta \text{ satisfy } \gamma \in u^\alpha_i\}. \end{aligned}$$

[2]Actually, we can make this case cover Case 1 too: for $\delta_* \in S^\lambda_{\kappa^+}$ choose C'_{δ_*} a club of δ_* of order type κ^+. Now for each δ we can repeat the construction of names from the proof of Case 2, for each $p \in \mathbb{P}_\kappa$ for some δ_* we succeed to show $\circledast$ below.

So $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha\rangle$ is a sequence of pairwise disjoint subsets of κ and for $\varepsilon < \kappa$ let

$$Z_{\eta,\delta,\alpha,\beta,\varepsilon} = \cup\{Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma \in S_\varepsilon \cap \alpha\}.$$

Clearly

$\boxdot_1$ (a) for $(\eta,\delta,\alpha,\beta)$ as above, $\langle Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of pairwise disjoint subsets of κ

(b) for (η,δ,α) as above such that $(\forall\varepsilon < \chi)(S_\varepsilon \cap \alpha \neq \emptyset)$, for every large enough $\beta < \kappa^+$ we have $\varepsilon < \kappa \Rightarrow Z_{\eta,\delta,\alpha,\beta,\varepsilon} \in [\kappa]^\kappa$. is a sequence of pairwise disjoint subsets of κ

We shall show during the proof of (1) that

$$< \langle Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon < \kappa\rangle : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \lambda, \beta < \lambda >$$

exemplify part (2) and in the present case, part (3) is equivalent to part (2).

Let $\langle X^*_\xi : \xi < \chi\rangle$ be a family of sets from $[\kappa]^\kappa$ such that the intersection of any two have cardinality $< \kappa$, it exists as $\mathbb{P}_\kappa$ fail the χ-c.c.. For each $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define a $\mathbb{P}_\kappa$-name $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}$ as follows:

$\boxdot_2$ for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}, \underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}[\mathbf{G}] = \xi$ <u>iff</u>:

(α) for some $A \in \mathbf{G}$ we have

(a) $\varepsilon < \kappa \Rightarrow A \cap Z_{\eta,\delta,\alpha,\beta,\varepsilon}$ has at most one member

(b) $A \subseteq \cup\{Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon \in X^*_\xi\}$

(β) if for no $A \in \mathbf{G}$ does (a)+(b) hold then $\xi = 0$.

Clearly

$\boxdot_3$ $\underset{\sim}{\tau}_{\eta,\gamma,\alpha,\beta}$ is a well defined ($\mathbb{P}_\kappa$-name) (by $\boxdot_2$).

Now

$\boxdot_4$ for every $p \in \mathbb{P}_\kappa$, for some $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+, \beta < \kappa^+$ we have: for every $\xi < \chi$ for some $q \in \mathbb{P}_\kappa$ above p we have $q \Vdash$ "$\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta} = \xi$".

As in Case 1, this is enough for proving that $\mathbb{P}_\kappa$ collapse χ to $\lambda = \kappa^+$. But by 2.6 we already know that forcing with $\mathbb{P}_\kappa$ collapses κ^+ to $\aleph_0$ and so we are done except the "Moreover" in part (1).

Note: we can eliminate η from the $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}$, but not worth it. So we are left with proving $\boxdot_4$.

Why does $\boxdot_4$ hold? First, as in the earlier cases, find $\eta \in {}^{\omega>}\lambda$ and $\delta \in S^\lambda_\kappa$ such that $p \cap A_{\eta,\delta,i} \in [\kappa]^\kappa$ for κ ordinals $i < \kappa$. Second, as in the previous proof $W_p \subseteq \lambda = \sup(W_p)$ where $W_p := \{\beta < \lambda$: for some $\gamma \in (\beta,\lambda)$ we have $p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa\}$. Third, for some club C_p of λ we have $\beta < \gamma \wedge \beta,\gamma \in C_p \Rightarrow p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa$. As S_ε (for $\varepsilon < \kappa$) is a stationary subset of λ and C_p a club of λ for each $\varepsilon < \kappa$ we can choose $\gamma^*_\varepsilon \in S_\varepsilon \cap C_p$. Hence there is $\alpha^* < \kappa^+$ large enough such that $\varepsilon < \kappa \Rightarrow \gamma^*_\varepsilon < \alpha^* \in C_p$. Now define a function $h : \kappa \to \kappa$ by induction on i, as follows:

$$h(i) = \text{Min}\{j : \begin{array}{l} j \in (i,\kappa) \text{ and } i_1 < i \Rightarrow h(i_1) < j \text{ and if} \\ \gamma \in u^{\alpha^*}_i \cap S_\varepsilon \text{ then} \\ p \cap (i,j) \cap B^*_{\eta,\delta,\gamma} \backslash \cup \{B^*_{\eta,\delta,\beta} : \beta \in \gamma \cap u^{\alpha^*}_i\} \text{ is not empty}\}. \end{array}$$

It is well defined as for a given $i < \kappa$ the number of pairs (γ,ε) such that $\gamma \in u^{\alpha^*}_i \cap S_\varepsilon$ is $< \kappa$ and is increasing; next we define

$$C = \{j < \kappa : j \text{ is a limit ordinal such that } i < j \Rightarrow h(i) < j\}.$$

Clearly C is a club of κ and let $h' \in {}^\kappa\kappa$ be defined by $h'(i) = h(\text{Min}(C\backslash(i+1))$. By the choice of $\langle h_\beta : \beta < \lambda\rangle$ there is $\beta < \lambda$ such that for κ many ordinals $i < \kappa$ we have $h'(i) < h_\beta(i)$. Recall that $C_\beta = \{\delta < \kappa : \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta\}$.

So $W_1 = \{i < \kappa : h'(i) < h_\beta(i)\} \in [\kappa]^\kappa$. Let $\langle i^0_j : j < \kappa\rangle$ be an enumeration of $C \cap C_\beta$ in increasing order, so clearly $\mathscr{U} := \{j < \kappa : W_1 \cap [i^0_j, i^0_{j+1}) \neq \emptyset\}$ is unbounded in κ. For each $j \in \mathscr{U}$ let $i^1_j \in W_1 \cap [i^0_j, i^0_{j+1})$, and

$(*)$ $i^0_j \le i^1_j \le h(i^1_j) < h'(i^1_j) < h_\beta(i^1_j) < i^0_{j+1}$.

Now for each $\varepsilon < \kappa$ we know that $\gamma^*_\varepsilon \in \alpha^* \cap S_\varepsilon \cap C_p \subseteq \alpha^* = \cup\{u^{\alpha^*}_i : i < \kappa\}$ and $\langle u^{\alpha^*}_i : i < \kappa\rangle$ is $\subseteq$-increasing hence for some $j(\varepsilon) < \kappa$ if $j \in \mathscr{U}(j(\varepsilon))$ then $\gamma^*_\varepsilon \in u^{\alpha^*}_{i^0_j}$ hence by the choice of $h(i^1_j)$ and $(*)$ we have $p \cap (i^0_j, i^0_{j+1}) \cap B^*_{\eta,\delta,\gamma^*_\varepsilon} \backslash \cup \{B^*_{\eta,\delta,\beta} : \beta \in \gamma^*_\varepsilon \cap u^{\alpha^*}_{i^0_j}\}$ is not empty which by the definition of $Y_{\eta,\delta,\alpha^*,\beta,\gamma^*_\varepsilon}$ implies that $p \cap Y_{\eta,\delta,\alpha^*,\beta,\gamma^*_\varepsilon} \cap [i^0_j, i^0_{j+1}) \neq \emptyset$.

As this holds for every large enough $j \in \mathscr{U}\backslash j(\varepsilon)$ and $\mathscr{U} \in [\kappa]^\kappa$ it follows that $p \cap Y_{\eta,\delta,\alpha^*,\beta,\gamma^*_\varepsilon} \in [\kappa]^\kappa$. By the definition of $Z_{\eta,\delta,\alpha^*,\beta,\varepsilon}$ it follows that $p \cap Z_{\eta,\delta,\alpha^*,\beta,\varepsilon} \in [\kappa]^\kappa$.

Choose $\zeta_\varepsilon \in p \cap Z_{\eta,\delta,\alpha^*,\beta,\varepsilon}$. Now for each $\xi < \chi$ let

$$q_\xi = \{\zeta_\varepsilon : \varepsilon \in X^*_\xi\}.$$

So clearly:

$$\xi < \chi \Rightarrow \mathbb{P}_\kappa \models \text{“}p \le q_\xi\text{” and } q_\xi \Vdash_{\mathbb{P}_\kappa} \text{“}\mathcal{I}_{\eta,\delta,\alpha^*,\beta} = \xi\text{”}.$$

What about the “Moreover” (in part (1))?

It is not clear that for every n the sequence $\langle \mathcal{I}_{\eta,\delta,\alpha,\beta} : \eta \in {}^n\lambda, \delta \in S^\lambda_\kappa, \alpha < \lambda, \beta < \lambda\rangle$ has at most one non-zero entry. Clearly the following suffices (we could have used it in both cases).

$\boxplus_0$ If (A) then (B) where:

- (A)
  - (a) $\bar B = \langle B_\alpha : \alpha < \lambda\rangle$
  - (b) $B_\alpha \in [\kappa]^\kappa$ is $\subseteq^*_\kappa$-increasing
  - (c) $(\forall \alpha < \lambda)(\exists \beta)(\alpha < \beta < \lambda)$
- (B) there is a sequence $\langle \bar A_\alpha : \alpha < \kappa'\rangle$ such that:
  - (a) $\bar A_\alpha = \langle A_{\alpha_i} : i < \lambda\rangle$ is a partition of χ

(b) if $A \in [\kappa]^\kappa$ and $(\forall \beta < \lambda)[A \backslash (\forall \beta < \lambda)(\exists \gamma)[\beta < \gamma < \lambda \wedge A \cap (B_\gamma \backslash B_\beta) \in [\kappa]^\kappa]$ <u>then</u> for stationarily many $\delta \in S^\lambda_\kappa$ we have $(\forall i < \kappa)(A_\alpha \cap A_{\alpha,i} \in [\kappa]^\kappa)$

(c) if $\bar{B}$ is as in xyz, then the assumption of clause B means "$A \in [\kappa]^\kappa$ and $(\forall \beta < \lambda)[A \backslash B_\beta \in [\kappa]^\kappa]$

(d) if $\lambda = \kappa^+$, then in clause (b) the conclusion holds for every $\delta < \kappa^+$ large enough.

[Why? The main case is $\lambda = \kappa^+$.

$\oplus_1$ For $\delta \in S^\lambda_\kappa$ and $\gamma < \lambda$ we define $A^\bullet_{\delta,\gamma} = \cup\{(B_{\nu_\delta(i)} \cap B_\delta \backslash \cup \{B_{\nu_\delta(j)} : j < i\}) \cap h_\gamma(i) : i < \kappa\}$

$\oplus_2$ (a) if $\delta_1 < \delta_2$ are from S^λ_κ and $\gamma < \lambda$ then $A^\bullet_{\delta_1,\gamma} \cap A^\bullet_{\delta_2,\gamma} \in [\kappa]^{<\kappa}$

(b) if $\delta \in S^\lambda_\kappa$ and $\gamma_1 < \gamma_2 < \lambda$ then $A^\bullet_{\delta_1,\gamma_1} \subseteq A^\bullet_{\delta,\gamma_2} \mod [\kappa]^{<\kappa}$.

[Why? For clause (a), because $A^\bullet_{\delta_1,\gamma} \subseteq B_{\delta_1}$ whereas $A^\bullet_{\delta_2,\gamma} \cap B_{\delta_1} \subseteq \cup\{B_{\nu_{\delta_2}(i)} \cap B_\delta \backslash \{B_{\nu_{\delta_2}(j)} : j < i\} : i < i(\delta_1, \delta_2)\}$ where $i(\delta_1, \delta_2) = \min\{j : \nu_{\delta_2}(j) > \delta_1\}$ (for $\delta_1 < \delta_2$ from S^λ_κ).

Now the last set is the union of $\leq |i(\delta_2, \delta_2)| < \kappa$ sets, the i-th set of cardinality $\leq |h(i)| < \kappa$, so $A^\bullet_{\delta_1,\gamma} \cap A^\bullet_{\delta_2,\gamma} \in [\kappa]^{<\kappa}$. Clause (b) is easy, too.]

$\oplus_3$ for $\gamma < \lambda$ there is a function $g_\gamma : S^\lambda_\kappa \cap \gamma \to \kappa$ such that $\langle A^\bullet_{\delta,\gamma} \backslash g_\gamma(\delta) : \delta \in S^\lambda_\kappa \cap \gamma \rangle$ is a sequence of pairwise disjoint sets.

Lastly,

$\oplus_4$ for $\gamma \in S^\lambda_\kappa$ we let $\bar{A}_\gamma = \langle A_{\gamma,\varepsilon} : \varepsilon < \kappa \rangle$ be defined by $A_{\gamma,\varepsilon} = \cup\{A^\bullet_{\delta,\gamma} \backslash g_\gamma(\delta) : \delta \in S_\varepsilon \cap \gamma\}$.

Note that by $\oplus_2 + \oplus_3$

$\oplus_5$ $\langle A_{\gamma,\varepsilon} : \varepsilon < \kappa \rangle$ is a sequence of pairwise disjoint subsets of κ.

Now the main point is:

$\oplus_6$ if $A \in [\kappa]^\kappa$ is as in clause (b) of $\oplus$ then for a club of $\gamma \in S^\lambda_\kappa, (\forall \varepsilon < \kappa)[A \cap A_{\gamma,\varepsilon} \in [\kappa]^\kappa]$.

Why? First as earlier, there is a club E_i of λ such that: $\beta < \gamma \in E \Rightarrow A \cap B_\gamma \backslash B_\beta \in [\kappa]^\kappa$ and let $E_2 = \{\delta \in E_1 : \delta = \sup(\delta \cap E_1)\}$.

Second, for each $\varepsilon < \kappa$ choose $\delta(\varepsilon) \in E_1 \cap S_\varepsilon$ and let $\gamma(\varepsilon) < \lambda$ be such that $\gamma(\varepsilon) \leq \gamma < \lambda \Rightarrow A^\bullet_{\delta,\gamma} \cap A \in [\kappa]^\kappa$.

Third, let $\gamma_* = \cup\{\gamma_\varepsilon + 1 : \varepsilon < \kappa\} < \lambda$; and we shall show:

- if $\delta \in S^\lambda_\kappa \backslash \gamma_*$ then $(\forall \varepsilon < \kappa)(A_{\delta,\varepsilon} \cap A \in [\kappa]^\kappa)$.

Clearly this suffices.

Why $\bullet$ holds? For $\varepsilon < \kappa, A^\bullet_{\delta_\varepsilon,\gamma} \subseteq A_{\delta,\varepsilon} \mod [\kappa]^{<\kappa}$ and $A^\bullet_{\delta_\varepsilon,\gamma} \cap A \in [\kappa]^\kappa$ hence $A_{\delta,\varepsilon} \cap A \in [\kappa]^\kappa$, as promised. $\square_{2.7}$

Conclusion 2.8. *If κ is regular uncountable and $\mathbb{P}_\kappa$ fail the 2^κ-c.c.* <u>*then*</u> $\mathrm{comp}(\mathbb{P}_\kappa)$ *is isomorphic to the completion of* $\mathrm{Levy}(\aleph_0, 2^\kappa)$.

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