BOUNDED m-ARY PATCH-WIDTH ARE EQUIVALENT FOR $m \geq 3$

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Abstract. We consider the notion of bounded m-ary patch-width defined in [4], and its very close relative m-constructibility defined below. We show that the notions of m-constructibility all coincide for $m \geq 3$, while 1-constructibility is a weaker notion. The same holds for bounded m-ary patch-width. The case $m = 2$ is left open.

1. Introduction

1.1. **Background.** Our interest in this subject started from investigating spectra of monadic sentences, so let us begin with a short description of spectra. Let ϕ be a sentence in (a fragment of) second order logic (SOL). The spectrum of ϕ is the set $\{n \in \mathbb{N} : \phi \text{ has a model of size n}\}$. In 1952 Scholz defined the notion of spectrum and asked for a characterization of all spectra of first order (FO) sentences. In [1] Asser asked if the complement of a FO spectrum is itself a FO spectrum.

Definition 1.1. *A set $A \subseteq \mathbb{N}$ is eventually periodic if for some $n, p \in \mathbb{N}$, for all $m > n$, $m \in A$ iff $m + p \in A$.*

In [3] Durand, Fagin and Loescher showed that the spectrum of a FO sentence in a vocabulary with finitely many unary relation symbols and one function symbol is eventually periodic. In [5] Gurevich and Shelah generalized this for spectrum of monadic second order (MSO) sentence in the same vocabulary. Inspired by [5] Fisher and Makowsky in [4] showed that the spectrum of a CMSO sentence (a monadic sentence with counting quantifiers) is eventually periodic provided that all its models have bounded patch-width. The notion of patch-width of structures (usually graphs) is a complexity measure on structures, generalizing clique-width. Their proof remains valid if we consider m-ary patch-width, i.e. we allow m-ary relations as auxiliary relations. In [6] Shelah generalized the proof of [5] and showed eventual periodicity for a MSO sentence provided that all its models are constructible by recursion using operations that preserve monadic theory (see definitions below).

1.2. **summation of results.** The above results on eventual periodicity led us to ask: What are the relations between the different notions for which we have eventual periodicity of MSO spectra? In other words do we have three different results, or are they all equivalent? We give an answer here. In [2] Courcelle proved (using somewhat different notations) that a class of structures is constructible iff it is monadicly interpretable in trees, thus implying that two of the results coincide. We give a proof of Courcelle's result more coherent with our definition, which we use

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later on. We prove that the notions of bounded m-ary patch-width is very close to m-constructibility (constructibility where we allow m-ary relations as auxiliary relations) (see lemmas 2.9 and 2.10). Next we show that for $m \geq 3$ a class of modes is contained in a m-constructible class iff it is contained in a 3-constructible class (see Theorem 3.7). The same holds for classes of bounded m-ary patch-width. Finally we show that in the above theorem we can not replace 3-constructible by 1-constructible. That is there exists a 3-constructible class which is not contained in any 1-constructible class. We give a specific example (see 4.1). The case $m = 2$ is left open.

2. Preliminary definitions and previous results

Notation 2.1.

(1) *Let τ be a finite relational vocabulary.*
(2) *For $R \in \tau$ let $n(R)$ be the number of places of R. We say that R is $n(R)$-ary or $n(R)$ place. We allow $n(R) = 0$ i.e. the interpretation of R is in $\{\mathbb{T}, \mathbb{F}\}$. We call τ nice if $R \in \tau \Rightarrow n(R) > 0$.*
(3) *For $k \in \omega$, let τ_k be $\tau \cup \{P_1, ..., P_k\}$ with $P_1, ..., P_k$ unary predicates.*
(4) *A k-colored τ-structure is a τ_k-structure in which the interpretation of the P_i's is a partition of the set of elements of the model (but some P_i's may be empty).*
(5) *A k-const τ-structure is a τ_k-structure in which every predicate P_i is interpreted by a singleton. We denote such a structure by $(M, a_1, ..., a_k)$ where M is a τ-structure and $a_1, ..., a_k \in M$.*

Definition 2.2.

(1) *A monadic second order (MSO) formula in vocabulary τ is a second order formula in which every second order quantifier quantifies an unary relation symbol. The notion of quantifier depth extends, naturally to MSO formulas.*
(2) *Let M be a τ-structure, and q a natural number. The monadic q-theory of M, $Th_q^{MSO}(M)$, is the set of all sentences of quantifier depth $\leq q$ that hold in M.*
(3) *Let M be a τ-structure, and n, q natural numbers. Let $\bar{a} = (a_1, ..., a_n) \in {}^n|M|$. The q-type of $\bar{a}$ in M, $tp_q(\bar{a}, M)$, is the set of all τ formulas ϕ, of quantifier depth $\leq q$ in free variables $x_1, ..., x_n$, such that: $M \models \phi[a_1, ..., a_n]$. If $q = 0$ we sometimes write $tp_{qf}(\bar{a}.M)$.*
(4) *The notion of a q-type extends to MSO logic. We write $tp_q^{MSO}(\bar{a}, M)$ for the set of MSO formulas ϕ, of quantifier depth $\leq q$ in free variables $x_1, ..., x_n$, such that: $M \models \phi[a_1, ..., a_n]$.*
(5) *The set of all formally possible q-types in a vocabulary τ and in variables $\langle x_1, ..., x_n \rangle$, will be denoted be $TP_q(\langle x_1, ..., x_n \rangle, \tau)$, and similarly $TP_q^{MSO}(\langle x_1, ..., x_n \rangle, \tau)$. We may write $TP_q^{MSO}(n, \tau)$ instead of $TP_q^{MSO}(\langle x_1, ..., x_n \rangle, \tau)$.*

Definition 2.3 (Patch-width)**.**

(1) *Let τ be a nice vocabulary, M a τ-structure, k a natural number, and $\mathfrak{P}$ a finite set of k-colored τ-structures. We say that M have patch-width at most k (with respect to $\mathfrak{P}$) and denote $pwd_{\mathfrak{P}}(M) \leq k$, if M is the τ-redact of a k-colored τ-structure which is in the closer of $\mathfrak{P}$ under the operations:*
 (i) *disjoint union - $\sqcup$,*

(ii) *recoloring* - $\rho_{i \to j}$ *(change all the elements with color* P_i *to color* P_j*) and*

(iii) *modifications* - $\delta_{R,B}$ *(redefine the relation* $R \in \tau$ *by the quantifier free formula* B *in vocabulary* τ_k*).*

A class $\mathfrak{K}$ *of* τ*-structures is a* $PW(k)$*-class, if for some finite set of* k*-colored* τ*-structures* $\mathfrak{P}$ *the elements of* $\mathfrak{K}$ *are all the* τ*-redacts of structures of patch-width at most* k *with respect to* $\mathfrak{P}$*. We say* $\mathfrak{K}$ *is of bounded patch-width (BPW) if it is a* $PW(k)$*-class for some* $k \in \mathbb{N}$*.*

(2) *In the definition above we may instead of* k*-colored* τ*-structures, talk about* τ^+ *structures where* $\tau^+ \supseteq \tau$, $|\tau^+ \setminus \tau| = k$ *and every relation in* $\tau^+ \setminus \tau$ *is at most* m*-ary. We then talk about* m*-ary patch-width, where the rest of the definition remains unchanged. Note that the notions of patch-width and unary patch- width are close but not identical as in the former we demand that the colors are disjoint.*

In [4] it is proved that:

Theorem 2.4. *Let* ϕ *be a MSO(*τ*) sentence, and suppose* $Mod(\phi)$ *is contained in some class of bounded* m*-ary patch-width. Then spec(*ϕ*) is eventually periodic.*

Definition 2.5 (Addition operations)**.**

(1) *For* $k, k_1, k_2 \in \mathbb{N}$, *let* $\mathfrak{S}_{\tau,k,k_1,k_2}$ *be the set of all addition operations of a* k_1*-const* τ*-structure with a* k_2*-const* τ*-structure, resulting in a* k*-const* τ*-structure. Formally each* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$ *consists of:*
  (i) *Sets* $A_l = A_l^{\mathbf{s}} \subseteq \{1, ..., k_l\}$ *for* $l \in \{1, 2\}$*.*
  (ii) *For* $l \in \{1, 2\}$*, a 1-1 function* $g_l = g_l^{\mathbf{s}}$ *from* A_l *to* $\{1, ..., k\}$ *such that:* $Im(g_1) \cup Im(g_2) = \{1, ..., k\}$*.*
  (iii) *For* $l \in \{1, 2\}$ *a set* $B_l \subseteq \{1, ..., k_l\}^2$*, and a set* $B \subseteq \{1, ..., k_1\} \times \{1, ..., k_2\}$*.*
  (iv) *For each* $R \in \tau$ *with* $n(R) = n$ *and each* $w_l \subseteq \{1, ..., n\}$ *for* $l \in \{1, 2\}$*, a function* $f_{R,w_1,w_2} = f^{\mathbf{s}}_{R,w_1,w_2}$ *with range* $\{\mathbb{T}, \mathbb{F}\}$*, and domain: triplets of the form* (p, q_1, q_2) *where:*
    – $p \in TP_0(\langle x_1, ..., x_n \rangle, \sigma)$ *were* σ *is a vocabulary with* $k_1 + k_2$ *individual constants and two unary predicates,*
    – *For* $l \in \{1, 2\}$, $q_l \in TP_0(\langle x_i : i \in w_l \rangle, \tau)$*.*

(2) *Let* $k, k_1, k_2 \in \mathbb{N}$ *and* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$*. Let* $(M_l, a^l_1, ..., a^l_{k_l})$ *be* k_l*-const* τ*-structure for* $l \in \{1, 2\}$*. The addition* $(M_1, a^1_1, ..., a^1_{k_1}) \circledast_{\mathbf{s}} (M_2, a^2_1, ..., a^2_{k_2})$ *is defined whenever:*
  - $(|M_1| \cap |M_2|) \subseteq (\{a^1_1, ..., a^1_{k_1}\} \cap \{a^2_1, ..., a^2_{k_2}\})$ *and*
  - *For* $l \in \{1, 2\}$*:* $a^l_i = a^l_j \Leftrightarrow (i, j) \in B_l$ *and* $a^1_i = a^2_j \Leftrightarrow (i, j) \in B$*,*

  to be the k*-const* τ*-structure* $(M, b_1, ..., b_k)$ *defined by:*
  (i) $|M| = (|M_1| \setminus \{a^1_1, ..., a^1_{k_1}\}) \cup (|M_2| \setminus \{a^2_1, ..., a^2_{k_2}\}) \cup \{a^l_i : l \in \{1, 2\}, i \in A_l\})$*.*
  (ii) *For each* $l \in \{1, 2\}$ *and* $i \in A_l$, $a^l_i = b_{g_l(i)}$*.*
  (iii) *For all* $R \in \tau$ *with* $n(R) = n$ *and* $\bar{x} = (x_1, ..., x_n) \in {}^n|M|$*, let* $w_l = \{i : x_i \in |M_l|\}$ *for* $l \in \{1, 2\}$*. Let* p *be the quantifier free type of* $\bar{x}$ *in the model with* $\{a^1_1, ..., a^1_{k_1}\} \cup \{a^2_1, ..., a^2_{k_2}\}$ *as constants, and* $|M_1|, |M_2|$ *as unary predicates. For* $l \in \{1, 2\}$ *let* $q_l = tp_{qf}(\langle x_i : i \in w_l \rangle, M_l)$*. Now the value of* $R^M(\bar{x})$ *is defined to be* $f^{\mathbf{s}}_{R,w_1,w_2}(p, q_1, q_2)$*.*

(3) *For technical reasons we would like to allow empty structures. i.e. let* $\tau' := \{R \in \tau : n(R) = 0\}$, *and* $X \subseteq \tau'$. *Now* $Null_X$ *is the* τ'*-structure with* $|Null_X| = \emptyset$ *and* $R^{Null_X} = True \Leftrightarrow R \in X$. *Then if* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,0}$, *and* M *is a* τ_{k_1}*- structure then* $M \circledast_{\mathbf{s}} Null_X$ *is a well defined* τ_k*-structure. Furthermore for any* τ*-structure* M, $M \sqcup Null_\emptyset$ *is defined and equal to* M.

The important attributes of the addition operations are the following:

Theorem 2.6. *Let* $k, k_1, k_2 \in \mathbb{N}$. *Then:*

(1) $\mathfrak{S}_{\tau,k,k_1,k_2}$ *is finite.*
(2) The addition theorem:
Let M, M' *be* k_1*-const* τ*-structures such that* $Th^q_{MSO}(M) = Th^q_{MSO}(M')$, *and* N, N' *be* k_2*-const* τ*-structures such that* $Th^q_{MSO}(N) = Th^q_{MSO}(N')$, *and* $\mathbf{s} \in \mathfrak{S}_{\tau,k,k_1,k_2}$. *Assume that the additions* $M \circledast_{\mathbf{s}} N$ *and* $M' \circledast_{\mathbf{s}} N'$ *are defined. Then*
$$Th^q_{MSO}(M \circledast_{\mathbf{s}} N) = Th^q_{MSO}(M' \circledast_{\mathbf{s}} N').$$

Definition 2.7 (Constructibility)**.** *A class* $\mathfrak{K}$ *of* τ*-structures is* (m^*, k^*)*-constructible, if there exists: A finite relational vocabulary* $\tau^+ \supseteq \tau$, *a finite set of structures* $\mathfrak{P}$, *and a finite set of addition operations* $\mathfrak{S}$ *such that:*

(i) *Every relation in* $\tau^+ \setminus \tau$ *is at most* m^**-ary.*
(ii) *Every structure in* $\mathfrak{P}$ *is a* k*-const* τ^+*-structure for some* $k \leq k^*$.
(iii) *Every operation in* $\mathfrak{S}$ *is in* $\mathfrak{S}_{\tau^+,k,k_1,k_2}$ *for some* $k, k_1, k_2 \leq k^*$.
(iv) *The elements of* $\mathfrak{K}$ *are all the* τ*-redacts of structures in the closer of* $\mathfrak{P}$ *under the operations in* $\mathfrak{S}$.

We say that $\mathfrak{K}$ *is* m^**-constructible if it is* (m^*, k^*)*- constructible for some* k^*, *and that it is constructible if it is* m^**-constructible for some* m^*.

In [6] it is proved that:

Theorem 2.8. *Let* ϕ *be a MSO(*τ*) sentence, and suppose* $Mod(\phi)$ *is contained in some* m*-constructible class. Then* $spec(\phi)$ *is eventually periodic.*

This is a generalization of 2.4 as we have:

Lemma 2.9. *Let* τ *be a nice vocabulary, and* $\mathfrak{K}$ *be a* m*-ary* $PW(k)$*-class of* τ*-structures. Then* $\mathfrak{K}$ *is a* $(m, 0)$*-constructible class.*

Proof. First note that the disjoint union operation of τ^+-structures is in $\mathfrak{S}_{\tau^+,0,0,0}$. As for the recoloring and the modification operations, those are unary operations, so we look at the operation $\mathbf{s} \in \mathfrak{S}_{\tau^+,0,0,0}$ that acts as recoloring or modification on its left operand. So $M \circledast_{\mathbf{s}} Null_\emptyset$ is the desired recoloring or modification of M. □

In the addition operations we allow omitting marked elements, and the universe of the the two operands is not necessarily disjoint. This is not allowed in the operations of patch-width. It turns out though that these are the only essential differences between the two types of operations as suggested by the following:

Lemma 2.10. *Let* $\mathfrak{K}$ *be a* $(m, 0)$*-constructible class such that the vocabulary* τ^+ *associated with* $\mathfrak{K}$ *is nice. Then* $\mathfrak{K}$ *is of bounded* m*-ary patch-width.*

Proof. $\mathfrak{K}$ is $(m, 0)$-constructible so we have a vocabulary τ^+ and sets $\mathfrak{S}$ and $\mathfrak{P}$. Now the set of atomic structures for the patch-width definition will be the same $\mathfrak{P}$.

The vocabulary of the patch-width definition will be: $\tau^+ \dot\cup \{R' : R \in \tau^+\} \dot\cup \{P_1, P_2\}$. P_1, P_2 are new unary relation symbols. We now have to show for each operation in $\mathfrak{S}$ how to simulate it by operations of patch-width. Let $\mathbf{s} \in \mathfrak{S}$ and let M_1, M_2 be τ^+-structures. Denote by M_1', M_2' the trivial extensions to the new vocabulary. We will now describe a series of patch-width operations on M_1', M_2' resulting in a structure M^* such that $M^*|_{\tau^+} \cong M_1 \circledast_{\mathbf{s}} M_2$, this will complete the proof. First color all the elements of M_l' by P_l for $l \in \{1,2\}$. Next for each $R \in \tau^+$ redefine R' to be the same as R, do this for both M_1', M_2'. Now take the disjoint union of the to resulting structures. Finally we have to redefine the relations of τ^+ of the our disjoint union to be as in $M_1 \circledast_{\mathbf{s}} M_2$. Let $R \in \tau^+$ be n-ary and let $w_1, w_2 \subseteq \{1, ..., n\}$ satisfy $w_1 \dot\cup w_2 = \{1, ..., n\}$. Let p be the quantifier free type in the vocabulary with two unary relations S_1, S_2 "saying" that for $i \leq n$ and $l \in \{1,2\}$, $x_i \in S_l$ iff $i \in w_l$. Now define:

$$\varphi_{R,w_1,w_2}(x_1, ...x_n) := \bigwedge_{i \in w_1} P_1(x_i) \bigwedge_{i \in w_2} P_2(x_i) \wedge [\bigvee_{\substack{q_l \in TP_0(\langle x_i : i \in w_l \rangle, \tau^+) \\ f^{\mathbf{s}}_{R,w_1,w_2}(p,q_1,q_2) = \mathbb{T}}} \wedge q_1' \wedge q_2'].$$

Where $\wedge q_l'$ is the disjunction of all the formulas in q_l where we replace every relation $R \in \tau^+$ be R'. Now redefine the relation R using the modification $\delta_{R,B}$ for the formula:

$$B(x_1, ..., x_n) := \bigwedge_{\substack{w_1, w_2 \subseteq \{1,...,n\} \\ w_1 \dot\cup w_2 = \{1,...,n\}}} \varphi_{R,w_1,w_2}(x_1, ..., x_n).$$

Do this for all $R \in \tau^+$ and we are done. □

Notation 2.11 (Trees)**.**

(1) *The vocabulary of trees, τ_{trees}, is $\{\leq, c_{rt}\}$.*
(2) *The vocabulary of k-trees, $\tau_{k-trees}$, is $\{\leq, c_{rt}\} \dot\cup \{P_1, ..., P_k\}$ i.e $(\tau_{trees})_k$.*
(3) *A tree $\mathfrak{T}$ is a τ_{trees}-structure in which:*
    - *For every $t \in |\mathfrak{T}|$ the set $\{s \in |\mathfrak{T}| : s \leq^{\mathfrak{T}} t\}$ is linearly ordered by $\leq^{\mathfrak{T}}$.*
    - *For all $x \in |\mathfrak{T}|$, ${c_{rt}}^{\mathfrak{T}} \leq^{\mathfrak{T}} x$.*
(4) *A k-tree $\mathfrak{T}$ is a $\tau_{k-trees}$-structure, such that $\mathfrak{T}|\tau_{trees}$ is a tree.*
(5) *A 2-tree $\mathfrak{T}$ is directed binary (DB) if $(c^{\mathfrak{T}}_{rt}, P^{\mathfrak{T}}_1, P^{\mathfrak{T}}_2)$ is a partition of $|\mathfrak{T}|$, and each non maximal element of $\mathfrak{T}$ has exactly two immediate successors one in $P^{\mathfrak{T}}_1$ and the other in $P^{\mathfrak{T}}_2$. For $k \geq 2$, a k-tree $\mathfrak{T}$ is DB if $(|\mathfrak{T}|; \leq^{\mathfrak{T}}, c^{\mathfrak{T}}_{rt}, P^{\mathfrak{T}}_1, P^{\mathfrak{T}}_2)$ is.*

Definition 2.12 (Monadic interpretation)**.**

(1) *We call $\mathbf{c}$ a monadic k-interpretation scheme for a vocabulary τ if $\mathbf{c}$ consists of:*
    - *Natural numbers $k_1 = k^{\mathbf{c}}_1$ and $k_1 = k^{\mathbf{c}}_2$ both less then or equal to k.*
    - *For every $l \leq k_1$ a monadic $\tau_{k_2-trees}$-formula $\varphi^{\mathbf{c}}_{=,l}(x)$.*
    - *For every $R \in \tau$ n-place relation, and every $\eta \in {}^{\{1,...,n\}}\{0, ..., k_1\}$ a monadic $\tau_{k_2-trees}$-formula: $\varphi = \varphi^{\mathbf{c}}_{R,\eta}(x_1, ..., x_n)$.*
(2) *Let $\mathbf{c}$ be a monadic k-interpretation scheme for a vocabulary τ, and $\mathfrak{T}$ a $k^{\mathbf{c}}_2$-tree. The interpretation of $\mathfrak{T}$ by $\mathbf{c}$ denoted by $\mathfrak{T}^{[\mathbf{c}]}$ is the τ-model M defined by:*
    - $|M| = \{(t, l) \in |\mathfrak{T}| \times \{0, ..., k_1\} : \mathfrak{T} \models \varphi_{=,l}(t)\}$

- *For every $R \in \tau$ n-place relation:*

  $R^M = \{((t_i, l_i) : i \leq n) \in {}^n|M| : \mathfrak{T} \models \varphi_{R,(l_i:i\leq n)}(t_1, ..., t_n)\}$

(3) *A τ-model M is monadicly k-interpretable in trees if for some $\mathbf{c}$ a monadic k-interpretation scheme for τ, and some $k_2^{\mathbf{c}}$-tree, $\mathfrak{T}$, we have: $\mathfrak{T}^{[\mathbf{c}]} \cong M$. We denote the class of all the τ-structures monadicly k-interpretable in trees by $\mathfrak{K}^{mo}_{\tau,k}$.*

(4) *For $\mathbf{c}$ a monadic k-interpretation scheme for τ we denote by $\mathfrak{K}^{mo}_{\mathbf{c}}$ the class of all τ-structures M such that for some $k_2^{\mathbf{c}}$-tree, $\mathfrak{T}$, we have: $\mathfrak{T}^{[\mathbf{c}]} \cong M$. $\mathfrak{K}^{mo,db}_{\mathbf{c}}$ is the same as $\mathfrak{K}^{mo}_{\mathbf{c}}$ only we demand that $\mathfrak{T}$ is directed binary.*

(5) *We say that $\mathbf{c}$ has the leaf property if $k_1^{\mathbf{c}} = 0$ and for every $k_2^{\mathbf{c}}$-tree $\mathfrak{T}$, and every $t \in |\mathfrak{T}|$: $\mathfrak{T} \models \varphi^{\mathbf{c}}_{0,=}[t]$ implies that t is a maximal element in $\mathfrak{T}$.*

Without loss of generality we may assume that $k_1^{\mathbf{c}} = 0$. This is because of the following:

Lemma 2.13. *For every $\mathbf{c}$ a monadic k-interpretation scheme for a vocabulary τ, there exists $\mathbf{c}'$ be a monadic $(k+2)$-interpretation scheme for τ, such that:*

- $k_1^{\mathbf{c}'} = 0$.
- $k_2^{\mathbf{c}'} = k_2^{\mathbf{c}} + 2$.
- *For every $k_2^{\mathbf{c}}$-tree $\mathfrak{T}$, there exists a $k_2^{\mathbf{c}'}$-tree $\mathfrak{T}'$, such that: $\mathfrak{T}^{[\mathbf{c}]} \cong \mathfrak{T}'^{[\mathbf{c}']}$.*

Hence $\mathfrak{K}^{mo}_{\mathbf{c}} \subseteq \mathfrak{K}^{mo}_{\mathbf{c}'}$.

Proof. Let s_1 and s_2 be the two "new" unary predicates, and let $\mathfrak{T}$ be a $k_2^{\mathbf{c}}$-tree. Define $\mathfrak{T}'$ as follows: $|\mathfrak{T}'| = |\mathfrak{T}| \cup (|\mathfrak{T}| \times \{0, ..., k_1^{\mathbf{c}}\})$, $s_1^{\mathfrak{T}'} = |\mathfrak{T}|$, $s_2^{\mathfrak{T}'} = |\mathfrak{T}| \times \{0, ..., k_1^{\mathbf{c}}\}$, and if t_1 is the immediate successor of t_2 in $\mathfrak{T}$ then define, $t_1 <^{\mathfrak{T}'} (t_1, 0) <^{\mathfrak{T}'} (t_1, 1) <^{\mathfrak{T}'} ...(t_1, k_1^{\mathbf{c}}) <^{\mathfrak{T}'} t_2$. Now define:

$$\varphi^{\mathbf{c}'}_{=,0}(x) := s_2(x) \wedge \bigwedge_{l<k_1^{\mathbf{c}}} (\forall y)[s_1(y) \wedge (\psi_l(x,y)] \rightarrow (\varphi^{\mathbf{c}}_{=,l}(y))^{s_1}$$

Where $\psi_l(x,y)$ is a formula stating that there are exactly l elements between x and y and all of them are in s_2, and $(\varphi^{\mathbf{c}}_{=,l}(y))^{s_1}$ is the formula $\varphi^{\mathbf{c}}_{=,l}(y)$ relativized to s_1 i.e we replace every quantifier of the form $\exists x$ or $\forall x$ by $\exists x \in s_1$ or $\forall x \in s_1$ respectively. It should be clear that $\mathfrak{T} \models \varphi^{\mathbf{c}}_{=,l}[t]$ iff $\mathfrak{T}' \models \varphi^{\mathbf{c}'}_{=,0}[(t,l)]$. The relations are dealt with in a similar way. □

Lemma 2.14. *Let $\mathfrak{K}$ be a (m^*, k^*)-constructible class of τ-models. Then there exists a natural number k^{**} such that $\mathfrak{K} \subseteq \mathfrak{K}^{mo}_{\tau,k^{**}}$. Moreover for some monadic k^{**}-interpretation scheme $\mathbf{c}$ with the leaf property, we have $\mathfrak{K} \subseteq \mathfrak{K}^{mo,db}_{\mathbf{c}}$.*

We will not go into detail here especially as a similar result was proved by Courcelle in [2]. We do however give a sketch of a proof containing some definitions that will be useful later.

Sketch. Suppose $\mathfrak{P}$ and $\mathfrak{S}$ are the finite sets of structures and operations generating $\mathfrak{K}$, and τ^+ the vocabulary associated with $\mathfrak{K}$ (see 2.7). Now with every $M \in \mathfrak{K}$ we can associate a tree which represents the construction of M from the structures in $\mathfrak{P}$. Formally we define:

Definition 2.15. *We say that the pair $(\mathfrak{T}, \mathfrak{M})$ with $\mathfrak{T} = \langle T; \leq^{\mathfrak{T}}, c^{\mathfrak{T}}_{rt}, S^{\mathfrak{T}}_1, S^{\mathfrak{T}}_2 \rangle$ a DB tree and $\mathfrak{M} = \langle M_t : t \in T \rangle$, is a full representation of $M \in \mathfrak{K}$ when:*

(1) *Every M_t is a k_t-const τ^+-structure for some $k_t \leq k^*$.*
(2) *For every $t \in T$ $\leq^{\mathfrak{T}}$-maximal, $M_t \in \mathfrak{P}$.*
(3) *The τ-redact of $M_{c^{\mathfrak{T}}_{rt}}$ is M.*
(4) *For every t, a non-maximal element of T, let s_1, s_2 be its immediate successors with $s_l \in S^{\mathfrak{T}}_l$. Then $M_t = M_{s_1} \circledast_{\mathbf{s}} M_{s_2}$ for some $\mathbf{s} \in \mathfrak{S}_{\tau^+,k_{s_1},k_{s_2},k_t} \cap \mathfrak{S}$.*

Definition 2.16.

(1) *Let τ^* be the vocabulary $\tau_{k_2-trees}$ with the following unary predicates:*
  (a) *S_1 and S_2.*
  (b) *P_k for $k \leq k^*$.*
  (c) *$Q_{\mathbf{s}}$ for $\mathbf{s} \in \mathfrak{S}$.*
  (d) *R_N for $N \in \mathfrak{P}$.*
  k_2 is the total number of unary predicates in τ^*, i.e $k_2 = |\mathfrak{P}| + |\mathfrak{S}| + k^* + 2$.
(2) *A τ^*-structure $\mathfrak{T}$ is a representation of $M \in \mathfrak{K}$, if we can find $\mathfrak{M} = \langle M_t : t \in |\mathfrak{T}|\rangle$ such that:*
  (a) *$((|\mathfrak{T}|, \leq^{\mathfrak{T}}, c^{\mathfrak{T}}_{rt}, S^{\mathfrak{T}}_1, S^{\mathfrak{T}}_2), \mathfrak{M})$ is a full representation of M.*
  (b) *$\langle P^{\mathfrak{T}}_k : k \leq k^*\rangle$ is a partition of $|\mathfrak{T}|$. If $t \in P^{\mathfrak{T}}_k$, then $k_t = k$ i.e. M_t is a k-const τ^+-structure. We write $k^{\mathfrak{T}}(t) = k$ iff $t \in P^{\mathfrak{T}}_k$.*
  (c) *$\langle Q^{\mathfrak{T}}_{\mathbf{s}} : \mathbf{s} \in \mathfrak{S}\rangle \cup \langle R^{\mathfrak{T}}_N : N \in \mathfrak{P}\rangle$ is a partition of $|\mathfrak{T}|$.*
  (d) *For every $t \in T$ $\leq^{\mathfrak{T}}$-maximal, $t \in R^{\mathfrak{T}}_{M_t}$.*
  (e) *For every $t \in T$ non-maximal, let s_1, s_2 be its immediate successors with $s_l \in S^{\mathfrak{T}}_l$. Suppose $M_t = M_{s_1} \circledast_{\mathbf{s}} M_{s_2}$ for some $\mathbf{s} \in \mathfrak{S}_{\tau^+,k_{s_1},k_{s_2},k_t} \cap \mathfrak{S}$. Then $t \in Q^{\mathfrak{T}}_{\mathbf{s}}$.*

Note that:

Observation 2.17.

(1) *Every $M \in \mathfrak{K}$ has a full representation, and hence a representation.*
(2) *If $M_l \in \mathfrak{K}$ are represented by $\mathfrak{T}_l$ for $l \in \{1, 2\}$, and $\mathfrak{T}_1 \cong \mathfrak{T}_2$. Then $M_1 \cong M_2$.*

Now define: $k_1 = max\{|N| : N \in \mathfrak{P}\}$, k_2 is the number of unary predicates in τ^* (see 2.16(1)), and let $k^{**} = max\{k_1, k_2\}$. We can define a k^{**}-interpretation scheme $\mathbf{c}$ with $k^{\mathbf{c}}_1 = k_1$ and $k^{\mathbf{c}}_2 = k_2$ such that for all $M \in \mathfrak{K}$, and $\mathfrak{T}$ a representation of M we have $M \cong \mathfrak{T}^{[\mathbf{c}]}$. Note that indeed $\mathfrak{T}$ is a DB $k^{\mathbf{c}}_2$-tree. We will not specify all the formulas of $\mathbf{c}$ as they tend to be very long and complicated, but do note that all the information about M can be decoded from the representation of M using monadic formulas. Finally by an argument very close to that of 2.13 we may assume that $\mathbf{c}$ has the leaf property. $\square$

3. Equivalence of m-ary patch-width for $m \geq 3$

We come now to the main part of our result. Basically what we do here is proving the reverse inclusion of 2.14. It turns out that in our constructible class we only need 3-ary relations as auxiliary relations, thus we can replace constructible by 3-constructible. It follows that a class $\mathfrak{K}$ is contained in a constructible class, iff it is contained in a 3-constructible class, and similarly for m-ary path-width. We start with an investigation of directed binary trees that will be useful later.

Notation 3.1. *Let $\mathfrak{T}$ be a DB k-tree. Let $n \in \mathbb{N}$ and $x_1, ..., x_n \in T$ be fixed maximal elements of $\mathfrak{T}$.*

(1) *For $x, y \in T$ denote by $x \wedge y$ the minimal element z with $z \geq x, y$.*
(2) *For $x, y \in T$ with $x \leq y$ denote $[x, y) := \{z \in Tx \leq z < y\}$ and similarly $(x, y), (x, y]$ and $[x, y]$.*
(3) *Define $Y := \{x_1, ..., x_n\} \cup \{x_i \wedge x_j : i, j \leq n\} \cup \{c^{\mathfrak{T}}_{rt}\}$. Note that $|Y| \leq 2n$.*
(4) *For any non-maximal $x \in T$ let $F_R(x) \in T$ (resp. $F_L(x) \in T$) be the unique immediate successor of x which is in $P^{\mathfrak{T}}_1$ (resp. $P^{\mathfrak{T}}_2$).*
(5) *For $y, y' \in Y$ with $y < y'$ define,*

$$\begin{aligned} T^3_{y,y'} := \quad & [y, y'] \cup \\ & \{z \in T : (\exists s \in (y, y'))F_R(s) \leq y' \wedge F_L(s) \leq z\} \cup \\ & \{z \in T : (\exists s \in (y, y'))F_L(s) \leq y' \wedge F_R(s) \leq z\} \end{aligned}$$

(6) *Let $T_R = \{c^{\mathfrak{T}}_{rt}\} \cup \{t \in T : F_R(c^{\mathfrak{T}}_{rt}) \leq t\}$, and similarly T_L.*

Lemma 3.2. *Let $R_R(y, y')$ and $R_L(y, y')$ be binary relations meaning $F_R(y) \leq y'$ and $F_L(y) \leq y'$ respectively. The type $tp^{MSO}_q((x_1, ..., x_n), \mathfrak{T})$ is computable from the structure $\langle Y; \leq^{\mathfrak{T}}, R_R, R_L\rangle$, the types $\{tp^{MSO}_q((y, y'), \mathfrak{T}|_{T^3_{y,y'}}) : y, y' \in Y, y < y', (y, y') \cap Y = \emptyset\}$, and the types $tp^{MSO}_q(c^{\mathfrak{T}}_{rt}, \mathfrak{T}|_{T_L})$, $tp^{MSO}_q(c^{\mathfrak{T}}_{rt}, \mathfrak{T}|_{T_R})$.*

Proof. Without going into detail note that from the sets $T^3_{y,y'}$ with y, y' as above, T_L and T_R, we can choose a decomposition of $|\mathfrak{T}|$, in which only the elements of Y belong to more then one set. Hence we can reconstruct the structure $\mathfrak{T}$ with the elements of Y as marked elements from the reduced structures: $\mathfrak{T}|_{T^3_{y,y'}}$ with y, y' as marked elements, $\mathfrak{T}|_{T_L}$ and $\mathfrak{T}|_{T_L}$ with $C^{\mathfrak{T}}_{rt}$ as marked element, in a way that the q theory of the resulting structure depends only on the q theory of the operands. The structure $\langle Y; \leq^{\mathfrak{T}}, R_R, R_L\rangle$ determines the order of the construction. □

Claim 3.3. *Let k^* be a natural number, and* **c** *a monadic k^*-interpretation scheme with the leaf property for a vocabulary τ. Then there exists a natural number k^{**}, and a $(3, k^{**})$-constructible class of τ structures, $\mathfrak{K}$, such that: $\mathfrak{K}^{mo,db}_{\mathbf{c}} \subseteq \mathfrak{K}$.*

Proof. Let q^* be the maximal quantifier rank of the formulas $\{\varphi_{Q,0} : Q \in \tau\}$. Define the vocabulary τ^+ to consist of:

- τ.
- $\tau_{k_2-trees}$.
- Two 3-place relations R_R and R_L.
- For each $\mathbf{t} \in TP^{MSO}_{q^*}(2, \tau_{k_2-trees})$, a 3-place relation $R^3_{\mathbf{t}}$.
- For each $\mathbf{t} \in TP^{MSO}_{q^*}(2, \tau_{k_2-trees})$, a 2-place relation $R^2_{\mathbf{t}}$.
- For each $\mathbf{t} \in TP^{MSO}_{q^*}(1, \tau_{k_2-trees})$ two 0-place relations $R^R_{\mathbf{t}}$ and $R^L_{\mathbf{t}}$.

Before we define the set of addition operations $\mathfrak{S}$, and the set $\mathfrak{P}$, let us define:

Definition 3.4. *A τ^+-structure, $\mathfrak{T}$, is called a "correct" k_2-tree if:*

- *For each $Q \in \tau$, $Q^{\mathfrak{T}} = \emptyset$.*
- *$\mathfrak{T}|\tau_{k_2-trees}$ is a DB k_2-tree.*
- *For each x_1, x_2, x_3 maximal elements of $|\mathfrak{T}|$, let $y = x_1 \wedge x_2$ and $y' = y \wedge x_3$ then we have, $R^{\mathfrak{T}}_R(x_1, x_2, x_3) \Leftrightarrow F_R(y) \leq y'$, and similarly for R_L.*
- *For each $\mathbf{t} \in TP^{MSO}_{q^*}(2, \tau_{k_2-trees})$, and x_1, x_2, x_3 maximal elements of $|\mathfrak{T}|$, let $y = x_1 \wedge x_2$ and $y' = y \wedge x_3$ then we have, $(R^3_{\mathbf{t}})^{\mathfrak{T}}(x_1, x_2, x_3) \Leftrightarrow tp^{MSO}_{q^*}((y, y'), \mathfrak{T}|_{T^3_{y,y'}}) = \mathbf{t}$.*

- *For each* $\boldsymbol{t} \in TP^{MSO}_{q^*}(2, \tau_{k_2-trees})$, *and* x_1, x_2 *maximal elements of* $|\mathfrak{T}|$, *let* $y = x_1 \wedge x_2$ *then we have,* $(R^2_{\boldsymbol{t}})^{\mathfrak{T}}(x_1, x_2) \Leftrightarrow tp^{MSO}_{q^*}((c^{\mathfrak{T}}_{rt}, y), \mathfrak{T}|_{T^3_{c^{\mathfrak{T}}_{rt},y}}) = \boldsymbol{t}$.
- *For each* $\mathbf{t} \in TP^{MSO}_{q^*}(1, \tau_{k_2-trees})$, $(R^R_{\mathbf{t}})^{\mathfrak{T}} = \mathbb{T}$ *iff* $tp^{MSO}_{q^*}(c^{\mathfrak{T}}_{rt}, \mathfrak{T}|_{T_R}) = \mathbf{t}$, *and similarly for* $R^L_{\mathbf{t}}$.

Note that every DB k_2-tree can be uniquely extended to a correct DB k_2-tree. Now define Our $\mathfrak{P}$ to consist of all singleton correct models (models with one element) of the vocabulary τ^+, plus all the Null τ^+-structures (see definition 2.5(5)) .

We now turn to the definition of the operations in $\mathfrak{S}$. Let u be a possible "color" of a singleton k_2-tree. Formally $u \subseteq \{P_3, ..., P_{k_2}\}$. We define the operation $\oplus_u$ on DB k_2-trees as the addition of two trees with root of color u. Formally Let $\mathfrak{T}_1, \mathfrak{T}_2$ be DB k_2-trees define $\mathfrak{T} = \mathfrak{T}_1 \oplus_u \mathfrak{T}_2$ by:

- $|\mathfrak{T}| = |\mathfrak{T}_1| \cup |\mathfrak{T}_2| \cup \{c\}$.
- c is the root of $\mathfrak{T}$ i.e. $c^{\mathfrak{T}}_{rt} = \{c\}$ and $\forall t \in |\mathfrak{T}|, c <^{\mathfrak{T}} x$.
- c has color u i.e. for all $i \geq 3$, $c \in P^{\mathfrak{T}}_i$ iff $i \in u$.
- $c^{\mathfrak{T}_1}_{rt} \in P^{\mathfrak{T}}_1$ and $c^{\mathfrak{T}_2}_{rt} \in P^{\mathfrak{T}}_2$.
- The rest of the relations on $\mathfrak{T}_1$ and $\mathfrak{T}_2$ remain unchanged.

Note that indeed $\mathfrak{T}_1 \oplus_u \mathfrak{T}_2$ is a DB k_2-tree whenever $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are, and hence $\oplus_u$ extends uniquely to an operation on <u>correct</u> k_2-trees.

Now for $l \in \{1, 2\}$ let $\mathfrak{A}_l$ be a τ^+ structure such that there exists a correct k_2-tree with $|\mathfrak{A}_l| \subseteq |\mathfrak{T}_l|$, $\mathfrak{T}_l|_{|\mathfrak{A}_l|} = \mathfrak{A}_l$, and every element of $\mathfrak{A}_l$ is maximal in $\mathfrak{T}_l$. Define an operation $\mathbf{s}_u$ on such structures by: $\mathfrak{A}_1 \circledast_{\mathbf{s}_u} \mathfrak{A}_2 = (\mathfrak{T}_1 \oplus_u \mathfrak{T}_2)|_{|\mathfrak{A}_1| \cup |\mathfrak{A}_2|}$. It is easy to verify that $\circledast_{\mathbf{s}_u}$ is well defined and indeed belongs to $\mathfrak{S}_{\tau^+,0,0,0}$. We now have:

Lemma 3.5. *For every correct k_2-tree, $\mathfrak{T}$ and every set $A \subseteq |\mathfrak{T}|$ of maximal elements, the restriction $\mathfrak{T}|_A$ is in the closer of $\mathfrak{P}$ under the operations $\{\mathbf{s}_u : u \subseteq \{1, ..., k_2\}\}$.*

Proof. First it is obvious that we can construct $\mathfrak{T}$ from $\mathfrak{P}$ using the operations $\{\oplus_u : u \subseteq \{3, ..., k_2\}\}$. Now use the same construction only replace in each step the operation $\mathfrak{T}_1 \oplus_u \mathfrak{T}_2$ by the operation $\mathfrak{T}_1|_A \circledast_u \mathfrak{T}_2|_A$. □

The last thing we need now is to "decode" the relations in the correct structure into the relations in our vocabulary τ. For this we use:

Lemma 3.6. *There exist $\mathbf{s}^* \in \mathfrak{S}_{\tau^+,0,0,0}$ such that For every correct k_2-tree, $\mathfrak{T}$ and every set $A \subseteq |\mathfrak{T}|$ of maximal elements, the structure $\mathfrak{A}' = \mathfrak{T}|_A \circledast_{\mathbf{s}^*} Null_\emptyset$ satisfies for each $Q \in \tau$ with $n(Q) = n$,*

$$(*) \quad Q^{\mathfrak{A}'} = \{(x_1, ..., x_n) \in {}^n A : \mathfrak{T} \models \varphi_{Q,0}(x_1, ..., x_n)\}.$$

Proof. Let $Q \in \tau$ be an n-place relation symbol, and $w_1, w_2 \subseteq \{1, ..., n\}$. We should define $f^{\mathbf{s_3}}_{Q,w_1,w_2}$ in such a way that $(*)$ will hold. As we have $k^{\mathbf{s}^*}_1 = k^{\mathbf{s}^*}_2 = k^{\mathbf{s}^*} = 0$ and we are only interested in $Null_\emptyset$ as the right operand, the only relevant case is $w_1 = \{1, ..., n\}$ and $w_2 = \emptyset$. In order to have $(*)$ We need to define a function:

$$f^{\mathbf{s}^*}_Q : \{p : \text{p is a quantifier free type of n variables in vocabulary } \tau^+\} \to \{\mathbb{T}, \mathbb{F}\}$$

such that for all $(x_1, ..., x_n) \in {}^n A$, $f_Q(tp_{qf}((x_1, ..., x_n), \mathfrak{A}')) = \mathbb{T}$ iff $\mathfrak{T} \models \varphi_{Q,0}(x_1, ..., x_n)$. Recall that by lemma 3.2 the value of $\mathfrak{T} \models \varphi_{Q,0}(x_1, ..., x_n)$, is determined by $\langle Y; \leq^{\mathfrak{T}}$

$, R_R, R_L\rangle$, the types $\{tp_q^{MSO}((y,y'), \mathfrak{T}|_{T^3_{y,y'}}) : y, y' \in Y, y < y', (y,y') \cap Y = \emptyset\}$, and the types $tp_q^{MSO}(c_{rt}^{\mathfrak{T}}, \mathfrak{T}|_{T_L})$, $tp_q^{MSO}(c_{rt}^{\mathfrak{T}}, \mathfrak{T}|_{T_R})$ (see 3.2 and notation 3.1). But as $\mathfrak{T}$ is correct these all are determined by p so we are done. $\square$

We can now conclude the proof of lemma 3.3. Define $\mathfrak{S} = \{\mathbf{s}_u : u \subseteq \{1, ..., k_2\}\} \cup \{\mathbf{s}^*\}$, and let $\mathfrak{K}$ be the constructible class of τ-structures defined by $\mathfrak{P}$ and $\mathfrak{S}$. Let M be a τ-structure in $\mathfrak{K}_{\mathbf{c}}^{mo,db}$. So we have $M \cong \mathfrak{T}_1^{[\mathbf{c}]}$ for some DB k_2-tree $\mathfrak{T}_1$. Let $\mathfrak{T}_2$ be the correct extension of $\mathfrak{T}_1$. Let $A = \{x \in |\mathfrak{T}_2| : \mathfrak{T}_1 \models \varphi_{=,0}(x)\}$, and $\mathfrak{A} = \mathfrak{T}_2|_A \circledast_{\mathbf{s}^*} Null_\emptyset$. From lemma 3.5 we have that $\mathfrak{T}_2|_A$ is in the closer of $\mathfrak{P}$ under the operations in $\mathfrak{S}$ and hence so is $\mathfrak{A}$. From lemma 3.6 and the definition of $\mathfrak{T}_1^{[\mathbf{c}]}$, we have that $\mathfrak{A}|_\tau = \mathfrak{T}_1^{[\mathbf{c}]} \cong M$, so $M \in \mathfrak{K}$ as desired.

$\square$

From lemmas 3.3 and 2.14 we conclude our main:

Theorem 3.7. *Let $\mathfrak{K}$ be a class of τ-structures. Then $\mathfrak{K}$ is contained in a m-constructible class for some $m \in \mathbb{N}$ iff $\mathfrak{K}$ is contained in a 3-constructible class.*

The same holds for patch-width:

Corollary 3.8. *Let τ be a nice vocabulary and $\mathfrak{K}$ a class of τ-structures. Then $\mathfrak{K}$ is contained in a class of bounded m-ary patch-width for some $m \in \mathbb{N}$ iff $\mathfrak{K}$ is contained in a class of bounded 3-ary patch-width.*

Proof. Assume $\mathfrak{K} \subseteq \mathfrak{K}'$ for some $\mathfrak{K}'$ of bounded m-ary patch-width. By lemma 2.9 $\mathfrak{K}'$ is $(m,0)$-constructible. By theorem 3.7 $\mathfrak{K}'$ is contained in some 3-constructible $\mathfrak{K}''$. Notice that that the set $\mathfrak{S}$ defined in the proof of 3.3 satisfies that $\mathfrak{S} \subseteq \mathfrak{S}_{\tau^+,0,0,0}$ so $\mathfrak{K}''$ is in fact $(3,0)$-constructible. Notice further that in the proof of 3.3 as we do not need null structures in the construction, hence we may replace τ^+ by a nice vocabulary. So by lemma 2.10 $\mathfrak{K}''$ is a bounded 3-ary patch-width class as desired. $\square$

4. A counter example for the unary case

It turns out that we can not replace the number 3 in theorem 3.7 by 1. This is because of the following:

Theorem 4.1. *There exists a nice vocabulary τ, and a class of τ-structures $\mathfrak{K}$, contained in some 3-constructible class, that is not contained in any 1-constructible class.*

Proof. Let $\tau = \{R\}$ with $n(R) = 4$. Set $p \in \mathbb{N}$ be large enough (to be defined later). Let $\mathfrak{T}$ be a tree. For $x, y \in T$ Define:

- $x \wedge^{\mathfrak{T}} y = x \wedge y =$ the $<^{\mathfrak{T}}$ minimal $z \in T$ such that $z \geq^{\mathfrak{T}} x$ and $z \geq^{\mathfrak{T}} x$.
- $d^{\mathfrak{T}}(x,y) = d(x,y) = min\{|S| : S \subseteq T, x, x \wedge y \in S,$ S is dense in $(T, <^{\mathfrak{T}})\}$.
- $d_p^{\mathfrak{T}}(x,y) = d_p(x,y) = d(x,y) \pmod p$.

Let $q : \{0, ..., p-1\}^2 \to \{0,1\}$ be some function that will be defined later. We now define $\mathbf{c}$ a 0-interpretation scheme for τ:

- $k_1^{\mathbf{c}} = k_2^{\mathbf{c}} = 0$.
- $\varphi_{=,0}^{\mathbf{c}}(x) = \neg \exists y y > x$ i.e. the elements of the interpreted structure are the leafs of the tree.
- $\varphi_{R,0}^{\mathbf{c}}(x_1, x_2, x_3, x_4) =$ "$q(d_p(x_1,x_2), d_p(x_3,x_4)) = 0$".

We have to show that $\varphi_{R,0}$ is indeed a monadic formula in τ_{trees}. Note that there exists a monadic formula $\varphi_{d_p=0}(x,y)$ such that for any tree $\mathfrak{T}$, $\mathfrak{T} \models \varphi_{d_p=0}(x,y)$ iff $d_l^{\mathfrak{T}}(x,y)=0$. $\varphi_{d_p=0}(x,y)$ will "say" that there exists a set X such that:

- $x, x \wedge y \in X$,
- if $z, z' \in X$ and $z < z'' < z'$ then $z'' \in X$,
- if z' is the immediate successor in X of $z \in X$, then there exist exactly $p-1$ elements (of T) between them.

similarly we have formulas $\varphi_{d_p=i}(x,y)$ for $0<i<p$. Now define:

$$\varphi_{R,0}(x_1,x_2,x_3,x_4) = \bigvee_{\substack{n_1,n_2 \in \{0,\ldots,p-1\} \\ n_1 \equiv n_2 \pmod p}} \varphi_{d_p=n_1}(x_1,x_2) \wedge \varphi_{d_p=n_2}(x_3,x_4).$$

This gives us $\mathbf{c}$ as desired. Define $\mathfrak{K} = \mathfrak{K}_{\mathbf{c}}^{mo,db}$. By 3.3 $\mathfrak{K}$ is contained in a 3-constructible class (in fact in a 3-ary BPW class).
For each $n \in \mathbb{N}$ let $M_n = ({}^{n\geq}2, \triangleleft)$ i.e. M_n is the complete binary tree of depth n, and $N_n = M_n^{[\mathbf{c}]}$. Let $\mathfrak{K}'$ be a constructible class of τ-structures, so $\tau^+ = \tau_k$ for some $k \in \mathbb{N}$. Towards contradiction assume that $N_n \in \mathfrak{K}'$ for all $n \in \mathbb{N}$. Let $\mathfrak{P}$ be the set of "atomic" structures associates with $\mathfrak{K}'$. w.l.o.g. we may assume that $\mathfrak{P}$ consists of singleton structures only. Otherwise increase k by $max\{|M| : M \in \mathfrak{P}\}$ and construct each $M \in \mathfrak{P}$ from singletons of distinct colors. Now let $K \in \mathfrak{K}'$, and let $(\mathfrak{T}, \mathfrak{M})$ be a full representation of K (see 2.15). Assume $K \cong N_n$ for some n. So we have a 1-1 function f, from ${}^n2$ to the leafs of $\mathfrak{T}$, as every $\eta \in {}^n2$ corresponds to a unique element $a \in K$ under the isomorphisms, and for every element of $a \in K$ there exist a unique t a leaf of $\mathfrak{T}$ such that $a = |M_t|$. Define $f(\eta)=t$. Note that f is not onto, as some of the leafs of $\mathfrak{T}$ may be omitted during the creation process. For each $t \in T$ let $A_t = \{f^{-1}(s) : s \leq^{\mathfrak{T}} \wedge s \in range(f)\}$. So $A_t \subseteq {}^n2$. For each $\eta \in A_t$ let $a = a_\eta = |M_{f(\eta)}|$. a_η is an element of M_t, so A_t is divided into 2^k parts according to the color of a_η in M_t, (more formally according to the type $tp_{qf}^{\tau^+ \setminus \tau}(a_\eta, M_t)$. We therefore have $B_t \subseteq A_t$ such that $|B_t| \geq \frac{|A_t|}{2^k}$, and all the elements of $f^{-1}(B_t)$ have the same color. Now define:

$$C_t = \{d_p^{N_n}(\eta, \eta \wedge \nu) : \eta, \nu \in B_t\} \subseteq \{0,...,p-1\}.$$

We have $\frac{|A_t|}{2^k} \leq |B_t| \leq 2^{|C_t|}$. For the right-hand inequality use induction on $|C_t|$. Hence we conclude

$$|A_t| \leq |B_t| \cdot 2^k \leq 2^{|C_t|+k}.$$

Now note that if $C_t \neq \{0,...,p-1\}$, then $C_t \leq n - \lfloor \frac{n}{p} \rfloor$ and hence $|A_t| \leq 2^{|C_t|+k} \leq 2^{n-\lfloor \frac{n}{p} \rfloor + k}$. We now consider two cases:

Case 1 There exist $s \in T$ with two immediate successors $t_1, t_2 \in T$ such that: $|A_{t_1}|, |A_{t_2}| > 2^{n-\lfloor \frac{n}{p} \rfloor + k}$.

According to what we saw above we have $C_1 = C_2 = \{0,...,p-1\}$. So for $l \in \{1,2\}$ we have $\langle (\rho_{t_l,i}, \nu_{t_l,i} : i \in \{0,...,p-1\} \rangle$ such that:

(α) $\{\rho_{t_l,i}, \nu_{t_l,i} : i \in \{0,...,p-1\}\}$ all have the same color in M_{t_l}.
(β) $d_p^{N_n}(\rho_{t_l,i}, \nu_{t_l,i}) = i$ for all $i < p$.

Denote by m the number of quantifier free types of couples in the vocabulary τ (actually in our case $m = 2^{(2^4)}$). Note that m does not depend on p. So for each $l \in \{1,2\}$, $\{0,...,p-1\}$ is deviled into m parts according to the type

$tp_{qf}((\rho_{t_l,i},\nu_{t_l,i}),M_{t_l})$. We claim that we can (a priori) choose p (large enough) and q in such a way that we can find i_1,i_2,j_1,j_2 such that for each $l\in\{1,2\}$: $(\rho_{t_l,i_l},\nu_{t_l,i_l})$ and $(\rho_{t_l,j_l},\nu_{t_l,j_l})$ have the same quantifier free type in vocabulary τ in M_{t_l}, and on the other hand: $q(i_1,j_1)\neq q(i_2,j_2)$. This is of course a contradiction as the quantifier free type of $(\rho_{t_l,i_l},\nu_{t_l,i_l})$ and $(\rho_{t_l,j_l},\nu_{t_l,j_l})$ in vocabulary τ_k in M_{t_l} determines the value of $R(\rho_{t_l,i_l},\nu_{t_l,i_l},\rho_{t_l,j_l},\nu_{t_l,j_l})$ in M_s and hance in $M_{c^{\mathfrak{T}}_{rt}}$. But this value is true iff $q(i_l,j_l)=0$ in contradiction with $q(i_1,j_1)\neq q(i_2,j_2)$. Why can we choose p and q as desired? For a given p the number of functions from $\{0,...,p-1\}^2$ to $\{0,1\}$ such that we can not choose as above (i.e. functions that "respects" some partition of $\{0,...,p-1\}$ into m parts is the number of partitions $m^p\cdot m^p$, time the number of functions that "respect" that partition $2^{m\cdot m}$, or $2^{2p\log(m)+m^2}$. The total number of functions is 2^{p^2}. So if we choose (a priori) p such that $p^2>2p\log(m)+m^2$ we can choose a function q as desired.

Assume now that the assumption of **Case 1** does not hold. Assume also that we have chosen n large enough such that $2^{\lfloor\frac{n}{p}\rfloor-k}>4$. In this case we can find $t_0,t_1,...,t_d\in T$ such that :

- $d\geq 5$.
- $t_0=c^{\mathfrak{T}}_{rt}$.
- t_d is a leaf of $\mathfrak{T}$.
- For $0\leq i<d$, t_{i+1} is an immediate successor in $\mathfrak{T}$, of t_1.
- For $0\leq i<d$, denote by s_{i+1} the immediate successor of t_i different from t_{i+1}, then $|A_{s_{i+1}}|\leq 2^{\lfloor\frac{n}{p}\rfloor-k}$.

Note that for any $0<i\leq d$: $\bigcup_{0<j\leq i}A_{s_j}$ and A_{t_i} is a partition of $A_{c^{\mathfrak{T}}_{rt}}$, and that $|A_{c^{\mathfrak{T}}_{rt}}|=2^n$. So we can find $0<i^*\leq d$ such that $|\bigcup_{0<j\leq i^*}A_{s_j}|,|A_{t_{i^*}}|>2^{\lfloor\frac{n}{p}\rfloor-k}$. We proceed similarly to **Case 1**. As there we can find $\langle(\rho_{t_{i^*},i},\nu_{t_{i^*},i}\in A_{t_{i^*}}:i\in\{0,...,p-1\}\rangle$ that satisfy (α) and (β) above, and the same for $\langle(\rho_i,\nu_i:i\in\{0,...,p-1\}\rangle$ where $\rho_i,\nu_i\in\bigcup_{0<j\leq i^*}A_{s_j}$. Again let m denote the number of quantifier free types of couples in the vocabulary τ. This time we want to choose p and q in such a way that we can find: i,j_1,j_2 such that: $(\rho_{t_{i^*},j_1},\nu_{t_{i^*},j_1})$ and $(\rho_{t_{i^*},j_2},\nu_{t_{i^*},j_2})$ have the same quantifier free type in vocabulary τ in $M_{t_{i^*}}$, and on the other hand: $q(i,j_1)\neq q(i,j_2)$. Again this is a contradiction as the quantifier free type of $(\rho_{t_{i^*},j_l},\nu_{t_{i^*},j_l})$ for $l\in\{1,2\}$ determines the value of $R(\rho_{t_{i^*},j_l},\nu_{t_{i^*},j_l},\rho_i,\nu_i)$ in $M_{c^{\mathfrak{T}}_{rt}}$. Again this value is true iff $q(i,j_l)=0$ in contradiction with $q(i,j_1)\neq q(i,j_2)$. Why can we choose p and q as desired? For a given p the number of functions from $\{0,...,p-1\}^2$ to $\{0,1\}$ such that we can not choose as above is the number of partitions m^p, times the number of functions that "respect" that partition $2^{m\cdot p}$, or $2^{p\log(m)+m\cdot p}$. So if we choose p such that $p^2>p\log(m)+m\cdot p$ we can choose a function q as desired. Note that the function we used for the second case will also work for the first case so we can use one definition of q. In both cases we get a contradiction and the proof is complete.

□

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