# WHEN FIRST ORDER $T$ HAS LIMIT MODELS
# SH868

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ABSTRACT. We to a large extent sort out when does a (first order complete theory) $T$ have a superlimit model in a cardinal $\lambda$. Also we deal with related notions of being limit.

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## Anotated Content

## 0. Introduction

§(0A) Background and Content

Recall that ([She90, Ch.III]). If $T$ is (first order complete and) superstable then for $\lambda \geq 2^{|T|}, T$ has a saturated model $M$ of cardinality $\lambda$ and moreover

- $(*)$ if $\langle M_\alpha : \alpha < \delta \rangle$ is $\prec$-increasing, $\delta$ a limit ordinal $< \lambda^+$ and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $\cup\{M_\alpha : \alpha < \delta\}$ is isomorphic to $M$.

When investigating categoricity of an a.e.c. (abstract elementary classes) $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$, the following property turns out to be central: $M$ is $\leq_{\mathfrak{k}}$-universal model of cardinality $\lambda$ with the property $(*)$ above (called superlimit) - possibly with addition parameter $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$); we also consider some relatives, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [She87a, 3.1] or see the revised version [Shea, 3.3] and see [She09] or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete $T$, we know $\{\lambda : T$ has a saturated model of $T$ of cardinality $\lambda\}$, that is, it is $\{\lambda : \lambda^{<\lambda} \geq |D(T)|$ or $T$ is stable in $\lambda\}$, on the definitions of $D(T)$ and other notions see §(0B) below. What if we replace saturated by superlimit (or some relative)? Let $\mathrm{EC}_\lambda(T)$ be the class of models $M$ of $T$ of cardinality $\lambda$.

If there is a saturated $M \in \mathrm{EC}_\lambda(T)$ we have considerable knoweldge on the existence of limit model for cardinal $\lambda$, this was as mentioned in [Shea, 3.6] by [She90], see 0.9(1),(2). E.g. for superstable $T$ in $\lambda \geq 2^{|T|}$ there is a superlimit model (the saturated one). It seems a natural question on [Shea, 3.6] whether it exhausts the possibilities of $(\lambda, *)$-superlimit and $(\lambda, \kappa)$-superlimit models for elementary classes.

Clearly the cases of the existence of such models of a (first order complete) theory $T$ where there are no saturated (or special) models are rare, because even the weakest version of Definition [She87a, 3.1] = [Shea, 3.3] or here Definitino 0.7 for $\lambda$ implies that $T$ has a universal model of cardinality $\lambda$, which is rare (see Kojman Shelah [KS92] which includes earlier history and recently Djamonza [Mirar]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for $\lambda < \lambda^{<\lambda}$), i.e., those for $\lambda = \aleph_1$.

E.g. a sufficient condition for some versions is the existence of $T' \supseteq T$ of cardinality $\lambda$ such that $\mathrm{PC}(T', T)$ is categorical in $\lambda$, see 0.4(3). By [She80] we have consistency results for such $T_1$ so naturally we first deal with the consistency results from [She80]. In §1 we deal with the case of the countable superstable $T_0$ from [She80] which is not $\aleph_0$-stable. By [She80] consistently $\aleph_1 < 2^{\aleph_0}$ and for some $T_0' \supseteq T_0$ of cardinality $\aleph_1$, $\mathrm{PC}(T_0', T_0)$ is categorical in $\aleph_1$. We use this to get the consistency of "$T_0$ has a superlimit model of cardinality $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$".

In §2 we prove that for some stable not superstable countable $T_1$ we have a parallel but weaker result. We relook at the old consistency results of "some $\mathrm{PC}(T_1', T_1), |T_1'| = \aleph_1 > |T_1|$, is categorical in $\aleph_1$" from [She80]. From this we deduce that in this universe, $T_1$ has a strongly $(\aleph_1, \aleph_0)$-limit model.

It is a reasonable thought that we can similarly have a consistency result on the theory of linear order, but this is still not clear.

In §3 we show that if $T$ has a superlimit model in $\lambda \geq |T| + \aleph_1$ then $T$ is stable and $T$ is superstable except possibly under some severe restrictions on the cardinal $\lambda$ (i.e., $\lambda < \beth_\omega$ and $\lambda < \lambda^{\aleph_0}$). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by 1.1 + 3.1 we have:

**Conclusion 0.1.** *Assume $\lambda \geq |T| + \beth_\omega$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ is superstable and $\lambda \geq |D(T)|$.*

In subsequent work we shall show that for some unstable $T$ (e.g. the theory of linear orders), if $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$, then $T$ has a medium $(\lambda, \kappa)$-limit model, whereas if $T$ has the independence property even weak $(\lambda, \kappa)$-limit models do not exist; see [She14] and more in [She15], [She11], [Sheb], [S+].

We thank Alex Usvyatsov for urging us to resolve the question of the superlimit case and John Baldwin for comments and complaints.

## §(0B) Basic Definitions

*Notation* 0.2. 1) Let $T$ denote a complete first order theory which has infinite models but $T_1, T'$, etc. are not necessarily complete.
2) Let $M, N$ denote models, $|M|$ the universe of $M$ and $\|M\|$ its cardinality and $M \prec N$ means $M$ is an elementary submodel of $N$.
3) Let $\tau_T = \tau(T), \tau_M = \tau(M)$ be the vocabulary of $T, M$ respectively.
4) Let $M \models$ "$\varphi[\bar{a}]^{\underline{if}(\mathrm{stat})}$" means that the model $M$ satisfies $\varphi[\bar{a}]$ iff the statement stat is true (or is 1 rather than 0)).

**Definition 0.3.** 1) For $\bar{a} \in {}^{\omega>}|M|$ and $B \subseteq M$ let $\mathrm{tp}(\bar{a}, B, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{b} \in {}^{\ell g(\bar{y})}B$ and $M \models \varphi[\bar{a}, \bar{b}]\}$.
2) Let $D(T) = \{\mathrm{tp}(\bar{a}, \emptyset, M) : M$ a model of $T$ and $\bar{a}$ a finite sequence from $M\}$.
3) If $A \subseteq M$ then $\mathbf{S}^m(A, M) = \{\mathrm{tp}(\bar{a}, A, N) : M \prec N$ and $\bar{a} \in {}^m N\}$, if $m = 1$ we may omit it.
4) A model $M$ is $\lambda$-saturated when: if $A \subseteq M, |A| < \lambda$ and $p \in \mathbf{S}(A, M)$ then $p$ is realized by some $a \in M$, i.e. $p \subseteq \mathrm{tp}(a, A, M)$; if $\lambda = \|M\|$ we may omit it.
5) A model $M$ is special <u>when</u> letting $\lambda = \|M\|$, there is an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ of cardinals with limit $\lambda$ and a $\prec$-increasing sequence $\langle M_i : i < \mathrm{cf}(\lambda) \rangle$ of models with union $M$ such that $M_{i+1}$ is $\lambda_i$-saturated of cardinality $\lambda_{i+1}$ for $i < \mathrm{cf}(\lambda)$.

**Definition 0.4.** 1) For any $T$ let $\mathrm{EC}(T) = \{M : M$ is a $\tau_T$-model of $T\}$.
2) $\mathrm{EC}_\lambda(T) = \{M \in \mathrm{EC}(T) : M$ is of cardinality $\lambda\}$.
3) For $T \subseteq T'$ let

$$\mathrm{PC}(T', T) = \{M \restriction \tau_T : M \text{ is model of } T'\}$$

$$\mathrm{PC}_\lambda(T', T) = \{M \in \mathrm{PC}(T', T) : M \text{ is of cardinality } \lambda\}.$$

4) We say $M$ is $\lambda$-universal for $T_1$ when it is a model of $T_1$ and every $N \in \mathrm{EC}_\lambda(T)$ can be elementarily embedded into $M$; if $T_1 = \mathrm{Th}(M)$ we may omit it.
5) We say $M \in \mathrm{EC}(T)$ is universal <u>when</u> it is $\lambda$-universal for $\lambda = \|M\|$.

We are here mainly interested in

**Definition 0.5.** Given $T$ and $M \in \mathrm{EC}_\lambda(T)$ we say that $M$ is a superlimit or $\lambda$-superlimit model <u>when</u>: $M$ is universal and if $\delta < \lambda^+$ is a limit ordinal, $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\prec$-increasing continuous, and $M_\alpha$ is isomorphic to $M$ for every $\alpha < \delta$ <u>then</u> $M_\delta$ is isomorphic to $M$.

*Remark* 0.6. Concerning the following definition we shall use strongly limit in 2.14(1), medium limit in 2.14(2).

**Definition 0.7.** Let $\lambda$ be a cardinal $\geq |T|$. For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice and $\mathbf{F}$ is a class function; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$.
1) For non-empty $\Theta \subseteq \{\mu : \aleph_0 \leq \mu < \lambda$ and $\mu$ is regular$\}$ and $M \in \mathrm{EC}_\lambda(T)$ we say that $M$ is a $(\lambda, \Theta)$-superlimit <u>when</u>: $M$ is universal and
if $\langle M_i : i \leq \mu \rangle$ is $\prec$-increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$
then $\cup\{M_i : i < \mu\} \cong M$.
2) If $\Theta$ is a singleton, say $\Theta = \{\theta\}$, we may say that $M$ is $(\lambda, \theta)$-superlimit.
3) Let $S \subseteq \lambda^+$ be stationary. A model $M \in \mathrm{EC}_\lambda(T)$ is called $S$-strongly limit or $(\lambda, S)$-strongly limit <u>when</u> for some function: $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ we have:

- $(a)$ for $N \in \mathrm{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
- $(b)$ if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is a $\prec$-increasing continuous sequence [1] in $\mathrm{EC}_\lambda(T)$ and $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$, <u>then</u> $M \cong \cup\{M_i : i < \delta\}$.

4) Let $S \subseteq \lambda^+$ be stationary. $M \in \mathrm{EC}_\lambda(T)$ is called $S$-limit or $(\lambda, S)$-limit <u>if</u> for some function $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ we have:

- $(a)$ for every $N \in \mathrm{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
- $(b)$ if $\langle M_i : i < \lambda^+ \rangle$ is a $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ such that $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ for $i < \lambda^+$ <u>then</u> for some closed unbounded [2] subset $C$ of $\lambda^+$,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

5) We define[3] "$S$-weakly limit", "$S$-medium limit" like "$S$-limit", "$S$-strongly limit" respectively by demanding that the domain of $\mathbf{F}$ is the family of $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ of length $< \lambda^+$ and replacing "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$".
6) If $S = \lambda^+$ then we may omit $S$ (in parts (3), (4), (5)).
7) For non-empty $\Theta \subseteq \{\mu : \mu \leq \lambda$ and $\mu$ is regular$\}$, $M$ is $(\lambda, \Theta)$-strongly limit[4] <u>if</u> $M$ is $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strongly limit. Similarly for the other notions. If we do not write $\lambda$ we mean $\lambda = \|M\|$.

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[1]no loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$, i.e., only $\mathbf{F}(M')$ for $M' \cong M$ is required

[2]alternatively, we can use as a parameter a filter on $\lambda^+$ extending the co-bounded filter

[3]Note that $M$ is $(\lambda, S)$-strongly limit iff $M$ is $(\{\lambda, \mathrm{cf}(\delta) : \delta \in S\})$-strongly limit.

[4]in [Shea] we consider: we replace "limit" by "limit$^-$" if "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$", "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$" are replaced by "$\mathbf{F}(M_i) \prec M_{i+1}$", "$M_i \prec \mathbf{F}(\langle M_j : j \leq i \rangle) \prec M_{i+1}$" respectively. But $(\mathrm{EC}(T), \prec)$ has amalgamation.

8) We say that $M \in K_\lambda$ is invariantly strong limit <u>when</u> in part (3), $\mathbf{F}$ is just a subset of $\{(M,N)/\cong : M \prec N$ are from $\mathrm{EC}_\lambda(T)\}$ and in clause (b) of part (3) we replace "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M,N)/\cong) \in \mathbf{F})$". But abusing notation we still write $N = \mathbf{F}(M)$ instead $((M,N)/\cong) \in \mathbf{F}$. Similarly with the other notions, so we use the isomorphism type of $\bar{M}\hat{\ }\langle N\rangle$ for "weakly limit" and "medium limit".
9) In the definitions above we may say "$\mathbf{F}$ witness $M$ is ..."

**Observation 0.8.** *1) Assume $\mathbf{F}_1, \mathbf{F}_2$ are as above and $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$ (or $\mathbf{F}_1(\bar{N}) \prec \mathbf{F}_2(\bar{N})$) whenever defined. If $\mathbf{F}_1$ is a witness then so is $\mathbf{F}_2$.*
*2) All versions of limit models implies being a universal model in $\mathrm{EC}_\lambda(T)$.*

3) <u>The Obvious implications diagram</u>: For non-empty $\Theta \subseteq \{\theta : \theta$ is regular $\le \lambda\}$ and stationary $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$:

$$\begin{array}{ccc}
 & \text{superlimit } = (\lambda, \{\mu : \mu \le \lambda \text{ regular}\})\text{-superlimit} & \\
 & \downarrow & \\
 & (\lambda,\Theta)\text{-superlimit} & \\
 & \downarrow & \\
 & S_1\text{-strongly limit} & \\
 & \downarrow \qquad \downarrow & \\
S_1\text{-medium limit}, & & S_1\text{-limit} \\
 & \downarrow \qquad \downarrow & \\
 & S_1\text{-weakly limit.} &
\end{array}$$

**Lemma 0.9.** *Let $T$ be a first order complete theory.*
*1) If $\lambda$ is regular, $M$ a saturated model of $T$ of cardinality $\lambda$, <u>then</u> $M$ is $(\lambda,\lambda)$-superlimit.*
*2) If $T$ is stable, and $M$ is a saturated model of $T$ of cardinality $\lambda \ge \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \le \mu \le \lambda$ and $\mu$ is regular$\})$, <u>then</u> $M$ is $(\lambda,\Theta)$-superlimit (on $\kappa(T)$-see* [She90, III,§3]*).*
*3) If $T$ is stable in $\lambda$ and $\kappa = \mathrm{cf}(\kappa) \le \lambda$ <u>then</u> $T$ has an invariantly strongly $(\lambda,\kappa)$-limit model.*

*Remark* 0.10. Concerning 0.9(2), note that by [She90] if $\lambda$ is singular or just $\lambda < \lambda^{<\lambda}$ and $T$ has a saturated model of cardinality $\lambda$ <u>then</u> $T$ is stable (even stable in $\lambda$) and $\mathrm{cf}(\lambda) \ge \kappa(T)$).

*Proof.* 1) Let $M_i$ be a $\lambda$-saturated model of $T$ of cardinality $\lambda$ for $i < \lambda$ and $\langle M_i : i < \lambda\rangle$ is $\prec$-increasing and $M_\lambda = \bigcup\limits_{i<\lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $< \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$ hence every $p \in \mathbf{S}(A, M_\lambda)$ is realized in $M_i$ hence in $M_\lambda$; so clearly $M_\lambda$ is $\lambda$-saturated. Remembering the uniqueness of a $\lambda$-saturated model of $T$ of cardinality $\lambda$ we finish.
2) Use [She90, III,3.11]: if $M_i$ is a $\lambda$-saturated model of $T, \langle M_i : i < \delta\rangle$ increasing $\mathrm{cf}(\delta) \ge \kappa(T)$ <u>then</u> $\bigcup\limits_{i<\delta} M_i$ is $\lambda$-saturated.
3) Let $\mathbf{K}_{\lambda,\kappa} = \{\bar{M} : \bar{M} = \langle M_i : i \le \kappa\rangle$ is $\prec$-increasing continuous, $M_i \in \mathrm{EC}_\lambda(T)$ and $(M_{i+2}, c)_{c \in M_{i+1}}$ is saturated for every $i < \kappa\}$. Clearly $\bar{M}, \bar{N} \in \mathbf{K}_{\lambda,\kappa} \Rightarrow M_\kappa \cong$

$N_\kappa$. Also for every $M \in \mathrm{EC}_\lambda(T)$ there is $N$ such that $M \prec N$ and $(N, c)_{c \in M}$ is saturated, as also $\mathrm{Th}((M, c)_{c \in M})$ is stable in $\lambda$; so there is an invariant $\mathbf{F}$ : $\mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ such that $M \prec \mathbf{F}(M)$ and $(\mathbf{F}(M), c)_{c \in M}$ is saturated; such $\mathbf{F}$ witness the desired conclusion. $\square_{0.9}$

**Definition 0.11.** 0) For regular $\kappa < \lambda$ let $S^\lambda_\theta = \{\delta < \lambda : \mathrm{cf}(\delta) = \lambda\}$.
1) For a regular uncountable cardinal $\lambda$ let $\check{I}[\lambda] = \{S \subseteq \lambda$: some pair $(E, \bar{a})$ witnesses $S \in \check{I}[\lambda]$, see below$\}$.
2) We say that $(E, \bar{u})$ is a witness for $S \in \check{I}[\lambda]$ iff:

- $(a)$ $E$ is a club of the regular cardinal $\lambda$
- $(b)$ $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle, u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$
- $(c)$ for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of $\delta$ of order-type $\mathrm{cf}(\delta)$ (and $\delta$ is a limit ordinal).

By [She93, §1]

**Claim 0.12.** *If $\kappa^+ < \lambda$ and $\kappa, \lambda$ are regular then some stationary $S \subseteq \{\delta < \lambda : cf(\delta) = \kappa\}$ belongs to $\check{I}[\lambda]$.*

By [She79]

**Claim 0.13.** *If $\lambda = \mu^+, \theta = \mathrm{cf}(\theta) \leq \mathrm{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\theta} \leq \mu$ then $S^\lambda_\theta \in \check{I}[\lambda]$.*

## 1. On superstable not $\aleph_0$-stable $T$

We first note that superstable $T$ tend to have superlimit models.

**Claim 1.1.** *Assume $T$ is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ has a saturated model of cardinality $\lambda$ iff $T$ has a universal model of cardinality $\lambda$ iff $\lambda \geq |D(T)|$.*

*Proof.* By [She90, III,§5] we know that $T$ is stable in $\lambda$ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of $T$ of cardinality $\lambda$ hence no saturated model and no superlimit model, etc., recalling 0.8(2). If $\lambda \geq |D(T)|$, then $T$ is stable in $\lambda$ hence has a saturated model of cardinality $\lambda$ by [She90, III] (hence universal) and the class of $\lambda$-saturated models of $T$ is closed under increasing elementary chains by [She90, III] so we are done. $\square_{1.1}$

The following are the prototypical theories which we shall consider.

**Definition 1.2.** 1) $T_0 = \text{Th}({}^\omega 2, E^0_n)_{n<\omega}$ when $\eta E^0_n \nu \Leftrightarrow \eta \restriction n = \nu \restriction n$.
2) $T_1 = \text{Th}({}^\omega(\omega_1), E^1_n)_{n<\omega}$ where $\eta E^1_n \nu \Leftrightarrow \eta \restriction n = \nu \restriction n$.
3) $T_2 = \text{Th}(\mathbb{R}, <)$.

Recall

**Observation 1.3.** *0) $T_\ell$ is a countable complete first order theory for $\ell = 0, 1, 2$.*
*1) $T_0$ is superstable not $\aleph_0$-stable.*
*2) $T_1$ is strictly stable, that is, stable not superstable.*
*3) $T_2$ is unstable.*
*4) $T_\ell$ has elimination of quantifiers for $\ell = 0, 1, 2$.*

**Claim 1.4.** *It is consistent with* ZFC *that $\aleph_1 < 2^{\aleph_0}$ and some $M \in \text{EC}_{\aleph_1}(T_0)$ is a superlimit model.*

*Proof.* By [She80], for notational simplicity we start with $\mathbf{V} = \mathbf{L}$.

So $T_0$ is defined in 1.2(1) and it is the $T$ from Theorem [She80, 1.1] and let $S$ be the set of $\eta \in ({}^\omega 2)^{\mathbf{L}}$. We define $T'$ (called $T_1$ there) as the following theory:

- $\circledast_1$ $(i)$ $T_0$, or just for each $n$ the sentence saying $E_n$ is an equivalence relation with $2^n$ equivalence classes, each $E_n$ equivalence class divided to two by $E_{n+1}, E_{n+1}$ refine $E_n, E_0$ is trivial
  - $(ii)$ the sentences saying that
    - $(\alpha)$ for every $x$, the function $z \mapsto F(x,z)$ is one-to-one and
    - $(\beta)$ $xE_n(F(x,z))$ for each $n < \omega$
  - $(iii)$ $E_n(c_\eta, c_\nu)^{\text{if}(\eta \restriction n = \nu \restriction n)}$ for $\eta, \nu \in S$.

In [She80] it is proved that in some forcing[5] extension $\mathbf{L}^{\mathbb{P}}$ of $\mathbf{L}$, $\mathbb{P}$ an $\aleph_2$-c.c. proper forcing of cardinality $\aleph_2$, in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$, the class $\text{PC}(T', T_0) = \{M \restriction \tau_{T_0} : M$ is a $\tau$-model of $T'\}$ is categorical in $\aleph_1$.

However, letting $M^*$ be any model from $\text{PC}(T', T_0)$ of cardinality $\aleph_1$, it is easy to see that (in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$):

- $\circledast_2$ the following conditions on $M$ are equivalent
  - $(a)$ $M$ is isomorphic to $M^*$

[5] We can replace $\mathbf{L}$ by any $\mathbf{V}_0$ which satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$.

(b) $M \in \mathrm{PC}(T', T_0)$

(c) ($\alpha$) $M$ is a model of $T_0$ of cardinality $\aleph_1$

($\beta$) $M^*$ can be elementarily embedded into $M$

($\gamma$) for every $a \in M$ the set $\cap\{a/E_n^M : n < \omega\}$ has cardinality $\aleph_1$.

But

$\circledast_3$ every model $M_1$ of $T$ of cardinality $\leq \aleph_1$ has a proper elementary extension to a model satisfying (c), i.e., $(\alpha), (\beta), (\gamma)$ of $\circledast_2$ above

$\circledast_4$ if $\langle M_\alpha : \alpha < \delta \rangle$ is an increasing chain of models satisfying (c) of $\circledast_2$ and $\delta < \omega_2$ then also $\cup\{M_\alpha : \alpha < \delta\}$ does.

Together we are done. $\square_{1.4}$

Naturally we ask

*Question* 1.5. What occurs to $T_0$ for $\lambda > \aleph_1$ but $\lambda < 2^{\aleph_0}$?

*Question* 1.6. Does the theory $T_2$ of linear order consistently have an $(\aleph_1, \aleph_0)$-superlimit? (or only strongly limit?) but see §3.

*Question* 1.7. What is the answer for $T$ when $T$ is countable superstable not $\aleph_0$-stable and $D(T)$ countable for $\aleph_1 < 2^{\aleph_0}$ for $\aleph_2 < 2^{\aleph_0}$?

So by the above for some such $T$, in some universe, for $\aleph_1$ the answer is yes, there is a superlimit.

## 2. A strictly stable consistent example

We now look at models of $T_1$ (redefined below) in cardinality $\aleph_1$; recall

**Definition 2.1.** $T_1 = \mathrm{Th}({}^\omega(\omega_1), E_n)_{n<\omega}$ where $E_n = \{(\eta,\nu) : \eta,\nu \in {}^\omega(\omega_1)$ and $\eta \restriction n = \nu \restriction n\}$.

*Remark* 2.2.

- $(a)$ Note that $T_1$ has elimination of quantifiers.
- $(b)$ If $\lambda = \Sigma\{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, <u>then</u> $T_1$ has a $(\lambda,\aleph_0)$-superlimit model in $\lambda$ (see 2.15).

**Definition/Claim 2.3.** 1) Any model of $T_1$ of cardinality $\lambda$ is isomorphic to $M_{A,h} := (\{(\eta,\varepsilon) : \eta \in A, \varepsilon < h(\eta))\}, E_n)_{n<\omega}$ for some $A \subseteq {}^\omega\lambda$ and $h : {}^\omega\lambda \to (\mathrm{Car} \cap \lambda^+)\backslash\{0\}$ where $(\eta_1,\varepsilon_1)E_n(\eta_2,\varepsilon_2) \Leftrightarrow \eta_1 \restriction n = \eta_2 \restriction n$, pedantically we should write $E_n^{M_{A,h}} = E_n\restriction|M_{A,n}|$.
2) We write $M_A$ for $M_{A,h}$ when $A$ is as above and $h : A \to \{|A|\}$, so constantly $|A|$ when $A$ is infinite.
3) For $A \subseteq {}^\omega\lambda$ and $h$ as above the model $M_{A,h}$ is a model of $T_1$ <u>iff</u> $A$ is non-empty and $(\forall\eta \in A)(\forall n < \omega)(\exists^{\aleph_0}\nu \in A)(\nu\restriction n = \eta\restriction n \wedge \nu(n) \neq \eta(n))$.
4) Above $M_{A,h}$ has cardinality $\lambda$ iff $\Sigma\{h(\eta) : \eta \in A\} = \lambda$.

**Definition 2.4.** 1) We say that $A$ is a $(T_1,\lambda)$-witness <u>when</u>

- $(a)$ $A \subseteq {}^\omega\lambda$ has cardinality $\lambda$
- $(b)$ if $B_1, B_2 \subseteq {}^\omega\lambda$ are $(T_1,A)$-big (see below) of cardinality $\lambda$ <u>then</u> $(B_1 \cup {}^{\omega>}\lambda, \triangleleft)$ is isomorphic to $(B_2 \cup {}^{\omega>}\lambda, \triangleleft)$.

2) A set $B \subseteq {}^\omega\lambda$ is called $(T_1,A)$-big <u>when</u> it is $(\lambda,\lambda) - (T_1,A)$-big; see below.
3) $B$ is $(\mu,\lambda) - (T_1,A)$-big means: $B \subseteq {}^\omega\lambda, |B| = |A| = \mu$ and for every $\eta \in {}^{\omega>}\lambda$ there is an isomorphism $f$ from $({}^{\omega\geq}\lambda, \triangleleft)$ onto $(\{\eta\hat{\ }\nu : \nu \in {}^{\omega\geq}\lambda\}, \triangleleft)$ mapping $A$ into $\{\nu : \eta\hat{\ }\nu \in B\}$.
4) $A \subseteq {}^\omega(\omega_1)$ is $\aleph_1$-suitable <u>when</u>:

- $(a)$ $|A| = \aleph_1$
- $(b)$ for a club of $\delta < \omega_1, A \cap {}^\omega\delta$ is everywhere not meagre in the space ${}^\omega\delta$, i.e., for every $\eta \in {}^{\omega>}\delta$ the set $\{\nu \in A \cap {}^\omega\delta : \eta \triangleleft \nu\}$ is a non-meagre subset of ${}^\omega\delta$ (that is what really is used in [She80]).

**Claim 2.5.** *It is consistent with ZFC that $2^{\aleph_0} > \aleph_1 +$ there is a $(T_1,\aleph_1)$-witness; moreover every $\aleph_1$-suitable set is a $(T_1,\aleph_1)$-witness.*

*Proof.* By [She80, §2]. $\square_{2.5}$

*Remark* 2.6. The witness does not give rise to an $(\aleph_1,\aleph_0)$-limit model, as for the union of any "fast enough" $\prec$-increasing $\omega$-chain of members of $\mathrm{EC}_{\aleph_1}(T_1)$, the relevant sets are meagre.

**Definition 2.7.** Let $A$ be a $(T_1,\lambda)$-witness. We define $K^1_{T_1,A}$ as the family of $M = (|M|, <^M, P^M_\alpha)_{\alpha\leq\omega}$ such that:

- $(\alpha)$ $(|M|, <^M)$ is a tree with $(\omega+1)$ levels
- $(\beta)$ $P^M_\alpha$ is the $\alpha$-th level; let $P^M_{<\omega} = \cup\{P^M_n : n < \omega\}$

($\gamma$) $M$ is isomorphic to $M^1_B$ for some $B \subseteq {}^\omega\lambda$ of cardinality $\lambda$ where $M^1_B$ is defined by $|M^1_\beta| = ({}^{\omega>}\lambda) \cup B, P^{M^1_B}_n = {}^n\lambda, P^{M^1_B}_\omega = B$ and $<^{M^1_B} = \triangleleft\restriction|M^1_B|$, i.e., being an initial segment

($\delta$) moreover $B$ is such that some $f$ satisfies:

$\circledast$ ($a$) $f : {}^{\omega>}\lambda \to \omega$ and $f(<>) = 0$ for simplicity

($b$) $\eta \trianglelefteq \nu \in {}^{\omega>}\lambda \Rightarrow f(\eta) \leq f(\nu)$

($c$) if $\eta \in B$ then $\langle f(\eta \restriction n) : n < \omega\rangle$ is eventually constant

($d$) if $\eta \in {}^{\omega>}\lambda$ then $\{\nu \in {}^\omega\lambda : \eta^\frown\nu \in B$ and $m < \omega \Rightarrow f(\eta^\frown(\nu \restriction m)) = f(\eta)\}$ is $(T_1, A)$-big

($e$) for $\eta \in {}^{\omega>}\lambda$ and $n \in [f(\eta), \omega)$ for $\lambda$ ordinals $\alpha < \lambda$, we have $f(\eta^\frown\langle\alpha\rangle) = n$.

**Claim 2.8.** *[The Global Axiom of Choice] If $A$ is a $(T_1, \aleph_1)$-witness <u>then</u>*

($a$) $K^1_{T_1,A} \neq \emptyset$

($b$) *any two members of $K^1_{T_1,A}$ are isomorphic*

($c$) *there is a function $\mathbf{F}$ from $K^1_{T_1,A}$ to itself (up to isomorphism, i.e., $(M, \mathbf{F}(M))$ is defined only up to isomorphism) satisfying $M \subseteq \mathbf{F}(M)$ such that $K^1_{T_1,A}$ is closed under increasing unions of sequence $\langle M_n : n < \omega\rangle$ such that $\mathbf{F}(M_n) \subseteq M_{n+1}$.*

*Proof.* <u>Clause (a)</u>: Trivial.

<u>Clause (b)</u>: By the definition of "$A$ is a $(T_1, \aleph_1)$-witness" and of $K^1_{T_1,A}$.

<u>Clause (c)</u>:

We choose $\mathbf{F}$ such that

$\circledast$ if $M \in K^1_{A,T_1}$ then $M \subseteq \mathbf{F}(M) \in K^1_{A,T_1}$ and for every $k < \omega$ and $a \in P^M_k$, the set $\{b \in P^{\mathbf{F}(M)}_{k+1} : a <_{\mathbf{F}(M)} b$ and $b \notin M\}$ has cardinality $\aleph_1$.

Assume $M = \cup\{M_n : n < \omega\}$ where $\langle M_n : n < \omega\rangle$ is $\subseteq$-increasing$\}$, $M_n \in K^1_{A,T_1}, \mathbf{F}(M_n) \subseteq M_{n+1}$. Clearly $M$ is as required in the beginning of Definition 2.7, that is, satisfies clauses $(\alpha), (\beta), (\gamma)$ there. To prove clause $(\delta)$, we define $f : P^M_{<\omega} \to \omega$ by $f(a) =$ Min$\{n : a \in M_n\}$. Pendantically, $\mathbf{F}$ is defined only up to isomorphism.
So we are done. $\square_{2.8}$

**Claim 2.9.** *[The Global Axiom of Choice]*
*If $A$ is a $(T_1, \lambda)$-witness <u>then</u>*

($a$) $K^1_{T_1,A} \neq \emptyset$

($b$) *any two members of $K^1_{T_1,A}$ are isomorphic*

($c$) *if $M_n \in K^1_{T_1,A}$ and $n < \omega \Rightarrow M_n \subseteq M_{n+1}$ <u>then</u> $M := \cup\{M_n : n < \omega\} \in K^1_{T_1,A}$.*

*Remark* 2.10. If we omit clause (b), we can weaken the demand on the set $A$.

*Proof.* Assume $M = \cup\{M_n : n < \omega\}, M_n \subseteq M_{n+1}, M_n \in K^1_{T_1,A}$ and $f_n$ witnesses $M_n \in K^1_{T_1,A}$. Clearly $M$ satisfies clauses $(\alpha), (\beta), (\gamma)$ from Definition 2.7, we just have to find a witness $f$ as in clause $(\delta)$ there.

For each $a \in M$ let $n(a) =$ Min$\{n : a \in M_n\}$, clearly if $M \models$ "$a < b < c$" then $n(a) \le n(b)$ and $n(a) = n(c) \Rightarrow n(a) = n(b)$. Let $g_n : M \to M$ be defined by: $g_n(a) = b$ iff $b \le^M a, b \in M_n$ and $b$ is $\le^M$-maximal under those restrictions; clearly it is well defined. Now we define $f'_n : M_n \to \omega$ by induction on $n < \omega$ such that $m < n \Rightarrow f'_m \subseteq f'_n$, as follows.

If $n = 0$ let $f'_n = f_n$.

If $n = m + 1$ and $a \in M_n$ we let $f'_n(a)$ be $f'_m(a)$ if $a \in M_m$ and be $(f_n(a) - f_n(g_m(a))) + f'_m(g_m(a)) + 1$ if $a \in M_n \backslash M_m$. Clearly $f := \cup\{f'_n : n < \omega\}$ is a function from $M$ to $\omega, a \le^M b \Rightarrow f(a) \le f(b)$, and for any $a \in M$ the set $\{b \in M : a \le^M b$ and $f(b) = f(a)\}$ is equal to $\{b \in M_{n(a)} : f_{n(a)}(a) = f_{n(a)}(b)$ and $a \le^M b\}$.

So we are done. $\square_{2.9}$

**Definition 2.11.** Let $A$ be a $(T_1, \lambda)$-witness. We define $K^2_{T_1,A}$ as in Definition 2.7 but $f$ is constantly zero.

**Claim 2.12.** *[The Global Axiom of Choice] If $A$ is a $(T_1, \aleph_1)$-witness* <u>*then*</u>

- $(a)$ $K^2_{T_1,A} \neq \emptyset$
- $(b)$ *any two members of $K^2_{T_1,A}$ are isomorphic*
- $(c)$ *there is a function* $\mathbf{F}$ *from* $\cup\{{}^{\alpha+2}(K^2_{T_0,A}) : \alpha < \omega_1\}$ *to* $K^2_{T_1,A}$ *which satisfies:*
  - $\boxtimes$ $(\alpha)$ *if $\bar{M} = \langle M_i : i \le \alpha + 1\rangle$ is an $\prec$-increasing sequence of models of $T$ then $M_{\alpha+1} \subseteq \mathbf{F}(\bar{M}) \in K^2_{T_1,A}$*
  - $(\beta)$ *the union of any increasing $\omega_1$-sequence $\bar{M} = \langle M_\alpha : \alpha < \omega_1\rangle$ of members of $K^2_{T_1,A}$ belongs to $K^2_{T_1,A}$ when $\omega_1 = \sup\{\alpha : \mathbf{F}(\bar{M} \restriction (\alpha + 2)) \subseteq M_{\alpha+2})$ and is a well defined embedding of $M_\alpha$ into $M_{\alpha+2}\}$.*

*Remark* 2.13. Instead of the global axiom of choice, we can restrict the models to have universe a subset of $\lambda^+$ (or just a set of ordinals).

*Proof.* <u>Clause (a)</u>: Easy.

<u>Clause (b)</u>: By the definition.

<u>Clause (c)</u>: Let $\langle \mathscr{U}_\varepsilon : \varepsilon < \omega_1\rangle$ be an increasing sequence of subsets of $\omega_1$ with union $\omega_1$ such that $\varepsilon < \omega_1 \Rightarrow |\mathscr{U}_\varepsilon \backslash \bigcup\limits_{\zeta<\varepsilon} \mathscr{U}_\zeta| = \aleph_1$. Let $M^* \in K^2_{T_1,A}$ be such that ${}^{\omega>}(\omega_1) \subseteq |M^*| \subseteq {}^{\omega\ge}(\omega_1)$ and $M^*_\varepsilon =: M^* \restriction {}^{\omega\ge}(\mathscr{U}_\varepsilon)$ belongs to $K^2_{T_1,A}$ for every $\varepsilon < \omega_1$.

We choose a pair $(\mathbf{F}, \mathbf{f})$ of functions with domain $\{\bar{M} : \bar{M}$ an increasing sequence of members of $K^2_{T_1,A}$ of length $< \omega_1\}$ such that:

- $(\alpha)$ $\mathbf{F}(\bar{M})$ is an extension of $\cup\{M_i : i < \ell g(\bar{M})\}$ from $K^2_{T_1,A}$
- $(\beta)$ $\mathbf{f}(\bar{M})$ is an embedding from $M^*_{\ell g(\bar{M})}$ into $\mathbf{F}(\bar{M})$
- $(\gamma)$ if $\bar{M}^\ell = \langle M_\alpha : \alpha < \alpha_\ell\rangle$ for $\ell = 1, 2$ and $\alpha_1 < \alpha_2, \bar{M}^1 = \bar{M}^2 \restriction \alpha_1$ and $\mathbf{F}(\bar{M}^1) \subseteq M_{\alpha_1}$ then $\mathbf{f}(\bar{M}^1) \subseteq \mathbf{f}(\bar{M}^2)$
- $(\delta)$ if $a \in \mathbf{F}(\bar{M})$ and $n < \omega$ then for some $b \in M^*_{\ell g(\bar{M})}$ we have $\mathbf{F}(M) \models aE_n(\mathbf{f}(\bar{M})(b))$.

Now check. $\square_{2.12}$

**Conclusion 2.14.** *Assume there is a $(T_1, \aleph_1)$-witness (see Definition 2.4) for the first-order complete theory $T_1$ from 2.1:*
*1) $T_1$ has an $(\aleph_1, \aleph_0)$-strongly limit model.*
*2) $T_1$ has an $(\aleph_1, \aleph_1)$-medium limit model.*
*3) $T_1$ has a $(\aleph_1, \aleph_0)$-superlimit model.*

*Proof.* 1) By 2.8 the reduction of problems on $(\mathrm{EC}(T_1), \prec)$ to $K^1_{T_1,A}$ (which is easy) is exactly as in [She80].
2) By 2.12.
3) Like part (1) using claim 2.9. $\square_{2.14}$

**Claim 2.15.** *If $\lambda = \Sigma\{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, <u>then</u> $T_1$ has a $(\lambda, \aleph_0)$-superlimit model in $\lambda$.*

*Proof.* Let $M_n$ be the model $M_{A_n,h_n}$ where $A_n = {}^\omega(\lambda_n)$ and $h_n : A_n \to \lambda_n^+$ is constantly $\lambda_n$.

Clearly

$(*)_1$ $M_n$ is a saturated model of $T_1$ of cardinality $\lambda_n$
$(*)_2$ $M_n \prec M_{n+1}$
$(*)_3$ $M_\omega = \cup\{M_n : n < \omega\}$ is a special model of $T_1$ of cardinality $\lambda$.

The main point:

$(*)_4$ $M_\omega$ is $(\lambda, \aleph_0)$-superlimit model of $T_1$.

[Why? Toward this assume

(a) $N_n$ is isomorphic to $M_\omega$ say $f_n : M_\omega \to N_n$ is such isomorphic
(b) $N_n \prec N_{n+1}$ for $n < \omega$.

Let $N_\omega = \cup\{N_n : n < \omega\}$ and we should prove $N_\omega \cong M_\omega$, so just $N_\omega$ is a special model of $T_1$ of cardinality $\lambda$ suffice.

Let $N'_n = N_\omega {\restriction} (\cup\{f_n(M_k) : k \leq n\})$. Easily $N'_n \prec N'_{n+1} \prec N_\omega$ and $\cup\{N'_n : n < \omega\} = N_{\omega_*}$ and $\|N'_n\| = \lambda_n$. So it suffices to prove that $N'_n$ is saturated and by direct inspection shows this. $\square_{2.15}$

## 3. On non-existence of limit models

Naturally we assume that non-existence of superlimit models for unstable $T$ is easier to prove. For other versions we need to look more. We first show that for $\lambda \geq |T|+\aleph_1$, if $T$ is unstable then it does not have a superlimit model of cardinality $\lambda$ and if $T$ is unsuperstable, we show this for "most" cardinals $\lambda$. On "$\Phi$ proper for $K_{\rm or}$ or $K^\omega_{\rm tr}$", see [She90, VII] or [Shec] or hopefully some day in [Shear, III]. We assume some knowledge on stability.

**Claim 3.1.** *1) If $T$ is unstable, $\lambda \geq |T|+\aleph_1$, then $T$ has no superlimit model of cardinality $\lambda$.*
*2) If $T$ is stable not superstable and $\lambda \geq |T|+\beth_\omega$ or $\lambda = \lambda^{\aleph_0} \geq |T|$ then $T$ has no superlimit model of cardinality $\lambda$.*

*Remark* 3.2. 1) We assume some knowledge on EM models for linear orders $I$ and members of $K^\omega_{\rm tr}$ as index models, see, e.g. [She90, VII].
2) We use the following definition in the proof, as well as a result from [She00] or [She06].

**Definition 3.3.** For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal $\mu$ such that for some, equivalently for every set $A$ of cardinality $\lambda$ there is $\mathscr{P}_A \subseteq [A]^{\leq\kappa} = \{B \subseteq A : |B| \leq \kappa\}$ of cardinality $\lambda$ such that any $B \in [\lambda]^{\leq\kappa}$ is the union of $< \kappa$ members of $\mathscr{P}_A$.

*Proof.* 1) Towards a contradiction assume $M^*$ is a superlimit model of $T$ of cardinality $\lambda$. As $T$ is unstable we can find $m, \varphi(\bar{x},\bar{y})$ such that

- $(*)$ $\varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\tau(T)}$ linearly orders some infinite $\mathbf{I} \subseteq {}^m M, M \models T$ so $\ell g(\bar{x}) = \ell g(\bar{y}) = m$.

We can find a $\Phi$ which is proper for linear orders (see [She90, VII]) and $F_\ell(\ell < m)$ such that $F_\ell \in \tau_\Phi \backslash \tau_T$ is a unary function symbol for $\ell < m, \tau_T \subseteq \tau(\Phi)$ and for every linear order $I$, ${\rm EM}(I,\Phi)$ has Skolem functions and its $\tau_T$-reduct ${\rm EM}_{\tau(T)}(I,\Phi)$ is a model of $T$ of cardinality $|T|+|I|$ and $\tau(\Phi)$ is of cardinality $|T|+\aleph_0$ and $\langle a_s : s \in I\rangle$ is the Skeleton of ${\rm EM}(I,\Phi)$, that is, it is an indiscernible sequence in ${\rm EM}(I,\Phi)$ and ${\rm EM}(I,\Phi)$ is the Skolem hull of $\{a_s : s \in I\}$, and letting $\bar{a}_s = \langle F_\ell(a_s) : \ell < m\rangle$ in ${\rm EM}(I,\Phi)$ we have ${\rm EM}_{\tau(T)}(I,\Phi) \models \varphi[\bar{a}_s,\bar{a}_t]^{{\rm if}(s<t)}$ for $s,t \in I$.

Next we can find $\Phi_n$ (for $n < \omega$) such that:

- $\boxplus$ (a) $\Phi_n$ is proper for linear order and $\Phi_0 = \Phi$
  - (b) ${\rm EM}_{\tau(\Phi)}(I,\Phi_n) \prec {\rm EM}_{\tau(\Phi)}(I,\Phi_{n+1})$ for every linear order $I$ and $n < \omega$; moreover
  - $(b)^+$ $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and ${\rm EM}(I,\Phi_n) \prec {\rm EM}_{\tau(\Phi_n)}(I,\Phi_{n+1})$ for every $n < \omega$ and linear order $I$
  - (c) if $|I| \leq n$ then ${\rm EM}_{\tau(\Phi)}(I,\Phi_n) = {\rm EM}_{\tau(\Phi)}(I,\Phi_{n+1})$ and ${\rm EM}_{\tau(T)}(I,\Phi_n) \cong M^*$
  - (d) $|\tau(\Phi_n)| = \lambda$.

This is easy. Let $\Phi_\omega$ be the limit of $\langle \Phi_n : n < \omega\rangle$, i.e. $\tau(\Phi_\omega) = \cup\{\tau(\Phi_n) : n < \omega\}$ and if $k < \omega$ then ${\rm EM}_{\tau(\Phi_k)}(I,\Phi_\omega) = \cup\{{\rm EM}_{\tau(\Phi_k)}(I,\Phi_n) : n \in [k,\omega)\}$. So as $M^*$ is a superlimit model, for any linear order $I$ of cardinality $\lambda, {\rm EM}_{\tau(T)}(I,\Phi_\omega)$ is the direct limit of $\langle {\rm EM}_{\tau(T)}(J,\Phi_\omega) : J \subseteq I \text{ finite}\rangle$, each isomorphic to $M^*$, so as we have assumed that $M^*$ is a superlimit model it follows that ${\rm EM}_{\tau(T)}(I,\Phi_\omega)$ is isomorphic

to $M^*$. But by [She87b, III] or [Shec] which may eventually be [Shear, III] there are $2^\lambda$ many pairwise non-isomorphic models of this form varying $I$ on the linear orders of cardinality $\lambda$, contradiction.
2) First assume $\lambda = \lambda^{\aleph_0}$. Let $\tau \subseteq \tau_T$ be countable such that $T' = T \cap \mathbb{L}(\tau)$ is not superstable. Clearly if $M^*$ is $(\lambda,\aleph_0)$-limit model then $M^* \restriction \tau'$ is not $\aleph_1$-saturated. [Why? As in [She78, Ch.VI,§6], but we shall give full details. There are $N_* \models T, p = \{\varphi_n(\lambda,\bar a_n) : n < \omega\}$ a type in $N_*, \bar a_n \triangleleft \bar a_{n+1}, \bar a_{<>}$ empty and $\varphi_{n+1}(x,\bar a_{n+1})$ forks over $\bar a_n$. Let $\mathbf{F}(M)$ be such that if $n < \omega$ and $\bar b_n \subseteq M$ realizes $\mathrm{tp}(\bar a_n,\emptyset,N_*)$ then for some $\bar b_{n+1}$ from $\mathbf{F}, M$ realizing $\mathrm{tp}(\bar a_{n+1},\emptyset,N_*)$, the type $\mathrm{tp}(\bar b_{n+1},M,\mathbf{F}(M))$ does not fork over $b_n$.] But if $\kappa = \mathrm{cf}(\kappa) \in [\aleph_1,\lambda]$ and $M^*$ is a $(\lambda,\kappa)$-limit then $M^* \restriction \tau'$ is $\aleph_1$-saturated, contradiction.

The case $\lambda \geq |T| + \beth_\omega$ is more complicated (the assumption $\lambda \geq \beth_\omega$ is to enable us to use [She00] or see [She06] for a simpler proof; we can use weaker but less transparent assumptions; maybe $\lambda \geq 2^{\aleph_0}$ suffices).

As $T$ is stable not superstable by [She90] for some $\bar\Delta$:

- $\circledast_1$ for any $\mu$ there are $M$ and $\langle a_{\eta,\alpha} : \eta \in {}^\omega\mu$ and $\alpha < \mu\rangle$ such that
  - $(a)$ $M$ is a model of $T$
  - $(b)$ $\mathbf{I}_\eta = \{a_{\eta,\alpha} : \alpha < \mu\} \subseteq M$ is an indiscernible set (and $\alpha < \beta < \mu \Rightarrow a_{\eta,\alpha} \neq a_{\eta,\beta}$)
  - $(c)$ $\bar\Delta = \langle \Delta_n : n < \omega\rangle$ and $\Delta_n \subseteq \mathbb{L}_{\tau(T)}$ infinite
  - $(d)$ for $\eta,\nu \in {}^\omega\mu$ we have $\mathrm{Av}_{\Delta_n}(M,\mathbf{I}_\eta) = \mathrm{Av}_{\Delta_n}(M,\mathbf{I}_\nu)$ iff $\eta \restriction n = \nu \restriction n$.

Hence by [She90, VIII], or see [Shec] assuming $M^*$ is a universal model of $T$ of cardinality $\lambda$ :

- $\circledast_{2.1}$ there is $\Phi$ such that
  - $(a)$ $\Phi$ is proper for $K^\omega_{\mathrm{tr}}, \tau_T \subseteq \tau(\Phi), |\tau(\Phi)| = \lambda \geq |T| + \aleph_0$
  - $(b)$ for $I \subseteq {}^{\omega\geq}\lambda$, $\mathrm{EM}_{\tau(\Phi)}(I,\Phi)$ is a model of $T$ and $I \subseteq J \Rightarrow \mathrm{EM}(I,\Phi) \prec \mathrm{EM}(J,\Phi)$
  - $(c)$ for some two-place function symbol $F$ if for $I \in K^\omega_{\mathrm{tr}}$ and $\eta \in P^I_\omega, I$ a subtree of ${}^{\omega\geq}\lambda$ for transparency we let $\mathbf{I}_{I,\eta} = \{F(a_\eta,a_\nu) : \nu \in I\}$ then $\langle \mathbf{I}_{I,\eta} : \eta \in P^I_\omega\rangle$ are as in $\circledast_1(b),(d)$.

Also

- $\circledast_{2.2}$ if $\Phi_1$ satisfies (a),(b),(c) of $\circledast_{2.1}$ and $M$ is a universal model of $T$ then there is $\Phi^*_2$ satisfying (a),(b),(c) of $\circledast_{2.1}$ and $\Phi_1 \leq \Phi^*_2$ see $\circledast_{2.3}(a)$ and for every finitely generated $J \in K^\omega_{\mathrm{tr}}$, see $\circledast_{2.3}(b)$ below, there is $M' \cong M$ such that $\mathrm{EM}_{\tau(T)},(J,\Phi_1) \prec M' \prec \mathrm{EM}_{\tau(T)}(J,\Phi^*_2)$
- $\circledast_{2.3}$ $(a)$ we say $\Phi_1 \leq \Phi_2$ when $\tau(\Phi_1) \subseteq \tau(\Phi_2)$ and $J \in K^\omega_{\mathrm{tr}} \Rightarrow$ $\mathrm{EM}(J,\Phi_1) \prec \mathrm{EM}_{\tau(\Phi_1)}(J,\Phi_2)$
  - $(b)$ we say $J \subseteq I$ is finitely generated if it has the form $\{\eta_\ell : \ell < n\} \cup \{\rho$: for some $n,\ell$ we have $\rho \in P^I_n$ and $\rho <^I \eta_\ell\}$ for some $\eta_0,\dots,\eta_{n-1} \in P^I_\omega$
- $\circledast_{2.4}$ if $M_* \in \mathrm{EC}_\lambda(T)$ is superlimit (or just weakly $S$-limit, $S \subseteq \lambda^+$ stationary) <u>then</u> there is $\Phi$ as in $\circledast_{2.1}$ above such that $\mathrm{EM}_{\tau(T)}(J,\Phi) \cong M_*$ for every finitely generated $J \in K^\omega_{\mathrm{tr}}$
- $\circledast_{2.5}$ we fix $\Phi$ as in $\circledast_{2.4}$ for $M_* \in \mathrm{EC}_\lambda(T)$ superlimit.

Hence (mainly by clause (b) of $\circledast_{2.1}$ and $\circledast_{2.4}$ as in the proof of part (1))

$\circledast_3$ if $I \in K^\omega_{\rm tr}$ has cardinality $\le \lambda$ then $\mathrm{EM}_{\tau(\Phi)}(I,\Phi)$ is isomorphic to $M^*$.

Now by [She00], we can find regular uncountable $\kappa < \beth_\omega$ such that $\lambda = \lambda^{[\kappa]}$, see Definition 3.3.

Let $S = \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ and $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ be such that $\eta_\delta$ an increasing sequence of length $\omega$ with limit $\delta$.

For a model $M$ of $T$ let $\mathrm{OB}_{\bar\eta}(M) = \{\bar{\mathbf{a}} : \bar{\mathbf{a}} = \langle a_{\eta_\delta,\alpha} : \delta \in W$ and $\alpha < \kappa\rangle, W \subseteq S$ and in $M$ they are as in $\circledast_1(b),(d)\}$.
For $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ let $W[\bar{\mathbf{a}}]$ be $W$ as above and let

$$\begin{aligned}\Xi(\bar{\mathbf{a}},M) = \{\eta \in {}^\omega\kappa : \ & \text{there is an indiscernible set}\\ & \mathbf{I} = \{a_\alpha : \alpha < \kappa\} \text{ in } M \text{ such that for every } n\\ & \text{for some } \delta \in W[\bar{\mathbf{a}}], \eta \restriction n = \eta_\delta \restriction n \text{ and}\\ & \mathrm{Av}_{\Delta_n}(M,\mathbf{I}) = \mathrm{Av}_{\Delta_n}(M,\{a_{\eta_\delta,\alpha} : \alpha < \kappa\})\}.\end{aligned}$$

Clearly

$\circledast_4$ (a) if $M \prec N$ then $\mathrm{OB}_{\bar\eta}(M) \subseteq \mathrm{OB}_{\bar\eta}(N)$

(b) if $M \prec N$ and $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ then $\Xi(\bar{\mathbf{a}},M) \subseteq \Xi(\bar{\mathbf{a}},N)$.

Now by the choice of $\kappa$ it should be clear that

$\circledast_5$ if $M \models T$ is of cardinality $\lambda$ then we can find an elementary extension $N$ of $M$ of cardinality $\lambda$ such that for every $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of $\kappa$, for some stationary $W' \subseteq W[\bar{\mathbf{a}}]$ the set $\Xi[\bar{\mathbf{a}},N]$ includes $\{\eta \in {}^\omega\kappa : (\forall n)(\exists \delta \in W')(\eta \restriction n = \eta_\delta \restriction n)\}$, (moreover we can even find $\varepsilon^* < \kappa$ and $W_\varepsilon \subseteq W$ for $\varepsilon < \varepsilon^*$ satisfying $W[\bar{\mathbf{a}}] = \cup\{W_\varepsilon : \varepsilon < \varepsilon^*\}$)

$\circledast_6$ we can find $M \in \mathrm{EC}_\lambda(T)$ isomorphic to $M^*$ such that for every $\bar{\mathbf{a}} \in \mathrm{OB}_{\bar\eta}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of $\kappa$, we can find a stationary subset $W'$ of $W[\bar{\mathbf{a}}]$ such that the set $\Xi[\bar{\mathbf{a}},M]$ includes $\{\eta \in {}^\omega\mu : (\forall n)(\exists \delta \in W')(\eta \restriction n = \eta_\delta \restriction n)\}$.

[Why? We choose $(M_i,N_i)$ for $i < \kappa^+$ such that

(a) $M_i \in \mathrm{EC}_\lambda(T)$ is $\prec$-increasing continuous

(a) $M_{i+1}$ is isomorphic to $M^*$

(a) $M_i \prec N_i \prec M_{i+1}$

(a) $(M_i,N_i)$ are like $(M,N)$ in $\circledast_5$.

Now $M = \cup\{M_i : i < \kappa^+\}$ is as required.
Now the model $M$ is isomorphic to $M^*$ as $M^*$ is superlimit.]

Now the model from $\circledast_6$ is not isomorphic to $M' = \mathrm{EM}_{\tau(T)}({}^{\omega>}\lambda \cup \{\eta_\delta : \delta \in S\},\Phi)$ where $\Phi$ is from $\circledast_{2.1}$. But $M' \cong M^*$ by $\circledast_3$.
Together we are done. $\square_{3.1}$

The following claim says in particular that if some not unreasonable pcf conjectures holds, the conclusion holds for every $\lambda \ge 2^{\aleph_0}$.

**Claim 3.4.** *Assume $T$ is stable not superstable, $\lambda \geq |T|$ and $\lambda \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$.*
*1) $T$ has no $(\lambda, \kappa)$-superlimit model provided that $\kappa = \mathrm{cf}(\kappa) > \aleph_0, \lambda \geq \kappa^{\aleph_0}$ and $\lambda = \mathbf{U}_D(\lambda) := \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^\kappa$ and for every $f : \kappa \to \lambda$ for some $u \in \mathscr{P}$ we have $\{\alpha < \kappa : f(\alpha) \in u\} \in D^+$, where $D$ is a normal filter on $\kappa$ to which $\{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ belongs.*
*2) Similarly if $\lambda \geq 2^{\aleph_0}$ and letting $J_0 = \{u \subseteq \kappa : |u| \leq \aleph_0\}, J_1 = \{u \subseteq \kappa : u \cap S^\kappa_{\aleph_0}$ non-stationary$\}$ we have $\lambda = \mathbf{U}_{J_1,J_0}(\lambda) := \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{\aleph_0}$, if $u \in J_1, f : (\kappa \backslash u) \to \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathscr{P}$, $\mathrm{Rang}(f \restriction w) \subseteq v\}$.*

*Proof.* Like 3.1(2). $\square$

**Claim 3.5.** *1) Assume $T$ is unstable and $\lambda \geq |T| + \beth_\omega$. Then for at most one regular $\kappa \leq \lambda$ does $T$ have a weakly $(\lambda, \kappa)$-limit model and even a weakly $(\lambda, S)$-limit model for some stationary $S \subseteq S^\lambda_\kappa$.*
*2) Assume $T$ is unsuperstable and $\lambda \geq |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \mathrm{cf}(\kappa_2)$. Then $T$ has no model which is a weak $(\lambda, S)$-limit where $S \subseteq \lambda$ and $S \cap S^\lambda_{\kappa_\ell}$ is stationary for $\ell = 1, 2$.*

*Proof.* 1) Assume $\kappa_1 \neq \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{[\kappa]}$, see Definition 3.3 and $\kappa \notin \{\kappa_1, \kappa_2\}$. Let $m, \varphi(\bar{x}, \bar{y})$ be as in the proof of 3.1

- $(*)$ if $M \in \mathrm{EC}_\lambda(T)$ then there is $N$ such that
  - $(a)$ $N \in \mathrm{EC}_\lambda(T)$
  - $(b)$ $M \prec N$
  - $(c)$ if $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa \rangle \in {}^\kappa({}^m M)$ for $\alpha < \kappa$ then for some $\mathscr{U} \in [\kappa]^\chi$ for every uniform ultrafilter $D$ on $\kappa$ to which $\mathscr{U}$ belongs there is $\bar{a}_D \in {}^n N$ such that $\mathrm{tp}(\bar{a}_D, N, N) = \mathrm{Av}(\bar{\mathbf{a}}/D, M) = \{\psi(\bar{x}, \bar{c}) : \psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{c} \in {}^{\ell g(\bar{z})}M$ and $\{\{\alpha < \kappa : N \models \psi[\bar{a}_{i_\alpha}, \bar{c}]\} \in D\}$.

Similarly

- $\boxplus_1$ for every function $\mathbf{F}$ with domain $\{\bar{M} : \bar{M}$ an $\prec$-increasing sequence of models of $T$ of length $< \lambda^+$ each with universe $\in \lambda^+\}$ such that $M_i \prec \mathbf{F}(\bar{M})$ for $i < \ell g(\bar{M})$ and $\mathbf{F}(\bar{M})$ has universe $\in \lambda^+$ there is a sequence $\langle M_\varepsilon : \varepsilon < \lambda^+ \rangle$ obeying $\mathbf{F}$ such that: for every $\varepsilon < \lambda^+$ and $\bar{\mathbf{a}} \in {}^\kappa({}^m(M_\varepsilon))$ for $\alpha < \kappa$, there is $\mathscr{U} \in [\kappa]^\kappa$ such that for every ultrafilter $D$ on $\kappa$ to which $\mathscr{U}$ belongs, for every $\zeta \in (\varepsilon, \lambda^+)$ there is $\bar{\mathbf{a}}_{D,\zeta} \in {}^m(M_{\zeta+1})$ realizing $\mathrm{Av}(\bar{\mathbf{a}}/D, M_\zeta)$ in $M_{\zeta+1}$.

Hence

- $\boxplus_2$ for $\langle M_\alpha : \alpha < \lambda^+ \rangle$ as in $\boxplus_1$ for every limit $\delta < \lambda^+$ of cofinality $\neq \kappa$ for every $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa \rangle \in {}^\kappa({}^m(M_\delta))$, there is $\mathscr{U} \in [\kappa]^\kappa$ such that for every ultrafilter $D$ on $\kappa$ to which $\mathscr{U}$ belongs, there is a sequence $\langle \bar{b}_\varepsilon : \varepsilon < \mathrm{cf}(\delta) \rangle \in {}^{\mathrm{cf}(\delta)}({}^m(M_\delta))$ such that for every $\psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}(M_\delta)$ for every $\varepsilon < \mathrm{cf}(\delta)$ large enough, $M_\delta \models \psi[\bar{b}_\varepsilon, \bar{c}]$ iff $\psi(\bar{x}, \bar{c}) \in \mathrm{Av}(\bar{\mathbf{a}}/D, M_\delta)$.

The rest should be clear.
2) Combine the above and the proof of 3.1(2). $\square_{3.5}$

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