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DEPENDENT T AND EXISTENCE OF LIMIT MODELS SH877

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Abstract. Does the class of linear orders have (one of the variants of) the so called (λ, κ)-limit model? It is necessarily unique, and naturally assuming some instances of G.C.H. we get some positive, i.e. existence results. More generally, letting T be a complete first order theory and for simplicity assume G.C.H., for regular $\lambda > \kappa > |T|$ does T have (variants of) a (λ, κ)-limit models, except for stable T? For some, yes, the theory of dense linear order, for some, no. Moreover, for independent T we get negative, i.e. non-existence results. We deal more with linear orders.

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§ 0. Introduction

The first part of the introduction is intended for a general mathematical reader. Cantor proved that the structure "the rationals as a linear order" is characterized up to isomorphism by being "a dense linear order with neither first nor last element which is countable". Hausdorff generalizes this as follows. For transparency assume the G.C.H., the generalized continuum hypothesis then for every cardinal λ there is a unique linear order I of cardinality λ^+ which is λ^+-dense (i.e. if $A < C$ are subsets of cardinality $\leq \lambda$ then for some $b \in I$ we have $A < b < C$) with neither first nor last elements). This canonical linear order is, in later model theoretic notions, the unique saturated model of the theory $T_{\rm ord} = {\rm Th}(\mathbb{Q}, <)$ of cardinality λ^+ (also the universal homogeneous model); note $T_{\rm ord}$ is the first order theory of the rational order.

Later Bjarni Jónsson [Jón56], [Jón60] introduced and proved the existence of homogeneous-universal models in cardinality λ, for a quite general class of structures. Morley and Vaught [MV62] introduced the notion of saturated models and investigate such models (which are homogeneous universal if we use elementary submodels instead of substructures). Saturated models become a central notion in model theory.

The author in [She87] or [Shea] = [She09a, Ch.I], introduce abstract elementary classes and there define some variants of (λ, κ)-limit models which are again (like the homogeneous universal ones) unique but for the pair of cardinals λ, κ; note that for $\lambda = \kappa = \mu^+$ this is the previous case. So natural questions are: what about elementary classes, i.e. first order theories? and what about the class of linear orders?

By [She12] if T is low enough (so called stable) there are existence theorems but, e.g. the theory of linear order is not stable.

What are our main results? First, a result meaningful also to one with very little set theoretic background. If $\lambda = \lambda^{<\lambda}$, e.g. $\lambda = \mu^+ = 2^\mu$, then in addition to the unique (up to isomorphisms) linear order which Hausdorff discovers, for $\kappa = \aleph_0$ or just $\kappa \leq \lambda$ which is a successor (or just a so called regular) there is a (λ, κ)-limit linear order and it is unique up to isomorphisms. We can also have a characterization (as in the case of Hausdorff), though not so elegant; see §1 which do not require model theoretic background, see Theorem 1.1. There are stronger versions of "(λ, κ)-limit models" ((λ, κ)-superlimit) for which we show non-existence, see §3.

Second, in model theoretic terms this shows that having (λ, κ)-limit model is satisfied by some (complete first order) theories T which are not stable; all this in §1. So does every T have such models? In §2 comes another major result of this work: the answer in general, is no, e.g. for (the first order theory) Peano arithmetic, see Theorem 2.3. Moreover, there is a reasonable natural sufficient condition: the theory T is so called dependent, this is Theorem 2.9.

Those complementary results lead to the main conjectures arising from this work on existence of (λ, κ)-limit models and to the generic pair conjecture. They essentially say that the above mentioned sufficient condition, "T is dependent" is the right one, each dealing with a variant of the question (the first: any relevant κ, the second: the parallel for $\kappa = 2$).

The question can be rephrased (under G.C.H., restricting ourselves to successor cardinality $\aleph_{\varepsilon+1}$) as follows: assume $\langle M_\alpha : \alpha < \aleph_{\varepsilon+2} \rangle$ is a $\prec$-increasing continuous

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sequence of models of the first order complete T, $\|M_\alpha\| = \aleph_{\varepsilon+1}$ and $M = \bigcup\{M_\alpha : \alpha < \aleph_{\varepsilon+2}\}$ is saturated (e.g. Hausdorff linear order of cardinality $\aleph_{\varepsilon+2}, T = T_{\rm ord}$). Let $\mathbf{n}_{\aleph_{\varepsilon+1}}(T) = \text{Min}\{|\{M_\alpha/\cong : \alpha \in E\}| : E$ a closed unbounded subset of $\aleph_{\varepsilon+2}\}$. Now the existence of $(\aleph_{\varepsilon+1}, \kappa)$-limit model for every regular $\kappa < \aleph_{\varepsilon+2}$ implies $\mathbf{n}_T(\aleph_{\varepsilon+1}) = |\varepsilon + 2|$, in fact for some such E for any κ all the models $\{M_\delta : \delta \in E$ has cofinality $\kappa\}$ are pairwise isomorphic. Our non-existence results give $\mathbf{n}_{\aleph_{\varepsilon+1}}(T) = \aleph_{\varepsilon+2}$.

In the rest of the introduction we assume more background.

* * *

We continue [She12] and [She09b]

The problem in [She12] is when does (a first order theory) T have a model M of cardinality λ which is (one of the variants of) a limit model for cofinality κ, in the cases not covered by [She12, 0.8] (or [She87, 3.3,3.2], [Shea, 3.6,3.5]). More accurately, there are some versions of limit models, "M is a (λ, κ)-x-limit model of T" mainly "(λ, κ)-i.md. limit", see Definition 0.8; (though we deal with others, too) the most natural case to try is $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa) > |T|$.

Note that if T has (any version of) a limit model of cardinality λ then there is a universal $M \in \text{Mod}_\lambda(T)$. Now we know that if $\lambda = 2^{<\lambda} > |T|$ then there is a universal $M \in \text{Mod}_\lambda(T)$ (see e.g. [Hod93]). But for other cardinals it is "hard to have a universal model", see history [KS92] and [Mir05]. E.g. if T has the strict order property, then, by Kojman-Shelah [KS92] there are ZFC non-existence results (a major case, for regular λ is when $(\exists\mu)(\mu^+ < \lambda \wedge 2^\mu > \lambda)$. In at least one case, $\lambda = \aleph_1 < 2^{\aleph_0}$ consistently we do not have a universal model, see [She80].

Stable theories have limit models (in many cases); hence it is natural to ask:

<u>Question 1</u>: Assume $\lambda = \lambda^{<\kappa} > \kappa > |T|$. Does the existence of a (λ, κ)-md.-limit model of T imply T is stable?

This is quite reasonable but in Theorem 1.1 we find a counterexample, in fact, one everyone knows about: the theory $T_{\rm ord}$ of dense linear orders (see 0.12). This per se is a continuation of Hausdorff result, revealing some canonical linear ordres. Returning to the family of elementary classes, i.e. first order theories, it is natural to ask:

<u>Question 2</u>: Does T have a (λ, κ)-i.md.-limit model whenever $\lambda = \lambda^{<\lambda} > \kappa + |T|$ for every unstable T?

For non-existence results it is natural to look at T dissimilar to $T_{\rm ord}$.
As $T_{\rm ord}$ is prototypical of dependent theories, it is natural to look for independent theories. A strong, explicit version of T being independent is having the strong independence property (see Definition 2.4), e.g. Peano arithmetic has. We prove that for such T there are no limit models (2.3). But the strong independence property does not seem a good dividing line. The independence property is a good candidate for being a meaningful dividing line.

<u>Question 3</u>: If T is independent, does T have a (λ, κ)-i.md.-limit model (with $\lambda = \lambda^{<\lambda} > \kappa > |T|$)?

We work harder (than in 2.3) to prove (in 2.9) the negative answer for every independent T (for many cardinals), i.e. with the independence property though a

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weaker version meaning we prove non-existence of a stronger version of "(λ,κ)-limit model".

This makes us

Conjecture 0.1. Any dependent T has (λ,κ)-i.md.-limit model.

Toward this end we intend to continue the investigation of types for dependent T.

We shall also consider a property $\mathrm{Pr}_{\lambda,\kappa}(T)$ (and the stronger $\mathrm{Pr}^2_{\lambda,\kappa}(T)$), see Definition 2.5, which are relatives of "there is no (λ,κ)-x-limit model"; i.e. non-existence results for independent T holds for $\lambda = \lambda^{<\lambda} \geq \kappa = \mathrm{cf}(\kappa), \lambda > |T|$. For $\lambda > \kappa$ this strengthens "there is no (λ,κ)-i.md.-limit model". But $\lambda = \kappa$ is a new non-trivial case and it is also a candidate to be "an outside equivalent condition for T being dependent".

The most promising among the relatives (for having a dichotomy) is the following conjecture (the assumption $2^\lambda = \lambda^+$ is just for simplicity).

Conjecture 0.2. <u>The generic pair conjecture</u>

Assume $\lambda = \lambda^{<\lambda} > |T|$ and $2^\lambda = \lambda^+$ (for transparency) and $M_\alpha \in \mathrm{EC}_\lambda(T)$ is $\prec$-increasing continuous for $\alpha < \lambda^+$ with $\bigcup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_{\lambda^+}(T)$ saturated. Then T is dependent iff for some club E of λ^+ for all pairs $\alpha < \beta < \lambda^+$ from E both of cofinality λ, (M_β, M_α) has the same isomorphism type (we denote this property of T by $\mathrm{Pr}^2_\lambda(T)$), see Definition 2.5).

Here we prove that for independent T, a strong version of the conjecture holds.

In §2, we also prove the parallel of what we say above. In §3 we prove that (λ,κ)-superlimit models does not exist even for $T = T_{\mathrm{ord}}$. This work is continued in [She11], [She15], [Sheb], [S+] and Kaplan-Lavi-Shelah [KLS16].

* * *

Now we define some versions of "M is a (λ,S)-x-limit model" and for them "$\bar{M}$ obeys a (λ,T)-x-function".

Notation 0.3. 1) Let T denote a complete first order theory.
2) Let $\tau_T = \tau(T), \tau_M = \tau(M)$ be the vocabulary of T, M respectively.

Definition 0.4. 1) For any T let $\mathrm{EC}(T) = \{M : M \text{ a } \tau_T\text{-model of } T\}$.
2) $\mathrm{EC}_\lambda(T) = \{M \in \mathrm{EC}(T) : M \text{ is of cardinality } \lambda\}$ and $\mathrm{EC}_{\lambda,\kappa}(T) = \{M \in \mathrm{EC}_\lambda(T) : M \text{ is } \kappa\text{-saturated}\}$.
3) We say $M \in \mathrm{EC}(T)$ is λ-universal when every $N \in \mathrm{EC}_\lambda(T)$ can be elementarily embedded into M.
4) We say $M \in \mathrm{EC}(T)$ is universal when it is λ-universal for $\lambda = \|M\|$.
5) For $T \subseteq T'$ let

$$\mathrm{PC}(T',T) = \{M \restriction \tau_T : M \text{ is model of } T'\}$$

$$\mathrm{PC}_\lambda(T',T) = \{M \in \mathrm{PC}(T',T) : M \text{ is of cardinality } \lambda\}.$$

Definition 0.5. Given T and $M \in \mathrm{EC}_\lambda(T)$ we say that M is a (λ,κ)-superlimit model when: M is a λ-universal model of cardinality λ and if $\delta < \lambda^+$ is a limit ordinal such that $\mathrm{cf}(\delta) = \kappa, \langle M_\alpha : \alpha \leq \delta\rangle$ is $\prec$-increasing continuous, and $M_{\alpha+1}$ is isomorphic to M for every $\alpha < \delta$ <u>then</u> M_δ is isomorphic to M.

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Remark 0.6. We shall use:

(a) (λ, κ)-i.md.-limit in 1.1, (existence for $T_{\rm ord}$)
(b) (λ, κ)-wk-limit in 2.3, (non-existence from "T is strongly independent")
(c) (λ, κ)-md.-limit in 2.9, (non-existence for independent T)
(d) (λ, κ)-i.st.-limit for $T_{\rm ord}$: 3.12 and 3.5(3), 3.7(3), (on characterization) for $T_{\rm ord}$)
(e) (λ, κ)-superlimit in 3.10 (non-existence).

Recall the definition of some versions of "(λ, κ)-limit model".

Convention 0.7. In this work let "M is (λ, S)-limit" mean "M is $(\lambda, S)-$md-limit, see Definition below; similarly for (λ, κ).

Definition 0.8. Let λ be a cardinal $\geq |T|$. For parts 3) - 5) but not 6), for simplifying the presentation we assume the axiom of global choice; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$. Below if $S = \{\delta < \lambda^+ : {\rm cf}(\delta) = \kappa\}$ then instead (λ, S) we may write (λ, κ), this is the main case.
1) Let $S \subseteq \lambda^+$ be stationary. A model $M \in {\rm EC}_\lambda(T)$ is called (λ, S)-st-limit (or S-strongly limit or (λ, S)-strongly limit) when for some function: $\mathbf{F} : {\rm EC}_\lambda(T) \to {\rm EC}_\lambda(T)$ we have:

(a) for $N \in {\rm EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
(b) if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is a $\prec$-increasing continuous sequence [1] in ${\rm EC}_\lambda(T)$ obeying $\mathbf{F}$ which means $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$, then $M \cong \cup\{M_i : i < \delta\}$.

2) Let $S \subseteq \lambda^+$ be stationary. $M \in {\rm EC}_\lambda(T)$ is called (λ, S)-nr-limit (or S-normally limit, or may omit nr/normally) when for some function $\mathbf{F} : {\rm EC}_\lambda(T) \to {\rm EC}_\lambda(T)$ we have:

(a) for every $N \in {\rm EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
(b) if $\langle M_i : i < \lambda^+ \rangle$ is a $\prec$-increasing continuous sequence of members of ${\rm EC}_\lambda(T), \mathbf{F}(M_{i+1}) \prec M_{i+2}$ then for some closed unbounded [2] subset C of λ^+,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

2A) $M \in {\rm EC}_\lambda(T)$ is (λ, S)-limit$^+$ when if $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is $\subseteq$-increasing and continuous and $\alpha < \lambda^+ \Rightarrow M_{\alpha+1} \cong M$ then for some club E of λ we have $\alpha \in E \cap S \to M_\alpha \cong M$. Notice that being a (λ, S)-limit$^+$ implies being a (λ, S)-nr-limit.
3) We define "M is (λ, S)-wk-limit", "(λ, S)-md-limit" like "(λ, S)-nr-limit", "(λ, S)-st-limit" respectively by demanding that the domain of $\mathbf{F}$ is the family of $\prec$-increasing continuous sequences of members of ${\rm EC}_\lambda(T)$ of length $< \lambda^+$ and replacing "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$". (They are also called S-weakly limit, S-medium limit, respectively.)
3A) We replace "limit" by "limit$^-$" if "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$", "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$" are replaced by "$\mathbf{F}(M_i) \prec M_{i+1}$", "$M_i \prec \mathbf{F}(\langle M_j : j \leq i \rangle) \prec M_{i+1}$" respectively.

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[1] No loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$, i.e., only $\mathbf{F}(M)$ is required
[2] We can use a filter as a parameter

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4) If $S = \lambda^+$ then we may omit S (in parts (3), (4), (5)).
5) For $\Theta \subseteq \{\mu : \mu \leq \lambda$ and μ is regular$\}$, M is (λ, Θ)-strongly limit if M is $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strongly limit in the sense of part (1). Similarly for the other notions (where $\Theta \subseteq \{\mu : \mu$ regular $\leq \lambda\}$ is non-empty and $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$ is a stationary subset of λ^+). If we do not write λ we mean $\lambda = \|M\|$.
6) We say that $M \in K_\lambda$ is (λ, S)-i.st-limit (or S-invariantly strong limit) when in part (3), $\mathbf{F}$ is just a subset of $\{(M,N)/\cong : M \prec N$ are from $\mathrm{EC}_\lambda(T)\}$ and in clause (b) of part (3) we replace "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M_{i+1}, N)/\cong) \in \mathbf{F})$". But abusing notation we still write $N = \mathbf{F}(M)$ instead $((M,N)/\cong) \in \mathbf{F}$. Similarly with the other notions, i.e., we use the isomorphism type of $\bar{M}\char`^\langle N \rangle$.

Observation 0.9. 1) If $\mathbf{F}_1, \mathbf{F}_2$ are as above and $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$ (or $\mathbf{F}_1(\bar{M}) \prec \mathbf{F}_2(\bar{M})$) whenever defined then if $\mathbf{F}_1$ is a witness so is $\mathbf{F}_2$.
2) All versions of limit models imply being a universal model in $\mathrm{EC}_\lambda(T)$.

3) Obvious implication diagram: For stationary $S \subseteq S^{\lambda^+}_\kappa$ as in 0.8(7):

(λ, κ)-superlimit
↓
S-strong limit
↓ ↓
S-medium limit, S-normally limit
↓ ↓
S-weakly limit.

Claim 0.10. *Assume $\lambda = \lambda^{<\kappa} \geq |T|$ and κ is regular and M is a model of T of cardinality λ. Then the following conditions are equivalent (assuming the universal axiom of choice or restrict ourselves below to models with universe $\subseteq \lambda^+$):*

- (a) *M is (λ, κ)-md-limit*
- (b) *in the following game the isomorphism player has a winning strategy. A play last κ-moves, in the i-th move the anti-isomorphism player chooses $M_\alpha \in \mathrm{EC}_\lambda(T)$ such that $\langle M_\beta : \beta \leq \alpha \rangle$ is $\prec$-increasing continuous and $\alpha = \beta + 1 \Rightarrow M'_\beta \prec M_\alpha$ and the isomorphism player chooses M'_α such that $M_\alpha \prec M'_\alpha \in \mathrm{EC}_\lambda(T)$. The isomorphism player wins a play when $\bigcup\{M_\alpha : \alpha < \kappa\}$ is isomorphic to M*
- (c) *there is a function $\mathbf{F}$ with domain $\{\bar{M} : \bar{M}$ a $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ of length $< \kappa\}$ such that $i < \ell g(\bar{M}) \Rightarrow M_i \prec \mathbf{F}(\bar{M})$ and if $\bar{M} = \langle M_i : i \leq \kappa \rangle$ is $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ and $i < \kappa \Rightarrow \mathbf{F}(\bar{M} \restriction (2i+2) \prec M_{2i+2}$ then $M_\kappa \cong M$*
- (d) *there is a function $\mathbf{F}$ such that: if $\langle M_i : i \leq \kappa \rangle$ is $\prec$-increasing continuous in $\mathrm{EC}_\lambda(T)$ and for some sequence $\langle M'_i : i < \kappa \rangle$ we have $M_i \prec M'_i \in \mathrm{EC}_\lambda(T)$ and $i < \kappa \Rightarrow M'_i \prec \mathbf{F}(\langle M_j : j \leq i \rangle\char`^\langle M'_i \rangle) = M_{i+1}$ (we say $\bar{M}$ obeys $\mathbf{F}$) then $\cup\{M_i : i < \kappa\} \cong M$*

(e) *in* $\mathbf{V}^{\mathrm{Levy}(\lambda^+,2^\lambda)}$ *we have: if* $\langle M_\alpha : \alpha < \lambda^+\rangle$ *is* $\prec$*-increasing continuous,* $M_\alpha \in \mathrm{EC}_\lambda(T)$*, and* $\bigcup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_{\lambda,\lambda}(T)$ *then for some club* E *of* λ^+ *we have* $\delta \in E \wedge \mathrm{cf}(\delta) = \kappa \Rightarrow M_\delta \cong M$

(f) *like (e) for any* λ^+*-complete forcing notion* $\mathbb{P}$ *such that* $\Vdash_{\mathbb{P}}$ "$2^\lambda = \lambda^+$".

Proof. As $(\mathrm{EC}_\lambda(T), \prec)$ has the JEP (joint embedding property) and the amalgamation property this is straightforward.
E.G.

$(f) \Rightarrow (b)$:

Let $\langle \underset{\sim}{M}_\alpha : \alpha < \lambda^+\rangle$ be a $\mathbb{P}$-name of a $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ with union in $\mathrm{EC}_{\lambda,\lambda}(T)$ and $\underset{\sim}{E}$ a $\mathbb{P}$-name of a club of λ^+ such that $\delta \in \underset{\sim}{E} \wedge \mathrm{cf}(\delta) = \kappa \Rightarrow M_\delta \cong M$; clearly it exists by clause (f) which we are assuming. We now define a strategy **st** for the isomorphic player: together with choosing M'_α the isomorphic player chooses $(\gamma_\alpha, p_\alpha, h_\alpha)$ such that

- $\bullet_1$ $M_\alpha \prec M'_\alpha \in \mathrm{EC}_\lambda(T)$ as demanded in (b)
- $\bullet_2$ $p_\alpha \in \mathbb{P}$ and $\beta < \alpha \Rightarrow \mathbb{P} \models$ "$p_\beta \le p_\alpha$"
- $\bullet_3$ $\gamma_\alpha < \lambda^+$ and $\beta < \alpha \Rightarrow \gamma_\beta < \gamma_\alpha$
- $\bullet_4$ $p_\alpha \Vdash_{\mathbb{P}}$ "h_α is an isomorphism from M'_α onto $\underset{\sim}{M}_{\gamma_\alpha}$"
- $\bullet_5$ if $\beta < \alpha$ then $h_\beta \subseteq h_\alpha$.

$\square_{0.10}$

Like 0.10 but for the invariant version we note

Claim 0.11. *For* $M \in \mathrm{EC}_\lambda(T)$ *the following are equivalent (and seemingly stronger than the conditions in 0.10):*

$(a)'$ *M is* (λ,κ)*-i.md-limit (that is invariantly medium* (λ,κ)*-limit)*

$(d)'$ *there is a class* $\mathbf{F}$ *such that:*

- (α) $\mathbf{F} \subseteq \{\bar M : \bar M = \langle M_i : i \le \alpha\rangle$ *for some* $\alpha \le \kappa$ *is* $\prec$*-increasing continuous,* $\{M_i : i \le \alpha\} \subseteq \mathrm{EC}_\lambda(T)\}$ *and* $\mathbf{F}$ *is closed under isomorphisms*
- (β) *if* $\bar M = \langle M_i : i \le \alpha\rangle \in \mathbf{F}$ *and* $M_\alpha \prec M'_\alpha \in \mathrm{EC}_\lambda(T)$ *then for some* $M_{\alpha+1}$ *we have* $M'_\alpha \prec M_{\alpha+1}$ *and* $\bar M\hat{\ }\langle M_{\alpha+1}\rangle \in \mathbf{F}$
- (γ) *for* α *limit* $\langle M_i : i \le \alpha\rangle \in \mathbf{F}$ *iff* $j < \alpha \Rightarrow \langle M_i : i \le j\rangle \in \mathbf{F}$ *and* $M_\alpha = \cup\{M_i : i < \alpha\}$
- (δ) *if* $\langle M_i : i \le \kappa\rangle \in \mathbf{F}$ *then* $M_\kappa \cong M$

$(d)''$ *there is* $\mathbf{F}$ *such that:*

- (α) • $\mathbf{F}$ *is a subset of* $\{\bar M : \bar M = \langle M_i : i \le \alpha\rangle$ *for some* $\alpha < \kappa$ *is* $\prec$*-increasing continuous in* $\mathrm{EC}_\lambda(T)\}$
  - • *[α odd* $\Rightarrow M_\alpha \prec \mathbf{F}(\bar M) \in \mathrm{EC}_\lambda(T))$ *so we can ignore the member* $\alpha\rangle)]$
- (β) • *if* $\bar M \in K$ *has length* $2\alpha+1 < \kappa$ *then for some* $M', \bar M\hat{\ }\langle (M')\rangle \in \mathbf{F}$ *and* M' *is unique up to isomorphism, i.e. if* $\bar M^\ell = \bar M\hat{\ }\langle M^\ell\rangle \in \mathscr{F}$ *for* $\ell = 1,2$ *then* $\bar M^1, \bar M^2$ *are isomorphic, so abusing notation we may write* $M' = \mathbf{F}(\bar M)$

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- *if* $\bar{M} \in K$ *has length* $2\alpha < \kappa$ *and* $M_{2\alpha} \prec M' \in \mathrm{EC}_\lambda(T)$ *then* $\bar{M}\char`^\langle \bar{M}' \rangle \in \mathbf{F}$ *and we may write* $\mathbf{F}(\bar{M}) = M_\alpha$

$(\gamma), (\delta)$ *as in* $(d)'$.

Definition 0.12. 1) T_{ord} is the theory of dense linear order with neither first nor last element.
2) T_{rd} is the theory of linear orders.

Definition 0.13. 1) We say that (C_1, C_2) is a cut of $M \in \mathrm{EC}(T_{\mathrm{rd}})$ when:

- (a) C_1 is an initial segment of M
- (b) C_2 is an end-segment of M
- (c) $C_1 \cap C_2 = \emptyset$
- (d) $C_1 \cup C_2 = M$.

2) For a cut (C_1, C_2) of M, let $\mathrm{cf}(C_1, C_2)$, the cofinality of the cut (C_1, C_2), be the pair (θ_1, θ_2) when

- (a) θ_1 is the cofinality of C_1, i.e. of $M \restriction C_1$ (can be $0, 1$ or a regular cardinal $\in [\aleph_0, \lambda)$
- (b) θ_2 is the cofinality of C_2 inverted.

Definition 0.14. $\dot{I}(\lambda, T)$ is the number of $M \in \mathrm{EC}_\lambda(T)$ up to isomorphism.

Definition 0.15. 1) Fixing $T, \varphi(\bar{x}, \bar{y})$ is an independent formula when for every n and $M \models T$ for some $\bar{a}_\ell \in {}^{\ell g(\bar{y})}M$ for $\ell < n$, for every $u \subseteq n, M \models (\exists \bar{x}) \bigwedge\limits_{\ell < n} \varphi(\bar{x}, \bar{a}_\ell)^{\mathrm{if}(\ell \in u)}$.
2) T is independent iff some $\varphi(x, \bar{y})$ is independent.

Notation 0.16. $\varphi^{\mathrm{if}(\mathbf{t})}$ is φ if $\mathbf{t}$ is true or $1, \neg\varphi$ if $\mathbf{t}$ is false or 0.

Definition 0.17. 1) $\lambda^{<\kappa>_{\mathrm{tr}}}$ is $\sup\{|\mathscr{T} \cap {}^\kappa\lambda| : \mathscr{T} \subseteq {}^{\kappa \geq}\lambda$ is closed under initial segments and $\varepsilon < \kappa \Rightarrow |\mathscr{T} \cap {}^\varepsilon\lambda| \leq \lambda\}$.
2) $\mathbf{U}_\kappa(\lambda) = \min\{|\mathscr{F}| : \mathscr{F}$ is a set of functions from κ to λ such that $f \neq g \in \mathscr{F} \Rightarrow \kappa > \{i < \kappa : f(i) = g(i)\}|$.

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§ 1. Dense linear order has medium limit models

Theorem 1.1. *Assume* $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$. *Then* T_{ord} *has an invariantly medium* (λ,κ)*-limit model.*

Remark 1.2. 1) We use condition $(d)'$ from 0.11, we may use it as a definition.
2) So a model of T_{ord} is a dense linear order with neither first nor last element and $\prec$ for models of T_{ord} is just $\subseteq$ and saturated means λ-dense for models of T_{ord} of cardinality λ.
3) We actually prove a result with $\mathbf{F}$ of a simple kind, dealing with $\mathbf{F}$ acting on pairs of models, $\cup\{M_i : i < \kappa\}$ is isomorphic to the (λ,κ)-i.md.-limit model when $\langle M_i : i < \kappa\rangle$ is $\prec$-increasing continuous sequence of linear orders such that for any $i_1 < i_2 < \kappa$ for some $i_3 \in (i_2,\kappa)$ we have $\mathbf{F}(M_{i_1}, M_{i_3}) \prec M_{i_3+1}$.
4) On cuts and their cofinalities 0.13.

Remark 1.3. Concerning $\circledast^\kappa_{\bar\mu}$ in the beginning of the proof of 1.1.
1) It is a characterization of the invariantly medium (λ,κ)-model. We shall return to this in §3.
2) Concerning the clause inside $\circledast^\kappa_\mu$ in the proof of 1.1 note the following: Clause (f) almost implies clause (d).

Clause (f) implies $(\mathrm{h})_1$; why? use $A = \emptyset, B = M_i$. Also clause (f) implies $(\mathrm{h})_2$; why? use $A = M_i, B = \emptyset$.

Lastly, clause (f) implies $(\mathrm{i})_1$; why? use $A' = A, B' = \{c \in M_i : A < c\}$ and clause (f) implies $(\mathrm{i})_2$ similarly.
2) Note that clause (f) is equivalent to $(\mathrm{i})_1 + (\mathrm{i})_2$.

Proof. First we say that $\bar M$ is a fast (λ,κ)-sequence (of models of T_{ord}) when:

$\circledast^\kappa_{\bar M}$ (a) $\bar M = \langle M_i : i \le \kappa\rangle$

(b) M_i is $\prec$-increasing continuous

(c) M_i is a model of T_{ord} of cardinality λ

(d) M_i is saturated if i is a non-limit ordinal

(e) if $i < \kappa$ and $a \in M_{i+1}\backslash M_i$ then $M_{i+1} \restriction \{b \in M_{i+1}\backslash M_i : (\forall c \in M_i)$ $[(c < b) \equiv (c < a)]\}$ is a saturated model of T_{ord} of cardinality λ

(f) if $i < \kappa$ and $A, B \subseteq M_i$ and $A < B$ (i.e. $(\forall a \in A)(\forall b \in B)(a <_{M_i} b)$) and A or B has cardinality $< \lambda$, then for some $c \in M_{i+1}\backslash M_i$ we have $A < c < B$; this includes A, B singletons but it is enough to have this when $c \in M_i \Rightarrow \neg(A < c < B)$; note that we say "$A$ or $B \ldots$"

$(g)_1$ if $i < j < \kappa$ and $a \in M_j\backslash M_i$, then for some $d \in M_{j+1}\backslash M_j$ we have

- (α) if $b \in M_i$ and $b <_{M_j} a$ then $b <_{M_{j+1}} d$
- (β) if $c \in M_j$ and $(\forall b \in M_i)(b <_{M_j} a \Rightarrow b <_{M_j} c)$ then $d <_{M_{j+1}} c$

$(g)_2$ if $i < j < \kappa$ and $a \in M_j\backslash M_i$ then for some $d \in M_{j+1}\backslash M_j$ we have

- (α) if $b \in M_i$ and $a <_{M_j} b$ then $d <_{M_{j+1}} b$
- (β) if $c \in M_j$ and $(\forall b \in M_i)(a <_{M_j} b \Rightarrow c <_{M_j} b)$ then $c <_{M_{j+1}} d$

$(h)_1$ for $i < \kappa$ there is $b \in M_{i+1}\backslash M_i$ such that $a \in M_i \Rightarrow a <_{M_{i+1}} b$

$(h)_2$ for $i < \kappa$ there is $b \in M_{i+1}\backslash M_i$ such that $a \in M_i \Rightarrow b <_{M_{i+1}} a$

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$(i)_1$ if $A \subseteq M_i, i < \kappa$ and $|A| < \lambda$ then for some $c \in M_{i+1} \backslash M_i$ we have $(\forall d \in M_i)(d <_{M_{i+1}} c \leftrightarrow (\exists a \in A)(d \le_{M_i} a))$

$(i)_2$ if $A \subseteq M_i, i < \kappa$ and $|A| < \lambda$ then for some $c \in M_{i+1} \backslash M_i$ we have $(\forall d \in M_i)(c <_{M_{i+1}} d \leftrightarrow (\forall a \in A)(a \le_{M_i} d))$

(j) if $i < \kappa$ and $a <_{M_i} b$ then for some $c \in (a,b)_{M_{i+1}} \backslash M_i$ the orders $M_i \restriction \{d \in M_i : d <_{M_{i+1}} c\}$ and the inverse of $M_i \restriction \{d \in M_i : c <_{M_{i+1}} d\}$ have cofinality λ.

Clearly:

$\boxtimes$ it is enough to prove $\boxtimes_1 + \boxtimes_2$ where

$\boxtimes_1$ there is $\mathbf{F}$ such that

(α) $\mathrm{Dom}(\mathbf{F}) = \{\bar{M}$: for some $\alpha \le \kappa, \bar{M} = \langle M_i : i \le \alpha\rangle$ is $\prec$-increasing continuous, $M_i \in \mathrm{EC}_\lambda(T_{\mathrm{ord}})\}$

(β) for $\bar{M} = \langle M_i : i \le \alpha\rangle \in \mathbf{F}$

- if α is odd then $M_\alpha \prec \mathbf{F}(\bar{M}) \in \mathrm{EC}_\lambda(T_{\mathrm{Ord}})$ and $\bar{M}\hat{}\langle \mathbf{F}(\bar{M})\rangle \in \mathbf{F}$
- if α is even and $M_\alpha \prec M' \in \mathrm{EC}_\lambda(T_{\mathrm{Ord}})$ then $M_\alpha = \mathbf{F}(\bar{M})$ and $\bar{M}\hat{}\langle M'\rangle \in \mathbf{F}$

(γ) $\mathbf{F}$ is invariant, i.e. if $\bar{M}_1 \cong \bar{M}_2$ then $(\bar{M}_1, \mathbf{F}(M_1)) \cong (M_2, \mathbf{F}(\bar{M}_2))$

(δ) if $\bar{M} = \langle M_i : i \le \kappa\rangle$ is an $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T_{\mathrm{ord}})$ belonging to, i.e. $\mathbf{F}$ then $\circledast^\kappa_{\bar{M}}$

$\boxtimes_2$ if $\circledast^\kappa_{\bar{M}^1}$ and $\circledast^\kappa_{\bar{M}^2}$ then $M^1_\kappa \cong M^2_\kappa$.

Why is clause $\boxtimes_1$ true?:

How do we choose $\mathbf{F}$?

Reading the definition of $\circledast^\kappa_{\bar{M}}$ this should be clear: all our demands on M_{j+1} when we are given $\langle M_i : i \le j\rangle$ and M'_j can be fulfilled. We first choose $\mathscr{P}_{\langle M_i : i \le j\rangle} = \{(A,B) : (A,B)$ a cut of M_j such that A has cofinality $< \lambda$ or the inverse of B has cofinality $< \lambda$ or for some $i < j$ and $a \in M_j \backslash M_i$ the set $\{b \in M_i : b <_{M_j} a\}$ is unbounded in A or for some $i < j$ and $a \in M_j \backslash M_i$ the set $\{b \in M_i : a <_{M_j} b\}$ is unbounded from below in the set $B\}$.

Second, choose $M_{j+1} = \mathbf{F}(\langle M_i : i \le j\rangle\hat{}\langle M'_j\rangle)$ such that $M'_j \prec M_{j+1}$ and any cut $(A,B) \in \mathscr{P}_{\langle M_i : i \le j\rangle}$ is realized in M_{j+1} and for each $c \in M_{j+1} \backslash M_j$ we have $M_{j+1} \restriction \{a \in M_{j+2} \backslash M_j : a, c$ realize the same cut of $M_j\}$ is a saturated model of T_{ord}.

Having chosen $\mathbf{F}$, clauses $(\alpha), (\beta), (\gamma)$ of $\boxtimes_1$ follow and clause (δ) follows too.

Why is clause $\boxtimes_2$ true?:

Suppose $\circledast^\kappa_{\langle M^\ell_i : i \le \kappa\rangle}$ for $\ell = 1, 2$.

For $\ell = 1, 2$ let

$$Y_\ell = \{a \in M^\ell_\kappa \backslash M^\ell_0 : \text{ for every } A \subseteq M^\ell_0 \text{ of cardinality } < \lambda \text{ we have } A < a \Rightarrow (\exists b \in M^\ell_0)(A < b < a) \text{ and } a < A \Rightarrow (\exists b \in M^\ell_0)(a < b < A)\}$$

$$E_\ell = \{(a,b) : a, b \in M^\ell_\kappa \backslash M^\ell_0 \text{ and } (\forall c \in M_0)(c < a \equiv c < b)\}.$$

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Now E_ℓ is an equivalence relation on $M^\ell_\kappa \backslash M^\ell_0$ and Y_ℓ is a union of some equivalence classes of E_ℓ. Let $Z_\ell \subseteq Y_\ell$ be a set of representatives of $E_\ell \restriction Y_\ell$. Now we define N_ℓ: it is the model with universe $M^\ell_0 \cup Z_\ell$, the relation $<^{N_\ell} = <^{M^\ell_\kappa} \restriction (M^\ell_0 \cup Z_\ell)$ and the relation $P^{N_\ell} = \{a : a \in M^\ell_0\}$.

Now it is easy to check that N_ℓ has first and last elements both from $N_\ell \backslash P^{N_\ell}$ and is dense. Also if $A, B \subseteq N_\ell$ have cardinality $< \lambda$ and $A < B$ then we can find a', a'' such that $A <_{N_\ell} \{a', a''\} <_{N_\ell} B$ and $a' \in P^{N_\ell}, a'' \in N_\ell \backslash P^{N_\ell}$. Hence N_ℓ is a saturated model (not of $T_{\rm ord}$ but of a variant). So easily N_1, N_2 are isomorphic and let g_0 be such an isomorphism and $f_0 = g_0 \restriction M^1_0$.

Now

$(*)_0$ f_0 induces a mapping $\hat f_0$ from the class of E_1-equivalence classes onto the class of E_2-equivalence classes.

[Why? Check the cases.]

Now we have to separately deal with each case of $M^1_\kappa \restriction (a_1/E_1), M^2_\kappa \restriction (a_2/E_2)$ where $\hat f_0(a_1/E_1) = a_2/E_2$. But this is similar to the original problem, i.e., choose $i < \kappa$ large enough such that $(a_1/E_1) \cap M^1_i \ne \emptyset$ and $(a_2/E_2) \cap M^2_i \ne \emptyset$. It is not hard to understand that we can continue and in the end we exhaust the models, but we shall elaborate; without loss of generality $M^1_\kappa \cap M^2_\kappa = \emptyset$. For a set $A \subseteq M^\ell_\kappa$ we define

$(*)_1$ $E^\ell_A := \{(a,b) : a,b \in M^\ell_\kappa \backslash A$ and $(\forall c \in A)(a <_{M^\ell_\kappa} c \equiv b <_{M^\ell_\kappa} c)\}$.

Note

$(*)_2$ E^ℓ_A is an equivalence relation on $M^\ell_\kappa \backslash A$.

Define

$(*)_3$ Y^ℓ_A is the set $\{a \in M^\ell_\kappa \backslash A$: the cut that a induces on A has cofinality $(\lambda, \lambda)\}$.

So

$(*)_4$ Y^ℓ_A is a subset $M^\ell_\kappa \backslash A$ closed under E^ℓ_A.

Define

$(*)_5$ We say that $A \subseteq M^\ell_\kappa$ is ℓ-nice <u>when</u> for every $a \in M^\ell_\kappa \backslash A$, for some $i = i_\ell(a, A) = i_\ell(a/E^\ell_A) < \kappa$ we have
- (α) the set a/E^ℓ_A is disjoint to M^ℓ_i but not to M^ℓ_{i+1}
- (β) the set $\{b \in A : b <_{M^\ell_\kappa} a$ and $b \in M^\ell_i\}$ is unbounded in $\{b \in A : b <_{M^\ell_\kappa} a\}$
- (γ) the set $\{b \in A : a <_{M^\ell_\kappa} b$ and $b \in M^\ell_i\}$ is unbounded from below in $\{b \in A : a <_{M^\ell_\kappa} b\}$

$(*)_6$ in $(*)_5$, $i_\ell(a, A)$ is uniquely defined by (a, A), actually just by a/E^ℓ_A

$(*)_7$ if $\delta < \kappa$ is a limit ordinal, $\ell \in \{1, 2\}$ and $\langle A_\alpha : \alpha < \delta \rangle$ is an $\subseteq$-increasing sequence of ℓ-nice sets such that $[\alpha < \beta < \delta \wedge a \in M^\ell_\kappa \backslash A^\ell_\beta \Rightarrow i(a, A_\alpha) < i(a, A_\beta)]$ then $A_\delta =: \cup\{A_\alpha : \alpha < \delta\}$ is an ℓ-nice set.

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[Why? Trivially $A_\delta \subseteq M^\ell_\kappa$, so let $a \in M^\ell_\kappa \backslash A_\delta$ then for each $\alpha < \delta$ we have $a \in M^\ell_\kappa \backslash A_\alpha$ hence $i_\ell(a, A_\alpha) < \kappa$ is well defined and it is $\le$-increasing with α because $(a/E^\ell_{A_\beta}) \subseteq (a/E^\ell_\alpha)$ by clause (α) of $(*)_5$.

Recall that $\langle i_\ell(a, A_\alpha) : \alpha < \delta\rangle$ is not eventually constant. We claim $i(*) = \bigcup\{i_\ell(a, A_\alpha) : \alpha < \delta\}$ is as required. First of all, as the union of an $\le$-increasing not eventually constant sequence of length $\delta < \kappa$ of ordinals $< \kappa$ it is an ordinal $< \kappa$, in fact a limit ordinal $< \kappa$.

Clearly, $a/E^\ell_{A_\delta}$ is the intersection of the $\subseteq$-decreasing sequence $\langle a/E^\ell_{A_\alpha} : \alpha < \delta\rangle$. Now if $i < i(*)$ then for some $\alpha < \delta$ we have $i \le i_\ell(a, A_\alpha)$ hence $a/E^\ell_{A_\alpha}$ is disjoint to M^ℓ_i hence $a/E^\ell_{A_\delta} \subseteq a/E^\ell_{A_\alpha}$ is disjoint to M_i. As this holds for every $i < i(*)$ it follows that also $\bigcup\{M^\ell_i : i < i(*)\}$ is disjoint to $a/E^\ell_{A_\delta}$, but $\bigcup\{M^\ell_i : i < i(*)\} = M^\ell_{i(*)}$ because $i(*)$ is a limit ordinal. So really $(a/E^\ell_{A_\delta}) \cap M^\ell_{i(*)} = \emptyset$.

It is also clear that $(\{b \in M^\ell_{i(*)} : b <_{M^\ell_\kappa} a\}, \{b \in M^\ell_{i(*)} : a <_{M^\ell_\kappa} b\})$ is a cut of $M^\ell_{i(*)}$ whose cofinality (λ_1, λ_2) is not equal to (λ, λ), hence by clauses $(i)_1 + (i_2)$ of $\circledast^\kappa_{\bar M^\ell}$ we have $(a/E^\ell_{A_i}) \cap M_{i(*)+1} \ne \emptyset$ so $i(*)$ satisfies demand (α) from $(*)_5$ on $i(a, A_\delta)$. The other two clauses should be clear, too.]

Define

$(*)_8$ $\mathscr{F}$ is the set of f such that

- (a) for some 1-nice $A_1 \subseteq M^1_\kappa$ and 2-nice set $A_2 \subseteq M^2_\kappa$, f is an isomorphism from the linear order $M^1_\kappa \restriction A_1$ onto the linear order $M^2_\kappa \restriction A_2$
- (b) for every $a_1 \in M^1_\kappa \backslash A_1$ there is $a_2 \in M^2_\kappa \backslash A_2$ such that f maps $\{b \in A_1 : b < a_1\}$ onto $\{b \in A_2 : b < a_2\}$; it follows that $a_1 \in Y^1_A$ iff $a_2 \in Y^2_A$
- (c) for every $a_2 \in M^2_\kappa \backslash A_2$ for some $a_1 \in M^1_\kappa \backslash A_1$ the conclusion of clause (b) holds.

Define

$(*)_9$ $<_*$ is the following two-place relation of $\mathscr{F}$: $f <_* f'$ <u>iff</u> $(f, f' \in \mathscr{F}$ and)

- (a) $f \subseteq f'$
- (b) if $a_1 \in M^1_\kappa \backslash \mathrm{Dom}(f')$ then $i_1(a_1/E^1_{\mathrm{Dom}(f')}) > i_1(a_1/E^1_{\mathrm{Dom}(f)})$
- (c) if $a_2 \in M^2_\kappa \backslash \mathrm{Rang}(f')$ then $i_2(a_2/E^2_{\mathrm{Rang}(f')}) > i_2(a_2/E^2_{\mathrm{Rang}(f)}$
- (d) if $a \in M^1_\kappa \backslash \mathrm{Dom}(f')$ then there are $b, c \in (a/E^1_{\mathrm{Dom}(f)}) \cap \mathrm{Dom}(f')$ such that $b <_{M^1_\kappa} a <_{M^1_\kappa} c$
- (e) if $a \in M^2_\kappa \backslash \mathrm{Rang}(f')$ then there are $b, c \in (a/E^2_{\mathrm{Rang}(f)}) \cap$ $\mathrm{Rang}(f')$ such that $b <_{M^2_\kappa} a <_{M^2_\kappa} c$.

Note

$(*)_{10}$ $(\mathscr{F}, <_*)$ is a non-empty partial order.

[Why? We have in $(*)_0$ above proved that there is an isomorphism from M^1_0 onto M^2_0 which belongs to $\mathscr{F}$. Being a partial order is obvious.]

$(*)_{11}$ if $\delta < \kappa$ is a limit ordinal and $\langle f_\alpha : \alpha < \delta\rangle$ is a $<_*$-increasing sequence in $\mathscr{F}$, <u>then</u> $f_\delta := \bigcup\{f_\alpha : \alpha < \delta\}$ belongs to $\mathscr{F}$ and $\alpha < \delta \Rightarrow f_\alpha <_* f_\delta$.

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[Why? Clearly f_δ is an isomorphism from the linear order $M^1_\kappa \restriction A^1_\delta$ where $A^1_\delta =: \bigcup\{\mathrm{Dom}(f_\alpha) : \alpha < \delta\}$ onto the linear order $M^2_\kappa \restriction A^2_\delta$ where $A^2_\delta =: \bigcup\{\mathrm{Rang}(f_\alpha) : \alpha < \delta\}$. Now $\mathrm{Dom}(f_\delta) = \bigcup\{\mathrm{Dom}(f_\alpha) : \alpha < \delta\}$ is 1-nice by $(*)_7$ recalling clause (a) of $(*)_9$ and similarly $\mathrm{Rang}(f_\delta) = \bigcup\{\mathrm{Rang}(f_\alpha) : \alpha < \delta\}$ is 2-nice. So from the demands for "$f_\delta \in \mathscr{F}$" in $(*)_8$, clause (a) holds. Concerning clause (b) there, let $a_1 \in M^1_\kappa \backslash \mathrm{Dom}(f_\delta)$. For each $\alpha < \delta$ by $(*)_{10}(d)$ applied to $f_\alpha <_* f_{\alpha+1}$ there is a pair (b_α, c_α) satisfying $b_\alpha, c_\alpha \in (a/E^1_{\mathrm{Dom}(f_\alpha)}) \cap \mathrm{Dom}(f_{\alpha+1})$ such that $b_\alpha <_{M^1_\kappa} a_1 <_{M^1_\kappa} c_\alpha$. Note that as $b_\alpha, c_\alpha \in (a/E^1_{\mathrm{Dom}(f_\alpha)})$ necessarily $b_\alpha, c_\alpha \notin \mathrm{Dom}(f_\alpha)$ and clearly $d \in \mathrm{Dom}(f_\alpha) \Rightarrow (d < b_\alpha \equiv d < c_\alpha)$. Hence $\langle b_\alpha : \alpha < \delta\rangle$ is increasing, $\langle c_\alpha : \alpha < \delta\rangle$ is decreasing, and: $d \in \mathrm{Dom}(f_\delta)$ implies that for some $\alpha < \delta, d \in \mathrm{Dom}(f_\alpha)$ hence for every $\beta < \delta$ large enough $d < b_\beta \equiv d < c_\beta$. Recall $b_\alpha, c_\alpha \in \mathrm{Dom}(f_{\alpha+1}) \backslash \mathrm{Dom}(f_\alpha)$ so $\langle i_1(b_\alpha, \mathrm{Dom}(f_\beta)) : \beta \le \alpha\rangle$ is increasing, $i_1(b_\alpha, \mathrm{Dom}(f_\beta)) = i_1(c_\alpha, \mathrm{Dom}(f_\beta))$. So $(\{b_\alpha; \alpha < \delta\}, \{c_\alpha : \alpha < \delta\})$ determine the cut a_1 induces on $\mathrm{Dom}(f_\delta)$ and they are $\subseteq M^1_{i_1(a,\mathrm{Dom}(f_\delta))}$. Now $(\{f_{\alpha+1}(b_\alpha) : \alpha < \delta\}, (\{f_{\alpha+1}(c_\alpha) : \alpha < \delta\})$, behave similarly in M^2_κ and they induce a cut of $M_i, i = \bigcup\{i_2(f_{\alpha+1}(b_\alpha), \mathrm{Rang}(f_\alpha)) : \alpha < \delta\}$ which is realized by some $a_2 \in M_{i+1}$ by clause (f) of $\circledast^2_{\bar M^2}$. Now a_2 is as required.

Clause (c) is proved similarly using $(*)_{10}(e)$.]

$(*)_{12}$ if $\langle f_\alpha : \alpha < \kappa\rangle$ is an $<_*$-increasing sequence in $\mathscr{F}$ then $f_\kappa := \bigcup\{f_\alpha : \alpha < \kappa\}$ is an isomorphism from M^1_κ onto M^2_κ.

[Why? Toward contradiction first assume $\mathrm{Dom}(f_\kappa) \subset M^1_\kappa$ so choose $a_1 \in M^1_\kappa \backslash \mathrm{Dom}(f_\kappa)$ hence $\langle i_1(a_1/E^1_{\mathrm{Dom}(f_\alpha)}) : \alpha < \kappa\rangle$ is a (strictly) increasing sequence of ordinals $< \kappa$, hence its sup is κ. Now for $\alpha < \kappa, a_1 \in (a_1/D^1_{\mathrm{Dom}(f_\alpha)})$ but $a_1/E^1_{\mathrm{Dom}(f_\alpha)}$ is disjoint to $M^1_{i_1(a_2,\mathrm{Dom}(f_\alpha))}$. Hence $a_1 \notin \cup\{M^1_{i_1(a_2,\mathrm{Dom}(f_\alpha))} : \alpha < \kappa\} = \{M_\beta : \beta < \kappa\} = M_\kappa$ which is impossible. Similarly $\mathrm{Rang}(f_\kappa) \subset M^2_\kappa$ leads to contradiction, so we are done.]

$(*)_{13}$ for every $f \in \mathscr{F}$ there is f' such that $f <_* f' \in \mathscr{F}$.

[Why? Let $\langle a^1_t : t \in I\rangle$ be a set of representatives of $(M^1_\kappa \backslash \mathrm{Dom}(f))/E_{\mathrm{Dom}(f)}$. For $t \in I$ choose $a^2_t \in M^2_\kappa \backslash \mathrm{Rang}(f)$ such that f maps $\{b \in \mathrm{Dom}(f) : b < a^1_t\}$ onto $\{b \in \mathrm{Rang}(f) : b < a^2_t\}$ and let $i_{1,t} := i_1(a^1_t/E^1_{\mathrm{Dom}(f)}), i_{2,t} = i_2(a^2_t/E^2_{\mathrm{Rang}(f)})$. It is enough to choose for each $t \in I$ an isomorphism g_t from $M^1_{i_1(a^1_2,\mathrm{Dom}(f))+1} \restriction (a^1_t/E^1_{\mathrm{Dom}(f)})$ onto $M^2_{i_2(a^2_t,\mathrm{Rang}(f))+1} \restriction (a^2_t/E^2_{\mathrm{Rang}(f)})$ such that: if (A, B) is a cut of $M^1_{i_1(a^1_t,\mathrm{Dom}(f))} \restriction (a_1/E^1_{\mathrm{Dom}(f)})$ of cofinality (λ, λ) then for some $a \in M^0_1$ we have $A < a < B$ iff for some $b \in M^2_2$ we have $g_t(A) <_{M^1_2} b <_{M_2} g_t(B)$. This is done as in the proof of $(*)_0$ above.]

Together it follows that $M^1_\kappa \cong M^2_\kappa$ as required. $\square_{1.1}$

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§ 2. Independent theories lack limit models

Considering §1 it is a natural to ask:

Question 2.1. 1) Is there an unstable T for which the conclusion of 1.1 fails?
2) For which unstable T does the conclusion of 1.1 fail?

Remark 2.2. 1) We shall consider also relatives $\mathrm{Pr}_{\lambda,\kappa}(\bar{M}), \mathrm{Pr}_{\lambda,\kappa}(T)$.
2) In Definition 2.5 below if $2^\lambda = \lambda^+$ we can restrict ourselves to $\bar{M}$ such that $\bigcup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_{\lambda^+}(T)$ is saturated. The union is unique (for λ) and there is $\mathbf{F}$ as in 0.8(3) guaranteeing this.

We first note that for some T's there are non-existence result (see definitions after the claim).

Theorem 2.3. *1) If T has the strong independence property (see below, e.g. T is the theory of random graphs), $|T| \le \lambda$ and $\lambda^\kappa < 2^\lambda$ <u>then</u> T does not have a (λ,κ)-wk-limit model.*
2) Moreover for every $\mathbf{F}$ as in Definition 0.8(3), there is a $\prec$-increasing continuous sequence $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ of members of $\mathrm{EC}_\lambda(T)$ obeying $\mathbf{F}$ such that if $\mathrm{cf}(\delta_1) = \kappa = \mathrm{cf}(\delta_2)$ <u>then</u> $M_{\delta_1} \cong M_{\delta_2} \Leftrightarrow \delta_1 = \delta_2$.

Definition 2.4. T has the strong independence property (or is strongly independent) <u>when</u>: for some $\varphi(\bar{x},y) \in \mathbb{L}(\tau_T)$ for every $M \in \mathrm{EC}(\tau_T)$ and pairwise distinct $a_0,\dots,a_{2n-1} \in M$ for some $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $M \models$ "$\varphi[\bar{a},a_\ell]^{\mathrm{if}(\ell \text{ is even})}$".

Definition 2.5. Recall $S^\lambda_\kappa =: \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.
1) Let $\mathrm{Pr}_{\lambda,\kappa}(\bar{M})$ mean that $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ is $\prec$-increasing continuous, each M_α is of cardinality λ and for some club E of λ^+, if $\alpha \in S^{\lambda^+}_\lambda \cap E$ and $\delta_1 \ne \delta_2 \in S^{\lambda^+}_\kappa \cap E$ but $\alpha < \delta_1 < \delta_2$ <u>then</u> there is no automorphism π of M_α which maps $\{\mathrm{tp}(\bar{a},M_\alpha,M_{\delta_1}) : \bar{a} \in {}^{\omega>}(M_{\delta_1})\}$ onto $\mathrm{tp}(\bar{a},M_\alpha,M_{\delta_2}) : \bar{a} \in {}^{\omega>}(M_{\delta_2})\}$ (actually even demanding just $\alpha \in E$ is O.K., i.e. we can prove it); note that π acts of M_α hence on $\mathbf{S}^{<\omega}(M_\alpha)$ and π is not necessarily the identity.
1A) Let $\mathrm{Pr}_\lambda(\bar{M})$ mean $\mathrm{Pr}_{\lambda,\lambda}(\bar{M})$, similarly for the versions below.
2) Let $\mathrm{Pr}_{\lambda,\kappa}(T)$ means: for some $\mathbf{F}$ as in 0.8(3), if $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ obeys $\mathbf{F}$ then $\mathrm{Pr}_{\lambda,\kappa}(\bar{M})$.
3) Let $\mathrm{Pr}^2_{\lambda,\kappa}(\bar{M})$ be defined as in part (1) but π is an isomorphism from M_{δ_1} onto M_{δ_2} mapping M_α onto itself. We define $\mathrm{Pr}^2_{\lambda,\kappa}(T)$ as in part (2) using $\mathrm{Pr}^2_{\lambda,\kappa}(\bar{M})$.
4) Let $\mathrm{Pr}^1_{\lambda,\kappa}(-)$ mean $\mathrm{Pr}_{\lambda,\kappa}(-)$.

Remark 2.6. 1) Clearly $\mathrm{Pr}^2_{\lambda,\kappa}(\bar{M}) \Rightarrow \mathrm{Pr}^1_{\lambda,\kappa}(\bar{M})$ and $\mathrm{Pr}^2_{\lambda,\kappa}(T) \Rightarrow \mathrm{Pr}^1_{\lambda,\kappa}(T)$.
2) Also there is no point (in 2.5(1)) to use α_1,α_2 as some $\mathbf{F}$ guarantee that $\alpha_1 < \alpha_2 < \delta \in S^\lambda_\kappa$ implies there is an automorphism of M_δ mapping M_{α_1} onto M_{α_2}.

Proof. 1) Assume that $\varphi(\bar{x},y)$ exemplifies the strong independence property.

For every $M \in \mathrm{EC}_\lambda(T)$ and function $\mathbf{F}$ as in 0.8(3) we can find a sequence $\langle M_\alpha : \alpha < \lambda^+\rangle$ obeying $\mathbf{F}$ such that $M \prec M_0$ and:

- ⊛ if $\alpha < \lambda^+$ then for some $\bar{c}^\alpha \in {}^{\ell g(\bar{x})}(M_{\alpha+1})$ we have: in $M_{\alpha+1}$ every $a \in M_\alpha$ satisfies $\varphi(\bar{c}^\alpha, a) \Leftrightarrow a \in M$.

Now for any $\delta < \lambda^+$ of cofinality κ let $\langle \alpha^\delta_\varepsilon : \varepsilon < \kappa\rangle$ be increasing with limit δ then $\bar{\mathbf{c}}^\delta = \langle \bar{c}^{\alpha^\delta_\varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of $\ell g(\bar{x})$-tuples from M_δ of length κ, and for every $a \in M_\delta$ we have:

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(∗) a realizes the type $p(y, \bar{\mathbf{c}}^\delta) = \{\varphi(\bar{c}^{\alpha_\varepsilon}, y) : \varepsilon < \kappa\}$ in M_δ iff $a \in M$.

The number of isomorphism types of τ_T-models M' of cardinality λ is 2^λ whereas the number of $\langle \bar{c}^\alpha_i : i < \kappa\rangle$ for a given M' is $\le \lambda^\kappa < 2^\lambda$.

For a given $\mathbf{F}$ the construction above works for every $M \in \mathrm{EC}_\lambda(T)$, but $\dot{I}(\lambda, T) = 2^\lambda$, see 0.14 as $\lambda \ge |T| + \aleph_1$ so we can finish easily, or see more in part (2).

2) We can make the counterexample more explicit. For a model M and $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}M$ for $\varepsilon < \kappa$ we define $N = N[M, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle]$ as the following submodel of M (if well defined): it is the submodel with universe the set $A = \{d \in M : M \models \varphi[\bar{c}^\varepsilon, d]$ for every $\varepsilon < \kappa\}$; (note that N is not necessarily an elementary submodel of M or even well defined, e.g. $A = \emptyset$ or A not closed under functions of M). For $M \in \mathrm{EC}_\lambda(T)$ let $\mathscr{M}[M] = \{N \prec M : N$ is $N[M, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle]$ for some $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}M$ for $\varepsilon < \kappa\}$. Fixing $\mathbf{F}$ as in 0.8(3) we can choose $M_\alpha \in \mathrm{EC}_\lambda(T)$ with universe $\lambda \times (1 + \alpha)$ such that

- $(*)_1$ if $\alpha = 4\beta + 3$ and $\delta \le 4\beta$ then M_α is not isomorphic to $N \prec M_\delta$ whenever there are $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}(M_\delta)$ for $\varepsilon < \kappa$ such that $N = N[M_\delta, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle]$
- $(*)_2$ for $\alpha < \beta < \lambda^+$ there is $\bar{c}^\beta_\alpha \in {}^{(\ell g(\bar{x}))}(M_{\beta+1})$ such that for every $a \in M_\beta$ we have $M_{\beta+1} \models \varphi[\bar{c}^\beta_\alpha, a] \Leftrightarrow a \in M_\alpha$
- $(*)_3$ the sequence $\langle M_{2\alpha} : \alpha < \lambda^+\rangle$ obeys $\mathbf{F}$.

As $\dot{I}(\lambda, T) = 2^\lambda$ and moreover for any theory $T_1 \supseteq T$ of cardinality λ we have $\dot{I}(\lambda, T_1, T) = 2^\lambda$ and for every $M \in \mathrm{EC}_\lambda(T)$, the number of $N \in \mathscr{M}[M]$ is $\le \lambda^\kappa < 2^\lambda$ we get

⊠ for every appropriate $\mathbf{F}$ there is a $\prec$-increasing continuous sequence $\langle M_\alpha : \alpha < \lambda^+\rangle$ of models of T as above such that if $\delta_1 \ne \delta_2 < \lambda^+$ has cofinality κ <u>then</u> $M_{\delta_1}, M_{\delta_2}$ are not isomorphic.

[Why? Without loss of generality $\delta_1 < \delta_2$, let $\langle \alpha^{\delta_2}_\varepsilon : \varepsilon < \kappa\rangle$ be increasing with limit δ_2, all $> \delta_1 + 4$. Now by $(*)_2$ we know that $\langle \bar{c}^{\alpha_\varepsilon}_{\delta_1+3} : \varepsilon < \kappa\rangle$ exemplified that in M_{δ_2} there is a sequence $\langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle$ which define M_{δ_1+3}, i.e. $M_{\delta_1+3} = N[M_{\delta_2}, \langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle]$.

So if $M_{\delta_1} \cong M_{\delta_2}$ then there are $\bar{d}^\varepsilon \in {}^{\ell g(\bar{x})}(M_{\delta_1})$ for $\varepsilon < \kappa$ such that $N[M_{\delta_1}, \langle \bar{d}^\varepsilon : \varepsilon < \kappa\rangle]$ is well defined and isomorphic to M_{δ_1+3}. But consider the choice of M_{δ_1+3}, clearly $(*)_1$ says that this is impossible.] $\square_{2.3}$

Observation 2.7. If, inside the proof of 2.3, in the definition of $\mathscr{M}[M]$ we restrict ourselves to $\langle \bar{c}^\varepsilon : \varepsilon < \kappa\rangle$ such that $(\forall a \in M)(\exists \varepsilon < \kappa)(\forall \zeta)(\varepsilon < \zeta < \kappa \to M \models \varphi[\bar{c}^\varepsilon, a] \equiv \varphi[\bar{c}^\zeta, a])$ <u>then</u> we can replace $\lambda^\kappa < 2^\lambda$ by $\mathbf{U}_\kappa(\lambda) < 2^\lambda$, see 0.17.

Considering 2.7 (and 1.1), it is natural to ask:

Question 2.8. Is the independence property enough to imply no limit models?

The problem was that the independence we can get may be "hidden", "camouflaged" by other "parts" of the model.

Working harder (than in 2.3), the answer is yes.

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Theorem 2.9. *Assume T is independent.*
1) If $|T| \leq \lambda = \lambda^\theta = 2^\kappa > \theta = \mathrm{cf}(\theta)$ then T has no $(\lambda,\theta)-$ md-limit model.
2) Moreover, there is $\mathbf{F}$ such that

- (a) *$\mathbf{F}$ is a function with domain $\bigcup\{K_\alpha : \alpha < \lambda^+ \text{odd}\}$ where $K_\alpha = \{M : M$ a model of T with universe $\lambda \times (1+\alpha)\}$*
- (b) *if $\alpha < \lambda^+$ is odd and $M \in K_\alpha$ then $M \prec \mathbf{F}(M) \in K_{\alpha+1}$*
- (c) *if $M_\alpha \in K_\alpha$ for $\alpha < \lambda^+$ is $\prec$-increasing continuous and $M_{2\alpha+2} = \mathbf{F}(M_{2\alpha+1})$ for $\alpha < \lambda^+$ then for no $\alpha < \lambda^+$ is the set $\{\delta : M_\delta \cong M_\alpha$ and $\mathrm{cf}(\delta) = \theta\}$ stationary.*

3) We can strengthen part (2) by adding in clause (c):

- $(*)$ *there are $\bar{c}_\alpha \in {}^\kappa(M_{2\alpha+2})$ for $\alpha < \lambda^+$ such that: if $\langle \alpha_{\ell,\varepsilon} : \varepsilon < \theta\rangle$ is an increasing continuous sequence of ordinals $< \lambda^+$ with limit α_ℓ for $\ell = 1,2$ and $\alpha_1 \neq \alpha_2$ then there is no isomorphism f from M_{α_1} onto M_{α_2} mapping $\bar{c}_{\alpha_{1,\varepsilon}}$ to $\bar{c}_{\alpha_{2,\varepsilon}}$ for $\varepsilon < \theta$.*

4) In part (2) we can replace K_α (for $\alpha < \lambda^+$) by $K_{<\lambda^+} := \cup\{K_\alpha : \alpha < \lambda^+\}$.

Remark 2.10. 1) How does $2^\kappa = \lambda$ help us?

We shall consider $M_\alpha \in K_\alpha$ for $\alpha < \lambda^+$ which is $\prec$-increasing. We fix a sequence $\langle \bar{a}_t : t \in I\rangle$ in M_0 such that $\langle \varphi(x,\bar{a}_t) : t \in I\rangle$ is an independent set of formulas (actually $I = \lambda$). Now for any sequence $\langle \eta_i : i < \kappa+\kappa\rangle$ of members of ${}^I 2$, and $\prec$-extension M of M_0 we can find $N,\bar{c}$ such that $M \prec N, \bar{c} = \langle c_i : i < \kappa+\kappa\rangle$ and $N \models \varphi[c_i,\bar{a}_t]^{\mathrm{if}(\eta_i(t))}$. Specifically if $M_{2\alpha+1}$ is already chosen then when choosing $M_{2\alpha+2}$ we choose also a sequence $\langle \eta^\alpha_i : i < \kappa+\kappa\rangle$, of members of ${}^I 2$ and $\langle c^\alpha_i : i < \kappa+\kappa\rangle$ such that $M_{2\alpha+2} \models \varphi[c^\alpha_i,\bar{a}_t]^{\mathrm{if}(\eta^\alpha_i(t))}$.

We may look at it as coding a sequence of λ subsets of κ. We essentially like to gain some information on $\langle \eta^\alpha_i : i < \kappa+\kappa\rangle$ from $(M_{2\alpha+1}, M_{2\alpha+2}, \bar{c}^\alpha)$, but we are not given who are the $\bar{a}_t$'s. We shall try to use $\langle c^\alpha_i : i < \kappa\rangle$, to distinguish between the "true" $\bar{a}_t$'s and "fakers". We do an approximation: some will be "exposed fakes", which we can discard, and the others are "perfect fakers", i.e., they immitate perfectly some a_t, so it does not matter.

Clearly it suffices to prove part (3) of 2.9 for having parts (1),(2) because $\lambda = \lambda^\kappa$ and the proof of part (4) is similar. The proof is broken to some definitions and claims.

Definition 2.11. 1) Assume $\varphi = \varphi(x,\bar{y}) \in \mathbb{L}(\tau_T)$ has the independence property in T. We say $(M,\bar{\mathbf{a}})$ is a (T,φ)-candidate or an (I,T,φ)-candidate <u>when</u>:

- (a) M is a model of T
- (b) $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I\rangle, \bar{a}_t \in {}^{\ell g(\bar{y})}M$ and I is an infinite linear order
- (c) $\bar{\mathbf{a}}$ is an indiscernible sequence in M
- (d) $\{\varphi(x,\bar{a}_t) : t \in I\}$ is independent in M; that is for every $\eta \in \mathrm{fin}(I) := \{\eta : \eta \in {}^J 2$ for some finite $J \subseteq I\}$, there is $b \in M$ such that $t \in \mathrm{Dom}(\eta) \Rightarrow M \models \varphi[b,\bar{a}_t]^{\mathrm{if}(\eta(t))}$.

2) If $(M,\bar{\mathbf{a}})$ is an (I,T,φ)-candidate <u>then</u> let $\Gamma_{M,\bar{\mathbf{a}}} = \Gamma^{T,\varphi}_{M,\bar{\mathbf{a}}} = \Gamma^{T,\varphi,1}_{M,\bar{\mathbf{a}}} \cup \Gamma^{T,\varphi,2}_{M,\bar{\mathbf{a}}}$ be the following set of first order sentences and τ^+_M be the following vocabulary

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(a) $\tau^+_M = \tau_T \cup \{c : c \in M\} \cup \{P\}$ where P is a unary predicate ($\notin \tau_T$ of course) and each $c \in M$ serves as an individual constant ($\notin \tau_T$)

(b) $\Gamma^{T,\varphi,1}_{M,\bar{\mathbf{a}}} = \text{Th}(M,c)_{c\in M}$

(c) $\Gamma^{T,\varphi,2}_{M,\bar{\mathbf{a}}} = \{(\exists x)[P(x) \wedge \bigwedge_{t\in J} \varphi(x,\bar{a}_t)^{\text{if}(\eta(t))}]$: for some finite $J \subseteq I$ and $\eta \in {}^J 2\}$ (so the vocabulary is $\subseteq \tau^+_M$).

3) In (2) let $\Omega_{M,\bar{\mathbf{a}}} = \Omega^{T,\varphi}_{M,\bar{\mathbf{a}}}$ be the family of consistent sets Γ of sentences in $\mathbb{L}(\tau^+_M)$ such that Γ is of the form $\Gamma_{M,\bar{\mathbf{a}}}$ union with a subset of $\Phi_{M,\bar{\mathbf{a}}} = \{\neg(\exists x)[P(x) \wedge \psi(x,\bar{c}) \wedge \bigwedge_{t\in J} \varphi(x,\bar{a}_t)^{\eta(t)}] : J \subseteq I$ is finite, $\eta \in {}^J 2, \bar{c} \in {}^{\ell g(\bar{z})}M$ and $\psi(x,\bar{z}) \in \mathbb{L}(\tau_T)\}$.

4) For $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}$ let

(a) $\mathbf{S}_\Gamma = \{p : p \in \mathbf{S}(M)$ and $\Gamma \cup \{(\exists x)(P(x) \wedge \psi(x,\bar{c})) : \psi(x,\bar{c}) \in p(x)\}$ is consistent$\}$

(b) for $J \subseteq I$ and $\eta \in {}^J 2$ let $\mathbf{S}_{\Gamma,\eta} = \{p \in \mathbf{S}_\Gamma : p$ include $q^\eta_{M,\bar{\mathbf{a}}}\}$

where

(c) $q^\eta_{M,\bar{\mathbf{a}}} := q^{T,\varphi,\eta}_{M,\bar{\mathbf{a}}} = \{\varphi(x,\bar{a}_t)^{\text{if}(\eta(t))} : t \in \text{Dom}(\eta)\}$.

5) For $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}, \psi(x,\bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}M$ let

$$\Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c})) = \{\eta \in \text{fin}(I) : \ \Gamma \text{ is consistent with } (\exists x)[P(x) \wedge \psi(x,\bar{c}) \wedge \bigwedge_{t\in \text{Dom}(\eta)} \varphi(x,\bar{a}_t)^{\text{if}(\eta(t))}]\}.$$

Remark 2.12. 1) $\text{fin}(I) = \{\eta : \eta$ is a function from some finite $J \subseteq I$ to $\{0,1\}\}$.
2) In parts (3) and (4) we could have used only $\psi(x,\bar{z}) \in \{\varphi(x,\bar{y}), \neg\varphi(x,\bar{y})\}$.

Observation 2.13. Let $(M,\bar{\mathbf{a}})$ be a (T,φ)-candidate.
1) $\Gamma_{M,\bar{\mathbf{a}}} \in \Omega_{M,\bar{\mathbf{a}}}$, i.e., $\Gamma_{M,\bar{\mathbf{a}}}$ is consistent so $\Omega_{M,\bar{a}}$ is non-empty.
2) $\Omega_{M,\bar{\mathbf{a}}}$ is closed under increasing (by $\subseteq$) unions.
3) Any member of $\Omega_{M,\bar{\mathbf{a}}}$ can be extended to a maximal member of $\Omega_{M,\bar{\mathbf{a}}}$.
4) If $M \prec N$ <u>then</u> $(N,\bar{\mathbf{a}})$ is a (T,φ)-candidate and for every $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}$ the set $\Gamma \cup \Gamma_{N,\bar{\mathbf{a}}}$ belongs to $\Omega_{N,\bar{a}}$.
5) If $\langle I_\alpha : \alpha \leq \delta\rangle$ is an increasing continuous sequence of linear orders and $\langle N_\alpha : \alpha \leq \delta\rangle$ is $\prec$-increasing continuous sequence of models of $T, \bar{\mathbf{a}} = \langle \bar{a}_t : t \in I_\delta\rangle$ and $(N_\alpha, \bar{\mathbf{a}} \restriction I_\alpha)$ is a (T,φ)-candidate for $\alpha < \delta$ <u>then</u> $(N_\delta,\bar{\mathbf{a}})$ is a (T,φ)-candidate.
6) In part (5), if $\Gamma_\alpha \in \Omega_{N_\alpha,\bar{\mathbf{a}}}$ for $\alpha < \delta$ is increasing continuous with α <u>then</u> $\Gamma_\delta := \bigcup\{\Gamma_\alpha : \alpha < \delta\}$ belongs to $\Omega_{N_\delta,\bar{\mathbf{a}}}$.
7) In part (6) if Γ_α is maximal in $\Omega_{N_\alpha,\mathbf{a}}$ for each $\alpha < \delta$ <u>then</u> Γ_δ is maximal in $\Omega_{N_\delta,\bar{\mathbf{a}}}$.
8) If $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}, \psi(x,\bar{z}) \in \mathbb{L}(\tau_T), \bar{c} \in {}^{\ell g(\bar{z})}M$ and $M \models (\exists x)\psi(x,\bar{c})$ <u>then</u>

(a) the empty function belongs to $\Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c}))$

(b) if $I_1 \subseteq I_2$ are finite subsets of I and $\eta \in \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c})) \cap {}^{(I_1)}2$ <u>then</u> there is $\nu \in \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x,\bar{c})) \cap {}^{(I_2)}2$ extending η.

Proof. Straightforward. $\square_{2.13}$

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Claim 2.14. *Assume that* $(M, \bar{\mathbf{a}})$ *is a* (T, φ)*-candidate and* $\Gamma \in \Omega^{T,\varphi}_{M,\bar{\mathbf{a}}}$ *is maximal.*
1) If $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$ *and* $\bar{c} \in {}^{\ell g(\bar{z})}M$ *and* $\eta \in \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x, \bar{c})) \subseteq \mathrm{fin}(I)$ *then for some* ν *we have* $\eta \subseteq \nu \in \mathrm{fin}(I)$ *and* $\nu \notin \Xi_{M,\bar{\mathbf{a}},\Gamma}(\neg\psi(x, \bar{c}))$.
2) For every $\eta \in {}^I 2$ *there are* N, b *such that:*

- (a) $M \prec N$ *(and* $\|N\| \le \|M\| + |T|$*)*
- (b) $b \in N$
- (c) *if* $t \in I$ *then* $N \models \varphi[b, \bar{a}_t]^{\mathrm{if}(\eta(t))}$
- (d) *if* $\bar{a} \in {}^{\ell g(\bar{z})}M, \psi = \psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$ *and* $\psi(x, \bar{a}) \in \mathrm{tp}(b, M, N)$ *then* Γ *is disjoint to* $\{\neg(\exists x)[P(x) \wedge \psi(x, \bar{a}) \wedge \bigwedge_{t \in J} \varphi(x, \bar{a}_t)^{\mathrm{if}(\eta(t))}] : J \subseteq I$ *finite*$\}$.

Proof. 1) Assume that the conclusion fails. Consider the formula $\psi'(x, \bar{c}) := \bigwedge_{t \in \mathrm{Dom}(\eta)} \varphi(x, \bar{a}_t)^{\mathrm{if}(\eta(t))} \rightarrow \neg\psi(x, \bar{c})$.

By the assumption of the claim + the assumption toward contradiction it follows that "$\rho \in \mathrm{fin}(I) \Rightarrow \Gamma \cup \{(\exists x)[P(x) \wedge \bigwedge_{t \in \mathrm{Dom}(\rho)} \varphi(x, \bar{a}_t)^{\mathrm{if}(\rho(t))} \wedge \psi'(x, \bar{c})]\}$ is consistent).
[Why? Just note that it is enough to consider $\rho \in \mathrm{fin}(I)$ such that $\mathrm{Dom}(\eta) \subseteq \rho$ and we split to two cases: first when $\rho \restriction \mathrm{Dom}(\eta) \neq \eta$ then $\psi'(x, \bar{c})$ adds nothing in the conjunction (and use 2.11(2)(c)); second when $\rho \restriction \mathrm{Dom}(\eta) = \eta$ and we use the assumption toward the contradiction.]

So if N' is a model of Γ and we define N'' as N' by replacing $P^{N'}$ by $P^{N''} = \{b \in P^{N'} : N' \models \psi'[b, \bar{c}]\}$ we see that $\Gamma \cup \{\neg(\exists x)[P(x) \wedge \neg\psi'(x, \bar{c})]\} \in \Omega_{M,\bar{\mathbf{a}}}$. By the maximality of Γ it follows that $\neg(\exists x)[P(x) \wedge \neg\psi'(x, \bar{c})] \in \Gamma$. But this contradicts the assumption $\eta \in \Xi_{M,\bar{\mathbf{a}},\Gamma}(\psi(x, \bar{c}))$.
2) Easy. $\square_{2.14}$

Claim 2.15. *Assume that*

- (a) $(M, \bar{\mathbf{a}})$ *is an* (I, T, φ)*-candidate*
- (b) $\bar{\eta} = \langle \eta_i : i < i(*) \rangle$ *and* $\eta_i \in {}^I 2$ *for* $i < i(*)$
- (c) $j(*) \le i(*)$
- (d) $\{\eta_i : i < j(*)\}$ *is a dense subset of* ${}^I 2$.

Then we can find $N, \bar{\mathbf{c}}$ *such that*

- (α) $M \prec N$ *and* $\|N\| \le \|M\| + |T| + |i(*)|$
- (β) $\bar{\mathbf{c}} = \langle c_i : i < i(*) \rangle$ *and* $c_i \in N$
- (γ) *if* $i < i(*)$ *and* $t \in I$ *then* $N \models \varphi[c_i, \bar{a}_t]^{\mathrm{if}(\eta_i(t))}$
- (δ) *for every* $\bar{a} \in {}^{\ell g(\bar{y})}M$ *at least one of the following holds:*
  - (i) *[the perfect fakers] for some* $t \in I$ *for every* $\rho_0 \in \mathrm{fin}(I \backslash \{t\})$ *we can find* $\rho_1 \in \mathrm{fin}(I \backslash \{t\})$ *extending* ρ_0 *such that:* $\rho_1 \subseteq \eta_i \wedge i < i(*) \Rightarrow N \models$ *"*$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$*", i.e. for "most"* $i < i(*), \bar{a}, \bar{a}_t$ *are similar*
  - (ii) *[the rejected* $\bar{a}$*'s] for no* $t \in I$ *do we have* $i < j(*) \Rightarrow N \models$ *"*$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$*".*

Proof. By 2.9(1), $\Gamma_{M,\bar{\mathbf{a}}} \in \Omega_{M,\bar{\mathbf{a}}}$ hence by 2.9(3) there is a maximal $\Gamma \in \Omega_{M,\bar{\mathbf{a}}}$.
Let $N, \langle c_i : i < i(*) \rangle$ be such that

$\circledast$ (a) $M \prec N$ and $\|N\| = \|M\| + |T| + |i(*)|$

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(b) for $i < i(*), c_i \in N$ realizes some $p_i \in \mathbf{S}_{\Gamma,\eta_i}$ (see Definition 2.11(4)(b)).

Clearly clauses $(\alpha), (\beta), (\gamma)$ of the desired conclusion hold, and let us check clause (δ). So assume that $\bar{a} \in {}^{\ell g(\bar{y})}M$ and clause (ii) there fails so we can choose $t \in I$ such that $i < j(*) \Rightarrow N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$".

So it is enough to prove clause (i) for t; toward this assume $\rho_0 \in \text{fin}(I)$ satisfies $t \notin \text{Dom}(\rho_0)$, i.e. $\rho_0 \in \text{fin}(I \backslash \{t\})$. Let $\rho_1 \in \text{fin}(I)$ extend ρ_0 be such that $\rho_1(t) = 0$. By clause (d) of the assumption we know that for some $i < j(*)$ we have $\rho_1 \subseteq \eta_i$ but (see above) $N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$" hence $\rho_1 \in \Xi_{M,\bar{\mathbf{a}},\Gamma}(\varphi(x,\bar{a})^{\text{if}(\eta_i(t))})$ which means that $\rho_1 \in \Xi_{M,\bar{a},\Gamma}(\varphi(x,\bar{a})^{\text{if}(\rho_1(t))})$. Now apply claim 2.14(1) to $\psi(x,\bar{c}) := \varphi(x,\bar{a})^{\text{if}(\rho_1(t))}$, so we know that for some ν we have $\rho_1 \subseteq \nu \in \text{fin}(I)$ and $\nu \notin \Xi_{M,\bar{\mathbf{a}},\Gamma}(\neg\varphi(x,a)^{\text{if}(\rho_1(t))})$ hence

$(*)_1$ if $i < i(*)$ satisfies $\nu \subseteq \eta_i$ then $N \models$ "$\varphi[c_i, \bar{a}]^{\text{if}(\rho_1(t))}$" which means $N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$".

Let $\rho_2 \in {}^{\text{Dom}(\nu)}2$ be such that $\rho_2(t) = 1$ and $s \in \text{Dom}(\nu) \backslash \{t\} \Rightarrow \rho_2(s) = \nu(s)$. We repeat the use of 2.14(1) for ρ_2 instead of ρ_1 and get ν' such that $\rho_2 \subseteq \nu' \in \text{fin}(I)$ and

$(*)_2$ if $i < i(*)$ satisfies $\nu' \subseteq \eta_i$ then $N \models$ "$\neg\varphi[c_i, \bar{a}]^{\text{if}(\rho_2(t))}$" which means that $N \models$ "$\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]$".

Let $\rho_3 = \nu' \restriction (\text{Dom}(\nu') \backslash \{t\})$ and by $(*)_1 + (*)_2$ the function $\rho_3 \in \text{fin}(I)$ is as required in subclause (i) (for our $\bar{a}, t, \rho_0$) in clause (δ) of the claim, so we are done. $\square_{2.15}$

Definition 2.16. 1) For a model M of T, formula $\varphi = \varphi(x,\bar{y}) \in \mathbb{L}(\tau_M), \bar{\mathbf{c}} = \langle c_i^\varepsilon : i < i(*), \varepsilon < \theta \rangle$ such that $c_i^\varepsilon \in M$ let $\mathscr{P}_\varphi(\bar{\mathbf{c}}, M) = \{\mathscr{U} \subseteq i(*)$: for some $\bar{a} \in {}^{\ell g(\bar{y})}M$ for every $\varepsilon < \theta$ large enough $\mathscr{U} = u_\varphi(\bar{a}, \langle c_i^\varepsilon : i < i(*) \rangle, M)\}$ where $u_\varphi(\bar{a}, \langle c_i^\varepsilon : i < i(*) \rangle, M) = \{i < i(*) : M \models \varphi[c_i^\varepsilon, \bar{a}]\}$.
2) For a model M and $\varphi = \varphi(x,\bar{y}) \in \mathbb{L}(\tau_M)$ let $\mathbf{P}^\varphi_{i(*),\theta}(M) = \{\mathscr{P}_\varphi(\bar{\mathbf{c}}, M) : \bar{\mathbf{c}}$ has the form $\langle c_i^\varepsilon : i < i(*), \varepsilon < \theta \rangle$ with $c_i^\varepsilon \in M\}$.
3) For $M_1 \prec M_2$ and $\varphi = \varphi(x,\bar{y}) \in \mathbb{L}(\tau_{M_\ell})$ and $\bar{c} = \langle c_i : i < i(*) \rangle \in {}^{i(*)}(M_2)$ let $\mathscr{P}_\varphi(\bar{\mathbf{c}}, M_1, M_2) = \{u_\varphi(\bar{a}, \bar{\mathbf{c}}, M_2) : \bar{a} \in {}^{\ell g(\bar{y})}M_1\}$.

Observation 2.17. For $M, \varphi(x,\bar{y}), i(*), \theta$ as in Definition 2.16, $\mathbf{P}^\varphi_{i(*),\theta}(M)$ has cardinality $\leq \|M\|^{|i(*)|+\theta}$.

Claim 2.18. *If I is a linear order of cardinality $\leq 2^\kappa$ <u>then</u> we can find a uniform filter D on κ and a sequence $\bar{\mathscr{U}}^* = \langle \mathscr{U}^*_{t,\ell} : t \in I, \ell \in \{0,1,2\} \rangle$ of members of $[\kappa]^\kappa$ such that:*

$\boxtimes_1$ (a) *for each $t \in I, \langle \mathscr{U}^*_{t,\ell} : \ell = 0,1,2 \rangle$ is a partition of κ*

(b) *$\mathscr{U}^*_{t,2} \in D$ for $t \in I$*

(c) *$\mathscr{P}(\kappa)/D$ has cardinality 2^κ, moreover extend some free Boolean Algebra of cardinality 2^κ*

$\boxtimes_2$ *if $(M, \bar{\mathbf{a}})$ is a (I,T,φ)-candidate and $\bar{\mathscr{U}} = \langle \mathscr{U}_t : t \in I \rangle$ satisfies $\mathscr{U}^*_{t,1} \subseteq \mathscr{U}_t \subseteq \mathscr{U}^*_{t,1} \cup \mathscr{U}^*_{t,2}$ <u>then</u> for some $(N, \bar{\mathbf{c}})$ we have*

(a) *$M \prec N$ and $\|N\| \leq \|M\| + |\tau_T| + \kappa$*

(b) *$\bar{\mathbf{c}} \in {}^\kappa N$*

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(c) *if* $t \in I, \bar{a} \in {}^{\ell g(\bar{y})}M$ *and* $\mathscr{U}^*_{t,1} \subseteq u(\bar{a}, \bar{\mathbf{c}}, N) \subseteq \mathscr{U}^*_{t,1} \cup \mathscr{U}^*_{t,2}$ *then* $u(\bar{a}, \bar{\mathbf{c}}, N) = \mathscr{U}_t \mod D$

(d) *if* $t \in I$ *then* $\mathscr{U}_t \in \mathscr{P}_\varphi(\bar{\mathbf{c}}, M, N)$.

Proof. We replace κ by $\kappa + \kappa$.

Let $\langle \eta^*_i : i < \kappa \rangle$ be a sequence of members of ${}^I\kappa$ which is dense possible by [EK65].

For $\ell = 0, 1, 2$ let $\mathscr{U}_{t,\ell} = \{i < \kappa : \eta_i(t) = \ell \text{ or } \eta_i(t) \geq 3 \wedge \ell = 2\}$. Notice that it is important that D is defined independently of $\mathscr{U}_t$ and we should therefore define it here. But for clarity of exposition we will only define it later.

Let (where $\alpha + \mathscr{U} = \{\alpha + \beta : \beta \in \mathscr{U}\}$)

$(*)_1$ $\mathscr{U}^*_{t,0} = \mathscr{U}_{t,0} \cup (\kappa + \mathscr{U}_{t,0})$

$(*)_2$ $\mathscr{U}^*_{t,1} = \mathscr{U}_{t,1} \cup \mathscr{U}_{t,2} \cup (\kappa + \mathscr{U}_{t,1})$

$(*)_3$ $\mathscr{U}^*_{t,2} = \kappa + \mathscr{U}_{t,2}$.

Assume $\bar{\mathscr{U}} = \langle \mathscr{U}_t : t \in I \rangle$ is such that

$(*)_4$ $\mathscr{U}^*_{t,1} \subseteq \mathscr{U}_t \subseteq \mathscr{U}^*_{t,1} \cup \mathscr{U}^*_{t,2} \subseteq \kappa + \kappa$.

Define $\eta_i = \eta^{\bar{\mathscr{U}}}_i \in {}^{\kappa+\kappa}2$ for $i < \kappa + \kappa$ by:

$(*)_5$ $\eta_i(t) = \begin{cases} 0 & i \notin \mathscr{U}_t \\ 1 & i \in \mathscr{U}_t \end{cases}$

Let $\bar{\eta} = \langle \eta_i : i < \kappa + \kappa \rangle$.
Notice that $\langle \eta_i : i < \kappa \rangle$ is dense in ${}^I 2$ by the choice of η_i in $(*)_5$ and $(*)_2$ because $\mathscr{U}_t \cap \kappa = \mathscr{U}^*_{t,1} \cup \mathscr{U}^*_{t,2}$ and $\langle \eta^*_i : i < \kappa \rangle$ was dense in ${}^I\kappa$. By 2.15 applied to $(M, \bar{\mathbf{a}}, \bar{\eta}), i(*) = \kappa + \kappa, j(*) = \kappa$ we can find $N, \bar{\mathbf{c}}$ as there and we should check that they are as required. Clauses $(\alpha), (\beta), (\gamma)$ of the conclusion of 2.15 give the "soft" demands.

More specifically clause (a) of $\boxtimes_1$ holds by the choice of the $\mathscr{U}^*_{t,\ell}$'s; clauses (b),(c) of $\boxtimes_2$ holds by the conclusion of 2.15.

Clearly

$(*)_6$ $u_\varphi(\bar{a}_t, \bar{\mathbf{c}}, N) = \{i < \kappa + \kappa : N \models \text{“}\varphi[c_i, \bar{a}_t]\text{”}\} = \{i < \kappa + \kappa : \eta_i(t) = 1\} = \mathscr{U}_t$

hence

$(*)_7$ $t \in I \Rightarrow \mathscr{U}_t \in \mathscr{P}_\varphi(\bar{\mathbf{c}}, M, N)$.

So we see that demand (d) of $\boxtimes_2$ is satisfied - all the $\mathscr{U}_t$ are included. We still need to prove clause (c) of $\boxtimes_2$, that is to show that there are no "fakers" and, of course, to define D.

So assume

$\odot_1$ $\mathscr{U}^*_{t_1} \subseteq u_\varphi(\bar{a}, \bar{c}, N) \subseteq \mathscr{U}^*_{t_1} \cup \mathscr{U}^*_{t_1}$ for some $t_1 \in I$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$.

Denote $\mathscr{U} = u_\varphi(\bar{a}, \bar{c}, N)$. We need to show $\mathscr{U} = \mathscr{U}_{t_1} \mod D$.

By clause (δ) of the conclusion of 2.15 for $\bar{a}$ one of the two clauses there (i),(ii) occurs.

Recall that

$\odot_2$ $\mathscr{U}^*_{t_1,1} \subseteq \mathscr{U} \subseteq \mathscr{U}^*_{t_1,1} \cup \mathscr{U}^*_{t_1,2}$.

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So $\mathscr{U} \cap \kappa = \mathscr{U}^*_{t_1,1} \cap \kappa = \mathscr{U}_{t_1,1} \cup \mathscr{U}_{t_1,2}$.

Now

$\odot_3$ for $\bar{a}$ clause (ii) of 2.15(δ) fails.

[Why? Because t_1 witnesses this by the above equality and for each $i < \kappa$

$$i \in \mathscr{U} \Leftrightarrow i \in \mathscr{U}_{t_1,1} \cup \mathscr{U}_{t_1,2} \Leftrightarrow \eta_i[t_1] = 1 \Leftrightarrow N \models \text{“}\varphi[c_i, \bar{a}_{t_1}]\text{”}.]$$

By 2.15(δ) and $\odot_3$ we can deduce:

$\odot_4$ for $\bar{a}$, clause (i) of 2.15(δ) holds so there is t_2 witnessing it.

Next

$\odot_5$ $t_1 = t_2$.

Why? Toward contradiction assume $t_1 \neq t_2$ hence we can find $\rho_1 \in \text{fin}(I \backslash \{t_2\})$ such that

$\circledast_{5.1}$ $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_2}]\text{”}$.

Without loss of generality $t_1 \in \text{Dom}(\rho_1)$ and define $\rho_2 = \rho \cup \{(t_2, 1 - \rho_1(t_1))\}$, so $\rho_1 \subseteq \rho_2 \in \text{fin}(I)$. As $\{\eta_i : i < \kappa\}$ was chosen as a dense subset of ${}^{\{0,1,2\}}I$, there is $i < \kappa$ such that $\rho_2 \subseteq \eta^*_i$, hence by $\circledast_{5.1}$

$\circledast_{5.2}$ $N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_2}]\text{”}$

but by the choice of η_i we have:

$\circledast_{5.3}$ $N \models \text{“}\varphi[c_i, \bar{a}_{t_2}]^{\text{if}(\eta_i(t_2))}\text{”}$

but $\eta_i(t_2) = 1 - \rho_1(t_1)$ hence together

$\circledast_{5.4}$ $N \models \text{“}\varphi[c_i, \bar{a}]^{\text{if}(1-\rho_1(t_1))}\text{”}$

but by the choice of c_i we have:

$\circledast_{5.5}$ $N \models \text{“}\varphi[c_i, \bar{a}_{t_1}]^{\text{if}(\eta_i(t_1))}\text{”}$

hence by $\circledast_{5.1}$

$\circledast_{5.6}$ $N \models \text{“}\varphi[c_i, \bar{a}_{t_1}]^{\text{if}(\rho_1(t_1))}\text{”}$.

But $\circledast_{5.5} + \circledast_{5.6}$ contradict the choice of t_1 as $i < \kappa$ using $\odot_2$ so $\odot_5$ holds, i.e. $t_1 = t_2$.]

Now subclause (i) of 2.15(δ) tells us

$\odot_6$ for every $\rho_0 \in \text{fin}(I \backslash \{t_1\})$ there is $\rho_1 \in \text{fin}(I \backslash \{t_1\})$ extending ρ_0 such that
- (a) $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow N \models \varphi[\bar{c}_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_1}]$ hence
- (b) if $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow i \in u_\varphi(\bar{a}, \bar{\mathbf{c}}, N) \Leftrightarrow i \in \mathscr{U}_{t_1}$.

So let

$$D = \{\mathscr{U} \subseteq \kappa + \kappa : \text{ for every } \rho_0 \in \text{fin}(I) \text{ there is } \rho_1, \rho_0 \subseteq \rho_1 \in \text{fin}(I) \text{ such that } \kappa \leq i < \kappa + \kappa \wedge \rho_1 \subseteq \eta_i \Rightarrow i \in \mathscr{U}\}.$$

Clearly the filter D satisfies clause $\boxtimes_1(c)$ so we are done. $\square_{2.18}$

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Proof. Proof of the Theorem 2.9(3) Like the proof 2.3 of the case "T has the strong independence property." $\square_{2.9}$

Remark 2.19. 1) The $\mathbf{F}$ we construct works for all $\theta = \text{cf}(\theta) < \lambda$ for which $\lambda = \lambda^\theta$ simultaneously.

Discussion 2.20. Can we prove 2.9 also for λ strongly inaccessible? Toward this

(a) we have to use $\bar{\mathbf{c}}_\alpha = \langle c_{\alpha,i} : i < \lambda \rangle$, instead $\langle c_{\alpha,i} : i < \kappa \rangle$
(b) each M_α has a presentation $\langle M_{\alpha,\zeta} : \zeta < \lambda \rangle$
(c) for a club E of $\mu < \lambda$, we use $\langle c_{\alpha,i} : i < \mu \rangle \hat{} \langle c_\mu \rangle$ to code $\mathscr{U}_\alpha \cap \mu$
(d) instead $i, \kappa + i$ we use $2i, 2i+1$.

So the problem is: arriving to μ, we have already committed ourselves for the coding of $\mathscr{U}_\alpha \cap \mu'$ for $\mu' \in E_\alpha \cap \mu$, what freedom do we have in μ?

Essentially we have a set $\Lambda_\mu \subseteq {}^{2^\mu}2$ quite independent, and for $\mu_1 < \mu_2$, there is a natural reflection, the set of possibilities in ${}^\lambda 2$ is decreasing. But the amount of freedom left should be enough to code. We shall deal in [She11] with the inaccessible case.

Question 2.21. Can we improve 2.9(3) in the case of T not strongly dependent?

Claim 2.22. *1) Assume T has the strong independence property. If $\lambda \ge \kappa = \text{cf}(\kappa), 2^{\min\{2^\kappa,\lambda\}} > \lambda^\kappa$ and $\lambda > |T|$, then $\text{Pr}_{\lambda,\kappa}(T)$.*
2) Assume T is independent. If λ, κ are as above, then $\text{Pr}(\lambda,\kappa)$.

Proof. 1) Let $\varphi(\bar{x}, y)$ exemplify "T has the strong independent property", see Definition 2.4.

We choose $\mathbf{F}$ such that:

(∗) if $\mathbf{F}(\langle M_i : i \le \alpha+1 \rangle) \prec M_{\alpha+2}$ then for every $i \le \alpha$ for some $\bar{c} = \bar{c}_{\alpha,i} \in {}^{\ell g(\bar{x})}(M_{\alpha+2})$ the set $\{a \in M_i : M_{\alpha+1} \models \varphi[\bar{c}, a]\}$ does not belong to $\{\{a \in M_i : M_{\alpha+1} \models \varphi[\bar{d}, a]\} : \bar{d} \in {}^{\ell g(\bar{x})}(M_{\alpha+1})\}$.

We continue as in the proof of 2.3.
2) Similarly (recalling the proof of 2.9). $\square_{2.22}$

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§ 3. More on (λ,κ)-limit for $T_{\rm ord}$

It is natural to hope that a (λ,κ)-i.md.-limit model is (λ,κ)-superlimit but in Theorem 3.10. we prove that there is no (λ,κ)-superlimit model for $T_{\rm rd}$, see Definition 0.12(2).

We conclude by showing that the (λ,κ)-i.md.-limit model has properties in the direction of superlimit. By 3.12 it is (λ,S)-limit$^+$, that is if $\langle M_\alpha : \alpha < \lambda^+\rangle$ is a $\subseteq$-increasing sequence of (λ,κ)-i.md-limit models for a club of $\delta < \lambda^+$ of cofinality κ the model $\cup\{M_i : i < \delta\}$ is a (λ,κ)-i.md.-limit model. Also in §1 the function $\mathbf{F}$ does not need memory.

Hypothesis 3.1. 1) $\lambda = \lambda^{<\lambda} > \kappa = {\rm cf}(\kappa)$.
2) We deal with ${\rm EC}_{T_{\rm rd}}(\lambda)$, ordered by $\subseteq$, so M,N denotes members of ${\rm EC}_\lambda(T_{\rm rd})$.

Recall $T_{\rm rd}$ is from Definition 0.12(2) and recalling Definition 0.13.

Definition 3.2. 1) If $M \subseteq N$ and (C_1,C_2) is a cut of M let $N^{[(C_1,C_2)]} = N \restriction \{a \in N : a$ realizes the cut (C_1,C_2) of M which means $c_1 \in C_1 \Rightarrow c_1 <_N a$ and $c_2 \in C_2 \Rightarrow a <_N c_2\}$.
2) For a cut (C_1,C_2) of M,A is unbounded in the cut if $A \cap C_1$ is unbounded in C_1 and $A \cap C_2$ is unbounded from below in C_2.
3) Let ${\rm cut}_\kappa(M) = \{(C_1,C_2) : (C_1,C_2)$ a cut of M such that ${\rm cf}(C_1,C_2) = (\kappa,\kappa)\}$ for any $M \in {\rm EC}_\lambda(T_{\rm rd})$.

* * *

Definition 3.3. 1) We say $\bar M$ is a (λ,κ)-sequence <u>when</u>:

- (a) $\bar M = \langle M_i : i \le \kappa\rangle$ is $\subseteq$-increasing continuous sequence of members of ${\rm EC}_\lambda(T_{\rm rd})$
- (b) if $i < \kappa$ and (C_1,C_2) is a cut of M_i then (α) or (β) hold but not both where
  - (α) ${\rm cf}(C_1,C_2) = (\lambda,\lambda)$ and no $a \in M_\kappa \backslash M_i$ realizes (C_1,C_2)
  - (β) $M_\kappa \restriction \{a \in M_i : a$ realizes $(C_1,C_2)\}$ is infinite, moreover has neither first nor last member
- (c) for every $a <_{M_i} b$ the model $(M_\kappa \restriction (a,b)_{M_\kappa})$ is universal (for ${\rm EC}_\lambda(T_{\rm rd})$, usual embedding).

Remark 3.4. Compared to §1 we do not require

- (d) if $i < \kappa$ and (C_1,C_2) is a cut of M_i not realized by any $a \in M_\kappa$ <u>then</u>: for every $j < i$, either M_j is unbounded in (C_1,C_2), or for some $a_1 \in C_1, a_2 \in C_2$ the interval $(a_1,a_2)_{M_i}$ is disjoint to M_j.

Claim 3.5. *1) If $M = \langle M_i : i \le \kappa\rangle$ is a (λ,κ)-sequence <u>then</u> M_κ is (λ,κ)-i.md.-limit (and so for some $\bar M' = \langle M'_i : i \le \kappa\rangle$ the statement $\circledast_{\bar M'}$ from the proof of 1.1 holds and $M_\kappa \cong M'_\kappa$).*
2) If (C_1,C_2) is a cut of $M_i, i < \kappa$ and $(b)(\beta)$ of Definition 3.3 holds, <u>then</u> for some $j \in (i,\kappa), M_j \restriction \{a \in M_j : a$ realizes $(C_1,C_2)\}$ is a universal model of $T_{\rm rd}$.
3) If M is (λ,κ)-i.md.-limit, <u>then</u> there is a (λ,κ)-sequence $\langle M_i : i \le \kappa\rangle$ such that $M_\kappa = M$.
4) If $M \in {\rm EC}_\lambda(T_{\rm rd})$ then:

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(a) *if* $\lambda = \|M\| = \lambda^{<\lambda}$ <u>then</u> *the number of cuts of* M *of cofinality* $\neq (\lambda,\lambda)$ *is at most* λ

(b) *if* $\lambda = \|M\| = \|M\|^\kappa$ <u>then</u> *the number of cuts of* M *of cofinality* (κ,κ) *is at most* λ

(c) *if* $\lambda = \|M\|$ <u>then</u> *the number of cuts of cofinality* (σ_1,σ_2) *where* $\sigma_1 \neq \sigma_2$ *is* $\leq \lambda$.

Proof. 1) As in the proof of 1.1, using parts (2),(3) see 3.7(1).
2) There are $j \in (i,\kappa)$ and $c \in M_j^{[c_1,c_2]}$ and $d \in M_j$ such that $c < d, (c,d)_{M_j} \cap M_i = \emptyset$. Now use 3.7 below.
3) Should be clear.
4) Clauses (a),(b) are easy and clause (c) holds by [She90, VIII,§0]. $\square_{3.5}$

Remark 3.6. A difference between Definition 3.3 and the earlier one is that we do not ask that a dense set of cuts of cofinality (λ,λ) of M_i is realized in $\bigcup\{M_j : j < i\}$.

Observation 3.7. 1) If $M \in \mathrm{EC}_\lambda(T_{\mathrm{rd}})$ is universal, $\lambda = \lambda^\kappa$ and $M = \bigcup\limits_{i<\kappa} I_i$ <u>then</u> for at least one $i < \kappa, M \restriction I_i$ is universal for $\mathrm{EC}_\lambda(T_{\mathrm{rd}})$.
2) If M is (λ,κ)-i.md.-limit or just weakly (λ,κ)-i.md-limit and $a <_M b$ <u>then</u> for some N:

(a) $N \subseteq M \restriction (a,b)_M$
(b) $N \in \mathrm{EC}_\lambda(T_{\mathrm{rd}})$ is universal
(c) every $(C_1,C_2) \in \mathrm{cut}_\kappa(N)$ is realized in M, (but not used).

Proof. 1) Let $N = {}^\kappa M$ ordered lexicographically, so $N \in \mathrm{EC}_\lambda(T_{\mathrm{rd}})$ hence there is an embedding f of N into M. We try to choose $\nu_i \in {}^i|M|$ by induction on $i < \kappa$ such that $j < i \Rightarrow \nu_j \triangleleft \nu_i$ and $\nu_i \triangleleft \eta \in {}^\kappa M \Rightarrow f(\eta) \notin I_i$ and for $i = 0$ or i limit there is no problem to choose ν_i. We cannot succeed as then $f(\bigcup\limits_{i<\kappa} \nu_i) \in M \backslash \bigcup_{j<i} I_j$, contradiction. So for some $i < \kappa, \nu_i$ has been chosen but we cannot choose ν_{i+1}. So for each $a \in M$ there is $\eta_a \in {}^\kappa M$ such that $\nu_i{}^\frown\langle a\rangle \triangleleft \eta_a \wedge f(\eta_a) \in I_i$. So $a \mapsto f(\eta_a)$ is an embedding of M into I_i, so we are done.

Alternatively, let $N \subseteq M$ be a saturated model of T_{ord}. Try to choose $c_i <_N d_i$ by induction on $i < \kappa$ such that $j < i \Rightarrow c_j <_N < c_i <_N d_i <_N d_j$ and $(c_{i+1},d_{i+1})_\mu \cap I_i = \emptyset$. For some i we have (c_i,d_i) well defined but we cannot choose (c_{i+1},d_{i+1}) hence $I_i \cap (c_i,d_i)_N$ is dense in $(c_i,d_i)_N$.
2) Should be clear. $\square_{3.7}$

Claim 3.8. *If* $S \subseteq S^{\lambda^+}_\kappa$ *is stationary and* $M \in \mathrm{EC}_\lambda(T_{\mathrm{rd}})$ *is* (λ,S)*-wk-limit* <u>then</u> M *is* (λ,κ)*-i.md.-limit.*

Proof. Let $\mathbf{F}_1$ witness that M is (λ,S)-wk-limit. We can find $\bar M = \langle M_\alpha : \alpha < \lambda^+\rangle$ so $M_\alpha \in \mathrm{EC}_\lambda(T_{\mathrm{or}})$ is a $\subseteq$-increasing continuous sequence such that $\bar M$ obeys $\mathbf{F}_1$, such that in addition the sequence is as in the proof of 1.1. So by the choice of the set $S' = \{\delta \in S : M_\delta \cong M\}$ is stationary, and by 1.1 the set $S'' = \{\delta : M_\delta$ is (λ,κ)-i.md.-limit$\}$ is $\equiv S^{\lambda^+}_\kappa \mod \mathscr{D}_{\lambda^+}$. Together $S' \cap S'' \neq \emptyset$ hence M is (λ,κ)-i.md.-limit. $\square_{3.8}$

Definition 3.9. 1) We say that $\bar M$ witnesses that M is (λ,κ)-i.md.-limit <u>when</u>:

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(∗) $\bar{M} = \langle M_\alpha : \alpha \le \kappa \rangle$ is such that $\circledast_{\bar{M}}$ from the proof of 1.1 holds and $M = M_\kappa$.

Claim 3.10. *For $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$ then there is no (λ,κ)-superlimit model of T_{ord}.*

Remark 3.11. It is trivial to show that there is no superlimit $M \in \text{EC}_\lambda(T)$, but we deal with (λ,κ)-superlimit.

Proof. Assume there is one, then by §1 it is a (λ,κ)-i.md.-limit model so there is $\bar{M} = \langle M_i : i \le \kappa \rangle$ which witnesses this (i.e. such that $\circledast^\kappa_{\bar{M}}$ from the proof of 1.1) hence each M_{i+1} is saturated. As M_0 is universal for $\text{EC}_\lambda(T_{\text{rd}})$, we can find $c_\eta \in M_0$ for $\eta \in {}^{\kappa \ge}(\lambda+1)$ such that $\eta <_{\text{lex}} \nu \Rightarrow c_\eta <_{M_0} c_\nu$. For $\zeta < \kappa$ let $\Lambda_\zeta = \{\eta \in {}^\kappa(\lambda+1)$: for every $\varepsilon \in [\zeta,\kappa)$ we have $\eta(\varepsilon) = \lambda\}$ and let $\Lambda_\kappa = \Lambda = \bigcup\{\Lambda_\zeta : \zeta < \kappa\}$ so $\langle \Lambda_\zeta : \zeta < \kappa \rangle$ is $\subseteq$-increasing. For $\eta \in \Lambda_\kappa$ let $(C_{1,\eta}, C_{2,\eta})$ be the cut of M_κ with $C_{1,\eta} = \{a \in M_\kappa : a <_{M_\kappa} c_{\eta \restriction i}$ for some $i < \kappa\}$. So $\text{cf}(C_{1,\eta}, C_{2,\eta}) = (\kappa,\kappa)$ recalling clause $(i)_1$ of $\circledast^\kappa_{\bar{M}}$ from the proof of 1.1.

Let $\langle d_j : j < \lambda \rangle$ be a decreasing sequence in M_0 and let

$\circledast_0$ $M'_i = M_i \restriction \{d : d_j <_M d$ for some $j < \lambda\}$ for $i \le \kappa$.

We can choose M_* such that:

$\circledast_1$ (a) $M_\kappa \subseteq M_* \in \text{EC}_{T_{\text{rd}}}(\lambda)$
(b) if $c \in M_* \backslash M_\kappa$ then some $\eta \in \Lambda_\kappa, c$ realizes the cut $(C_{1,\eta}, C_{2,\eta})$
(c) for every $\eta \in \Lambda$ there is an isomorphism f_η from M'_κ onto $M_*^{[(C_{1,\eta},C_{2,\eta})]}$

$\circledast_2$ for $\zeta \le \kappa$ let $M^*_\zeta = M_* \restriction \{c : c \in M_\kappa$ or $c \in M_*$ realizes the cut $(C_{1,\eta}, C_{2,\eta})$ for some $\eta \in \Lambda_\zeta\}$.

So

$\circledast_3$ $\langle M^*_\zeta : \zeta \le \kappa \rangle$ is $\subseteq$-increasing (notice that we didn't demand continuity) and $M^*_\kappa = M_*$.

So it is enough to prove that M^*_ζ is (λ,κ)-i.md.-limit for $\zeta < \kappa$ but not for $\zeta = \kappa$.

$\odot_1$ $M^*_\kappa = M_*$ is not a (λ,κ)-i.md.-limit model.

Why? Assume toward contradiction that there is an isomorphism g from M_κ onto M^*_κ and let $N_i := g(M_i)$ for $i < \kappa$, and let $h : M^*_\kappa \to \kappa$ be $h(c) = \min\{i < \kappa : c \in N_{i+1}\}$. Fix $\eta \in \Lambda_\kappa$ for a while and let $(C'_{1,\eta}, C'_{2,\eta})$ be the cut of $M^*_\kappa = M_*$ with $C'_{1,\eta} := \{c \in M_* : c <_{M_*} c_{\eta \restriction \zeta}$ for some $\zeta < \kappa\}$. Clearly $\langle c_{\eta \restriction \zeta} : \zeta < \kappa \rangle$ is an increasing unbounded sequence of members of $C'_{1,\eta}$ and $\langle f_\eta(d_\alpha) : \alpha < \lambda \rangle$ (f_η is from $\circledast_1(c)$) is a decreasing sequence of members of $C'_{2,\eta}$ unbounded from below in it. So $\text{cf}(C'_{1,\eta}, C'_{2,\eta}) = (\kappa,\lambda)$. This implies that for some $i = i(\eta) < \kappa$, the set $C'_{2,\eta} \cap N_i$ is unbounded from below in $C'_{2,\eta}$. Hence recalling the choice of $\bar{M}$ there is an increasing continuous function $h_\eta : \kappa \to \kappa$ such that: $\cup\{(c_{\eta \restriction h_\eta(i)}, c_{\eta \restriction j})_{M^*_\kappa} : j \in [h_\eta(i),\kappa)\}$ is disjoint to N_i. All this holds for any $\eta \in \Lambda_\kappa$. Now we choose (η_ζ, ξ_ζ) by induction on $\zeta < \kappa$ such that:

$\circledast_4$ (a) $\xi_\zeta < \kappa$ and $\eta_\zeta \in \Lambda_{\xi_\zeta}$
(b) if $\zeta_1 < \zeta_2 < \kappa$ then $(\eta_{\zeta_1} \restriction \xi_{\zeta_1})\hat{\ }\langle 1 \rangle \triangleleft \eta_{\zeta_2}$ and $\xi_{\zeta_1} < \xi_{\zeta_2}$

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(c) the set $\bigcup\{[c_{\eta_\zeta \restriction \xi_\zeta}, c_{\eta_\zeta \restriction \xi})_{M^*_\kappa} : \xi \in (\xi_{\zeta+1}, \kappa)\}$ is disjoint to N_ζ

(d) if ζ is a successor then ξ_ζ is a successor

(e) if $\eta_{\zeta+1} \restriction \xi_{\zeta+1} \triangleleft \nu \in {}^{\kappa \geq}(\lambda+1)$ then $c_\nu \in (C_{\eta_{\zeta+1} \restriction (\xi_{\zeta+1}-1)\hat{\ }<1>}, C_{\eta_{\zeta+1} \restriction (\xi_{\zeta+1}-1)\hat{\ }<2>})_{M_\kappa}$ is disjoint to N_ζ.

There is no problem to carry the induction:

Case 1: $\zeta = 0$.
Choose $\xi_\zeta = 0, \eta_\zeta \in \Lambda_0$.

Case 2: $\zeta = \zeta_1 + 1$.
Choose $\xi_\zeta = h_{\eta_{\zeta_1}}(\xi_{\zeta_1}) + 6$.
Choose η_ζ such that

$$\eta_\zeta \restriction (h_{\eta_{\zeta_1}(\xi_{\zeta_1})+5)} \hat{\ } \langle 1 \rangle \trianglelefteq \eta_\zeta \in \Lambda_{\xi_\zeta}.$$

Case 3: ζ limit.
$\xi_\zeta = \cup\{\xi_\alpha : \alpha < \zeta\}$.
Choose $\eta_\zeta \in \Lambda_{\xi_\zeta+1}$ such that $\alpha < \zeta \Rightarrow \eta_\alpha \restriction \xi_\alpha \trianglelefteq \eta_\zeta$.
Let $\eta = \bigcup\{\eta_\zeta \restriction \xi_\zeta : \zeta < \kappa\}$. So $\eta \in {}^\kappa(\lambda+1)$ and $c_\eta \notin N_\zeta$ for every $\zeta < \kappa$ but $\cup\{N_\zeta : \zeta < \kappa\} = M^*_\kappa = M^*$, contradiction, so $\odot_1$ holds indeed.

$\boxdot$ M^*_ζ is a (λ,κ)-.i.md.-limit model for $\zeta < \kappa$.

Why? We define $M_{\zeta,i} \subseteq M^*_\zeta$ for $i < \kappa$ by: $c \in M_{\zeta,i}$ iff one of the following occurs:

(a) $c \in M_i$ but for no $\eta \in \Lambda_\zeta$ do we have $c \in B_\eta := \bigcup\{[c_{\eta\restriction(\zeta+i)}, c_{\eta\restriction\varepsilon})_{M_i} : \varepsilon \in (\zeta+i,\kappa)\}$

(b) $c \in f_\eta(M'_i)$ for some $\eta \in \Lambda_\zeta$.

Let

- $J_{\zeta,\eta} = \bigcup\{(C_{\eta\restriction\zeta}, C_{\eta\restriction\varepsilon})_{M^*_\kappa} : \varepsilon \in (\zeta,\kappa)\}$
- $J_{\zeta,\eta,\varepsilon} = (C_{\eta\restriction\zeta}, C_{\eta\restriction\varepsilon})$
- $\langle J_{\zeta,\eta} : \eta \in \Lambda_\zeta\rangle$ are pairwise disjoint
- $J_{\zeta,\eta,\varepsilon}$ is an initial segment of $J_{\zeta,\eta}$
- $J_{\zeta,\eta} = \bigcup\{J_{\zeta,\eta,\varepsilon} : \varepsilon \in (\zeta,\kappa)\}$.

We will make $M_{\zeta,i} \cap J_{\zeta,\eta}$ bounded in $J_{\zeta,\eta}$ for each $i < \kappa$.

Now $\langle M_{\zeta,i} : i < \kappa\rangle$ is a (λ,κ)-sequence, see Definition 3.3 hence by 3.5(1) the model M^*_ζ is a (λ,κ)-i.md.-limit model. $\square_{3.10}$

Claim 3.12. *If $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$ then T_{rd} has a (λ,κ)-limit$^+$model, i.e.: if $\langle M_\alpha : \alpha < \lambda^+\rangle$ is $\subseteq$-increasing continuous sequence of models $\in \mathrm{EC}_\lambda(T_{\mathrm{rd}})$ and $M_{\alpha+1}$ is (λ,κ)-i.md.-limit model for every $\alpha < \lambda^+$ then: for a club of $\delta < \lambda^+$ if $\mathrm{cf}(\delta) = \kappa$ then M_δ is a (λ,κ)-i.md.-limit.*

Proof. Let $\langle M_\alpha : \alpha < \lambda^+\rangle$ be as in the theorem and $M = \bigcup\limits_{\alpha<\lambda^+} M_\alpha$, without loss of generality $\|M\| = \lambda^+$. As $\lambda = \lambda^{<\lambda}$ by 3.5(4) we can find a club E of λ^+ such that:

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⊛ if $\alpha < \delta \in E$ and (C_1, C_2) is a cut of M_α of cofinality $\neq (\lambda, \lambda)$ and some $a \in M$ realizes the cut <u>then</u> some $a \in M_\delta$ realizes the cut.

Let $\langle \alpha_\varepsilon : \varepsilon \leq \kappa \rangle$ be an increasing continuous sequence of ordinals from E and we shall prove that M_{α_κ} is (λ, κ)-i.md.-limit; this suffices (really just $\alpha_\kappa \in E$ suffice).

Now $M_{\alpha_\kappa + 1}$ is (λ, κ)-i.md.-limit hence there is an $\subseteq$-increasing continuous sequence $\langle M_{\alpha_\kappa+1,i} : i < \kappa \rangle$ witnessing $M_{\alpha_\kappa+1}$ is (λ, κ)-i.md.-limit model, i.e. its union is $M_{\alpha_\kappa+1}$ and it is a (λ, κ)-sequence. Now $M_{\alpha_\kappa+1,i} \cap M_{\alpha_\kappa} = \bigcup \{ M_{\alpha_{\kappa+1,i}} \cap M_{\alpha_\zeta} : \zeta < \kappa \}$ but $\kappa < \lambda = \mathrm{cf}(\lambda)$ hence without loss of generality $M_{\alpha_\kappa,0} \cap M_{\alpha_0}$ has cardinality λ hence $N_i := M_{\alpha_\kappa+1,i} \cap M_{\alpha_i} \in \mathrm{EC}_\lambda(T_{\mathrm{rd}})$.

Clearly

$(*)_1$ $\langle N_i : i < \kappa \rangle$ is a $\subseteq$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T_{\mathrm{rd}})$ with union M_{α_κ}.

So it is enough to show that $\langle N_i : i < \kappa \rangle$ is a (λ, κ)-sequence by 3.5(1). By $(*)_1$, clause (a) from Definition 3.3 holds.

$⊛_2$ $\langle N_i : i < \kappa \rangle$ satisfies clause (b) of 3.3.

[Why? Let $i < \kappa$ and (C_1, C_2) be a cut of N_i. First, we assume (C_1, C_2) is of cofinality $\neq (\lambda, \lambda)$. As $C_1, C_2 \subseteq N_i \subseteq M_{\alpha_\kappa+1,i}$ by the properties of $\langle M_{\alpha_\kappa+1,i} : i < \kappa \rangle$ there is $a \in M_{\alpha_\kappa+1}$ such that $C_1 < a < C_2$.

If for some $b \in M_{\alpha_i}, C_1 < b < C_2$ then without loss of generality $a \in M_{\alpha_\kappa}$ and we are done. If not, a induces on M_{α_i} a cut $(C_1', C_2'), C_1 \subseteq C_1', C_2 \subseteq C_2'$, $\mathrm{cf}(C_1', C_2') = \mathrm{cf}(C_1, C_2) \neq (\lambda, \lambda)$. As $\alpha_i < \alpha_\kappa \in E$, by ⊛ there is $a \in M_{\alpha_i+1} \subseteq M_{\alpha_\kappa}$ such that $C_1 < a < C_2$. So clause (b) of Definition 3.3 really holds.

Second, we assume that (C_1, C_1) is of cofinality $(\lambda, \lambda), \kappa < \lambda = \mathrm{cf}(\lambda)$ so without loss of generality clause $(b)(\beta)$ of 3.3 holds so some $a \in M_{\alpha_\kappa}$ realizes (C_1, C_2) so for some $j \in (i, \kappa), a \in M_{\alpha_j}$ hence the cut $(\{b \in M_{\alpha_j} : b < a\}, \{c \in M_{\alpha_j} : a \leq c\})$ of M_{α_j} has cofinality $\neq (\lambda, \lambda)$ so is realized by infinitely many $a' \in M_{\alpha_j+1} \subseteq M_{\alpha_\kappa}$, hence also (C_1, C_2) is, so clause $(b)(\beta)$ of 3.3 holds. Together (C_1, C_1) satisfies (α) of (β) of 3.3 as promised.]

$⊛_3$ if $a <_{M_{\alpha_\kappa}} b$ then $M_{\alpha_\kappa} \restriction (a, b)$ is universal (for $(\mathrm{EC}_{T_{\mathrm{or}}}(\lambda), \subseteq)$).

[Why? As $\langle \alpha_\varepsilon : \varepsilon \leq \kappa \rangle$ is increasing continuous and $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is increasing continuous, clearly for some $i < \kappa$ we have $a, b \in M_{\alpha_i}$ hence $M_{\alpha_i+1} \restriction (a, b)$ is λ-universal but $M_{\alpha_i+1} \subseteq M_{\alpha_k}$ so $M_{\alpha_\kappa} \restriction (a, b)$ is universal so we are done.] $\square_{3.12}$

References

[EK65] Ryszard Engelking and Monika Karłowicz, *Some theorems of set theory and their topological consequences*, Fundamenta Math. **57** (1965), 275–285.

[Hod93] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[Jón56] Bjarni Jónsson, *Universal relational systems*, Mathematica Scandinavica **4** (1956), 193–208.

[Jón60] ______, *Homogeneous universal relational systems*, Mathematica Scandinavica **8** (1960), 137–142.

[KLS16] Itay Kaplan, Noa Lavi, and Saharon Shelah, *The generic pair conjecture for dependent finite diagrams*, Israel J. Math. **212** (2016), no. 1, 259–287, arXiv: 1410.2516. MR 3504327

[KS92] Menachem Kojman and Saharon Shelah, *Nonexistence of universal orders in many cardinals*, J. Symbolic Logic **57** (1992), no. 3, 875–891, arXiv: math/9209201. MR 1187454

Paper Sh:877, version 2015-06-02_12. See https://shelah.logic.at/papers/877/ for possible updates.

[Mir05] Džamonja Mirna, *Club guessing and the universal models*, Notre Dame J. Formal Logic **46** (2005), 283–300.
[MV62] M. D. Morley and R. L. Vaught, *Homogeneous and universal models*, Mathematica Scandinavica **11** (1962), 37–57.
[S$^+$] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F1124.
[Shea] Saharon Shelah, *Abstract elementary classes near $\aleph_1$*, arXiv: 0705.4137 Ch. I of [Sh:h].
[Sheb] ______, *Dependent dreams: recounting types*, arXiv: 1202.5795.
[She80] ______, *Independence results*, J. Symbolic Logic **45** (1980), no. 3, 563–573. MR 583374
[She87] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 419–497. MR 1033034
[She90] ______, *Classification theory and the number of nonisomorphic models*, second ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990. MR 1083551
[She09a] ______, *Classification theory for abstract elementary classes*, Studies in Logic (London), vol. 18, College Publications, London, 2009. MR 2643267
[She09b] ______, *Dependent first order theories, continued*, Israel J. Math. **173** (2009), 1–60, arXiv: math/0406440. MR 2570659
[She11] ______, *No limit model in inaccessibles*, Models, logics, and higher-dimensional categories, CRM Proc. Lecture Notes, vol. 53, Amer. Math. Soc., Providence, RI, 2011, arXiv: 0705.4131, pp. 277–290. MR 2867976
[She12] ______, *When a first order T has limit models*, Colloq. Math. **126** (2012), no. 2, 187–204, arXiv: math/0603651. MR 2924249
[She15] ______, *Dependent theories and the generic pair conjecture*, Commun. Contemp. Math. **17** (2015), no. 1, 1550004, 64, arXiv: math/0702292. MR 3291978

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