Paper Sh:887, version 2007-09-17_11. See https://shelah.logic.at/papers/887/ for possible updates.

GROUPWISE DENSITY CANNOT BE MUCH BIGGER THAN THE UNBOUNDED NUMBER

SAHARON SHELAH

ABSTRACT. We prove that $\mathfrak{g}$ (the groupwise density number) is smaller or equal to $\mathfrak{b}^+$, the successor of the minimal cardinality of an unbounded subset of ${}^\omega\omega$. This is true even for the version of $\mathfrak{g}$ for groupwise dense ideals.

1. INTRODUCTION

In the present note we are interested in two cardinal characteristics of the continuum, the unbounded number $\mathfrak{b}$ and the groupwise density number $\mathfrak{g}$. The former cardinal belongs to the oldest and most studied cardinal invariants of the continuum (see, e.g., van Douwen [vD84] and Bartoszyński and Judah [BJ95]) and it is defined as follows.

Definition 1.1. (a) The partial order $\le_{J^{\rm bd}_\omega}$ on ${}^\omega\omega$ is defined by

$$f \le_{J^{\rm bd}_\omega} g \text{ if and only if } (\exists N<\omega)(\forall n>N)(f(n)\le g(n)).$$

(b) The *unbounded number* $\mathfrak{b}$ is defined by

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F}\subseteq {}^\omega\omega \text{ has no } \le_{J^{\rm bd}_\omega}\text{–upper bound in } {}^\omega\omega\}.$$

The groupwise density number $\mathfrak{g}$, introduced in Blass and Laflamme [BL89], is perhaps less popular but it has gained substantial importance in the realm of cardinal invariants. For instance, it has been studied in connection with the cofinality $\mathrm{cf}(\mathrm{Sym}(\omega))$ of the symmetric group on the set ω of all integers, see Thomas [Tho98] or Brendle and Losada [BL03]. The cardinal $\mathfrak{g}$ is defined as follows.

Definition 1.2. (a) We say that a family $\mathcal{A}\subseteq[\omega]^{\aleph_0}$ is *groupwise dense* whenever:

- $B\subseteq A\in\mathcal{A}$, $B\in[\omega]^{\aleph_0}$ implies $B\in\mathcal{A}$, and
- for every increasing sequence $\langle m_i : i<\omega\rangle\in{}^\omega\omega$ there is an infinite set $\mathcal{U}\subseteq\omega$ such that $\bigcup\{[m_i,m_{i+1}) : i\in\mathcal{U}\}\in\mathcal{A}$.

(b) The *groupwise density number* $\mathfrak{g}$ is defined as the minimal cardinal θ for which there is a sequence $\langle\mathcal{A}_\alpha : \alpha<\theta\rangle$ of groupwise dense subsets of $[\omega]^{\aleph_0}$ such that

$$\big(\forall B\in[\omega]^{\aleph_0}\big)\big(\exists\alpha<\theta\big)\big(\forall A\in\mathcal{A}_\alpha\big)\big(B\not\subseteq^* A\big).$$

(Recall that for infinite sets A and B, $A\subseteq^* B$ means $A\setminus B$ is finite.)

The unbounded number $\mathfrak{b}$ and groupwise density number $\mathfrak{g}$ can be in either order, see Blass [Bla89] and more Mildenberger and Shelah [MS02], [MS07], the

---

Date: August 2007.

1991 *Mathematics Subject Classification.* Primary 03E17; Secondary: 03E05, 03E20.

The author acknowledges support from the United States-Israel Binational Science Foundation (Grant no. 2002323). Publication 887.

latter article gives a bound on $\mathfrak{g}$. However, as we show in Theorem 2.2, $\mathfrak{g}$ cannot be bigger than $\mathfrak{b}^+$.

We would like to thank Shimoni Garti and the anonymous referee for corrections.

Notation: Our notation is rather standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [BJ95]). We will keep the following rules concerning the use of symbols.

1. $A, B, \mathcal{U}$ (with possible sub- and superscripts) denote subsets of ω, infinite if not said otherwise.
2. m, n, ℓ, k, i, j are natural numbers.
3. $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \zeta$ are ordinals, θ is a cardinal.

2. The result

Lemma 2.1. *For some cardinal $\theta \le \mathfrak{b}$ there is a sequence $\langle B_{\zeta,t} : \zeta < \theta,\ t \in I_\zeta\rangle$ such that:*

- (a) $B_{\zeta,t} \in [\omega]^{\aleph_0}$
- (b) *if $\zeta < \theta$ and $s \neq t$ are from I_ζ, then $B_{\zeta,s} \cap B_{\zeta,t}$ is finite (so $|I_\zeta| \le 2^{\aleph_0}$),*
- (c) *for every $B \in [\omega]^{\aleph_0}$ the set*
$$\big\{(\zeta, t) : \zeta < \theta \ \&\ t \in I_\zeta \ \&\ B_{\zeta,t} \cap B \text{ is infinite } \big\}$$
is of cardinality $2^{\aleph_0}$.

Proof. This is a weak version of the celebrated base-tree theorem of Bohuslav Balcar and Petr Simon with $\theta = \mathfrak{h}$ which is known to be $\le \mathfrak{b}$, see Balcar and Simon [BS89, 3.4, pg.350]. However, for the sake of completeness of our exposition, let us present a proof.

Let $\langle f_\zeta : \zeta < \mathfrak{b}\rangle$ be a $\le_{J^{\rm bd}_\omega}$–increasing sequence of members of ${}^\omega\omega$ with no $\le_{J^{\rm bd}_\omega}$–upper bound in ${}^\omega\omega$. Moreover we demand that each f_ζ is increasing (clearly, this does not change $\mathfrak{b}$). By induction on $\zeta < \mathfrak{b}$ choose sets $\mathcal{T}_\zeta$ and systems $\langle B_{\zeta,\eta} : \eta \in \mathcal{T}_{\zeta+1}\rangle$ such that:

- (i) $\mathcal{T}_\zeta \subseteq {}^\zeta(2^{\aleph_0})$ and if $\eta \in \mathcal{T}_{\zeta+1}$ then $B_{\zeta,\eta} \in [\omega]^{\aleph_0}$,
- (ii) if $\eta \in \mathcal{T}_\zeta$ and $\varepsilon < \zeta$, then $\eta \restriction \varepsilon \in \mathcal{T}_\varepsilon$,
- (iii) if ζ is a limit ordinal, then

$$\mathcal{T}_\zeta = \big\{\eta \in {}^\zeta(2^{\aleph_0}) : \big(\forall \varepsilon < \zeta\big)\big(\eta \restriction \varepsilon \in \mathcal{T}_\varepsilon\big) \text{ and } \big(\exists A \in [\omega]^{\aleph_0}\big)\big(\forall \varepsilon < \zeta\big)\big(A \subseteq^* B_{\varepsilon,\eta\restriction(\varepsilon+1)}\big)\big\},$$

- (iv) if $\varepsilon < \zeta$ and $\eta \in \mathcal{T}_{\zeta+1}$, then $B_{\zeta,\eta} \subseteq^* B_{\varepsilon,\eta\restriction(\varepsilon+1)}$,
- (v) for $\eta \in \mathcal{T}_{\zeta+1}$ and $m_1 < m_2$ from $B_{\zeta,\eta}$ we have $f_\zeta(m_1) < m_2$,
- (vi) if $\eta \in \mathcal{T}_\varepsilon$, then the set $\{B_{\varepsilon,\nu} : \eta \vartriangleleft \nu \in \mathcal{T}_{\varepsilon+1}\}$ is an infinite maximal subfamily of
$$\big\{A \in [\omega]^{\aleph_0} : \big(\forall \xi < \varepsilon\big)\big(A \subseteq^* B_{\xi,\eta\restriction(\xi+1)}\big)\big\}$$
consisting of pairwise almost disjoint sets.

It should be clear that the choice is possible. Note that for some limit $\zeta < \mathfrak{b}$ we may have $\mathcal{T}_\zeta = \emptyset$ (and then also $\mathcal{T}_\xi = \emptyset$ for $\xi > \zeta$). Also, if we define $\mathcal{T}_{\mathfrak{b}}$ as in (iii), then it will be empty (remember clause (v) and the choice of $\langle f_\zeta : \zeta < \mathfrak{b}\rangle$).

The lemma will readily follow from the following fact.

- ($\circledast$) For every $A \in [\omega]^{\aleph_0}$ there is $\xi < \mathfrak{b}$ such that
$$\big|\big\{\eta \in \mathcal{T}_{\xi+1} : B_{\xi,\eta} \cap A \text{ is infinite } \big\}\big| = 2^{\aleph_0}.$$

To show ($\circledast$) let $A \in [\omega]^{\aleph_0}$ and define

$$S = \bigcup_{\zeta<\mathfrak{b}} \{\eta \in \mathcal{T}_\zeta : (\forall \varepsilon < \zeta)(A \cap B_{\varepsilon,\eta\restriction(\varepsilon+1)} \text{ is infinite })\}.$$

Clearly S is closed under taking the initial segments and $\langle\rangle \in S$. By the "maximal" in clause (vi), we have that

($\circledast$)$_1$ if $\eta \in S \cap \mathcal{T}_\zeta$ where $\zeta < \mathfrak{b}$ is non-limit or $\mathrm{cf}(\zeta) = \aleph_0$, then $(\exists \nu)(\eta \lhd \nu \in \mathcal{T}_{\zeta+1} \cap S)$.

Now,

($\circledast$)$_2$ if $\eta \in S$ and $\ell g(\eta)$ is non-limit or $\mathrm{cf}(\ell g(\eta)) = \aleph_0$, then there are $\lhd$–incomparable $\nu_0, \nu_1 \in S$ extending η, i.e., $\eta \lhd \nu_0$ and $\eta \lhd \nu_1$.

[Why? As otherwise $S_\eta = \{\nu \in S : \eta \trianglelefteq \nu\}$ is linearly ordered by $\lhd$, so let $\rho = \bigcup S_\eta$. It follows from ($\circledast$)$_1$ that $\ell g(\rho) > \ell g(\eta)$ is a limit ordinal (of uncountable cofinality). Moreover, by (iv)+(vi), we have that

$$\ell g(\eta) \le \varepsilon < \ell g(\rho) \quad \Rightarrow \quad A \cap B_{\ell g(\eta),\rho\restriction(\ell g(\eta)+1)} =^* A \cap B_{\varepsilon,\rho\restriction(\varepsilon+1)}.$$

Hence, by (iii)+(ii), $\rho \in \mathcal{T}_{\ell g(\rho)}$ so necessarily $\ell g(\rho) < \mathfrak{b}$. Using (vi) again we may conclude that there is $\rho' \in S$ properly extending ρ, getting a contradiction.]

Consequently, we may find a system $\langle \eta_\rho : \rho \in {}^{\omega>}2\rangle \subseteq S$ such that for every $\rho \in {}^{\omega>}2$:

- $k < \ell g(\rho) \quad \Rightarrow \quad \eta_{\rho\restriction k} \lhd \eta_\rho$, and
- $\eta_{\rho\frown\langle 0\rangle}, \eta_{\rho\frown\langle 1\rangle}$ are $\lhd$–incomparable.

For $\rho \in {}^{\omega>}2$ let $\zeta(\rho) = \sup\{\ell g(\eta_\nu) : \rho \trianglelefteq \nu \in {}^{\omega>}2\}$. Pick ρ such that $\zeta(\rho)$ is the smallest possible (note that $\mathrm{cf}(\zeta(\rho)) = \aleph_0$). Now it is possible to choose a perfect subtree T^* of ${}^{\omega>}2$ such that

$$\nu \in \lim(T^*) \quad \Rightarrow \quad \sup\{\ell g(\eta_{\nu\restriction n}) : n < \omega\} = \zeta(\rho).$$

We finish by noting that for every $\nu \in \lim(T^*)$ we have that $\bigcup\{\eta_{\nu\restriction n} : n < \omega\} \in \mathcal{T}_{\zeta(\rho)} \cap S$ and there is $\eta^* \in \mathcal{T}_{\zeta(\rho)+1} \cap S$ extending $\bigcup\{\eta_{\nu\restriction n} : n < \omega\}$. $\square$

Theorem 2.2. $\mathfrak{g} \le \mathfrak{b}^+$.

Proof. Assume towards contradiction that $\mathfrak{g} > \mathfrak{b}^+$.

Let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle \subseteq {}^\omega\omega$ be an $\le_{J^{\rm bd}_\omega}$–increasing sequence with no $\le_{J^{\rm bd}_\omega}$–upper bound. We also demand that all functions f_α are increasing and $f_\alpha(n) > n$ for $n < \omega$. Fix a list $\langle \bar m_\xi : \xi < 2^{\aleph_0}\rangle$ of all sequences $\bar m = \langle m_i : i < \omega\rangle$ such that $0 = m_0$ and $m_i + 1 < m_{i+1}$.

For $\alpha < \mathfrak{b}$ we define:

$(*)_1$ $n_{\alpha,0} = 0$, $n_{\alpha,i+1} = f_\alpha(n_{\alpha,i})$ (for $i < \omega$) and $\bar n_\alpha = \langle n_{\alpha,i} : i < \omega\rangle$;
$(*)_2$ $\bar n^0_\alpha = \langle 0, n_{\alpha,2}, n_{\alpha,4}, \ldots\rangle = \langle n^0_{\alpha,i} : i < \omega\rangle$ and $\bar n^1_\alpha = \langle 0, n_{\alpha,3}, n_{\alpha,5}, n_{\alpha,7}, \ldots\rangle = \langle n^1_{\alpha,i} : i < \omega\rangle$.

Observe that

$(*)_3$ if $\bar m \in {}^\omega\omega$ is increasing, then for every large enough $\alpha < \mathfrak{b}$ we have:
  (α) $(\exists^\infty i < \omega)(m_{i+1} < f_\alpha(m_i))$, and hence
  (β) for at least one $\ell \in \{0, 1\}$ we have

$$\big(\exists^\infty i < \omega\big)\big(\exists j < \omega\big)\big([m_i, m_{i+1}) \subseteq [n^\ell_{\alpha,j}, n^\ell_{\alpha,j+1})\big).$$

Now, for $\xi < 2^{\aleph_0}$ we put:

$(*)_4$ $\gamma(\xi)=\min\{\alpha<\mathfrak{b}:(\exists^\infty i<\omega)(f_\alpha(m_{\xi,i})>m_{\xi,i+1})\}$;
$(*)_5$ $\ell(\xi)=\min\{\ell\leq 1:(\exists^\infty i{<}\omega)(\exists j{<}\omega)([m_{\xi,i},m_{\xi,i+1})\subseteq[n^\ell_{\gamma(\xi),j},n^\ell_{\gamma(\xi),j+1}))\}$;
$(*)_6$ $\mathcal{U}^1_\xi=\{i<\omega:(\exists j<\omega)([m_{\xi,i},m_{\xi,i+1})\subseteq[n^{\ell(\xi)}_{\gamma(\xi),j},n^{\ell(\xi)}_{\gamma(\xi),j+1}))\}$.

Note that $\gamma(\xi)$ is well defined by (α) of $(*)_3$, and so also $\ell(\xi)$ is well defined (by (β) of $(*)_3$). Plainly, $\mathcal{U}^1_\xi$ is an infinite subset of ω. Now, for each $\xi<2^{\aleph_0}$, we may choose $\mathcal{U}^2_\xi$ so that

$(*)_7$ $\mathcal{U}^2_\xi\subseteq\mathcal{U}^1_\xi$ is infinite and for any $i_1<i_2$ from $\mathcal{U}^2_\xi$ we have

$$(\exists j<\omega)(m_{\xi,i_1+1}<n^{\ell(\xi)}_{\gamma(\xi),j}\ \&\ n^{\ell(\xi)}_{\gamma(\xi),j+1}<m_{\xi,i_2}).$$

Let a function $g_\xi:\mathcal{U}^2_\xi\longrightarrow\omega$ be such that

$(*)_8$ $i\in\mathcal{U}^2_\xi\ \&\ g_\xi(i)=j\quad\Rightarrow\quad[m_{\xi,i},m_{\xi,i+1})\subseteq[n^{\ell(\xi)}_{\gamma(\xi),j},n^{\ell(\xi)}_{\gamma(\xi),j+1})$.

Clearly, g_ξ is well defined and one-to-one. (This is very important, since it makes sure that the set $g_\xi[\mathcal{U}^2_\xi]$ is infinite.)

Fix a sequence $\bar{B}=\langle B_{\zeta,t}:\zeta<\theta,\ t\in I_\zeta\rangle$ given by Lemma 2.1 (so $\theta\leq\mathfrak{b}$ and $\bar{B}$ satisfies the demands in (a)–(c) of 2.1). By clause 2.1(c), for every $\xi<2^{\aleph_0}$, the set

$$\big\{(\zeta,t):\zeta<\theta\text{ and }t\in I_\zeta\text{ and }B_{\zeta,t}\cap g_\xi[\mathcal{U}^2_\xi]\text{ is infinite }\big\}$$

has cardinality continuum.

Now, for each $\beta<\mathfrak{b}^+$ and $\xi<2^{\aleph_0}$ we choose a pair $(\zeta_{\beta,\xi},t_{\beta,\xi})$ such that

$(*)_9$ $\zeta_{\beta,\xi}<\theta$ and $t_{\beta,\xi}\in I_{\zeta_{\beta,\xi}}$,
$(*)_{10}$ $B_{\zeta_{\beta,\xi},t_{\beta,\xi}}\cap g_\xi[\mathcal{U}^2_\xi]$ is infinite, and
$(*)_{11}$ $t_{\beta,\xi}\notin\{t_{\alpha,\varepsilon}:\varepsilon<\xi\text{ or }\varepsilon=\xi\ \&\ \alpha<\beta\}$.

To carry out the choice we proceed by induction *first* on $\xi<2^{\aleph_0}$, then on $\beta<\mathfrak{b}^+$. As there are $2^{\aleph_0}$ pairs (ζ,t) satisfying clauses $(*)_9+(*)_{10}$ whereas clause $(*)_{11}$ excludes $\leq(\mathfrak{b}^++|\xi|)\times\theta<2^{\aleph_0}$ pairs (recalling that towards contradiction we are assuming $\mathfrak{b}^+<\mathfrak{g}\leq 2^{\aleph_0}$), there is such a pair at each stage $(\beta,\xi)\in\mathfrak{b}^+\times 2^{\aleph_0}$.

Lastly, for $\beta<\mathfrak{b}^+$ and $\xi<2^{\aleph_0}$ we let

$(*)_{12}$ $\mathcal{U}_{\beta,\xi}=g^{-1}_\xi[B_{\zeta_{\beta,\xi},t_{\beta,\xi}}]\cap\mathcal{U}^2_\xi$

(it is an infinite subset of ω) and we put

$(*)_{13}$ $A^+_{\beta,\xi}=\bigcup\{[m_{\xi,i},m_{\xi,i+1}):i\in\mathcal{U}_{\beta,\xi}\}$, and
$(*)_{14}$ $\mathcal{A}_\beta=\{A\in[\omega]^{\aleph_0}$: for some $\xi<2^{\aleph_0}$ we have $A\subseteq A^+_{\beta,\xi}\}$.

By the choice of $\langle\bar{m}_\xi:\xi<2^{\aleph_0}\rangle$, $A^+_{\beta,\xi}$ and $\mathcal{A}_\beta$ one easily verifies that for each $\beta<\mathfrak{b}^+$:

$(*)_{15}$ $\mathcal{A}_\beta$ is a groupwise dense subset of $[\omega]^{\aleph_0}$.

Since we are assuming towards contradiction that $\mathfrak{g}>\mathfrak{b}^+$, there is an infinite $B\subseteq\omega$ such that

$$(\forall\beta<\mathfrak{b}^+)(\exists A\in\mathcal{A}_\beta)(B\subseteq^* A).$$

Hence for every $\beta<\mathfrak{b}^+$ we may choose $\xi(\beta)<2^{\aleph_0}$ such that $B\subseteq^* A^+_{\beta,\xi(\beta)}$. Now, since $\gamma(\xi(\beta))<\mathfrak{b}$ and $\zeta_{\beta,\xi(\beta)}<\theta\leq\mathfrak{b}$ and $\ell(\xi(\beta))\in\{0,1\}$, hence for some triple $(\gamma^*,\zeta^*,\ell^*)$ we have that

$(\odot)_1$ the set

$$W =: \big\{\beta < \mathfrak{b}^+ : \big(\gamma(\xi(\beta)), \zeta_{\beta,\xi(\beta)}, \ell(\xi(\beta))\big) = \big(\gamma^*, \zeta^*, \ell^*\big)\big\}$$

is unbounded in $\mathfrak{b}^+$.

Note that if $\beta \in W$ then

$(\odot)_2$ $B \subseteq^* A^+_{\beta,\xi(\beta)} = \bigcup\big\{[m_{\xi(\beta),i}, m_{\xi(\beta),i+1}) : i \in \mathcal{U}_{\beta,\xi(\beta)}\big\} \subseteq$
$\bigcup\big\{[n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j}, n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j+1}) : j = g_{\xi(\beta)}(i)$ for some $i \in \mathcal{U}_{\beta,\xi(\beta)}\big\} \subseteq$
$\bigcup\big\{[n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j}, n^{\ell(\xi(\beta))}_{\gamma(\xi(\beta)),j+1}) : j \in B_{\zeta_{\beta,\xi(\beta)},t_{\beta,\xi(\beta)}}\big\}$.

[Why? By the choice of $(\beta, \xi(\beta))$, by $(*)_{13}$, and by $(*)_8$ as $\mathrm{Dom}(g_{\xi(\beta)}) \subseteq \mathcal{U}_{\beta,\xi(\beta)} \subseteq \mathcal{U}^2_{\beta,\xi(\beta)}$; also remember $(*)_{12}$.]
Also, for $\beta \in W$ we have $\ell(\xi(\beta)) = \ell^*$, $\gamma(\xi(\beta)) = \gamma^*$ and $\zeta(\beta, \xi(\beta)) = \zeta^*$, so it follows from $(\odot)_2$ that

$(\odot)_3$ $B \subseteq^* \bigcup\big\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\beta,\xi(\beta)}}\big\}$ for every $\beta \in W$.

Consequently, if $\beta \neq \alpha$ are from W, then the sets

$$\bigcup\big\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\beta,\xi(\beta)}}\big\} \text{ and}$$
$$\bigcup\big\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\alpha,\xi(\alpha)}}\big\}$$

are *not* almost disjoint. Hence, as $\langle n^{\ell^*}_{\gamma^*,j} : j < \omega\rangle$ is increasing, necessarily the sets $B_{\zeta^*,t_{\beta,\xi(\beta)}}$ and $B_{\zeta^*,t_{\alpha,\xi(\alpha)}}$ are not almost disjoint. So applying 2.1(b) we conclude that $t_{\beta,\xi(\beta)} = t_{\alpha,\xi(\alpha)}$. But this contradicts $\beta \neq \alpha$ by $(*)_{11}$, and we are done. $\square$

Definition 2.3. We define a cardinal characteristic $\mathfrak{g}_{\mathfrak{f}}$ as the minimal cardinal θ for which there is a sequence $\langle \mathcal{I}_\alpha : \alpha < \theta\rangle$ of groupwise dense *ideals* of $\mathcal{P}(\omega)$ (i.e., $\mathcal{I}_\alpha \subseteq [\omega]^{\aleph_0}$ is groupwise dense and $\mathcal{I}_\alpha \cup [\omega]^{<\aleph_0}$ is an ideal of subsets of ω) such that

$$\big(\forall B \in [\omega]^{\aleph_0}\big)\big(\exists \alpha < \theta\big)\big(\forall A \in \mathcal{A}_\alpha\big)\big(B \not\subseteq^* A\big).$$

Observation 2.4. $2^{\aleph_0} \geq \mathfrak{g}_{\mathfrak{f}} \geq \mathfrak{g}$.

Theorem 2.5. $\mathfrak{g}_{\mathfrak{f}} \leq \mathfrak{b}^+$.

Proof. We repeat the proof of Theorem 2.2. However, for $\beta < \mathfrak{b}^+$ the family $\mathcal{A}_\beta \subseteq [\omega]^{\leq\aleph_0}$ does not have to be an ideal. So let $\mathcal{I}_\beta$ be an ideal on $\mathcal{P}(\omega)$ generated by $\mathcal{A}_\beta$ (so also $\mathcal{I}_\beta$ is the ideal generated by $\{A^+_{\beta,\xi} : \xi < 2^{\aleph_0}\} \cup [\omega]^{<\aleph_0}$). Lastly, let $\mathcal{I}'_\beta = \mathcal{I}_\beta \setminus [\omega]^{<\aleph_0}$.

Assume towards contradiction that $B \in [\omega]^{\aleph_0}$ is such that $(\forall \alpha < \mathfrak{b}^+)(\exists A \in \mathcal{I}_\alpha)(B \subseteq^* A)$. So for each $\beta < \mathfrak{b}^+$ we can find $k_\beta < \omega$ and $\xi(\beta, 0) < \xi(\beta, 1) < \ldots < \xi(\beta, k_\beta) < 2^{\aleph_0}$ such that $B \subseteq^* \bigcup\{A^+_{\beta,\xi(\beta,k)} : k \leq k_\beta\}$. Let D be a non-principal ultrafilter on ω to which B belongs. For each $\beta < \mathfrak{b}^+$ there is $k(\beta) \leq k_\beta$ such that $A^+_{\beta,\xi(\beta,k(\beta))} \in D$. As in the proof there for some $(\gamma^*, \zeta^*, \ell^*, k^*, k(*))$ the following set is unbounded in $\mathfrak{b}^+$:

$$W =: \big\{\beta < \mathfrak{b}^+ : \begin{array}{l} k(\beta) = k(*),\ k_\beta = k^*,\ \gamma_{\xi(\beta,k(*))} = \gamma^*, \\ \zeta_{\beta,\xi(\beta,k(*))} = \zeta^* \text{ and } \ell(\xi(\beta, k(*))) = \ell^* \end{array}\big\}.$$

As there it follows that:

$(\odot)$ if $\beta \in W$, then $\bigcup\big\{[n^{\ell^*}_{\gamma^*,j}, n^{\ell^*}_{\gamma^*,j+1}) : j \in B_{\zeta^*,t_{\beta,\xi(\beta,k(*))}}\big\}$ belongs to D.

But for $\beta \neq \alpha \in W$ those sets are not almost disjoint whereas $(\zeta^*, t_{\beta,\xi(\beta,k(*))}) \neq (\zeta^*, t_{\alpha,\xi(\alpha,k(*))})$ are distinct, giving us a contradiction. $\square$

REFERENCES

[BJ95] Tomek Bartoszyński and Haim Judah, *Set theory: On the structure of the real line*, A K Peters, Wellesley, Massachusetts, 1995.

[BL89] Andreas Blass and Claude Laflamme, *Consistency results about filters and the number of inequivalent growth types*, J. Symbolic Logic **54** (1989), 50–56.

[BL03] Jörg Brendle and Maria Losada, *The cofinality of the infinite symmetric group and groupwise density*, J. Symbolic Logic **68** (2003), 1354–1361.

[Bla89] Andreas Blass, *Applications of superperfect forcing and its relatives*, Set theory and its applications (Toronto, ON, 1987), Lecture Notes in Math., vol. 1401, Springer, Berlin, 1989, pp. 18–40.

[BS89] Bohuslav Balcar and Petr Simon, *Disjoint refinement*, Handbook of Boolean Algebras, vol. 2, North–Holland, 1989, Monk D., Bonnet R. eds., pp. 333–388.

[MS02] Heike Mildenberger and Saharon Shelah, *The relative consistency of* $\mathfrak{g} < \mathrm{cf}(\mathrm{Sym}(\omega))$, J. Symbolic Logic **67** (2002), no. 1, 297–314, arXiv: math/0009077. MR 1889552

[MS07] ______, *Increasing the groupwise density number by c.c.c. forcing*, Ann. Pure Appl. Logic **149** (2007), no. 1-3, 7–13, arXiv: math/0404147. MR 2364193

[Tho98] Simon Thomas, *Groupwise density and the confinality of the infinite symmetric group*, Arch. Math. Logic **37** (1998), 483–493.

[vD84] Eric K. van Douwen, *The integers and topology*, Handbook of Set-Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), Elsevier Science Publishers, 1984, pp. 111–167.

INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, 91904 JERUSALEM, ISRAEL, AND DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08854, USA

Email address: `shelah@math.huji.ac.il`

URL: `http://shelah.logic.at`