THEORIES WITH EF-EQUIVALENT NON-ISOMORPHIC MODELS
SH897

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ABSTRACT. Our "long term and large scale" aim is to characterize the first order theories T (at least the countable ones) such that: for every ordinal α there are λ, M_1, M_2 such that M_1, M_2 are non-isomorphic models of T of cardinality λ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent. We expect that as in the main gap ([She90, **?**]) we get a strong dichotomy, so in the non-structure side we have stronger, better examples, and in the structure side we have a parallel of [She90, Ch.XIII]. We presently prove the <u>consistency</u> of the non-structure side for T which is $\aleph_0$-independent (= not strongly dependent), even for $\mathrm{PC}(T_1, T)$.

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§ 0. Introduction

§ 0(A). **Motivation.**

We first give some an introduction for non-model theorists. A major theme in the author's work in model theory is to find "main gap theorems". This means, considering the family of elementary classes (e.g. the classes of the form $\mathrm{Mod}_T =$ the class of models of a (complete) first order theory T, each such class is either very "simple" or is very complicated; expecting that we have much knowledge to gain on the "very simple" ones and even on approximations to them.

Of course, this depends on the criterion for "simple". Essentially the main theorem of [She90] does this for countable T, with "complicated" interpreted as "$\dot{I}(\lambda, T)$", the number of models in Mod_T of cardinality λ, is maximal, i.e. 2^λ, for every λ. See more e.g. in [Sheb]. Here we are interested with interpreting "complicated" as "for arbitrarily large cardinals, there are models $M_1, M_2 \in \mathrm{Mod}_T$ of cardinality λ which are "very similar" but not isomorphic", where "very similar" is interpreted as a relative of the game of the following form. The isomorphism player constructs during the play, partial isomorphism of cardinality $< \lambda$, in each move the anti-isomorphism player demands some elements to be in the domain or the range, the isomorphism player has to extend the partial isomorphism accordingly; in the play there are α moves, $\alpha < \lambda$; and the isomorphism player wins the play if he has a legal move in each stage (see Definition 1.5, 1.7).

In the present paper we try to deal with suggesting the "right" variant of the game, (see Definition 1.6), and give quite weak sufficient conditions for Mod_T being complicated.

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Our aim is to prove (on $\mathrm{PC}(T_1, T)$, see Definition 0.3(3))

$\boxtimes$ if $T \subseteq T_1$ are complete first order theories such that T is not strongly stable, α is an ordinal and $\lambda > |T|$ (or at least for many such λ's) <u>then</u>

- $(*)$ there are $M_1, M_2 \in \mathrm{PC}(T_1, T)$ of cardinality λ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent for every $\alpha < \lambda$ but not isomorphic (for the definition of $\mathrm{EF}^+_{\alpha,\lambda}$, see Definition 1.7 below, it is a somewhat stronger relative of $\mathrm{EF}_{\alpha,\lambda}$-equivalent).

§ 0(B). **Related Works.**

Concerning constructing non-isomorphic $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent models M_1, M_2 (with no relation to T) we have intended to continue [She06], or see more Havlin-Shelah [HS07] and see history in Vaananen in [Vaa95]. Those works leave the case $\lambda = \aleph_1$ open; a recent construction [She08] resolve this but whereas it applies to every regular uncountable λ, it seems less amenable to generalizations.

By [She90] we essentially know for T a countable complete first order <u>when</u> there are $\mathbb{L}_{\infty,\lambda}(\tau_T)$-equivalent non-isomorphic models of T of cardinality λ for some λ, see §4; this is exactly when T is superstable with NDOP, NOTOP; (see [She87a]).

On restricting ourselves to models of T for "$\mathrm{EF}_{\alpha,\lambda}$-equivalent non-isomorphic", Hyttinen-Tuuri [HT91] started, then Hyttinen-Shelah [HS94], [HS95], [HS99]. The notion "$\mathrm{EF}^+_{\alpha,\lambda}$-equivalent" is introduced here, in Definition 1.7.

By [HS94], if T is stable unsuperstable, complete first order theory, $\lambda = \mu^+, \mu = \mathrm{cf}(\mu) \geq |T|$, <u>then</u> there are $\mathrm{EF}_{\mu\times\omega,\lambda}$-equivalent non-isomorphic models of T (even in $\mathrm{PC}(T_1, T)$) of cardinality λ. But by the variant $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent, such results are excluded; by it we define our choice test problem the version of being fat/lean, see Definition 0.1.

Why EF^+? See Discussion 1.6.

Concerning variants of strongly dependent theories see [She09b, §3],[She14] (and maybe [S$^+$a]), most relevant is [She14, §5], part (F). The best relative for us is "strongly$_4$ dependent", a definition of it is given below but we delay the treatment to a subsequent paper, [S$^+$b]. There we also deal with the relevant logics and more.

We prove here that if T is not strongly stable then T is consistently fat. More specifically, for every $\mu = \mu^{<\mu} > |T|$ there is a μ-complete class forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ the theory T is fat. The result holds even for $\mathrm{PC}(T_1, T)$. This gives new cases even for $\mathrm{PC}(T)$ by 0.2.

Also if T is unstable or has the DOP or OTOP (see 0.7 below or [She90]) <u>then</u> it is fat, i.e. already in $\mathbf{V}$.

Of course, forcing the example is a drawback, but note that for proving there is no positive theory it is certainly enough. Hence it gives us an upper bound on the relevant dividing lines.

On Eherenfeucht-Mostowski models, see [Shear, Ch.III] or [She90, Ch.VII] or [She09a], [Shear, Ch.III,§1]. I thank a referee for pointing out on earlier version that Hyttinen-Shelah [HS94] was forgotten hence as Definition 1.7 was not yet written, the main result 3.1 had not said anything new.

I also thank referees for many helpful remarks.

§ 0(C). **Notations and Basic Definitions.**

Definition 0.1. Let T be a complete first order theory.
1) We say T is fat <u>when</u> for every ordinal κ, for some (regular) cardinality $\lambda > \kappa$ there are non-isomorphic models M_1, M_2 of T of cardinality λ which are $\mathrm{EF}^+_{\beta,\kappa,\kappa,\lambda}$-equivalent for every $\beta < \lambda$ (see Definition 1.7 below).
2) If T is not fat, we say it is lean.
3) We say the pair (T, T_1) is fat/lean <u>when</u> (T_1 is first order $\supseteq T$ and) $\mathrm{PC}(T_1, T) := \{M \restriction \tau_T : M \text{ a model of } T_1\}$ is as above.
4) We say $(T, *)$ is fat <u>when</u> for every first order $T_1 \supseteq T$ the pair (T, T_1) is fat. We say $(T, *)$ is lean otherwise.

Our claims (mainly 3.1) seem to make it clear that some stable T has NDOT and NOTOP which falls under 3.1, but a referee asks for an example, see [She14, §5(F)] for details.

Example 0.2. 1) There is a stable NDOP,NOTOP countable complete theory which is not strongly dependent; (moreover not is not strongly$_4$ stable), see [She14, §5(G)].
2) $T = \mathrm{Th}({}^{\omega_1}(\mathbb{Z}_2), E_n)_{n<\omega}$ is as above where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ as an additive group, $E_n = \{(\eta,\nu) : \eta,\nu \in {}^{\omega_1}(\mathbb{Z}_2)$ are such that $\eta\restriction(\omega n) = \nu\restriction(\omega n)$ where we interpret $\mathbb{Z}_2$ as the additive group $(\mathbb{Z}/2\mathbb{Z}, +, 0)$ and ${}^{\omega_1}(\mathbb{Z}_2)$ as its ω_1-th power as an abelian group.

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The reader may look at the definitions below only when used.

Definition 0.3. 1) $\mathrm{Mod}_T(\lambda) = \mathrm{EC}_T(\lambda)$ is the class of models of T of cardinality λ and $\mathrm{Mod}_T = \mathrm{EC}_T$ is $\cup\{\mathrm{EC}_T(\lambda) : \lambda \text{ a cardinality}\}$.

2) $\mathrm{PC}_\tau(T) = \{M \restriction \tau : M$ a model of $T\}$ where T is a theory or a sentence, in whatever logic, in a vocabulary $\tau_T \supseteq \tau$; if $\tau = \tau_T$ we may omit τ.
3) If $T \subseteq T_1$ are complete first order theories then $\mathrm{PC}(T_1, T) = \mathrm{PC}_{\tau(T)}(T_1)$.

Notation 0.4. 1) $\ell g(\bar{a})$ is the length of a sequence $\bar{a}$.
2) $\bar{a} \trianglelefteq \bar{b}$ means that $\bar{a}$ is an initial segment of $\bar{a}$.
3) $\bar{a} \restriction \alpha$ is the unique initial segment of $\bar{a}$ of length α for $\alpha \le \ell g(\bar{a})$.

Definition 0.5. 1) For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda$: some pair $(E, \bar{a})$ witnesses $S \in \check{I}(\lambda)$, see below$\}$.
2) We say that (E, u) is a witness for $S \in \check{I}[\lambda]$ <u>when</u>:

- (a) E is a club of the regular cardinal λ
- (b) $\bar{u} = \langle u_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \alpha$ and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$
- (c) for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal).

Notation 0.6. 1) For a model $M, \bar{a} \in {}^\alpha M, B \subseteq M$ and Δ a set of formulas, we are interested in formulas of the form $\varphi(\bar{x}, \bar{y}), \bar{x} = \langle x_i : i < \alpha\rangle$, so α may be infinite, but the formulas here are normally first order, so all but finitely many of the x_i's are dummy variables.
1A) $\mathrm{tp}_\Delta(\bar{a}, B, M) = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in \Delta$ and $\bar{b} \in {}^{\ell g(\bar{y})}A$ and $M \models \varphi[\bar{a}, \bar{b}]\}$.
2) If Δ_{qf} is the set of quantifier-free formulas in $\mathbb{L}(\tau_M)$, we may write $\mathrm{tp}_{\mathrm{qf}}$ instead of tp_Δ.
3) $\dot{I}(\lambda, T)$ is the number of isomorphic types of models of T of cardinality λ.
4) $\dot{I}_\tau(\lambda, T)$ is the number of isomorphic types of $M \restriction \tau, M$ a model of T of cardinality λ.
5) $\dot{I}\dot{E}_\tau(\lambda, T)$ is the supremum of $\{|K| : K \subseteq \mathrm{PC}_\tau(T)$ and $M \in K \Rightarrow \|M\| = \lambda$ no $M \in K$ has an elementary embeding into any $N \in K \backslash \{M\}$, writing $\dot{I}\dot{E}_\tau(\lambda, T) =^+ \chi$ we mean the supremum is obtained if not said otherwise.
6) $\dot{I}\dot{E}(\lambda, T) = \dot{I}\dot{E}_{\tau(T)}(\lambda, T)$.

Definition 0.7. Let T be a first order complete theory.
1) T has OTOP <u>when</u> T is stable and for some n, m letting $\bar{x} = \langle x_\ell : \ell < n\rangle, \bar{y} = \langle y_\ell : \ell < n\rangle, \bar{z} = \langle z_\ell : \ell < m\rangle$, there are complete types $p(\bar{x}, \bar{y}, \bar{z})$ such that: for every λ there is a model M of T and $\bar{a}_\alpha \in {}^n M$ for $\alpha < \lambda$ such that:

- (a) $\langle \bar{a}_\alpha : \alpha < \lambda\rangle$ is an indiscernible set
- (b) for $\alpha \ne B < \lambda$ the type $(p(\bar{a}_\alpha, \bar{a}_b, \bar{z})$ is realized in M iff $\alpha < \beta$.

1A) T has the NOTOP <u>when</u> it is stable but fail the OTOP.
2) T has NDOP <u>when</u> T is stable and we can find $|T|^+$-saturated models M_ℓ of T for $\ell \le 3$ such that $M_0 \prec M_\ell \prec M_3$ for $\ell = 1, 2$ and $\mathrm{tp}(M_1, M_2)$ does not fork over M_0, M_3 is $|T|^+$-prime over $M_1 \cup M_2$ but not $|T|^+$-minimal over it; equivalently for every $\bar{c} \in {}^{\omega >}(M_3)$ the type $\mathrm{tp}(\bar{c}, M_1 \cup M_2, M_3)$ is $|T|^+$-isolated but there is no infinite $\mathbf{I} \subseteq M_3$ which is indiscernible over $M_1 \cup M_2$.
2A) T has DOP <u>when</u> T is stable and fail to have the NDOP.

Definition 0.8. 1) For a complete first order theory T, we can say that ψ is a (μ, κ, T)-candidate when:

- (a) $\psi \in \mathbb{L}_{\kappa^+, \omega}(\tau_*)$ for some vocabulary $\tau_* \supseteq \tau_T$ of cardinality $\le \kappa$

(*b*) $\mathrm{PC}_{\tau(T)}(\psi) \subseteq \mathrm{EC}(T)$

(*c*) for some[1] $\Phi \in \Upsilon^{\omega-\mathrm{tr}}_\kappa$ satisfying $\tau_\Phi \supseteq \tau_\psi$ and $\mathrm{EM}({}^{\omega \geq}\lambda, \Phi) \models \psi$ for every (equivalent some) λ and Φ witness T is not superstable.

Recall that by [She90, Ch.VII]:

Claim 0.9. *If a first order complete theory T is not superstable, then for some $\Phi \in \Upsilon^{\omega-\mathrm{tr}}_{\tau_2}$, see Definition 2.2, $\tau_2 \supseteq \tau(\psi)$ of cardinality κ, Φ witness T is not superstable, i.e. for some formulas $\varphi_n(x, \bar{y}_n) \in \mathbb{L}(\tau_T)$, if $I = {}^\omega\lambda, M = \mathrm{EM}(I, \Phi)$ then for $\eta \in {}^\omega\lambda, n < \omega$ and $\alpha < \lambda$ we have $M \models \varphi_n[\bar{a}_\eta, a_{(\eta \restriction n)\hat{}<\alpha>}]$ iff $\alpha = \eta(n)$.*

Definition 0.10. 1) For any structure I we say that $\langle \bar{a}_t : t \in I \rangle$ is indiscernible (in the model $\mathfrak{C}$, over A, if $A = \emptyset$ we may omit it) when: ($\bar{a}_t \in {}^{\ell g(\bar{a}_t)}\mathfrak{C}$ and) $\ell g(\bar{a}_t)$, which is not necessarily finite depends only on the quantifier-free type of t in I and:

> if $n < \omega$ and $\bar{s} = \langle s_0, s_1, \ldots, s_{n-1} \rangle, \bar{t} = \langle t_0, \ldots, t_{n-1} \rangle$ realize the same quantifier-free type in I then $\bar{a}_{\bar{t}} := \bar{a}_{t_0} \hat{} \ldots \hat{} \bar{a}_{t_{n-1}}$ and $\bar{a}_{\bar{s}} = \bar{a}_{s_0} \hat{} \ldots \hat{} \bar{a}_{s_{n-1}}$ realizes the same type (over A) in $\mathfrak{C}$.

2) We say that $\langle \bar{a}_u : u \in [I]^{<\aleph_0} \rangle$ is indiscernible (in $\mathfrak{C}$, over A) similarly:

> if $n < \omega, w_0, \ldots, w_{m-1} \subseteq \{0, \ldots, n-1\}$ and $\bar{s} = \langle s_\ell : \ell < n \rangle, \bar{t} = \langle t_\ell : \ell < n \rangle$ realize the same quantifier-free types in I and $u_\ell = \{s_k : k \in w_\ell\}, v_\ell = \{t_k : k \in w_\ell\}$ then $\bar{a}_{u_0} \hat{} \ldots \hat{} \bar{a}_{u_{n-1}}, \bar{a}_{v_0} \hat{} \ldots \hat{} \bar{a}_{v_{n-1}}$ realize the same type in $\mathfrak{C}$ (over A).

3) If I is a linear order then we let $\mathrm{incr}({}^\alpha I) = \mathrm{incr}_\alpha(I) = \mathrm{incr}(\alpha, I)$ be $\{\rho : \rho$ is an increasing sequence of length α of members of $I\}$; similarly $\mathrm{incr}({}^{\alpha>}I)$-$\mathrm{incr}_{<\alpha}(I) = \mathrm{incr}(< \alpha, I) := \cup\{\mathrm{incr}_\beta(I) : \beta < \alpha\}$. So instead $[I]^{<\aleph_0}$ we may use $\mathrm{incr}_{<\omega}(I)$; clearly the difference is notational only.

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[1] i.e. Φ proper for K^ω_{tr}, i.e. normal trees with $\omega + 1$ level, with linear order on the successor of each node of finite level, see Definition 2.2(7) or [She90, Ch.VII]

§ 1. Games, equivalences and questions

What is the meaning in using $\mathrm{EF}^+_{\alpha,\lambda}$? Consider for various γ's the game $\Game_{\gamma,\lambda}(M_1,M_2)$ where $M_1,M_2 \in \mathrm{Mod}_T(\lambda), T$ complete first order $\mathbb{L}(\tau)$-theory. During a play we can consider dependence relations on "short" sequences from M_ℓ (where $\le 2^{|\tau|+\aleph_0}$ is the default value), definable in a suitable sense. So if T is a well understood unsuperstable T like $\mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ with $E_n := \{(\eta,\nu) : \eta,\nu \in {}^\omega\omega$ and $\eta \restriction n = \nu \restriction n\}$, then even for $\gamma = \omega+2$ we have $E^+_{\gamma,\lambda}$- equivalence implies being isomorphic. This fits the thesis:

Thesis 1.1. The desirable dichotomy characterized, on the family of first order T, by the property "$M_1,M_2 \in \mathrm{Mod}_T(\lambda)$ are long game EF-equivalent iff they are isomorphic", is quite similar to the one in [She90, Ch.XIII]; the structure side is e.g.: T is stable and every $M \in \mathrm{Mod}_T$ is prime over some $\cup\{M_\eta : \eta \in I\}$, where $\mathscr{T}$ is a subtree of ${}^{\kappa_r(T)>}\|M\|$ and $\eta \triangleleft \nu \Rightarrow M_\eta \prec M_\nu \prec M, \|M_\eta\| \le 2^{|T|}$ and $\eta \triangleleft \nu \in \mathscr{T} \Rightarrow \mathrm{tp}(M_\nu, \cup\{M_\rho : \rho \in \mathscr{T}, \rho \restriction (\ell g(\nu)+1) \ne \eta \restriction (\ell g(\nu)+1))$ does not fork over M_ν, i.e. $\bar{M} = \langle M_\eta : \eta \in \mathscr{T}\rangle$ is a non-forking tree of models with $\le \kappa_r(T)$ many levels.

We think the right (variant of the) question is from 1.2. Probably a reasonable analog is the situation in [She90, Ch.XII,XIII]: the original question was on the function $\lambda \mapsto \dot{I}(\lambda,T)$, the number of non-isomorphic models; but the answer is more transparent for $\lambda \mapsto \dot{I}\dot{E}(\lambda,T)$.

If $\lambda = \mu^+, \mu = \mu^{|T|} = \mathrm{cf}(\mu), T = \mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ then by Hyttinen-Shelah [HS94, Th4.4]; for $\gamma \ge \mu\omega$ we get equivalence $\Rightarrow$ isomorphic, but not for $\gamma < \mu\omega$; now 1.9 is parallel to that. This seems to indicate that $\mathrm{EF}^+_{\gamma,\lambda}$ is suitable for the questions we are asking: it uses the game EF^+, which is more complicated but the length of the game is much "smaller" in the relevant results.

So the natural question concerning such equivalences is (see [She90], [Sheb]):

Question 1.2. Classify first order complete T, or at least the countable ones by:

Version $(\mathrm{A})_1$: For every ordinal α, there are a cardinal λ and non-isomorphic $M_1,M_2 \in \mathrm{Mod}_T(\lambda)$ which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent (at least, e.g. in some $\mathbf{V}^{\mathbb{P}}, \mathbb{P}$ is $(2^{|T|+|\alpha|})^+$-complete forcing notion).

Version $(\mathrm{A})_0$: Similar version for $\mathrm{EF}_{\alpha,\lambda}$.

Version $(\mathrm{B})_1$: For every cardinal $\kappa > |T|$ and vocabulary $\tau_1 \supseteq \tau_T$ and $\psi \in \mathbb{L}_{\kappa,\omega}(\tau_1)$ such that $\mathrm{PC}_\tau(\psi) \subseteq \mathrm{EC}_T$ has members of arbitrarily large cardinality we have $(a) \Rightarrow (b)$ where

- (a) for every cardinal μ in $\mathrm{PC}_\tau(\psi) := \{M \restriction \tau : M$ a model of $\psi\}$ there is a μ-saturated member
- (b) for every α for arbitrarily large λ there are $M_1,M_2 \in \mathrm{PC}_\tau(\psi)$ of cardinality λ with non-isomorphic τ-reducts which are $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent.

Version $(\mathrm{B})_0$: Like $(\mathrm{B})_1$ for $\mathrm{EF}_{\alpha,\lambda}$.

Version $(\mathrm{C})_1$: Like $(\mathrm{B})_1$ using $\psi = \wedge T_1$ where T_1 is first order $\supseteq T$.

Version $(\mathrm{C})_0$: Like $(\mathrm{B})_0$ using $\psi = \wedge T_1$ where T_1 is a first order $\supseteq T$.

Discussion 1.3. 1) For reasons to prefer version (B) over (C) - see [Sheb].
2) Now by the works quoted above, (see [HS95, 3.19] quoted in 4.1 below): T satisfies $(A)_0$ iff T is superstable NDOP, OTOP iff $(B)_0$. Of course if we change the order of the quantifier (to "for aribitrarily large some λ for every $\alpha < \lambda$,...") this is not so, but we believe solving $(A)_1$ and/or $(B)_1$ will eventually do much also for this.

So all this means

Conjecture 1.4. 1) For a complete (first order) T the following are equivalent:

- (a) for every ordinal α for some λ there are non-isomorphic, $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent models $M_1, M_2 \in \mathrm{EC}_T(N)$
- (b) for arbitrarily large λ for every $\alpha < \lambda$ there are non-isomorphisms, $\mathrm{EF}^+_{\alpha,\lambda}$-equivalent models $M_1, M_2 \in \mathrm{EC}_T(\lambda)$
- (c) for every large enough regular λ there are non-isomorphisms $M_1, M_2 \in \mathrm{EC}_T(\lambda)$ which are $E_{\alpha,\lambda}$-equivalent for every $\alpha < \lambda$.

2) Similarly for "some $T_1 \supseteq T$, $\mathrm{PC}(T_1, T)$ is lean.

We conjecture that proving that if we prove that a (countable) fat T is close enough to superstable, will enable us to generalize proofs in [She90, Ch.XII] only now the tree has $\leq \omega_1$ levels rather than ω.

We can also return to the ordinals $\alpha \in (\lambda\omega, \lambda^+)$.

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Now we shall actually look at the games.

Definition 1.5. 1) We say that M_1, M_2 are EF_α-equivalent if M_1, M_2 are models (with same vocabulary) and α is an ordinal such that the isomorphism player has a winning strategy in the game $\mathscr{G}^\alpha_1(M_1, M_2)$ defined below.
1A) Replacing α by $< \alpha$ means: for every $\beta < \alpha$; similarly below.
2) We say that M_1, M_2 are $\mathrm{EF}_{\alpha,\mu}$-equivalent or $\mathscr{G}^\alpha_\mu$-equivalent <u>when</u> M_1, M_2 are models with the same vocabulary, α an ordinal, μ a cardinal such that the isomorphism player has a winning strategy in the game $\mathscr{G}^\alpha_\mu(M_1, M_2)$ defined below.
3) For M_1, M_2, α, μ as above and partial isomorphism f from M_1 into M_2 we define the game $\mathscr{G}^\alpha_\mu(f, M_1, M_2)$ between the players ISO, the isomorphism player and AIS, the anti-isomorphism player as follows:

- (a) a play lasts α moves
- (b) after β moves a partial isomorphism f_β from M_1 into M_2 is chosen, increasing continuous with β
- (c) in the $(\beta+1)$-th move, the player AIS chooses $A_{\beta,1} \subseteq M_1, A_{\beta,2} \subseteq M_2$ such that $|A_{\beta,1}| + |A_{\beta,2}| < 1 + \mu$ and then the player ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $A_{\beta,1} \subseteq \mathrm{Dom}(f_{\beta+1})$ and $A_{\beta,2} \subseteq \mathrm{Rang}(f_{\beta+1})$
- (d) if $\beta = 0$, ISO chooses $f_0 = f$; if β is a limit ordinal ISO chooses $f_\beta = \cup\{f_\gamma : \gamma < \beta\}$.

The ISO player loses if he had no legal move for some $\beta < \alpha$, otherwise he wins the play.
4) If $f = \emptyset$ we may write $\mathscr{G}^\alpha_\mu(M_1, M_2)$. If μ is 1 we may omit it. We may write $\leq \mu$ instead of μ^+.

Discussion 1.6. 1) Why do we need EF^+?

First, if we like a parallel of [She90, Ch.XIII], i.e. a game in which set of small cardinality are chosen, say $|T|$ or $2^{|T|}$ or whatever rather than just $< \lambda = \|M_\ell\|$, clearly $\mathrm{EF}_{\alpha,\mu}$ cannot help.

2) Also, consider $\lambda = \mu^+, \mu = \mathrm{cf}(\mu) > |T|$ and an ordinal $\alpha < \lambda$ and ask for which T: for any two models M_1, M_2 of T of cardinality λ, $\mathrm{EF}_{\alpha,\lambda}$-equivalence implies isomorphisms? (The $\mathrm{EF}_{\alpha,\lambda}$-equivalence means that the isomorphism player wins in the game of length α, in each step adding $\leq \mu$ elements to the domain and range of the partial isomorphism.)

Now we know (by earlier works, see 4.5) for countable T that if $\alpha \in [\omega, \mu \times \omega]$ that the answer (for the pair (α, λ)) is as in the main gap for $\dot{I}\dot{E}$ (T superstable with NDOP and NOTOP). But for larger $\alpha < \lambda$ this is not so, as e.g. for the prototypical stable unsuperstable T for $\alpha = \mu \times (\omega + 2)$ we get yes, "it is low".

3) Looking at the reason for this, i.e. why we need $\mu \times (\omega + 2)$ moves, not $(\omega + 2)$ moves we formulate EF^+. We think that with $\mathrm{EF}^+_{\alpha,\theta,\mu,\lambda}$ for small α, θ, μ and just $\lambda = \|M_\ell\|$ we get the desired dichotomy. In general, we expect the results will be robust under choosing such an exact game; and will resolve the case $\alpha \in (\mu \times (\omega, 2), \lambda)$ case above.

4) More specifically, the reason $\mathrm{EF}_{\alpha,\lambda}$-equivalence does not imply isomorphisms for $M_1, M_2 \in \mathrm{EC}_\lambda(T)$, even in the case $T = \mathrm{Th}({}^\omega\omega, E_n)_{m<\omega}$, is that: assume we fix a winning strategy $\mathbf{st}$ for $\mathscr{G}_{\alpha,\lambda}(M_1, M_2)$, if we let $\langle a^\ell_\alpha / E_1^{M_\ell} : \alpha < \lambda \rangle$ list $M_\ell / E_1^{M_\ell}$ and $\mathbf{R} = \{(\alpha, \beta)$: in some short initial segment $\mathbf{x}$ of a play of $\mathscr{G}_{\alpha,\lambda}(M_1, M_2)$ in which the player ISO uses the strategy $\mathbf{st}$, we have $f^{\mathbf{x}}_\alpha(a^1_\alpha) E_1^{M_2} a^2_\beta\}$, we have to find a function h from λ onto λ whose graph is $\subseteq \mathbf{R}$.

Now being in a winning position is enough to show the existence of such h, only when the game is long enough. For $\mathrm{EF}^+_{\alpha,\theta}$ this is different.

5) Note: we use the case $k = 1$ from 1.7. If we shall have good structure theorems then even $k = 2$ is O.K. For $k = \kappa$ it expresses the logic in [She90, Ch.XIII] when we add the game quantifier of appropriate length.

6) Of course, the case $k = 0$ is easier for ISO then the case $k = 2$ which is easier than $k = 1$, so the relevant implications holds.

Definition 1.7. 1) For $k \in \{0, 1, 2\}$ the models M_1, M_2 are $\mathrm{EF}^{+,k}_{\gamma,\theta,\mu,\lambda}$-equivalent, but if $k = 1$ we may omit it, <u>when</u> the isomorphism player, ISO, has a winning strategy in the game $\mathscr{G}^k_{\gamma,\theta,\mu,\lambda}(M_1, M_2)$ defined below.

We always assume $\aleph_0 \leq \theta \leq \mu$. If $\mu = \min\{\|M_1\|, \|M_2\|\}$ then we may omit it. If also $\theta = (2^{|\tau(M_\ell)| + \aleph_0})^+$ we may omit it, too.

2) For an ordinal γ, cardinals $\theta \leq \mu$, vocabulary τ and τ-models M_1, M_2 and partial isomorphism f from M_1 to M_2, we define a game $\mathscr{G}^k = \mathscr{G}^{+,k}_{\gamma,\theta,\mu,\lambda}(f, M_1, M_2)$, between the player ISO (isomorphism) and AIS (anti-isomorphism).

A play last γ moves; in the β-th move a partial isomorphism f_β from M_1 to M_2 is chosen by ISO, extending f_α for $\alpha < \beta$ such that $f_0 = f$ and for limit β we have $f_\beta = \cup\{f_\alpha : \alpha < \beta\}$ and for every $\beta < \alpha$ the set $\mathrm{Dom}(f_{\beta+1}) \setminus \mathrm{Dom}(f_\beta)$ has cardinality $< 1 + \mu$; let f^ℓ_β be f_β if $\ell = 1$, f^{-1}_β if $\ell = 2$.

During a play, the player ISO loses if he has no legal move and he wins in the end of the play iff he always had a legal move.

In the $(\beta + 1)$-th move, the AIS player does one of the following cases:

Case 1: The AIS player chooses $A_\ell = A^\ell_\beta \subseteq M_\ell$ for $\ell = 1,2$ such that $|A_1| + |A_2| < 1 + \mu$ and then ISO chooses f_β as above such that $A_\ell \subseteq \mathrm{Dom}(f^\ell_\beta)$ for $\ell = 1,2$.

Case 2: First the AIS player chooses[2] a pre-dependence relation $\mathbf{R}_\ell$ on ${}^{\theta>}(M_\ell)$ (see Definition 1.8 below) and $\mathscr{A}_\ell \subseteq {}^{\varepsilon}(M_\ell)$ of cardinality $\le \lambda$ for $\ell = 1,2$ such that:

$\odot$ (a) if $k = 0$ then $\mathbf{R}_\ell = [{}^{\theta>}(M_\ell)]^{<\aleph_0}$, so really an empty case

(b) if $k = 1,2$ then $\mathbf{R}_\ell$ is a 1-dependence relation (see 1.8(4)(b)(B) below)

(c) if $k = 1,2$ and $\ell = 1,2$ and $n < \omega$ and $\bar a_0, \dots, \bar a_{n-1} \in {}^{\varepsilon}(M_\ell)$ then the truth value of $\{\bar a_0, \dots, \bar a_{n-1}\} \in \mathbf{R}_\ell$ depends just on the complete first order type which $\langle \bar a_0, \dots, \bar a_{n-1}\rangle$ realizes on $\mathrm{Dom}(f^\ell_\beta)$ inside the model M_ℓ.

Second, the ISO does one of the following:

Subcase 2A: First, assume $k = 2$. The player ISO chooses $\langle (\bar a^1_\zeta, \bar a^2_\zeta) : \zeta < \lambda\rangle$ such that for $\ell = 1,2$:

(α) for each $\zeta < \lambda$ for some $\varepsilon < \theta$ we have $\bar a^\ell_\zeta \in {}^{\varepsilon}(M_\ell)$

(β) $\langle \bar a^\ell_\zeta : \zeta < \lambda\rangle$ is independent for $\mathbf{R}_\ell$

(γ) each $\bar a \in \mathscr{A}_\ell$ does $\mathbf{R}_\ell$-depend on $\{\bar a^\ell_\zeta : \zeta < \lambda\}$.

Then AIS chooses $\zeta < \lambda$ and ISO chooses $f_{\beta+1} \supseteq f_\beta$ such that $f_\beta(\bar a^1_\zeta) = \bar a^2_\zeta$.

Second, assume $k = 1$. Then the ISO player chooses equivalence relations E_ℓ on ${}^{\theta>}(M_\ell)$ which the dependence relation, i.e. $E_{\mathbf{R}_\ell}$, i.e. 1.8(6) and equality of length refine for $\ell = 1,2$ and choose a function h from the family of E_1-equivalence classes onto the family of E_2-equivalence classes which preserve cardinality up to λ; that is, if $h(\bar a_1/E_1) = \bar a_2/E_2$ then $\ell g(\bar a_1) = \ell g(\bar a_2) and$ $\min\{\dim(\bar a_1/E_1), \lambda\} = \min\{\dim(\bar a_2/E_2), \lambda\}$.

Then the AIS player chooses a pair $(\bar a_1, \bar a_2)$ such that $\bar a_\ell \in {}^{\theta>}(M_\ell)$ for $\ell = 1,2$ such that $h(\bar a_1/E_1) = (\bar a_2/E_2)$ and ISO has to choose $f_{\beta+1} \supseteq f_\beta$ such that $f(\bar a_1) = \bar a_2$.

Subcase 2B: The player ISO chooses $f_{\beta+1} \supseteq f_\beta$ as required such that for some $n < \omega$ and $\bar a^1_\ell \in {}^{\varepsilon}\mathrm{Dom}(f_\beta)$ for $\ell \le n$ we have: $\{\bar a^1_0, \dots, \bar a^1_{n-1}\}$ is $\mathbf{R}_1$-dependent iff $\{f_\beta(\bar a^1_0), \dots, f_\beta(\bar a^1_{n-1})\}$ is not $\mathbf{R}_2$-dependent.

Definition 1.8. 1) We say $\mathbf{R}$ is a pre-dependence relation on X when $\mathbf{R}$ is a subset of $[X]^{<\aleph_0}$.
2) For $X, \mathbf{R}$ as above, we say $Y \subseteq X$ is $\mathbf{R}$-independent when $[Y]^{<\aleph_0} \cap \mathbf{R} = \emptyset$; of course, an index set with repetitions is considered dependent.
3) We say $\mathbf{R}$ or $(X, \mathbf{R})$ has character $\le \kappa$ when for every $\mathbf{R}$-independent $Y \subseteq X$ and $\{x\} \subseteq X$ for some $Z \in [Y]^{<\kappa}$ the set $(Y \backslash Z) \cup \{x\}$ is $\mathbf{R}$-independent.
4) We say that $\mathbf{R}$ is a k-dependence relation on X (if $k = 1$ we may omit it) when:

[2]note that for $k = 0,1$ we require "$\mathbb{L}(\tau_T)$-definable $\mathbf{R}_\ell$ such that f maps the definition of $\mathbf{R}_1$ to the one of $\mathbf{R}_2$"; moreover we expect that we can demand it is as in the case of using regular types.

(a) $\mathbf{R}$ is a subset of $[X]^{<\aleph_0}$

(b) (α) if $k = 0$ then $\mathbf{R} = [X]^{<\aleph_0}$

(β) if $k = 1$ then $\mathbf{R}$-independence satisfies the exchange principle (so dimension is well defined, as for regular types).

5) We say R is trivial when for every $Y \subseteq X, Y$ is $\mathbf{R}$-independent iff every $Z \subseteq [Y]^{\leq 2}$, is $\mathbf{R}$-independent.
6) For $\mathbf{R}$ as in $(a), (b)(\beta)$ let $E_{\mathbf{R}} = \{\{x_1, x_2\} : x_1 = x_2 \in X$ on $\{\lambda_1\}, \{x_2\} \in \mathbf{R}$ or $\{x_1, x_2\} \in \mathbf{R} \wedge \{x_1\} \notin \mathbf{R} \wedge \{x_2\} \notin \mathbf{R}\}$ is an equivalence relation on X; pedantically we should write $E_X, \mathbf{R}$.

Claim 1.9. *M_1, M_2 are isomorphic when:*

(a) *M_1, M_2 are models of T of cardinality λ*

(b) *M_1, M_2 are $\mathrm{EF}^+_{\omega+2,\aleph_0,\aleph_0,\lambda}$-equivalent*

(c) *$T = \mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ and $E_n = \{(\eta,\nu) : \eta \in {}^\omega\omega, \nu \in {}^\omega\omega$ and $\eta{\restriction}n = \nu{\restriction}n\}$.*

Proof. Step A: We choose a winning strategy $\mathbf{st}$ of the isomorphism player in the game $\mathscr{G}_{\omega+2,\aleph_0,\aleph_0,\lambda}(M_1, M_2)$.

Step B: By the choice of T for $\ell = 1, 2$ we can find $\mathscr{T}_\ell, \bar{\mathbf{a}}_\ell$ such that:

$\boxtimes_\ell$ (a) $\mathscr{T}_\ell$ is a subtree of ${}^{\omega>}\lambda$

(b) $\bar{\mathbf{a}}_\ell = \langle a^\ell_\eta : \eta \in \mathscr{T}_\ell\rangle$

(c) $a^\ell_\eta \in M_\ell$

(d) if $\eta \in \mathscr{T}_\ell$ and $\ell g(\eta) = n$ then $\langle a^\ell_\nu / E^{M_\ell}_{n+1} : \nu \in \mathrm{suc}_{\mathscr{T}_\ell}(\eta)\rangle$ list $\{b/E^{M_\ell}_{n+1} : b \in M_\ell, b \in a^\ell_\eta / E^{M_\ell}_n\}$ without repetitions.

Let $\mathscr{T}_{\ell,n} = \{\eta \in \mathscr{T}_\ell : \ell g(\eta) = n\}$ and let $\mathscr{T}_{\ell,\omega} = \{\eta \in {}^\omega\lambda : \eta \restriction n \in \mathscr{T}_\ell$ for every $n < \omega\}$.

Lastly, let $\bar{\mu}_\ell = \langle \mu^\ell_\eta : \eta \in \mathscr{T}_{\ell,\omega}\rangle$, where

$$\mu^\ell_\eta = |\{b \in M_\ell : b \in a^\ell_{\eta\restriction n}/E^{M_\ell} \text{ for every } n < \omega\}|.$$

Step C:
Clearly

$\boxplus$ M_1, M_2 are isomorphic iff there is an isomorphism h from $\mathscr{T}_1$ onto $\mathscr{T}_2$ (i.e. h maps $\mathscr{T}_{1,n}$ onto $\mathscr{T}_{2,n}, h$ preserves the length, $\eta \triangleleft \nu$ and $\eta \ntriangleleft \nu$) such that letting $h_n = h \restriction \mathscr{T}_{1,n}$ and h_ω be the mapping from $\mathscr{T}_{1,\omega}$ onto $\mathscr{T}_{2,\omega}$ which h induces (so $h_\omega(\eta) = \bigcup\limits_{n<\omega} h_n(\eta \restriction n))$ we have $\eta \in \mathscr{T}_{1,\omega} \Rightarrow \mu^1_\eta = \mu^2_{h_\omega(\eta)}$.

Step D:
By induction on n we choose $h_n, \bar{\mathbf{x}}_n$ such that

$\circledast$ (a) h_n is a one-to-one mapping from $\mathscr{T}_{1,n}$ onto $\mathscr{T}_{2,n}$

(b) if $m < n$ and $\eta \in \mathscr{T}_{1,n}$ then $h_m(\eta \restriction m) = (h_n(\eta)) \restriction m$

(c) $\bar{\mathbf{x}}_n = \langle \mathbf{x}^n_\eta : \eta \in \mathscr{T}_{1,n}\rangle$

(d) (α) $\mathbf{x}^n_\eta$ is an initial segment of a play of the game $\partial_{\omega+2,\aleph_0,\aleph_0,\lambda}(M_1,M_2)$

(β) in $\mathbf{x}^n_\eta$ only finitely many moves have been played (can specify), the last one is $m(\mathbf{x}^n_\eta)$

(γ) in $\mathbf{x}^n_\eta$, the player ISO uses his winning strategy $\mathbf{st}$

(e) if $\eta_1 \in \mathscr{T}_{1,n}$ and $\eta_2 = h_n(\eta_1)$, then for some $b_1 \in \mathrm{Dom}(f^{\mathbf{x}^n_\eta}_{m(\mathbf{x}^n_\eta)})$ we have

(α) $b_1 \in a^1_\eta / E^{M_1}_n$

(β) $f^{\mathbf{x}^n_\eta}_{m(\mathbf{x}^n_\eta)}(b_1) \in a^2_{h_n(\eta)}/E^{M_2}_n$

(f) if $\nu \triangleleft \eta \in \mathscr{T}_{1,n}$ then $\mathbf{x}^{\ell g(\nu)}_\nu$ is an initial segment of $\mathbf{x}^n_\eta$.

Why can we carry the induction?

For $n=0$:
Note that h_0 is uniquely determined. As for $\mathbf{x}^0_{<>}$, any $\mathbf{x}$ as in $\circledast(d)$ is O.K., as long as at least one move was done (note that $E^{M_\ell}_0$ has one and only one equivalence class.

For $n=m+1$: So $h_m, \bar{\mathbf{x}}_m$ has been chosen.
Let $\eta_1 \in \mathscr{T}_{1,m}$ and let $\eta_2 = h_m(\eta_1)$ and

$$\mathbf{F}_{\eta_1} := \{(\nu_1,\nu_2) : \begin{array}{l} \nu_1 \in \mathrm{suc}_{\mathscr{T}_1}(\eta_1), \nu_2 \in \mathrm{suc}_{\mathscr{T}_2}(\eta_2) \\ \text{and there is } \mathbf{x} \text{ as in } \circledast(d) \text{ such that} \\ \mathbf{x}^m_{\eta_1} \text{ is an initial segment of } \mathbf{x} \text{ and for some} \\ b_1 \in \mathrm{Dom}(f^{\mathbf{x}}_{m(\mathbf{x})}) \text{ we have} \\ b_1 \in a^1_{\nu_1}/E^{M_1}_n \text{ and } f^{\mathbf{x}}(b_1) \in a^2_{\nu_2}/E^{M_2}_n\}. \end{array}$$

Now

$\odot$ to do the induction step, it suffices to prove that: if $\eta_1 \in \mathscr{T}_{1,m}$ then there is a one-to-one function h_{n,η_1} from $\mathrm{suc}_{\mathscr{T}_1}(\eta_1)$ onto $\mathrm{suc}_{\mathscr{T}_2}(\eta_2)$ such that $\nu \in \mathrm{suc}_{\mathscr{T}_1}(\eta_1) \Rightarrow (\nu, h_{n,\eta_1}(\nu)) \in \mathbf{F}_{\eta_1}$.

However by case 2 in Definition 1.7 this holds.

Stage E:
So we can find $\langle h_n : n<\omega\rangle, \langle \mathbf{x}_\eta : \eta \in \mathscr{T}_1\rangle$ as in $\circledast$. Let $h := \cup\{h_n : n<\omega\}$, clearly it is an isomorphism from $\mathscr{T}_1$ onto $\mathscr{T}_2$ and h_ω is well defined, see $\boxplus$ from Stage C.

So it is enough to check the sufficient condition for $M_1 \cong M_2$ then, i.e. $\eta \in \mathscr{T}_{1,\omega} \Rightarrow \mu_{1,\eta} = \mu_{2,h_\omega(\eta)}$. But if $\eta \in \mathscr{T}_{1,\omega}$ then $\langle \mathbf{x}_{\eta\restriction n} : n<\omega\rangle$ is a sequence of initial segments of a play of $\mathscr{G}$ with ISO using his winning strategy $\mathbf{st}$, increasing with n, each with finitely many moves. So $\mathbf{x}_\eta$, defined as the limit $\langle \mathbf{x}_{\eta\restriction n} : n<\omega\rangle$, is an initial segment of the play $\mathscr{G}$, with $\le \omega$ moves and $f^{\mathbf{x}_\eta}_{\mathbf{m}(\mathbf{x}_\eta)} = \cup\{f^{\mathbf{x}_{\eta\restriction n}}_{m(\mathbf{x}_{\eta\restriction n})} : n<\omega\}$.

Clearly $n<\omega \Rightarrow f(a^1_{\eta\restriction n}) E^{M_2}_n a^2_{h_n(\eta\restriction n)}$. As we have one move left and can use case 2 in Definition 1.7(2) we are done. $\square_{1.9}$

The following claim says that the game in 1.5, 1.7 when $\lambda = \mu^+, \alpha < \lambda$ divisible enough are equivalent, i.e. the ISO player wins one iff he wins the other.

Claim 1.10. *1) M_1, M_2 are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent <u>when</u>:*

(a) *M_1, M_2 are τ-models*

(b) *$\lambda = \lambda_1^+, \lambda_1 \geq \mu$ and $\theta \leq \mu \leq \lambda$ and $\gamma \leq \mu$ and $\mathrm{cf}(\mu) < \mu \Rightarrow \lambda_1 > \mu$ and $\lambda \in \check{I}[\lambda]$, see Definition 0.5*

(c) *M_1, M_2 are $\mathrm{EF}_{\gamma(*),\mu}$-equivalent where $\gamma(*) = \lambda_1 \times \gamma$ (see Definition 1.5(2))*

(d) *$\|M_\ell\| = \lambda = \lambda^{<\theta}$ for $\ell = 1, 2$*

2) M_1, M_2 are $\mathrm{EF}^+_{\gamma,\mu}$-equivalent <u>when</u> they are $\mathrm{EF}_{\gamma,\theta,\mu,\lambda}$-equivalent.
3) M_1, M_2 are $\mathrm{EF}^+_{\gamma_1,\theta_1,\mu_1,\lambda_1}$-equivalent <u>when</u> they are $\mathrm{EF}^+_{\gamma_2,\theta_2,\mu_2,\lambda_2}$-equivalent and $\gamma_1 \leq \gamma_2, \theta_1 \leq \theta_2, \mu_1 \leq \mu_2, \lambda_1 \leq \lambda_2$.

Proof. 1) First, we do not save on $\gamma(*)$, say use $\lambda_1 \times \lambda_1 \times \gamma$.

Let **st** be a winning strategy of the ISO player in the game $\mathscr{G}^{\gamma(*)}_\mu$. We try to use it as a winning strategy of the ISO player in the game $\mathscr{G}_{\gamma,\theta,\mu,\lambda}(M_1, M_0)$. Well, the $f^{\mathbf{x}}_\alpha$ may have too large a domain, so "on the side" in the β-th move ISO play $\mathbf{x}_\beta$ for $\mathscr{G}_{\gamma,\theta,\mu,\lambda}$ and $A^1_\beta \subseteq \mathrm{Dom}(f^{\mathbf{x}_\beta})$ of cardinality $< \mu$ (or $\leq \mu$ if $\mu > \mathrm{cf}(\mu) \wedge \beta \geq \mathrm{cf}(\mu)$) and he actually plays $f^{\mathbf{x}_\beta} \restriction A^1_\beta$, i.e. is an initial segment of a play of $\mathscr{G}_{\gamma,\mu}$ of length β in which the ISO player uses the strategy **st** such that $[\beta_1 < \beta \Rightarrow \mathbf{x}_{\beta_1}$ is an initial segment of $\mathbf{x}_\beta]$.

The only problem is when $\beta = \alpha + 1$ and in Definition 1.7, Case 2 occurs, i.e. with the AIS player choosing $\mathbf{R}^1_\beta, \mathbf{R}^2_\beta$. We may for notational simplicity choose $\varepsilon < \theta$ and deal only with $A_\ell \cap {}^\varepsilon(M_\ell)$ for $\ell = 1, 2$.

We can consider $\mathbf{x}_\beta$ extending $\mathbf{x}_\alpha$; if it is as required in subcase (2B) of Definition 1.7 we are done. Let

$$\mathbf{F}^1_\beta = \{(\bar{a}_1, \bar{a}_2) : \begin{array}{l} \text{for some } \varepsilon < \theta, \bar{a}_\ell \in {}^\varepsilon(M_\ell) \text{ for } \ell = 1, 2 \\ \text{and there is a candidate } \mathbf{x}_\beta \text{ for the} \\ \beta\text{-th move such that } f^{\mathbf{x}_\beta}(\bar{a}_1) = \bar{a}_2\}. \end{array}$$

Let

$$\mathbf{F}^2_\beta = \{(\bar{a}_2, \bar{a}_1) : (\bar{a}_1, \bar{a}_2) \in \mathbf{F}^1_\beta\}$$

$$\mathscr{A}^1_\ell = \mathscr{A}_\ell$$

$$\mathscr{A}^2_\ell = \{\bar{a} \in \mathscr{A}_\ell : \text{ the number of } \bar{b} \text{ such that } (\bar{a}, \bar{b}) \in \mathbf{F}^\ell_\beta \text{ is } \leq \lambda\}$$

$$\mathscr{A}^3_\ell = \mathscr{A}^2_\ell \cup \{\bar{a} : \text{ for some } \bar{b} \in \mathscr{A}^2_{3-\ell} \text{ we have } (\bar{a}, \bar{b}) \in \mathbf{F}^\ell_\beta\}.$$

So $|\mathscr{A}^3_\ell| \leq \lambda$ by clause (d) of the assumption and let $\langle \bar{a}^\ell_\zeta : \zeta < \lambda \rangle$ list $\mathscr{A}^3_\ell$ possibly with repetitions

$(*)$ it is enough to take care of $\mathscr{A}^3_\ell \cap \mathscr{A}_\ell$ for $\ell = 1, 2$.

[Why? By the basic properties of dependence relation.]

So we can continue.

Let S be the set of limit ordinals $\delta < \lambda$ such that: for a club of $\delta_* \in [\delta, \lambda)$ of cofinality $\aleph_0$ we can find $\langle \bar{b}^\ell_\zeta : \zeta \in [\delta, \delta_*) \rangle$ for $\ell = 1, 2$ such that:

(α) $\bar{b}^\ell_\zeta \in \{\bar{a}^\ell_\xi : \xi \in [\delta, \delta_*)\}$

(β) $(\bar{b}^1_\zeta, \bar{b}^2_\zeta) \in \mathbf{F}^1_\beta$

(γ) $\langle \bar{b}^\ell_\zeta : \zeta \in [\delta, \delta_*)\rangle$ is $\mathbf{R}_\ell$-independent over $\{\bar{a}^\ell_\zeta : \zeta < \delta\}$

(δ) if $\zeta < \delta_*$ and $\bar{a}^\ell_\zeta \in \mathscr{A}_\ell$ then $\bar{a}^\ell_\zeta$ does R_ℓ-depend on $\{\bar{a}^\ell_\zeta : \zeta < \delta\} \cup \{\bar{b}^\ell_\zeta : \zeta \in [\delta, \delta_*)\}$.

If S is not stationary we can easily finish (we start by playing ω moves in $\mathscr{G}^\gamma_\mu$). So assume S is stationary, hence for some regular $\sigma \le \lambda_1$ the set $S' = \{\delta \in S : \operatorname{cf}(\delta) = \sigma\}$ is stationary. By playing $\sigma + \omega$ moves (recalling $\lambda \in \check{I}[\lambda]$) we get a contradiction to the definition of S.

2),3) Obvious. $\square_{1.12}$

Remark 1.11. In 1.10(1), to get the exact $\gamma(*)$, we combine partial isomorphisms. So we simulate two plays and use the composition of the $f^{\mathbf{x}^i_\beta}$'s from two plays where in each ISO use a winning strategy **st**.

Claim 1.12. *We can use a variant of Definition 1.7(2) as follows: we can in case 2 make a $\mathbf{R}_\ell$ dependence relation on $\kappa \times {}^{\theta>}(M_\ell)$, but equivalently $C \times {}^{\theta>}(M_\ell)$ for a set C of cardinality $\le \kappa$, <u>but</u>*

(*a*) *it seems to help presently relevant only for $\kappa \le 2^{|\tau(M_1)|+\aleph_0}$*

(*b*) *if $\kappa \le 2^{<\theta}$ we get an equivalent game.*

Remark 1.13. 1) We can replace $2^{<\theta}$ by a larger cardinal in clause (b) for "interesting" cases of M_1, M_2.

2) Anyhow we use only (b).

Proof. Clause (a) is obvious.

For clause (b), without loss of generality $\|M_\ell\| > 1$, now let $\langle \eta_\alpha : \alpha < \kappa\rangle$ be a sequence of pairwise distinct members of ${}^{\kappa>}2$. now we define $F_\ell : {}^{\theta>}(M_\ell) \to (2^{<\theta}) \times {}^{\theta>}(M_\ell)$ as follows: for $\bar{a} \in {}^{\theta>}(M_\ell)$ let

$$\mathbf{i}(\bar{a}) = \min\{i : 2i \ge \ell g(\bar{a}) \text{ or } 2i+1 < \ell g(\bar{a}) \wedge a_{2i} \ne a_{2i+1}\}$$

$$\eta_{\bar{a}} = \langle T.V.(a_{2\mathbf{i}(\bar{a})+2+2j} = a_{2\mathbf{i}(\bar{a})+2+2j+1}) : j \ge 0 \text{ and } 2\mathbf{i}(\bar{a}) + 2 + 2j + 1 < \ell g(\bar{a})\rangle$$

where T.V. stands for "truth value".

$$\alpha(\bar{a}) = \operatorname{Min}\{\alpha \le \kappa : \text{ if } \alpha < \kappa \text{ then } \eta_\alpha = \eta_{\bar{a}}\}.$$

Finally, $F_\ell(\bar{a})$ <u>is</u> $(\alpha(\bar{a}), \langle a_{2j} : j < i(\bar{a})\rangle)$ if $\mathbf{i}(a) < \ell g(\bar{a}) \wedge \alpha(\bar{a}) < \kappa$, and is $(0, \bar{a})$ if otherwise.

Let

$$\begin{aligned}\mathbf{R}'_\ell := \{\mathscr{A} :\ & \mathscr{A} \subseteq {}^{\theta>}(M_\ell) \text{ and } \{F(\bar{a}) : \bar{a} \in \mathscr{A}\} \in \mathscr{R} \\ & \text{or for some } \bar{a}' \ne \bar{a}'' \in \mathscr{A} \text{ we have } F(\bar{a}') = \bar{a}''\}.\end{aligned}$$

Now check. $\square_{1.12}$

Claim 1.14. *M_1, M_2 are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent <u>when</u>:*

(a) *K a class of τ_0-structures and $\Phi \in \Upsilon[K]$, see 2.2(8), used here for $K = K_{or}$ = the class of linear orders and K_{oi}, see Definition 2.1*
(b) *the structures $I_1, I_2 \in K$ are $\mathrm{EF}^+_{\gamma,\theta,\mu,\lambda}$-equivalent*
(c) *$M_\ell = \mathrm{EM}_\tau(I_\ell, \Phi)$ for $\ell = 1, 2$ for some $\tau \subseteq \tau_\Phi$*
(d) *$\mu \geq \aleph_0$ and $|\tau_\Phi| < \theta$.*

Proof. Let **St** be a winning strategy of the ISO player in the game $\mathscr{G}^+_{\gamma,\theta,\mu,\lambda}(I_1, I_2)$. We define a strategy $\mathbf{st}_*$ of the ISO player in the game $\mathscr{G}^+_{\gamma,\theta,\mu,\lambda}(M_1, M_2)$ as follows.

During a play of it after β moves a partial isomorphism f^*_α from M_1 to M_2 has been chosen, but the ISO player also simulates a play of $\mathscr{G}^+_{\gamma,\theta,\mu,\lambda}(I_1, I_2)$ in which we call the function h_α, and in which he uses the winning strategy **st** and

$\boxplus$ $f_\alpha \subseteq \hat{h}_\alpha$ where $\hat{h}_\alpha$ is defined by
$\hat{h}_\alpha(\sigma^{M_1}(a_{t_0}, \ldots, a_{t_{n-1}})) = \sigma^{M_\ell}(a_{h_\alpha(t_0)}, \ldots, a_{h_\alpha(t_{n-1})})$ for $n < \omega, \sigma(x_0, \ldots, x_{n-1})$ a term of τ_Φ and $t_0, \ldots, t_{n-1} \in \mathrm{Dom}(h_\alpha)$.

Why can the player ISO carry this strategy $\mathbf{st}_*$? Suppose we arrive to the β-th move. The point to check is Case 2 in Definition 1.7(2), so the AIS player has chosen $\mathbf{R}_1, \mathbf{R}_2, \mathscr{A}_1, \mathscr{A}_2$ as there.

Let $\{\bar{\sigma}_\zeta(\bar{x}_\zeta) : \zeta < 2^{<\theta}\rangle$ list $\{\bar{\sigma}(\bar{x}) : \bar{\sigma}(\bar{x}) = \langle \sigma_i(\bar{x}) : i < \ell g(\bar{\sigma})\rangle, \ell g(\bar{\sigma}) < \theta, \ell g(\bar{x}) < \theta$ and each σ_i is a τ_K-term.

Clearly ${}^{\theta>}(M_\ell) = \{\bar{\sigma}^{M_\ell}_\zeta(\bar{t}) : \zeta < 2^{<\theta}$ and $\bar{t} \in {}^{\ell g(\bar{x}_\zeta)}(I_\ell)\}$, so by clause (b) of 1.12, we can assume "$\mathbf{R}_\ell$ is a dependence relation on $\{(\zeta, \bar{t}_\zeta) : \zeta < 2^{<\theta}, \bar{t}_\zeta \in {}^\theta(I_\theta)$ and $\ell g(\bar{t}) = \ell g(\bar{t}_\zeta)\}$.

That is

$$\mathbf{R}'_\ell = \{u : \ \{\sigma^{M_\ell}_\zeta(\bar{t}) : (\zeta, \bar{t}) \in u\} \in \mathbf{R}_\ell\} \text{ \underline{or} there are } (\zeta_1, \bar{t}_1) \neq (\zeta_2, \bar{t}_2) \text{ from } u \text{ such that } \sigma^{M_\ell}_{\zeta_1}(\bar{a}_{\bar{t}_1}) = \sigma^{M_\ell}_{\zeta_2}(\bar{a}_{\bar{t}_2})\}.$$

The rest should be clear. $\square_{1.14}$

§ 2. The properties of T and relevant indiscernibility

In [She90, Ch.VIII], [Shear, Ch.VI] we use as indiscernible sets trees with $\omega+1$ levels, suitable for dealing with unsuperstable (complete first order) theories.

Here instead we use a linear order and family of ω-sequences from it, suitable for our case. The letters "oi" stands for order + increasing (ω-sequences).

Definition 2.1. 1) $K^{\rm oi}_\lambda$ is the class of structures $\mathbf{J}$ of the form $(J,Q,P<,F_n)_{n<\omega} = (|\mathbf{J}|,P^{\mathbf{J}},Q^{\mathbf{J}},<^{\mathbf{J}},F^{\mathbf{J}}_n)$, where $J=|\mathbf{J}|$ is a set of cardinality $\lambda,<^{\mathbf{J}}$ a linear order on $Q^{\mathbf{J}}\subseteq J,P^{\mathbf{J}}=|\mathbf{J}|\setminus Q^{\mathbf{J}},F^{\mathbf{J}}_n$ a unary function, $F^{\mathbf{J}}_n\restriction Q^{\mathbf{J}}=$ the identity and $a\in J\setminus Q^{\mathbf{I}}\Rightarrow F^{\mathbf{J}}_n(a)\in Q^{\mathbf{J}}$ and $n\neq m\Rightarrow F^{\mathbf{J}}_n(a)\neq F^{\mathbf{J}}_m(a)$ and for simplicity $a\neq b\in P^M\Rightarrow \bigvee\limits_{n<\omega} F_n(a)\neq F_n(b)$; lastly, we add $n<m\Rightarrow F^{\mathbf{J}}_n(a)<^{\mathbf{J}} F^{\mathbf{J}}_m(a)$ (there is a small price). We stipulate $F^{\mathbf{J}}_\omega=$ the identity on $|\mathbf{J}|$ and $I^{\mathbf{J}}=(Q^{\mathbf{J}},<^{\mathbf{J}})$.
1A) $K_{\rm oi}=\cup\{K^{\rm oi}_\lambda:\lambda \text{ a cardinal}\}$.
2) For a linear order I and $\mathfrak{S}\subseteq\ {\rm inc}({}^\omega I)$ (see Definition 0.10(3)), we let $\mathbf{J}=\mathbf{J}_{I,\mathfrak{S}}$ be the derived member of $K_{\rm oi}$ which means: $|\mathbf{J}|=I\cup\mathfrak{S},(Q^{|\mathbf{J}|},<^{\mathbf{J}})=I,F^{\mathbf{J}}_n(\eta)=\eta(n)$ for $n<\omega,F^{\mathbf{J}}_n(t)=t$ for $t\in I$.
3) $K^{\rm oi}_\lambda$ is the class of linear order of cardinality $\lambda,K_{\rm or}=\cup\{K^{\rm or}_\lambda:\lambda \text{ a cardinal}\}$.

Definition 2.2. 1) For a vocabulary τ_1 let $\Upsilon^{\rm oi}_{\tau_1}$ be the class of functions Φ with domain $\{{\rm tp}_{\rm qf}(\bar t,\emptyset,\mathbf{J}):\bar t\in{}^{\omega>}|\mathbf{J}|,\mathbf{J}\in K^{\rm oi}\}$, see 0.3 and if $q(x_0,\dots,s_{m-1})\in\ {\rm Dom}(\Phi)$ then $\Phi(q)$ is a complete quantifier free n-type in $\mathbb{L}(\tau_1)$ with the natural compatibility functions.
2) Let $\Upsilon^{\rm oi}_\kappa=\{\Phi:\Phi\in\Upsilon^{\rm oi}_{\tau_1}$ for some vocabulary τ_1 of cardinality $\kappa\}$.
3) For $\Phi\in\Upsilon^{\rm oi}_\kappa$ let $\tau(\Phi)=\tau_\Phi$ be the vocabulary τ_1 such that $\Phi\in\Upsilon^{\rm oi}_{\tau_1}$.
4) For $\Phi\in\Upsilon^{\rm oi}_\kappa,\mathbf{J}\in K^{\rm oi}$ let ${\rm EM}(\mathbf{J},\Phi)$ be "the" τ_Φ-model M_1 generated by $\{a_t:t\in\mathbf{J}\}$ such that: $n<\omega,\bar t\in{}^n\mathbf{J}\Rightarrow\ {\rm tp}_{\rm qf}(\langle a_{t_0},\dots,a_{t_{n-1}}\rangle,\emptyset,M_1)=\Phi({\rm tp}_{\rm qf}(\langle t_0,\dots,t_{n-1}\rangle,\emptyset,\mathbf{J})$.
5) If $\tau\subseteq\tau_\Phi$ then ${\rm EM}_\tau(\mathbf{J},\Phi)$ is the τ-reduct of ${\rm EM}(J,\Phi)$.
6) Let $\Upsilon^{\rm or}_{\tau_1},\Upsilon^{\rm or}_\kappa$ and ${\rm EM}(I,\Phi)$, ${\rm EM}_\tau(I,\Phi)$ be defined similarly for $\mathbf{J}$ a linear order.
7) Let $\Upsilon^{\omega\text{-tr}}_{\tau_1},\Upsilon^{\omega\text{-tr}}_\kappa$ and ${\rm EM}(I,\Phi)$, ${\rm EM}_\tau(I,\Phi)$ be defined similarly for $\mathbf{J}\in K^\omega_{\rm tr}$, i.e. trees with $\omega+1$ levels (with a linear order on the successor of any member of level $<\omega$).
8) We can above replace $K_{\rm oi}$ by any class K of τ_K-structures.

Definition 2.3. 1) A (complete first order) T is $\aleph_0$-independent $\equiv$ not strongly dependent (this is from [She09b, §3], see [She14, §1]) <u>when</u>: there is a sequence $\bar\varphi=\langle\varphi_n(x,\bar y_n):n<\omega\rangle$, (may use finite $\bar x$, as usual does not matter by [She14, 2.1]) of (first order) formulas such that T is consist with Γ_λ for some ($\equiv$ every $\lambda\geq\aleph_0$) where

$$\Gamma_\lambda=\{\varphi_n(x_\eta,\bar y^n_\alpha)^{{\rm if}(\alpha=\eta(n))}:\eta\in{}^\omega\lambda,\alpha<\lambda,n<\omega\}.$$

2) T is strongly stable <u>when</u> it is stable and strongly dependent.

Claim 2.4. *If T is first order complete, $T_1\supseteq T$ is first order complete, without loss of generality with Skolem functions and T is not strongly dependent <u>then</u> we can find $\bar\varphi=\langle\varphi_n(x,\bar y_n):n<\omega\rangle,\bar y_n\trianglelefteq\bar y_{n+1}$ and $\varphi_n(x,\bar y_n)\in\mathbb{L}(\tau_T)$ for $n<\omega$ such that*

- $\circledast$ *for any $\mathbf{J}\in K_{oi}$ we can find $M,\langle\bar a_t:t\in\mathbf{J}\rangle$ such that*
  - *(a) M is the Skolem hull of $\{\bar a_t:t\in\mathbf{J}\}$*

(b) *$\bar a_t \in {}^\omega M$ for $t \in I^{\mathbf J}, \bar a_\eta = \langle a_\eta \rangle \in M_1$ for $\eta \in P^{\mathbf J}$*

(c) *for $\eta \in P^{\mathbf J}, t \in Q^{\mathbf J}$ and $n < \omega$ we have $M \models \varphi_n[\bar a_\eta, \bar a_t]$ iff $F_n(\eta) = t$; (pedantically we should write $\varphi_n(a_\eta, \bar a_t \restriction \ell g(\bar y_n)))$*

(d) *$\langle \bar a_t : t \in \mathbf J\rangle$ is indiscernible in M for the index model $\mathbf J$*

(e) *M is a model of T_1*

(f) *in fact (not actually used, see 2.2) there is $\Phi \in \Upsilon^{oi}_{|T_1|}$ depending on $T_1, \bar\varphi$ only such that $M = \mathrm{EM}(\mathbf J, \Phi)$, in fact if $n < \omega, \bar t = \langle t_\ell : \ell < n\rangle \in \mathbf J$ then $\mathrm{tp}_{qf}(\bar a_{t_0} \hat{} \ldots \hat{} \bar a_{t_{n-1}}, \emptyset, M) = \Phi(\mathrm{tp}_{qf}(\bar t, \emptyset, \mathbf J))$.*

Proof. Let $I = (Q^{\mathbf J}, <_{\mathbf J})$. By an assumption, i.e. 2.3 there is a sequence $\langle \varphi'_n(x, \bar y_n) : n < \omega\rangle$ as in Definition 2.3 and let $k_n = \ell g(\bar y_n)$.

Let I be an infinite linear order. Easily we can find $M_1 \models T_1$ and a sequence $\langle \bar a_t : t \in I\rangle$ with $\bar a_t \in {}^\omega(M_1)$ such that for every $\eta \in {}^\omega I$, the set $\{\varphi_n(\bar x, \bar a_t)^{\mathrm{if}(\eta(n)=t)} : t \in I, n < \omega\}$ is a type, i.e. finitely satisfiable in M_1.

Now by Ramsey theorem without loss of generality $\langle \bar a_t : t \in I\rangle$ is an indiscernible sequence in M_1. Without loss of generality M_1 is λ^+-saturated, we then expand M_1 to M_1^+ by function $F_n^{M_1^+} (n < \omega)$, (of finite arity) such that for $t_0 <_{\mathbf J} \ldots <_{\mathbf J} t_{n-1}$ from $Q^{\mathbf J}$ the element $F_n(\bar a_{t_0}, \bar a_{t_1}, \ldots \bar a_{t_{n-1}})$ or more exactly $F_n(\bar a_{t_0} \restriction k_0, \bar a_{t_1} \restriction k_1, \ldots, \bar a_{t_{n-1}} \restriction k_{n-1})$ realizes in M_1 the type $\{\varphi_\ell(x, \bar a_t)^{\mathrm{if}(\eta(\ell)=t)} : t \in I, \ell < n\}$. Let M_2^+ be an expansion of M_1^+ by Skolem functions such that $|\tau_{M_2^+}| = |T_1|$, (natural, though not strictly required). Without loss of generality $\langle \bar a_t : t \in I\rangle$ is an indiscernible sequence also in M_2^+.

Let D be a non-principal ultrafilter on ω and in $M_3^+ = (M_2^+)^\omega/D$, we let $\bar a'_t = \langle \bar a_t : n < \omega\rangle/D$ for $t \in I$, and $\bar a'_\eta = \langle F_n(\bar a_{\eta(0)}, \bar a_{\eta(1)}, \ldots, \bar a_{\eta(n-1)}) : n < \omega\rangle/D$ for $\eta \in \mathrm{incr}({}^\omega I)$ and $\bar a'_t = \bar a'_{<F^{\mathbf J}_n(t):n<\omega>}$ for $t \in P^{\mathbf J}$.

Let M_4^+ be the submodel of M_3^+ generated by $\{\bar a'_t : t \in \mathbf J\}$ and M be $M_4^+ \restriction \tau(T_1)$. Now $M, \langle \bar a_t : t \in \mathbf J\rangle$ are as required. $\square_{2.4}$

Claim 2.5. *Assume $\mathbf J_\ell \in K_{oi}$, and $M_\ell, \bar\varphi, T_1, T$ as in 2.4 for $\ell = 1,2$. A sufficient condition for $M_1 \restriction \tau_T \ncong M_2 \restriction \tau_T$ is:*

(∗) *if f is a function from $\mathbf J_1$ (i.e. its universe) into $\mathscr M_{|T_1|,\aleph_0}(\mathbf J_2)$ (i.e. the free algebra generated by $\{x_t : t \in \mathbf J_2\}$ in the vocabulary $\tau_{|T_1|,\aleph_0} = \{F^n_\alpha : n < \omega$ and $\alpha < |T_1|\}, F^n_\alpha$ has arity n, see more in* [Shear, Ch.III,§1] = [Shea]*) we can find $t \in P^{\mathbf J_1}, n < \omega$, and $s_1, s_2 \in Q^{\mathbf J_1}$ and $k, \sigma, r^\ell_i(\ell = 1,2$ and $i < k), m, \sigma^*$ such that:*

(α) *$F^{\mathbf J_1}_n(t) = s_1 \neq s_2$*

(β) *for $\ell \in \{1,2\}$ we have $f(s_\ell) = \sigma(r^\ell_0, \ldots, r^\ell_{k-1})$ so $k < \omega, r^\ell_t \in \mathbf J_2$ for $i < k$ and σ is a $\tau_{|T_1|,\aleph_0}$-term not dependent on ℓ*

(γ) *$f(t) = \sigma^*(r_0, \ldots, r_{m-1}), \sigma^*$ is a $\tau_{|T_1|,\aleph_0}$-term and $r_0, \ldots, r_{m-1} \in \mathbf J_2$*

(δ) *the sequences*

$$\langle r^1_i : i < k\rangle \hat{} \langle r_i : i < m\rangle$$

$$\langle r^2_i : i < k\rangle \hat{} \langle r_i : i < m\rangle$$

realize the same quantifier free type in $\mathbf J_2$ (note: we should close by the $F^{\mathbf J_2}_n$'s, but no need to iterate as $F^{\mathbf J_2}_n \restriction Q^{\mathbf J_2}$ is the identity so quantifier *free type mean the truth value of the inequalities $F_{n_1}(r') \neq F_{n_2}(r')$ (including F_ω) and the order between those terms).*

Proof. Straight (or as in [Shear, Ch.III] = [Shea]). $\square_{2.5}$

Remark 2.6. We could have replaced $Q^{\mathbf{J}}$ by the disjoint union of $\langle Q^{\mathbf{J}}_n : n < \omega\rangle, <^{\mathbf{J}}$ linearly order each $Q^{\mathbf{J}}_n$ (and $<^{\mathbf{J}} = \cup\{< \restriction Q^{\mathbf{J}_1}_n : n < \omega\}$ and use Q_n to index parameters for $\varphi_n(x, \bar y_n)$). Does not matter at present.

But for our aim we can replace "not strongly stable" by a weaker demand, though this will not be carried here, we present it. Recall (from [She14, §5(G)], i.e. [She14, 5.39=dw5.35tex(2A)]) the following, an equivalent definition to "a (complete first order theory) T is strongly$_4$ dependent".

Definition 2.7. 1) A (complete first order) T is not strongly$_4$ dependent <u>if</u> there is a sequence $\bar\varphi = \langle \varphi_n(\bar x, \bar y_n) : n < \omega\rangle$, (finite $\bar x$ of length $m < \omega$, as usual) of (first order) formulas from $\mathbb{L}(\tau_T)$, an infinite linear order I, a sequence $\langle \bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\rangle$ indiscernible in M with $\ell g(\bar a_\eta) \le \omega$ and letting $B = \cup\{\bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\rangle$ for some $m < \omega$ and $p \in \mathbf{S}^m(B, M)$ for every $k < \omega$ there is $n < \omega$, <u>satisfying</u>: for no linear order I^+ extending I and subset I_0 of I^+ with $\le k$ members, <u>do we have</u>:

- $\otimes$ if $\bar t^1, \bar t^2$ are increasing sequences from I of the same length n realizing the same quantifier-free type over I_0 in I^+ and for $i = 1, 2$ we let $\bar b^i = (\ldots{}^\frown(\bar a_{\langle t^i_{\eta(\ell)} : \ell < \ell g(\eta)\rangle} \restriction n)^\frown \ldots)_{\eta \in \text{inc}_{<\omega}(n)}$ <u>then</u> $\ell < n \wedge u \subseteq \ell g(\bar b^1) \wedge |u| = \ell g(\bar y_\ell) \Rightarrow \varphi_\ell(x, \bar b^1 \restriction u) \in p \Leftrightarrow \varphi_\ell(\bar x, \bar b^2 \restriction u) \in p$.

1A) In (1) without loss of generality $\bar y_n \triangleleft \bar y_{n+1}$ for $n < \omega$.
2) T is strongly$_4$ stable if it is stable and strongly dependent.

Remark 2.8. 1) We can write the condition in 2.7(1) without I^+ speaking on finite sets as done in $(*)$ in the proof of 2.9 below.
2) In 2.7 by compactness we can get such $\langle \bar a'_\rho : \rho \in \text{inc}_{<\omega}(I')\rangle$ for any infinite linear order I'.

Next we deduce a consequence of being non-strongly$_4$-dependent, see Definition 2.7 helpful in proving non-structure results.

Claim 2.9. *If T is first order complete, $T_1 \supseteq T$ is first order complete, without loss of generality with Skolem functions and T is not strongly$_4$ dependent as witnessed by $\bar\varphi = \langle \varphi_n(\bar x, \bar y_n) : n < \omega\rangle$, i.e. as in Definition 2.7(1A), <u>then</u> there is $\tau_1 \supseteq \tau_{T_1}, |\tau_1| = |T_1|$ and $\bar\sigma_n(\bar z_n) = \langle \sigma_{n,\ell}(\bar z_n) : \ell < \ell g(\bar\sigma_n)\rangle, \sigma_{n,\ell}$ is a τ_1-term such that:*

- $\circledast$ *if $I, \mathfrak{S}$ and $\mathbf{J} = \mathbf{J}_{I,\mathfrak{S}}$ are as in Definition 2.1(2), <u>then</u> there are M_1 and $\langle \bar a_t : t \in I\rangle$ and $\langle \bar a_\eta : \eta \in \mathfrak{S}\rangle$ such that:*
  - (α) *M_1 is a τ_1-model and is the Skolem hull of $\{\bar a_t : t \in I\} \cup \{\bar a_\eta : \eta \in \mathfrak{S}\}$ (we write $\bar a_t$ for $t \in \mathfrak{S} \subseteq \mathbf{J}$ for uniformity)*
  - (β) *$\langle \bar a_t : t \in \mathbf{J}\rangle$ is indiscernible in M_1,*
  - (γ) *if $\eta \in \mathfrak{S}$ and $k < \omega$ <u>then</u> for large enough $n(*)$ we have:*
    - $(*)$ *if $u \subseteq n(*), |u| \le k$, then we can find $\bar s, \bar t$ and $n_* < n(*)$ and $\bar\sigma$ such that*
      - (i) *$\bar s, \bar t$ are sequences of members of $\{F^{\mathbf{J}}_n(\eta) : n < n(*)\}$*

(ii) $\ell g(\bar s) = \ell g(\bar t) \le n(*)$
(iii) $s_i <_I s_j \Leftrightarrow t_i <_I t_j$ *for* $i, j < \ell g(\bar s)$
(iv) *if* $i < \ell g(\bar s) = \ell g(\bar t)$ *then* $(\forall n \in u)(F^{\mathbf{J}}_n(\eta) \le_I s_i \equiv F^{\mathbf{J}}_n(\eta) \le_I t_i)$
(v) $\bar\sigma = \langle \sigma_i(\bar y) : i < \ell g(\bar y_{n_*})\rangle, \sigma_i$ *a* τ_1*-term*
(vi) $M_1 \models \varphi_{n_*}[\bar a_\eta, \ldots, \sigma_i^{M_1}(\bar a_{t_0}, \bar a_{t_1}, \ldots), \ldots]_{i<\ell g(\bar y_{n_*})} \equiv \neg\varphi_{n_*}[\bar a_\eta, \ldots, \sigma_i^{M_1}(\bar a_{s_0}, \bar a_{s_1}, \ldots), \ldots]_{i<\ell g(\bar y_{n_*})}$

(δ) M_1 *is a model of* T_1 *so* $\tau_{M_1} \supseteq \tau_{T_1}$.

Proof. Fix $I, \mathfrak{S}$; without loss of generality I is dense with neither first nor last element and is $\aleph_1$-homogeneous hence there are infinite increasing sequences of members of I.

Let $I, \langle \varphi_n(\bar x, \bar y_n) : n < \omega\rangle, \langle \bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\rangle$ and $p \in \mathbf{S}^m(\cup\{\bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\})$ exemplify T is not strongly$_4$ dependent, i.e. be as in the Definition so $m = \ell g(\bar x)$. For notational simplicity (and even without loss of generality by [She14, §5]) assume $m = 1$.

Now in 2.7 we can add:

($*$) there is a sequence $\langle (n_k, m_k, I^*_k) : k < \omega\rangle$ such that $k < n_k < m_k, m_k < m_{k+1}, I^*_k \subseteq I$ has m_k members, for no $I_0 \subseteq I^*_k$ with $\le k$ members does $\otimes$ from 2.7 holds for $\bar t^1, \bar t^2 \in \text{incr}_{<n_k}(I^*_k)$ and (k, n_k) here standing for k, n.

[Why? By compactness.]

Without loss of generality I is the reduct to the vocabulary $\{<\}$, i.e. to just a linear order of an ordered field $\mathbb{F}$ and $t_q \in \mathbb{F}$ for $q \in \mathbb{Q}$ are such that $0 <_{\mathbb{F}} t_q, (t_{q_1})^2 <_{\mathbb{F}} t_{q_2}$ for $q_1 <_{\mathbb{F}} q_2$ (hence $n < \omega \Rightarrow n <_{\mathbb{F}} t^n_{q_1} <_{\mathbb{F}} t_{q_2}$). By easy manipulation without loss of generality $I^*_k = \{t_i : i = 0, 1, \ldots, m_k\}$.

Now for each $m < \omega$ and $\eta \in \text{incr}_m(I)$ we can choose c_η such that if $m = m_k$ then for some automorphism h of I mapping I^*_k onto $\text{Rang}(\eta)$, letting $\hat h$ be an automorphism of M_1 mapping $\bar a_\nu$ to $\bar a_{h(\nu)}$ for $\nu \in \text{incr}_{<\omega}(I)$, the element c_η realizes $\hat h(p)$ and $\langle c_\eta : \eta \in \text{incr}_{<\omega}(I)\rangle$ is without repetitions.

Now without loss of generality $\langle \langle c_\eta\rangle\hat{}\bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\rangle$ is an indiscernible sequence and let $a_t = c_{<t>}$ be such that M_0 be a model of T_1 satisfying $\cup\{\langle c_\eta\rangle\hat{}\bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\} \subseteq M_0 \restriction \tau \prec \mathfrak{C}$. Without loss of generality $\langle \langle c_\eta\rangle\hat{}\bar a_\eta : \eta \in \text{incr}_{<\omega}(I)\rangle$ is indiscernible in M_0 and we can find an expansion M_1 of M_0 such that $|\tau_{M_2}| = |T_1|$ such that $\bar a_\eta = \langle F_{\ell g(\eta), i}(\bar a_{\eta(0)}, \ldots, \bar a_{\eta(n-1)}) : i < \ell g(\bar a_\eta)\rangle, c_\eta = F_{\ell g(\eta)}(\bar a_{\eta(0)}, \ldots, \bar a_{\eta(n-1)})$ if $\eta \in \text{incr}_n(I)$ and M_1 has Skolem functions.

By manipulating I without loss of generality we can find $I_* \subseteq I$ of order type ω.

So for some $H_n \in \tau_1$ for $n < \omega$,

$\odot$ if $t_0 < t_1 < \ldots$ list I_*, for every $k < \omega$ large enough, for every $u \subseteq n(*)$ satisfying $|u| \le k$ for every n large enough $H_n^{M_1}(\bar a_{t_0}, \bar a_{t_1}, \ldots, \bar a_{t_{n-1}})$ satisfies the demand (on the singleton $\bar a_\eta$ from clause (γ) in the claim).

Let D be a non-principal ultrafilter on ω such that $\{m_k : k < \omega\} \in D$, let M_2 be isomorphic to M_1^ω/D over M_1, i.e. $M_1 \prec M_2$ and there is an isomorphism $\mathbf{f}$ from M_2 onto M_1^ω/D extending the canonical embedding.

If η is an increasing ω-sequence of members of I, we let

$$a^n_\eta = H^{M_1}_n(a_{\eta(0)}, \dots, a_{\eta(n-1)}) \in M_1$$

and let

$$a_\eta = \mathbf{j}^{-1}(\langle a^0_\eta, a^1_\eta, \dots, a^n_\eta, \dots : n < \omega\rangle / D) \in M_2.$$

Let M'_2 be the Skolem hull of $\{\bar{a}_t : t \in I\} \cup \{a_\eta : \eta \in \mathfrak{S}\}$ inside M_2. It is easy to check that it is as required. $\square_{2.9}$

Naturally it is helpful to have a sufficient condition for the non-isomorphism of two such models:

Claim 2.10. *Assume $\mathbf{J}_\ell \in K^{oi}$, and $M_\ell, \bar{\varphi}, T_1, T$ as in 2.9 for $\ell = 1, 2$. A sufficient condition for $M_1 \ncong M_2$ is*

(∗) *if f is a function from $\mathbf{J}_1$ (i.e. its universe) into $\mathscr{M}_{|T_1|,\aleph_0}(\mathbf{J}_2)$ (i.e. the free algebra generated by $\{x_t : t \in \mathbf{J}_1\}$ the vocabulary $\tau_{|T_1|,\aleph_0} = \{F^n_\alpha : n < \omega$ and $\alpha < |T_1|\}, F^n_\alpha$ has arity n), we can find $t \in P^{\mathbf{J}_1}$ and $k_* < \omega$ such that for every $n_* < \omega$ we can find $\bar{s}_1, \bar{s}_2$ such that:*
   - (α) *$\bar{s}_1, \bar{s}_2 \in^{k\ n_*}(Q^{\mathbf{J}})$ are increasing, $\bar{s}_1 = \langle F^{\mathbf{J}}_n(t) : n < n_*\rangle$ and $n < k_* \Rightarrow s_{2,n} = s_{1,n}$ and $s_{1,n_*-1} <_I s_{2,k_*}$*
   - (β) *$f(\bar{s}_\ell) = \bar{\sigma}(r^\ell_0, \dots, r^\ell_{k-1})$ so $k < \omega, r^\ell_t \in \mathbf{J}_2$ for $i < k$ so σ is a $\tau_{|T_1|,\aleph_0}$-term not dependent on ℓ*
   - (γ) *$f(t) = \sigma^*(r_0, \dots, r_{m-1}), \sigma^*$ is a $\tau_{|T_1|,\aleph_0}$-term and $r_0, \dots, r_{m-1} \in \mathbf{J}_2$*
   - (δ) *the sequences*

$$\langle r^1_i : i < k\rangle\hat{\ }\langle r_i : i < m\rangle$$

$$\langle r^2_i : i < k\rangle\hat{\ }\langle r_i : i < m\rangle$$

   realize the same quantifier free type in $\mathbf{J}_2$ (note: we should close by the $F^{\mathbf{J}_2}_n$, so type mean the truth value of the inequalities $F_{n_1}(r') \neq F_{n_2}(r')$ (including F_ω) and the order between those terms).

Proof. As in [She87b, Ch.III] or better in [Shear, Ch.III] = [Shea], called unembeddability. $\square_{2.10}$

§ 3. Forcing EF$^+$-equivalent Consistency non-isomorphic models

The following result is not optimal, but it is enough to prove necessary conditions on T for being lean and even on $(T,*)$. As for unstable T, see below in §4. So our main result is

Claim 3.1. *Assume $(\bar{\varphi},T,T_1,\Phi)$ is as in 2.4, T stable and $\lambda=\lambda^{<\lambda}\ge\aleph_1+|T_1|$ and $\mu=\lambda^+>\lambda$. Then for some λ-complete λ^+-c.c. forcing notion $\mathbb{Q}$ we have: $\Vdash_{\mathbb{Q}}$ "there are models M_1,M_2 of T of cardinality λ^+ such that $M_1\restriction\tau(T),M_2\restriction\tau(T)$ are $\mathrm{EF}^+_{\alpha,\lambda,\lambda^+}$-equivalent for every $\alpha<\lambda$ but are not isomorphic".*

Remark 3.2. 1) It should be clear that we can improve it allowing $\alpha<\lambda^+$ and replacing forcing by e.g. $2^\lambda=\lambda^+$ and $\lambda=\lambda^{<\lambda}$, but we shall continue in [S$^+$b].

Proof. We define $\mathbb{Q}$ as follows:

$\circledast_1$ $p\in\mathbb{Q}$ iff p consist of the following objects satisfying the following conditions:
- (a) $u=u^p\in[\mu]^{<\lambda}$ such that $\alpha+i\in u\wedge i<\lambda\Rightarrow\alpha\in u$
- (b) $<^p$ a linear order of u such that
$$\alpha,\beta\in u\wedge\alpha+\lambda\le\beta\Rightarrow\alpha<^p\beta$$
- (c) for $\ell=1,2$ $\quad\mathfrak{S}^p_\ell$ is a subset of $\{\eta\in{}^\omega u:\eta(n)+\lambda\le\eta(n+1)\text{ for }n<\omega\}$ such that $\eta\ne\nu\in\mathfrak{S}^p_\ell\Rightarrow\mathrm{Rang}(\eta)\cap\mathrm{Rang}(\nu)$ is finite; note that in particular $\eta\in\mathfrak{S}^p_\ell$ is without repetitions and is $<^p$-increasing
- (d) Λ^p a set of $<\lambda$ increasing sequences of ordinals from $\{\alpha\in u^p:\lambda|\alpha\}$ hence of length $<\lambda$
- (e) $\bar{f}^p=\langle f^p_\rho:\rho\in\Lambda^p\rangle$ such that
- (f) f^p_ρ is a partial automorphism of the linear order $(u^p,<^p)$ such that $\alpha\in\mathrm{Dom}(f^p_\rho)\Rightarrow\alpha+\lambda=f^p_\rho(\alpha)+\lambda$ and we let $f^{1,p}_\rho=f^p_\rho,f^{2,p}_\rho=(f^p_\rho)^{-1}$
- (g) if $\eta\in\mathfrak{S}^p_\ell,\rho\in\Lambda^p,\ell\in\{1,2\}$ then $\mathrm{Rang}(\eta)$ is included in $\mathrm{Dom}(f^{\ell,p}_\rho)$ or is almost disjoint to it (i.e. except finitely many "errors")
- (h) if $\rho\lhd\varrho\in\Lambda^p$ then $\rho\in\Lambda^p$ and $f^p_\rho\subseteq f^p_\varrho$
- (i) $f^{\ell,p}_{<>}$ is the empty function and if $\rho\in\Lambda^p$ has limit length then
$$f^p_\rho=\cup\{f^p_{\rho\restriction i}:i<\ell g(\rho)\}$$
- (j) if $\rho\in\Lambda^p$ has length $i+1$ then $\mathrm{Dom}(f^{\ell,p}_\rho)\subseteq\rho(i)$ for $\ell=1,2$
- (k) if $\rho\in\Lambda^p$ and $\eta\in{}^\omega(\mathrm{Dom}(f^p_\rho))$ then $\eta\in\mathfrak{S}^p_1\Leftrightarrow\langle f^p_\rho(\eta(n)):n<\omega\rangle\in\mathfrak{S}^p_2$
- (ℓ) if $\rho_n\in\Lambda^p$ for $n<\omega$ and $\rho_n\lhd\rho_{n+1}$ and $\lambda>\aleph_0$ then $\cup\{\rho_n:n<\omega\}\in\Lambda$

$\circledast_2$ We define the order $\le=\le_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows: $p\le q$ iff ($p,q\in\mathbb{Q}$ and)
- (a) $u^p\subseteq u^q$
- (b) $\le^p=\le^q\restriction u^p$
- (c) $\mathfrak{S}^p_\ell\subseteq\mathfrak{S}^q_\ell$ for $\ell=1,2$

(d) $\Lambda^p \subseteq \Lambda^q$

(e) if $\rho \in \Lambda^p$ then $f^p_\rho \subseteq f^q_\rho$

(f) if $\eta \in \mathfrak{S}^q_\ell \setminus \mathfrak{S}^p_\ell$ then $\mathrm{Rang}(\eta) \cap u^p$ is finite

(g) if $\rho \in \Lambda^p$ and $f^p_\rho \neq f^q_\rho$ <u>then</u> $u^p \cap \sup \mathrm{Rang}(\rho) \subseteq \mathrm{Dom}(f^{\ell,q}_\rho)$ for $\ell = 1,2$

(h) if $\rho \in \Lambda^p$ and $\ell \in \{1,2\}, \alpha \in u^p \setminus \mathrm{Dom}(f^{\ell,p}_\rho)$ and $\alpha \in \mathrm{Dom}(f^{\ell,q}_\rho)$ then $f^{\ell,p}_\rho(\alpha) \notin u^p$

Having defined the forcing notion $\mathbb{Q}$ we start to investigate it.

$\circledast_3$ $\mathbb{Q}$ is a partial order of cardinality $\mu^{<\lambda} = \lambda^+$.

[Why? Obviously.]

$\circledast_4$ (i) if $\bar p = \langle p_i : i < \delta\rangle$ is $\leq^{\mathbb{Q}}$-increasing, δ a limit ordinal $< \lambda$ of uncountable cofinality then $p_\delta := \cup\{p_i : i < \delta\}$ defined naturally is an upper bound of $\bar p$

(ii) if $\delta < \lambda$ is a limit ordinal of cofinality $\aleph_0$ and the sequence $\bar p = \langle p_i : i < \delta\rangle$ is increasing (in $\mathbb{Q}$), <u>then</u> it has an upper bound.

Why (i)? Think or see (ii); why the case $\mathrm{cf}(\delta) > \aleph_0$ is easier? Because of clause $\circledast_1(i)$ and $\circledast_1(\ell)$. Why (ii)? We define $p_\delta \in \mathbb{Q}$ as follows: $u^{p_\delta} = \cup\{u^{p_i} : i < \delta\}$, $<^{p_\delta} = \cup\{<^{p_i} : i < \delta\}, \Lambda^{p_\delta} = \cup\{\Lambda^{p_i} : i < \delta\} \cup \{\rho : \rho$ is an increasing sequence of ordinals from u^{p_δ} of length a limit ordinal of cofinality $\aleph_0$ such that $\varepsilon < \ell g(\rho) \Rightarrow \rho \restriction \varepsilon \in \cup\{\Lambda^{p_i} : i < \delta\}\}$.

Let $\bar f^{p_\delta} = \langle f^{p_\delta}_\rho : \rho \in \Lambda^q\rangle$ where: if $i < \delta$ and $\rho \in \Lambda^{p_i} \setminus \cup\{\Lambda^{p_j} : j < i\}$, then $f^q_\rho = \cup\{f^{p_j}_\rho : j \in [i,\delta)\}$ and if $\rho \in \Lambda^{p_\delta} \setminus \{\Lambda^{p_i} : i < \delta\}$ then $f^{p_\delta}_\rho = \cup\{f^{p_\delta}_{\rho\restriction\varepsilon} : \varepsilon < \ell g(\rho)\}$ is well defined as $\varepsilon < \ell g(\rho) \Rightarrow \rho\hat{\ }\langle\varepsilon\rangle \in \cup\{\Lambda^{p_j}; j < \delta\}$. Clearly clauses (a),(b),(d),(e),(f),(h),(i),(j),(ℓ) from $\circledast_1$ for $p_\delta \in \mathbb{Q}$ hold.

Lastly, let $\mathfrak{S}^{p_\delta}_\ell = \cup\{\mathfrak{S}^{p_\alpha}_\ell : \alpha < \delta\}$ for $\ell = 1,2$.

Note

$\odot_1$ if $\rho \in \Lambda^{p_\delta} \setminus \cup\{\Lambda^{p_\alpha} : \alpha < \delta\}$ then $\mathrm{Dom}(f^{p_\delta}_\rho) = u^{p_\delta} \cap \sup \mathrm{Rang}(\rho) = \mathrm{Rang}(f^{p_\delta}_\rho)$ and for every $\alpha < \delta$ for some $\beta < \delta$ we have $f^{p_\delta}_\rho \restriction u^{p_\alpha} \subseteq f^{p_\beta}_{\rho\restriction i}$ for some $i < \ell g(\rho)$.

[Why? Clearly, $\mathrm{cf}(\ell g(\rho)) = \aleph_0$.

Assume $\alpha < \delta$ and $i < \ell g(\rho)$. Clearly for some $\beta \in (\alpha,\delta)$ we have $\rho\restriction i \in \Lambda^{p_\beta}$. Also the set $\{j < \ell g(\rho) : \rho\restriction j \in \Lambda^{p_\beta}\}$ is an initial segment of $\ell g(\rho)$ and cannot be $\ell g(\rho)$ ecause $\rho \notin \Lambda^{p_\beta}$ by clause $\circledast_1(\ell)$. So for some $j < \ell g(\rho)$ we have $\rho\restriction j \notin \Lambda^{p_\beta}$ but by the choice of ρ for some $\gamma < \delta$ we have $\rho\restriction j \in \Lambda^{p_\gamma}$, so necessarily $\beta < \gamma$. As $p_\alpha \leq_{\mathbb{Q}} p_\beta \leq_{\mathbb{Q}} p_\gamma$ by clause (g) of $\circledast_2$, as $\rho\restriction i \in \Lambda^{p_\gamma} \setminus \Lambda^{p_\beta}$ we know that $u^{p_\beta} \cap \sup \mathrm{Rang}(\rho\restriction i)$ is included in $\mathrm{Dom}(f^{\ell,p_\gamma}_{\rho\restriction i})$ for $\ell = 1,2$ by $p_\alpha \leq_{\mathbb{Q}} p_\beta$ hence $u^{p_\alpha} \cap \sup \mathrm{Rang}(\rho\restriction i)$ is included in $\mathrm{Dom}(f^{\ell,p_\gamma}_{\rho\restriction i_\alpha})$ which $\subseteq \mathrm{Dom}(f^{\ell,p_\delta}_{\rho\restriction i})$ for $\ell = 1,2$.

As this holds for any $\alpha < \delta$ and $i < \ell g(\rho)$ and $u^{p_\delta} \cap \sup \mathrm{Rang}(\rho\restriction i) = \cup\{u^{p_\alpha} \cap \sup \mathrm{Rang}(\rho) : \alpha < \delta\}$ it follows that for $\ell = 1,2$ we have $\varepsilon \in u^{p_\delta} \cap \sup \mathrm{Rang}(\rho) \Rightarrow (\exists \alpha < \delta)(\varepsilon \in u^{p_\alpha} \cap \sup \mathrm{Rang}(\rho)) \Rightarrow (\exists \beta < \delta)[\varepsilon \in \mathrm{Dom}(f^{\ell,p_\delta}_\rho)] \Rightarrow \varepsilon \in \mathrm{Dom}(f^{\ell,p_\delta}_\rho)$ so are done.]

$\odot_2$ if $\rho \in \cup\{\Lambda^{p_\alpha} : \alpha < \delta\}$ then exactly one of the following occurs:

(a) there is a unique $\alpha = \alpha(\rho) < \delta$ such that $\rho \in \Lambda^{p_\alpha}, (\forall \beta)(\alpha \le \beta < \delta \Rightarrow f^{p_\beta}_\rho = f^{p_\alpha}_\rho)$ and $(\forall \beta < \alpha)(\rho \in \Lambda^{p_\beta} \rightarrow f^{p_\beta}_\rho \ne f^{p_\alpha}_\rho)$

(b) $\mathrm{Dom}(f^{p_\delta}_\rho) = u^{p_\delta} \cap \sup \mathrm{Rang}(\rho) = \mathrm{Rang}(f^{p_\delta}_\rho)$ and $(\forall \alpha < \delta)(\exists \beta < \delta)(f^{p_\delta}_\rho {\restriction} u^{p_\alpha} \subseteq f^{p_\beta}_\rho)$.

[Why? Similarly to the proof of $\odot_1$.]

To finish proving $p_\delta \in \mathbb{Q}$, i.e. verifying $\circledast_1$ holds, we have to check clauses (c),(g),(k).

Clause (c): Obvious by the choice of $\mathfrak{S}^{p_\delta}_1$.

Clause (g):

So let $\eta \in \mathfrak{S}^{p_\delta}_\ell, \rho \in \Lambda^{p_\delta}$ where $\ell \in \{1,2\}$ and we should prove that $\mathrm{Rang}(\eta) \subseteq \mathrm{Dom}(f^{\ell,p,\delta}_\rho)$ or $\mathrm{Rang}(\eta) \cap \mathrm{Dom}(f^{\ell,p}_\rho)$ is finite. For some $\alpha < \delta$ we have $\eta \in \mathfrak{S}^{p_\alpha}_\ell$. If $\rho \in \cup\{\Lambda^{p_\beta} : \beta < \delta\}$ then we apply $\odot_2$, now if clause (a) there holds so $\alpha = \alpha(\rho) < \delta$ is well defined and we use $p_\alpha \in \mathbb{Q}$ and if clause (b) there holds then trivially $\mathrm{Rang}(\eta) \subseteq u^{p_\delta} \subseteq \mathrm{Dom}(f^{\ell,p_\delta}_\rho)$ so assume $\rho \in \Lambda^{p_\delta} \setminus \cup \{\Lambda^{p_\beta} : \beta < \delta\}$.

By $\odot_1$ we finish as in the case $\odot_2(b)$ holds.

Clause (k):

By the choice of $\mathfrak{S}^{p_\delta}$ and the proof of clause (g).

Checking $p_\alpha \le_{\mathbb{Q}} p_\delta$: (where $\alpha < \delta$)

We should check that the pair (p_α, p_δ) satisfies the demands in $\circledast_2$ which is straight. $\circledast_2$.

So we have proved $\circledast_4$.

$\circledast_5$ if $\alpha < \mu$ then $\mathscr{I}^1_\alpha := \{p \in \mathbb{Q} : \alpha \in u^p\}$ is dense and open as well as $\mathscr{I}_* = \{p \in \mathbb{Q}$: if $\delta \in u^p, \lambda | \delta$ and $\mathrm{cf}(\delta) < \lambda$ then $\delta = \sup(\delta \cap u)\}$.

[Why? Straight. For the first, $\mathscr{I}^1_\alpha$, given $p \in \mathbb{Q}$ we define $q \in \mathbb{Q}$ by

(a) u^q is $u^p \cup \{\beta \le \alpha : \beta + \lambda = \alpha + \lambda\}$, so clause $\circledast_1(a)$ holds

(b) $<^q$ is the following linear order on u^q

$\alpha_1 < \alpha_2$ iff $\alpha_1 <^p \alpha_2$ or $\alpha_1 < \alpha_2 \wedge \{\alpha_1, \alpha_2\} \nsubseteq u^p \wedge \{\alpha_1, \alpha_2\} \subseteq u^q$

(c) $\mathfrak{S}^q_\ell = \mathfrak{S}^p_\ell$ for $\ell = 1, 2$

(d) $\Lambda^q = \Lambda^p$ and

(e) $f^q_\rho = f^p_\rho$ for $\rho \in \Lambda^q$.

Now check.

For the second, $\mathscr{I}_*$ use the first and $\circledast_4$.]

$\circledast_6$ if $\varrho \in \Lambda^* := \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by λ of length $< \lambda\}$ then $\mathscr{I}^2_\varrho = \{p \in \mathbb{Q} : \varrho \in \Lambda^p\}$ is dense open.

[Why? Let $p \in \mathbb{Q}$, by $\circledast_5 + \circledast_4$ there is $q \ge p$ (from $\mathbb{Q}$) such that $\mathrm{Rang}(\varrho) \subseteq u^q$. If $\varrho \in \Lambda^q$ we are done, otherwise define q' as follows: $u^{q'} = u^q, <^{q'} = <^q, \mathfrak{S}^{q'}_\ell = \mathfrak{S}^q_\ell, \Lambda^{q'} = \Lambda^q \cup \{\varrho \restriction \varepsilon : \varepsilon \le \ell g(\varrho\}$ and if $i \le \ell g(\varrho), \varrho \restriction i \notin \Lambda^q$ then we let $f^{q'}_{\varrho \restriction i} = \cup\{f^q_\rho : \rho \in \Lambda^q$ and $\rho \triangleleft \varrho \restriction i\}$. We should check all the clauses of $\circledast_1$ for "$q \in \mathbb{Q}$" and e.g. clause (k) there holds because q satisfies clause (ℓ). Then we should check all the clauses of $\circledast_2$ for "$q \le_{\mathbb{Q}} q'$"]

$\circledast_7$ if ϱ is as in $\circledast_8$ and $\alpha < \lambda^+$ and $\ell \in \{1,2\}$ <u>then</u>

$\mathscr{I}^3_{\varrho,\alpha,\ell} = \{p \in \mathbb{Q} : \alpha \in \mathrm{Dom}(f^{\ell,p}_\varrho)$ so $\varrho \in \Lambda^p, \alpha \in u^p\}$ is dense open.

[Why? By $\circledast_5 + \circledast_6$.]

$\circledast_8$ if $p \in \mathbb{Q}$ and $\varrho \in \Lambda^p$ <u>then</u> for some q we have $p \le_{\mathbb{Q}} q \wedge f^q_\varrho \ne f^p_\varrho \wedge \{\alpha+\lambda : \alpha \in u^q\} = \{\alpha + \lambda : \alpha \in u^p\}$.

Why? For each $\delta \in u \cap \sup \mathrm{Rang}(\varrho)$ divisible by λ let $u_\delta = u \cap [\delta, \delta+\lambda)$. So $g_\delta := f^p_\rho \restriction u_\delta$ is a partial function from u_δ into u_δ and $f^p_\rho = \cup\{g_\delta : \delta$ as above$\}$. Now, for δ as above we can find f_δ such that:

(a) f_δ is a one-to-one function
(b) $g_\delta = f^p_\varrho \restriction u_\delta \subseteq f_\delta$
(c) if $\alpha \in \mathrm{Dom}(f_\delta)$ iff $\alpha \in u_\delta \vee f_\delta(\alpha) \in u_\delta$
(d) $\mathrm{Dom}(f_\delta)\backslash u_\delta$ is an initial segment $[\alpha^1_\delta, \alpha^2_\delta)$ of $[\delta, \delta+\lambda)\backslash u_\delta$
(e) $\mathrm{Rang}(f_\delta)\backslash u$ is an initial segment $[\alpha^2_\delta, \alpha^3_\delta)$ of $[\delta,\delta+\lambda)\backslash u \backslash \mathrm{Dom}(f_\delta)$
(f) f_δ maps $[\alpha^1_\delta, \alpha^2_\delta)$ onto $u_\delta \backslash \mathrm{Rang}(f^p_\varrho \restriction u_\delta)$
(g) f_δ maps $u_\delta \backslash \mathrm{Dom}(f^p_\varrho \restriction u_1)$ onto $[\alpha^2_\delta, \alpha^3_\delta)$.

Now we can find a linear order $<_1$ on $u_\delta \cup [\alpha^1_\delta, \alpha^3_\delta]$ such that f_δ is order preserving (as the class of linear orders has amalgamation).

Lastly, we define q:

(α) $u^q = u^p \cup \{[\alpha^1_\delta, \alpha^3_\delta) : \delta$ as above$\}$
(β) $<^q$ is defined by $\alpha <^q \beta$ iff $(\exists \delta)(\alpha <_\delta \beta)$ or $\alpha + \lambda \le \beta$
(γ) $\Lambda^q = \Lambda^p$
(δ) $\mathfrak{S}^q_\ell = \mathfrak{S}^p_\ell \cup \{\langle f^{3-\ell}_\rho(\eta(n)) : n < \omega\rangle : \ell \in \{1,2\}, \rho \in \Lambda^p$ and $\eta \in \mathfrak{S}^p_{3-\ell}\}$.

Now we have to check $q \in \mathbb{Q}$, i.e. all the clauses of $\circledast_1$. This is straight; e.g. for clause (c), assume $\eta \ne \nu \in \mathfrak{S}^q_\ell$ and we have to prove that $\mathrm{Rang}(\eta) \cap \mathrm{Rang}(\nu)$ is finite.

Now we have four cases: first $\eta, \nu \in \mathfrak{S}^p_\ell$, so use $p \in \mathbb{Q}$, clause $\circledast_1(c)$ for ℓ. Second, $\eta, \nu \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell$, so η, ν are images by $f^{3-\ell,q}_\rho$ of members of $\mathfrak{S}^p_{3-\ell}$, as this function is one-to-one, this follows from $p, \mathfrak{S}^p_{3-\ell}$ satisfying clause $\circledast_1(c)$. Third, $\eta \in \mathfrak{S}^p_\ell \wedge \nu \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell$, then $\nu = \langle f^{3-\ell,q}_\rho(\nu'(n)) : n < \omega\rangle$ for some $\nu' \in \mathfrak{S}^p_{3-\ell}$ satisfying $\mathrm{Rang}(\nu') \nsubseteq \mathrm{Dom}(f^{3-\ell,p}_\rho)$, hence for some $n_* < \omega$ we have $n \in [n_*, \omega) \Rightarrow \nu'(n) \notin \mathrm{Dom}(f^{3-\ell,p}_\rho) \Rightarrow \nu(n) \notin u^p$ but $\mathrm{Rang}(\eta) \subseteq u^p$ so we are done. Fourth, $\eta \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell \wedge \nu \in \mathfrak{S}^p_\ell$ the proof is dual.

The proof of clause (g) is similar.

Also we have to check that $p \le_{\mathbb{Q}} q$, i.e. all the clauses of $\circledast_2$ for the pair (p,q). This is straight, clause (f) is sproved as in the proof of $\circledast_1(c)$ above and clause (h) holds by our choice of the f_δ's.

Now check that q is as required.]

Let

$\oplus_1$ $\mathbb{Q}^+ = \{p \in \mathbb{Q}$: if $\ell \in \{1,2\}$ and $\rho \in \Lambda^p$ then $\mathrm{Dom}(f^{\ell,p}_\rho) = u^p \cap \sup \mathrm{Rang}(\rho)\}$

$\oplus_2$ $\mathbb{Q}^+$ is a dense subset of $\mathbb{Q}$, moreover $(\forall p \in \mathbb{Q})(\exists q \in \mathbb{Q}^+)(p \le q \wedge \{\alpha+\lambda : \alpha \in u^q\} = \{\alpha+\lambda : \alpha \in u^p\}]$.

[Why? Let $p \in \mathbb{Q}, \kappa = |\Lambda|, \delta = \kappa \times \kappa$ and $\{\rho_i : i < i_* < \lambda\}$ list Λ^p each appearing unboundedly often. We choose p_i by induction on $i \le \delta$ such that

- (a) $p_i \in \mathbb{Q}$
- (b) $j < i \Rightarrow p_i \le_Q p_j$
- (c) $p_0 = p$
- (d) $\Lambda^{p_i} = \Lambda^p$
- (e) $f^{p_{i+1}}_{\rho_i} \ne f^{p_i}_{\rho_i}$
- (f) $\{\alpha+\lambda : \alpha \in u^{p_i}\} = \{\alpha+\lambda : \alpha \in u^p\}$.

For $i=0$ use clause (c) for i limit use $\circledast_4$, for $i = j+1$ use $\circledast_8$. Now p_δ is as required.]

$\oplus_3$ for $p \in \mathbb{Q}$ and $\delta < \lambda^+$ divisible by $\lambda, p \restriction \delta$ is naturally defined, belongs to $\mathbb{Q}$ and $u^p \subseteq \delta \Rightarrow p{\restriction}\delta = p$ and $p \restriction \delta \le_{\mathbb{Q}} p$, where $q = p \restriction \delta$ be defined by:
- (a) $u^q = u^p \cap \delta$
- (b) $<^q = <^p \restriction \delta$
- (c) $\mathfrak{S}^q_\ell = \{\eta \in \mathfrak{S}^p_\ell$:Rang$(\eta) \subseteq \delta\}$
- (d) $\Lambda^q = \{\rho \in \Lambda^p : \sup \text{Rang}(\rho) \le \delta\}$
- (e) $\bar{f}^q = \langle f^q_\rho : \rho \in \Lambda^q\rangle$ where $f^q_\rho = f^p_\rho$.

[Why? Check.]

$\oplus_4$ if $\delta < \lambda^+$ is divisible by $\lambda, p \in \mathbb{Q}^+$ and $(p{\restriction}\delta) \le_{\mathbb{Q}} q \in \mathbb{Q}^+$ but $u^q \subseteq \delta$ <u>then</u> p, q are compatible in $\mathbb{Q}$, moreover has a common upper bound $r = p+q$ such that $r{\restriction}\delta = q \wedge u^r = u^p \cup u^q$.

[Why? Note that if $\rho \in \Lambda^p \cap \Lambda^q$ then sup Rang$(\rho) \le \delta$ by clause (i) + (j) of $\circledast_1$; also $\Lambda^p \cap \Lambda^q = \Lambda_{p\restriction\delta}$. We define r as follows:

- (a) $u^r = u^p \cup u^q$
- (b) $\le^r$ is defined by: for $\alpha, \beta \in u^r$ we have $\alpha <^2 \beta$ iff $\alpha+\lambda \le \beta$ or $\alpha <^q \beta$ or $\alpha <^p \beta$
- (c) $\mathfrak{S}^r_\ell$ is $\mathfrak{S}^p_\ell \cup \mathfrak{S}^q_\ell$ for $\ell = 1,2$
- (d) $\Lambda^r = \Lambda^p \cup \Lambda^q$
- (e) $\bar{f}^r = \langle f^r_\rho : \rho \in \Lambda^r\rangle$ where f^r_ρ is:
  - f^q_ρ <u>when</u> $\rho \in \Lambda^q$
  - $f^p_\rho \cup \bigcup\{f^q_{\rho\restriction i} : i \le \ell g(\rho)$ and $\rho{\restriction}i \in \Lambda^q\}$ <u>when</u> $\rho \in \Lambda^p \backslash \Lambda^q$.

Why $r \in \mathbb{Q}$? We should check all the clauses in $\circledast_1$, which are easy. E.g. in clause (c), $\eta \ne \nu \in \mathfrak{S}^r_\ell \Rightarrow \aleph_0 > |\text{Rang}(\eta) \cap \text{ Rang}(\nu)|$, the only new case is $\eta \in \mathfrak{S}^p_\ell \Leftrightarrow \nu \notin \mathfrak{S}^p_\ell$ so without loss of generality $\eta \in \mathfrak{S}^p_\ell \backslash \mathfrak{S}^q_\ell \wedge \nu \in \{\mathfrak{S}\}^q_\ell$, hence $\sup(\eta) > \delta$ hence Rang$(\eta) \cap \delta$ is finite but Rang$(\nu) \subseteq u^q \subseteq \delta$.

Also clauses (g) + (k) should be checked only when f^r_ρ is new so necessarily $\rho \in \Lambda^p$ so $f^r_\rho = f^p_\rho \cup \bigcup\{f^q_{\rho\restriction i} : \rho{\restriction}i \in \Lambda^q\}$, but recalling that any $\eta \in \mathfrak{S}^r_\ell$ is an increasing ω-sequence, clearly if sup Rang$(\eta) > \delta$ we use "p satisfies clauses (g) + (k)" and if sup Rang$(\eta) \le \delta$ we use "q satisfies clauses (g) +(k) and (ℓ)".

Why $p \le_{\mathbb{Q}} r \wedge p \le_{\mathbb{Q}} r$? We should check all the clauses in $\circledast_2$ for both pairs. They are easy, e.g. clause (f) holds because: if $\eta \in \mathfrak{S}^r_\ell \backslash \mathfrak{S}^q_\ell$ then $\eta \in \mathfrak{S}^p_\ell \backslash S^q_\ell$ hence sup Rang$(\eta) > \delta$ and it should be clear; if $\eta \in \mathfrak{S}^r_\ell \backslash \mathfrak{S}^p_\ell$ then $\eta \in \mathfrak{S}^q_\ell \backslash \mathfrak{S}^p_\ell$ and we can use $p \restriction \delta \le_{\mathbb{Q}} q$, i.e. clause (f) for this pair.

Concerning clause (g) for $p \le_{\mathbb{Q}} r$, recall that $p, q \in \mathbb{Q}^+$ so $\ell \in \{1,2\} \wedge \rho \in \Lambda^p \Rightarrow u^p = \text{Dom}(f^{\ell,p}_\rho) \subseteq \text{Dom}(f^{\ell,r}_\rho)$ so clause (g) is O.K. and similarly clause (g) for $q \le_{\mathbb{Q}} r$.]

$\oplus_5$ $\mathbb{Q}$ satisfies the λ^+-c.c.

[Why? Let $p_\alpha \in \mathbb{Q}$ for $\alpha < \lambda^+$, so by $\circledast_{10}$ there are q_α such that $p_\alpha \le_{\mathbb{Q}} q_\alpha \in \mathbb{Q}^+$, now use the Δ-sytem lemma that is first $S^{\lambda^+}_\lambda = \{\delta < \lambda^+ \colon \text{cf}(\delta) = \lambda\}$; now $\delta \in S^{\lambda^+}_\lambda \Rightarrow p \restriction \delta \in \mathbb{Q} \wedge \sup(u^{p \restriction \delta}) < \delta$ and $\lambda \ge |\{p \in \mathbb{Q} : u^p = u\}$ for any u. Hence for some stationary $S \subseteq S^{\lambda^+}_\lambda$ and p_* we have $\delta \in S \Rightarrow q_\delta \restriction \delta = p_*$ and $\delta_1 < \delta_2 \in S \Rightarrow \sup(u^{q_{\delta_2}}) < \delta_2$. So for any $\delta_2 < \delta_2$ from S by $\oplus_4$ the condition $q_{\delta_1}, q_{\delta_2}$ are compatible.]

$\boxplus_1$ define $\underset{\sim}{\mathbf{J}}_\ell \in K^{\text{oi}}_\mu$, a $\mathbb{Q}$-name as follows:

- (a) $Q^{\underset{\sim}{\mathbf{J}}_\ell} = \mu$
- (b) $\mathfrak{S}^{\underset{\sim}{\mathbf{J}}_\ell} = \cup\{\mathfrak{S}^p_\ell : p \in \underset{\sim}{G}_{\mathbb{Q}}\}$
- (c) $<^{\underset{\sim}{\mathbf{J}}_\ell} = \cup\{<^p \colon p \in \underset{\sim}{G}_{\mathbb{Q}}\}$
- (d) $F^{\underset{\sim}{\mathbf{J}}_\ell}_n$ is a unary function, the identity on λ^+ and
- (e) $\eta \in \mathfrak{S}^{\underset{\sim}{\mathbf{J}}_\ell} \Rightarrow F^{\underset{\sim}{\mathbf{J}}_n}(\eta) = \eta(n)$

$\boxplus_2$ for $\ell \in \{1,2\}$ and $p \in \mathbb{Q}$ let $\mathbf{J}^p_\ell \in K_{\text{oi}}$ be defined as follows:

- (a) $\mathbf{J}^p_\ell$ has universe $u^p \cup \mathfrak{S}^p_\ell$
- (b) $<^{\mathbf{J}_\ell} = <^p$
- (c) $Q^{\mathbf{J}^p_\ell} = u^p$
- (d) $F^{\mathbf{J}^p_\ell}_n(\eta) = \eta(n)$

$\boxplus_3$ (a) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{J}}_\ell \in K^{\text{oi}}_{\lambda^+}$"

(b) $\Vdash_{\mathbb{Q}}$ "for each $\delta < \lambda^+$ divisible by λ the linear order $([\delta, \delta+\lambda), <^{\underset{\sim}{\mathbf{J}}_\ell} \restriction (\delta, \delta+\lambda))$ is a saturated linear order and $\alpha + \lambda \le \beta < \lambda^+ \Rightarrow \alpha <^{\underset{\sim}{\mathbf{J}}_\ell} \beta$"

(c) $p \in \mathbb{Q} \Rightarrow p \Vdash_{\mathbb{Q}}$ "$\mathbf{J}^p_\ell \subseteq \underset{\sim}{\mathbf{J}}_\ell$ for $\ell = 1,2$".

[Why? Think]

$\boxplus_4$ if $\delta < \lambda^+$ is divisible by λ then $\Vdash$ "$\underset{\sim}{\mathbf{J}}_\ell \restriction \delta \in K^{\text{oi}}_\lambda$ where $\mathbf{J}_\ell \restriction \delta = ((\delta \cup (P^{\mathbf{J}_\ell} \cap {}^\omega\delta), Q^{\mathbf{J}_\ell} \cap \delta, P^{\mathbf{J}_\ell} \restriction \delta, F^{\mathbf{J}_\ell}_n \restriction (\delta \cup (P^{\mathbf{J}_\ell} \cap {}^\omega\delta)))_{n<\omega}$"

$\boxplus_5$ $\Vdash_{\mathbb{Q}}$ "$\text{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_1, \Phi), \text{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_2, \Phi)$ are $\text{EF}^+_{\lambda,\lambda^+}$-equivalent

(so the games of length $< \lambda$, and the player INC chooses sets of cardinality $< \lambda^+$). [Why? To show the $\text{EF}^+_{\lambda,\lambda^+}$-equivalence, it suffices to show that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{J}}_1, \mathbf{J}_2$ are $\text{EF}_{\lambda,\lambda^+}$-equivalent" by 1.14 as $\lambda \ge \aleph_1 + |T_1|$. From $\circledast_6$, recall $\Lambda^* = \{\rho : \rho$ is an increasing sequence of ordinals $< \lambda^+$ divisible by λ of length $< \lambda\}$, (is the same in $\mathbf{V}$ and $\mathbf{V}^{\mathbb{Q}}$). For $\rho \in \Lambda^*$ let $\underset{\sim}{f}_\rho = \cup\{f^p_\rho : \rho \in \underset{\sim}{G}, p \in \Lambda^p\}$ and by $\circledast_1(f)+(j) and \circledast_2(e)$ easily $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{f}_\rho$ a partial isomorphism from $\underset{\sim}{\mathbf{J}}_1 \restriction \sup \text{Rang}(\rho)$ into $\underset{\sim}{\mathbf{J}}_2 \restriction \sup \text{Rang}(\rho)$", see Definition inside $\boxplus_4$.

Now $\Vdash_{\mathbb{Q}}$ "$\mathrm{Dom}(\underset{\sim}{f}_\rho) = \sup \mathrm{Rang}(\rho)$" as if $\underset{\sim}{G} \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$, for any $\alpha < \sup \mathrm{Rang}(\rho)$ for some $p \in \mathbf{G}$ we have $\alpha \in u^p \wedge \rho \in \Lambda^p$ by $\circledast_7$ and there is q such that $p \le q \in \mathbf{G}, p \ne q$ by $\circledast_8$, so recalling $\circledast_2(g)$ we are done.

Similarly $\Vdash_{\mathbb{Q}}$ "$\mathrm{Rang}(\underset{\sim}{f}_\rho) = \sup \mathrm{Rang}(\rho)$".

Also $\rho \triangleleft \varrho \Rightarrow \Vdash_{\mathbb{Q}} \underset{\sim}{f}_\rho \subseteq \underset{\sim}{f}_\varrho$. For the EF^+-version we have to analyze dependence relations, which is straight as in the proof in 4.2. So $\langle f_\rho : \rho \in \Lambda^* \rangle$ exemplify the equivalence.]

$\boxplus_6$ $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{M}_1 = \mathrm{EM}_{\tau(T)}(\mathbf{J}_1, \Phi), \underset{\sim}{M}_2 = \mathrm{EM}_{\tau(T)}(\underset{\sim}{\mathbf{J}}_2, \Phi)$ are not isomorphic".

Why? Let $\underset{\sim}{M}_\ell^+ = \mathrm{EM}(\mathbf{J}_\ell, \Phi)$ so $\underset{\sim}{M}_\ell^+ \restriction \tau(T) = \underset{\sim}{M}_\ell$ for $\ell = 1, 2$, and assume toward contradiction that $p \in \mathbb{Q}$, and $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{g}$ is an isomorphism from $\underset{\sim}{M}_1$ onto $\underset{\sim}{M}_2$". For each $\delta \in S^{\lambda^+}_\lambda := \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ by $\circledast_4$ we can find $p_\delta \in \mathbb{Q}$ above p and g_δ such that:

$\boxdot_1$ (a) $p \le p_\delta, \delta \in u^{p_\delta}$
(b) $p_\delta \Vdash$ "g_δ is $\underset{\sim}{g} \restriction \mathrm{EM}(\mathbf{J}_1^{p_\delta}, \Phi)$"
(c) g_δ is an isomorphism from $\mathrm{EM}_{\tau(T)}(\mathbf{J}_1^p, \Phi)$ onto $\mathrm{EM}_{\tau(T)}(\mathbf{J}_2^p, \Phi)$.

We can find stationary $S \subseteq S^{\lambda^+}_\lambda$ and p^* such that

$\boxdot_2$ (a) $p_\delta \restriction \delta$, defined in $\oplus_3$ is p^* for $\delta \in S$
(b) for $\delta_1, \delta_2 \in S$, $u^{p_{\delta_1}}, u^{p_{\delta_2}}$ has the same order type and the order preserving mapping π_{δ_1,δ_2} from $u^{p_{\delta_2}}$ onto $u^{p_{\delta_1}}$ induce an isomorphism from p_{δ_2} onto p_{δ_1}
(c) if $\delta_1 < \delta_2 \in S$ then $\sup(u^{p_{\delta_1}}) < \delta_2$.

Now choose $\eta^* = \langle \delta^*_n : n < \omega \rangle$ such that

$\boxdot_3$ (a) $\delta^*_n < \delta^*_{n+1}$
(b) $\delta^*_n = \sup(S \cap \delta^*_n)$ and $\delta^*_n \in S$
(c) let $\delta^* = \sup\{\delta^*_n : n < \omega\}$.

We define $q \in \mathbb{Q}$ as follows

$\boxdot_4$ (a) $u^q = \cup\{p_{\delta^*_n} : n < \omega\}$
(b) $<^q = \{(\alpha, \beta) : \alpha <^{p_{\delta^*_n}} \beta$ for some n or $\alpha + \lambda \le \beta \wedge \{\alpha, \beta\} \subseteq u^q$, equivalently for some $m < n, \alpha \in u^{p_{\delta^*_m}} \setminus \delta^*_m$ and $\beta \in u^{p_{\delta^*_n}} \setminus \delta^*_n\}$
(c) $\mathfrak{S}^q_1 = \cup\{\mathfrak{S}_1^{p_{\delta^*_n}} : n < \omega\} \cup \{\eta^*\}$
(d) $\mathfrak{S}^q_2 = \cup\{\mathfrak{S}_2^{p_{\delta^*_n}} : n < \omega\}$
(e) $\Lambda^q = \cup\{\Lambda^{p_{\delta^*_n}} : n < \omega\}$
(f) $f^q_\rho = f_\rho^{p_{\delta^*_n}}$ if $\rho \in \Lambda^{p_{\delta^*_n}}$.

So there is a pair (q_*, g^+) such that:

$\boxdot_5$ (a) $q \le_{\mathbb{Q}} q_*$
(b) $q_* \Vdash_{\mathbb{Q}}$ "$g^+ = \underset{\sim}{g} \restriction \mathrm{EM}(\mathbf{J}_1^{q_*}, \Phi)$
(c) g^+ is an isormorphism from $\mathrm{EM}_{\tau(T)}(\mathbf{J}_1^{q_*}, \Phi)$ onto $\mathrm{EM}_{\tau(T)}(\mathbf{J}_2^{q_*}, \Phi)$.

So $g^+(a_{\eta^*}) \in$ EM$(\mathbf{J}_2^{q_*}, \Phi)$ hence is of the form $\sigma^{M_2^+}(a_{t_0}, \dots, a_{t_{n-1}})$ for some $t_0, \dots, t_{n-1} \in \mathbf{J}_2^{q_*}$ and a τ_Φ-term $\sigma(x_0, \dots, x_{n-1})$.

Note that by the definition of $\leq_{\mathbb{Q}}$ in $\circledast_2$:

$\boxdot_6$ if $\eta \in \mathfrak{S}_2^{q_*}$ then $\mathrm{Rang}(\eta) \cap u^q$ is bounded in δ^*.

[Why? If $\eta \in \mathfrak{S}_2^q$ this holds by our choice of q and if $\eta \in \mathfrak{S}_2^{q_*} \backslash \mathfrak{S}_2^q$ then $\mathrm{Rang}(\eta) \cap u^q$ is finite so as $u^q \subseteq \delta$ it follows that $\mathrm{Rang}(\eta) \cap u^q$ is bounded in δ^*.]

We can find $n(*) < \omega$ such that:

$\boxdot_7$ for each $k < n$ and $\ell < n$ we have
- (a) if $t_\ell \in Q^{\mathbf{J}_2^{q_*}}$, i.e. $t_\ell \in u^{q_*} \subseteq \lambda^+$ then $t_\ell \leq^q \delta^*_{n(*)}$ or $\delta^* \leq t_\ell$ (hence $\bigwedge\limits_n \delta^*_n \leq^q t_\ell$)
- (b) if $t_\ell \in P^{\mathbf{J}_2^{q_*}}$, i.e. $t_\ell \in \mathfrak{S}_2^{q_*}$ then $\{F_n^{\mathbf{J}_2^{q_*}}(t_\ell) : n < \omega\}$ is disjoint to $[\delta^*_{n(*)}, \delta^*) \cap u^q$.

Now using "T is stable", the rest is as in 2.5, 2.9. $\square_{3.1}$

Discussion 3.3. (2012.11.23) 1) Can we do it in ZFC? It is natural to use $\langle W_\alpha : \alpha \in S^\mu_{\aleph_0}\rangle$ be stationary pairwise almost disjoint, see [S$^+$c].
2) Instead of "not strongly stable" it suffices to assume "not strongly^2 stable", see [She14]. In [S$^+$d] even much less.

§ 4. Theories with order

Recall from [HS95, 3.19]:

Claim 4.1. *If $\lambda = \mu^+$, $\mathrm{cf}(\mu), \lambda = \lambda^{<\kappa}, \kappa = \mathrm{cf}(\kappa) < \kappa(T)$ and T is unstable <u>then</u> there are $\mathrm{EF}_{\mu\times\kappa,\lambda^+}$-equivalent non-isomorphic models of T of cardinality λ.*

The new point in 4.2 is the EF^+ rather than EF.

Claim 4.2. *Assume $\lambda = \lambda^{<\theta}$ and λ is regular uncountable, $T \subseteq T_1$ are complete first order theories of cardinality $< \lambda$.*
1) If T is unstable <u>then</u> there are models M_1, M_2 of T_1 of cardinality λ^+, $\mathrm{EF}^+_{\lambda,\theta,\lambda^+}$-equivalent with non-isomorphic τ_T-reducts.
2) Assume $\Phi \in \Upsilon^{or}_\kappa$ is proper for linear orders, $\bar\sigma = \langle \sigma_i(x) : i < i(*)\rangle$ a sequence of terms from $\tau_\Phi, \bar x^\ell = \langle x^\ell_i : i < i(*)\rangle, i(*) < \lambda, \varphi(\bar x^1, \bar x^2)$ is a formula in $\mathbb{L}(\tau_T), \tau \le \tau_T$ (any logic) and for every linear order I letting $M = \mathrm{EM}(I,\Phi), \bar b_t = \langle \sigma^M_i(a_t) : i < i(*)\rangle$ we have $(M \restriction \tau) \models \varphi[\bar b_s, \bar b_t]^{\mathrm{if}(s<t)}$ for every $s,t \in I$. <u>Then</u> there are linear orders I_1, I_2 of cardinality λ^+ such that M_1, M_2 are $\mathrm{EF}^+_{\lambda,\theta,\lambda^+}$-equivalent but not isomorphic where $M_\ell = \mathrm{EM}_\tau(I_\ell,\Phi)$ for $\ell = 1,2$.
3) If every $\mathrm{EM}_\tau(I,\Phi)$ is a model of T_1 <u>then</u> in (2) the models M_1, M_2 are in $\mathrm{PC}(T_1,T)$.

Proof. 1) Let $\varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T)$ order some infinite subset of ${}^m M$ for some $M \models T$.

Let Φ be as in Definition 2.3, i.e. [She90, Ch.VII,VIII], i.e. proper for linear orders such that $\tau_{T_1} \subseteq \tau(\Phi), |\tau(\Phi)| = |T_1|$ and for every linear order I,$\mathrm{EM}(I,\Phi)$ (we allow the skeleton to consist of m-tuples rather than elements) is a model of T_1 satisfying $\varphi[\bar a_s, \bar a_t]$ iff $s <_I t$. Now we can apply part (2) with $i(*) = m$.
2) We choose I such that

⊛ (a) I is a linear order of cardinality λ (yes, not λ^+)
(b) if $\alpha,\beta \in (1,\lambda]$ then $(I\times\alpha) + (I\times\beta)^* \cong I$ (equivalently every α, $\beta \in [1,\lambda^+)$)
(c) I is isomorphic to its inverse
(d) I has cofinality λ.

For every $S \subseteq S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ we define $I_S = \sum\limits_{\alpha<\lambda^+} I_{S,\alpha}$ where $I_{S,\alpha}$ is isomorphic to I if $\alpha \in \lambda^+ \backslash S$ and isomorphic to the inverse of $I\times\omega$ otherwise. Now

⊛$_2$ if $S_1, S_2 \subseteq S^{\lambda^+}_\lambda$ <u>then</u> the models $\mathrm{EM}(I_{S_2},\Phi)$, $\mathrm{EM}(I_{S_1},\Phi)$ are $\mathrm{EF}^+_{\lambda,\theta,\lambda^+}$-equivalent.

[Why? Let $J_{\ell,\gamma} = \sum\limits_{\alpha<\gamma} I_{S_\ell,\alpha}$. Let $\mathscr{F} := \{f$: for some non-zero ordinal $\gamma < \lambda^+, f \in \mathscr{F}_\gamma$ and $[\gamma \in S_1 \Leftrightarrow \gamma \in S_2]\}$ where $\mathscr{F}_\gamma := \{f$ is an isomorphism from $\sum\limits_{\alpha<\gamma} I_{S_1,\alpha}$ onto $\sum\limits_{\alpha<\gamma} I_{S_2,\alpha}\}$.

Now

$(*)_1$ $\mathscr{F}_\gamma \ne \emptyset$ for $\gamma < \lambda^+$
$(*)_2$ if $f \in \mathscr{F}_\gamma$ and $[\gamma \in S_1 \equiv \gamma \in S_2]$ and $\gamma < \beta < \lambda$ then f can be extended to some $g \in \mathscr{F}_\beta\}$

$(*)_3$ if $\gamma < \lambda, X_\ell \subseteq I_\ell$ has cardinality $< \lambda^+$ for $\ell = 1,2$ then for some successor $\beta, \gamma < \beta < \lambda^+$ and $X_\ell \subseteq J_{\ell,\beta}$ for $\ell = 1,2$

$(*)_4$ if $\gamma_i \in S_1 \Leftrightarrow \gamma_i \in S_2$ for $i < \delta, \delta$ a limit ordinal $< \lambda$ and $\langle \gamma_i : i < \delta \rangle$ is increasing then $\gamma_\delta := \cup\{\gamma_i : i < \delta\}$ satisfies $\gamma_\delta \in S_1 \equiv \gamma_\delta \in S_1$.

Lastly, we have to deal with case 2 in Definition 1.7(2) so assume

$(*)_5$ $f_* \in \mathscr{F}_{\gamma_*}, [\gamma_* \in S_1 \equiv \gamma_* \in S_2]$ and $\mathbf{R}_\ell \subseteq {}^{\theta>}(M_\ell)$ for $\ell = 1,2$ are as there for f_*

This holds because the strategy is simple, e.g. with no memory. Now if f does not map the definition of $\mathbf{R}_1$ in M_1 to the definition of $\mathbf{R}_2$ in M_2 we can use subcase 2B there, so we assume this does not occur. Let $\ell \in \{1,2\}$.

$(*)_6$ Let $\mathbf{e}_\ell = \{(\bar{s},\bar{t}) : \bar{s},\bar{t} \in {}^{\theta>}(I_\ell)$ and some automorphism of I_ℓ over I_{ℓ,γ_*} maps $\bar{s}$ to $\bar{t}\}$.

$(*)_7$ Let Y_ℓ be the set of $\mathbf{e}_\ell$-equivalence classes.

Note

$\odot_1$ for $\ell \in \{1,2\}, n < \omega$ and $\mathbf{y}_0, \ldots, \mathbf{y}_n \in Y_\ell$ the following are equivalent:
- (a) some $\bar{a} \in \mathbf{y}_n$ depend (by $\mathbf{R}_1$) on $\mathbf{y}_0 \cup \ldots \cup \mathbf{y}_{n-1}$
- (b) every $\bar{a} \in \mathbf{y}_n$ depend (by $\mathbf{R}_1$) on $\mathbf{y}_0 \cup \ldots \cup \mathbf{y}_{n-1}$.

So $\mathbf{R}_1$ induce a 1-dependence relation on Y_1, so let $\langle \mathbf{y}_i : i < i(*) \rangle$ be a maximal independent subset of Y_1 such that $[i < i(*) \wedge \bar{a} \in \mathbf{y}_i \Rightarrow \bar{a}$ does not depend on $\cup\{\mathbf{y}_j : j < i(*), j \neq i\}$.

So

$\odot_2$ it is enough to deal with one $\mathbf{y}_i$.

Now we can find $\bar{t}_{i,\gamma} \in \mathbf{y}_i$ such that $\mathrm{Rang}(\bar{t}_{i,\gamma}) \backslash I_{\ell,\gamma_*} \subseteq I_{\ell,\gamma+2} \backslash I_{\ell,\gamma+1}$ for each $\gamma \in [\gamma_*, \lambda^+)$ as I_1 has enough automorphisms

$\odot_3$ if $\{\bar{t}_{i,\gamma} : \gamma \in [\gamma_*, \lambda^+)\}$ is not $\mathbf{R}_1$-independent, then $\dim(\mathbf{y}_i)$ is finite, in fact 1 or 0.

So we choose β_* such that

$\odot_4$ $\gamma_* < \beta_* < \lambda^+$ and $\beta_* \in S_1 \equiv \beta_* \in S_2$ and for every $i < i(*)$, if $\dim(X_{\mathbf{y}_i})$ is finite then $\mathbf{y}_i$ has a maximal $\mathbf{R}_1$-independent set included in ${}^{\varepsilon(\mathbf{y}_i)}(J_{1,\beta_*})$.

[Why possible? Because for any such β_* is an automorphism of I_2 over J_{1,γ_*} mapping I_{β_*+2} onto I_{γ_*+2}.]

Let $g \in \mathscr{F}_{\beta_*}$ extend f and using it we can choose $\langle (\bar{a}^1_\zeta, \bar{a}^2_\zeta) : \zeta < \zeta^* \rangle$ as required.

$\circledast_3$ if $S_1, S_2 \subseteq S^{\lambda^+}_\lambda$ and $S_1 \backslash S_2$ is stationary, then $\mathrm{EM}_\tau(I_{S_1}, \Phi)$, $\mathrm{EM}_\tau(I_{S_2}, \Phi)$ are not isomorphic.

[Why? By the proof in [She87b, Ch.III,§3] (or [Shear, Ch.III,§3] = [Shea, §3]) only easier. In fact, immitating it we can represent the invariants from there. If $k = 2$ we have to work somewhat more.]

2) As in [HS95].

3) Obvious. $\square_{4.2}$

Conclusion 4.3. *Assume T is a (first order complete) theory.*
1) If T is unstable, then $(T,*)$ is fat.
2) If T is unstable or stable with DOP, or stable with OTOP, then T is fat.
3) For every μ there is a μ-complete, class forcing $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ we have: if T is not strongly dependent or just not strongly stable then T is fat, moreover $(T,*)$ is fat.

Proof. 1) By 4.2.
2) Similar, the only difference is that the formula defining the "order" is not first order and the length of the relevant sequences may be infinite but still $\leq |T|$ (see [She90, Ch.XIII]).
3) By parts (1),(2) we should consider only stable - not strongly stable T. Choose a class $\mathbf{C}$ of regular cardinals such that $\lambda \in \mathbf{C} \Rightarrow (2^{<\lambda})^+ < \text{Min}(\mathbf{C} \backslash \lambda^+)$ and $\text{Min}(\mathbf{C}) > \mu$. We iterate with full support $\langle \mathbb{P}_\mu, \underset{\sim}{\mathbb{Q}}_\mu : \mu \in \mathbf{C}\rangle$ with $\underset{\sim}{\mathbb{Q}}_\mu$ as in 3.1. $\square_{4.3}$

Claim 4.4. *Assume $T \subseteq T_1, \lambda = \lambda^\kappa$ is not necessary regular and $\kappa = \text{cf}(\kappa) < \kappa(T)$, e.g. T is unstable. Then there are $\text{EF}^+_{\lambda\times\kappa,\lambda,\lambda^+}$-equivalent non-isomorphic models from $\text{PC}(T_1,T)$ of cardinality λ^+.*

Proof. As in [HS95], seeing the proof of 4.2. $\square_{4.4}$

As said in the introduction by the old results (note: 4.5 is on elementary classes and 4.6 on small enough pseudo elementary classes).

Conclusion 4.5. *(ZFC) For first order countable complete first order theory T the following conditions are equivalent:*

- (A) *T is superstable with NDOP and NOTOP*
- $(B)_1$ *if $\lambda = \text{cf}(\lambda) > |T|$ and $M_1, M_2 \in Mod_T(\lambda)$ are $\mathbb{L}_{\infty,\lambda}(\tau_T)$-equivalent then M_1, M_2 are isomorphic*
- $(B)_2$ *like $(B)_1$ for some $\lambda = \text{cf}(\lambda) > |T|$*
- (C) *if $\lambda = \text{cf}(\lambda) > |T|$ and $M_1, M_2 \in Mod_T(\lambda^+)$ are $EF_{\omega,\lambda}$-equivalent then M_1, M_2 are isomorphic*
- (D) *for some regular $\lambda > |T|$, if $M_1, M_2 \in Mod_T(\lambda^+)$ are EF_{λ,λ^+}-equivalent then they are isomorphic.*

Proof. Clause (A), clause $(B)_1$, clause $(B)_2$ are equivalent because: as proved in [She90, Ch.XIII,Th.1.11], we have $(A) \Rightarrow (B)_1 and (B)_2$ and the inverse implication holds by [She87a]. Now by the definitions trivially $(B)_1 \Rightarrow$ (C) $\Rightarrow$ (D).

Lastly, by [HS95], i.e. by 4.1 we have $\neg(A) \Rightarrow \neg(D)$, i.e. (D) $\Rightarrow$ (A) so we have the circle. $\square_{4.5}$

So 4.5 tells us what we know about Qustion $(A)_0$ of 1.2. Similarly concerning $(B)_0$ of Question 1.2.

Conclusion 4.6. *(ZFC) For first order countable complete first order theory T and $\kappa \geq 2^{\aleph_0}$ the following conditions are equivalent:*

- (A) *T is unsuperstable*
- $(B)_\kappa$ *for every $\lambda > \kappa \geq |T|$ and (κ, T)-candidate ψ (see Definition 0.8), and ordinal $\alpha < \lambda$ satisfying $|\alpha|^+ = \lambda \Rightarrow |\alpha \leq |\alpha| \times \omega$, there are $EF_{\alpha,\lambda}$-equivalent non-isomorphic models $M_1, M_2 \in PC_{\tau(T)}(\psi)$ of cardinality λ*

(C) *for some* $\lambda > \kappa \geq |T|$, *for no* (κ, T)*-candidate* ψ *is the class* $PC_{\tau(T)}(\psi)$ *categorical in* λ.

Proof. First, assume T is superstable, so clause (A) holds. By the proofs of [She90, Ch.VI,§4] there is a (κ, T)-candidate ψ, $\mathrm{PC}_{\tau(T)}(\psi)$ is the class of saturated models of T, (in details, if $n < \omega, \bar{a} \in {}^n\mathfrak{C}$, $\mathrm{tp}(b, \bar{a}, \mathfrak{C})$ is stationary, $q = \mathrm{tp}(\bar{b}, \emptyset, \mathfrak{C}), p = p(x, \bar{y}) = \mathrm{tp}(\langle a \rangle \hat{\ } \bar{b}, \emptyset, \mathfrak{C})$ then let $\psi_{p,q}$ be such that $M \models \psi_{p,q}$ $\underline{\text{iff}}$ for every $\bar{b}' \in {}^n M$ realizing the type $q(\bar{y})$, the function $c \mapsto F^M_{p,q}(c, \bar{b}')$ is one-to-one and if $k < \omega, c_0, \ldots, c_k \in M$ are pairwise distinct then $\mathrm{tp}_{\mathbb{L}(\tau(T))}(F^M_{p,q}(c_k), \{F^M_{p,q}(c_0), \ldots, F^M_{p,q}(c_{k-1})\} \cup \bar{b}', M)$ extends $p(x, \bar{b}')$ and does not fork over M.

Lastly, $\psi = \wedge\{\psi_{p,q} : p, q \text{ as above}\}$ so $\in \mathbb{L}_{\kappa^+,\omega}$. So in the present case also $(B)_\kappa, (C)_\kappa, (D)_\kappa$ holds.

Second, assume T is not superstable, so clause (A) holds and we shall prove the rest. Let ψ be a (κ, T)-candidate.

By 0.9 and let there is $\Phi \in \Upsilon^{\omega_1\text{-tr}}_\kappa$ witnessing this hence witnessing unsuperstability and now we can use Theorem 4.1 quoted above. $\square_{4.6}$

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