

RESOLVABILITY VS. ALMOST RESOLVABILITY

ISTVÁN JUHÁSZ, SAHARON SHELAH, AND LAJOS SOUKUP

ABSTRACT. A space X is κ -*resolvable* (resp. *almost κ -resolvable*) if it contains κ dense sets that are pairwise disjoint (resp. almost disjoint over the ideal of nowhere dense subsets of X).

Answering a problem raised by Juhász, Soukup, and Szentmiklóssy, and improving a consistency result of Comfort and Hu, we prove, in ZFC, that for every infinite cardinal κ there is an almost 2^κ -resolvable but not ω_1 -resolvable space of dispersion character κ .

A space X is said to be κ -*resolvable* if it contains κ dense sets that are pairwise disjoint. X is called *maximally resolvable* iff it is $\Delta(X)$ -resolvable, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\}$ is the *dispersion character* of X .

V. Malychin, in [4], was the first to suggest studying families of dense sets of a space X that, rather than disjoint, are merely *almost disjoint* with respect to the ideal $\mathcal{N}(X)$, where $\mathcal{N}(X)$ denotes the family of all nowhere dense subsets of the space X . He called a space X *extraresolvable* if it has $\Delta(X)^+$ many dense sets such that any two of them have nowhere dense intersection. This idea was generalized in [3], where the natural notion of *almost κ -resolvability* was introduced: A space X is called *almost κ -resolvable* if it contains κ dense sets that are pairwise almost disjoint over the ideal $\mathcal{N}(X)$ of nowhere dense subsets of X . (Actually, this concept was given a different name in [3], namely: “ κ -extraresolvable”, but we think the terminology given here is much better.)

Note that this makes good sense for $\kappa \leq \Delta(X)$ as well. But while “almost ω -resolvable” is clearly equivalent to “ ω -resolvable”, the analogous question for higher cardinals remained open. In particular, the following natural problem was formulated in [3]:

Problem 1. *Let X be an extraresolvable (T_2 , T_3 , or Tychonov) space with $\Delta(X) \geq \omega_1$. Is X then ω_1 -resolvable?*

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(The assumption $\Delta(X) \geq \omega_1$ is clearly necessary to make this problem non-trivial.)

Comfort and Hu, see [2, Corollary 3.6], gave a negative answer to this problem, assuming the failure of the continuum hypothesis, CH. More precisely they got the following result:

Theorem . *If κ is an infinite cardinal such that GCH first fails at κ then there is a 0-dimensional T_2 space X with $|X| = \Delta(X) = \kappa^+$ such that X is κ -resolvable, extraresolvable but not κ^+ -resolvable, hence not maximally resolvable and if $\kappa = \omega$ then not ω_1 -resolvable.*

Our aim in this note is to give the following “final” answer to the above problem, in ZFC.

Theorem 2. *For every cardinal κ there is a 0-dimensional T_2 space of dispersion character κ that is extraresolvable but not ω_1 -resolvable.*

We shall actually prove a bit more. Note that no space X can be almost $(2^{\Delta(X)})^+$ -resolvable, moreover “almost $2^{\Delta(X)}$ -resolvable” can be strictly stronger than “extraresolvable \equiv almost $\Delta(X)^+$ -resolvable”.

Theorem 3. *For every cardinal κ there is an almost 2^κ -resolvable (and so extraresolvable) but not ω_1 -resolvable 0-dimensional T_2 space of cardinality and dispersion character κ . In fact, our example is a κ -dense subspace of the Cantor cube of weight 2^κ .*

To prove this theorem we shall make use of the method of constructing \mathcal{D} -forced spaces that was introduced in [3]. Therefore, we first recall some definitions and results from [3].

Let \mathcal{D} be a family of dense subsets of a space X . A subset $M \subset X$ is called a \mathcal{D} -mosaic iff there is a maximal disjoint family \mathcal{V} of open subsets of X and for each $V \in \mathcal{V}$ there is $D_V \in \mathcal{D}$ such that

$$M = \cup\{V \cap D_V : V \in \mathcal{V}\}.$$

Clearly, every \mathcal{D} -mosaic is dense. We say that the space X (or its topology) is \mathcal{D} -forced iff every dense subset of X includes a \mathcal{D} -mosaic.

Let S be any set and $\mathbb{B} = \{\langle B_\zeta^0, B_\zeta^1 \rangle : \zeta < \mu\}$ be a family of 2-partitions of S . We denote by $\tau_{\mathbb{B}}$ the (obviously zero-dimensional) topology on S generated by the subbase $\{B_\zeta^i : \zeta < \mu, i < 2\}$, moreover we set $X_{\mathbb{B}} = \langle S, \tau_{\mathbb{B}} \rangle$.

Given a cardinal κ , we have $\Delta(X_{\mathbb{B}}) \geq \kappa$ iff \mathbb{B} is κ -independent, i.e.,

$$\mathbb{B}[\varepsilon] \stackrel{def}{=} \bigcap \{B_\zeta^{\varepsilon(\zeta)} : \zeta \in \text{dom } \varepsilon\}$$

has cardinality at least κ whenever $\varepsilon \in Fn(\mu, 2)$.

Note that $X_{\mathbb{B}}$ is Hausdorff iff \mathbb{B} is *separating*, i.e. for each pair $\{\alpha, \beta\} \in [S]^2$ there are $\zeta < \mu$ and $i < 2$ such that $\alpha \in B_{\zeta}^i$ and $\beta \in B_{\zeta}^{1-i}$.

A set $D \subset X$ is said to be κ -dense in the space X iff $|D \cap U| \geq \kappa$ for each nonempty open set $U \subset X$. Thus D is dense iff it is 1-dense. Also, it is obvious that the existence of a κ -dense set in X implies $\Delta(X) \geq \kappa$.

Theorem ([3, Main Theorem 3.3]). *Assume that κ is an infinite cardinal and we are given $\mathbb{B} = \{\langle B_{\xi}^0, B_{\xi}^1 \rangle : \xi < 2^{\kappa}\}$, a κ -independent family of 2-partitions of κ , moreover a non-empty family \mathcal{D} of κ -dense subsets of the space $X_{\mathbb{B}}$. Then there is a separating κ -independent family $\mathbb{C} = \{\langle C_{\xi}^0, C_{\xi}^1 \rangle : \xi < 2^{\kappa}\}$ of 2-partitions of κ such that*

- (1) every $D \in \mathcal{D}$ is also κ -dense in $X_{\mathbb{C}}$ (and so $\Delta(X_{\mathbb{C}}) = \kappa$),
- (2) $X_{\mathbb{C}}$ is \mathcal{D} -forced.

Actually, the space $X_{\mathbb{C}}$ has other interesting properties as well but we shall not make use of those here. We are now ready to prove our promised result.

Proof of Theorem 3. Let κ be an arbitrary infinite cardinal. It is well-known, see e. g. [3, Fact 3.2], that we can find two disjoint families $\mathbb{B} = \{\langle B_i^0, B_i^1 \rangle : i < 2^{\kappa}\}$ and $\mathbb{D} = \{\langle D_i^1, D_i^1 \rangle : i < 2^{\kappa}\}$ of 2-partitions of κ such that their union $\mathbb{B} \cup \mathbb{D}$ is κ -independent, that is, for any $\eta, \varepsilon \in \text{Fn}(2^{\kappa}, 2)$ we have

$$|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]| = \kappa.$$

In other words, this means that

$$\mathcal{D} = \{\mathbb{D}[\eta] : \eta \in \text{Fn}(2^{\kappa}, 2)\}$$

is a family of κ -dense subsets of $X_{\mathbb{B}}$, hence we may apply Theorem 4 to this \mathbb{B} and \mathcal{D} to get a family \mathbb{C} of 2^{κ} many 2-partitions of κ that satisfies conditions (1) and (2) above.

The space that we need will be a further refinement of $X_{\mathbb{C}}$. To obtain that, we next fix a 2-partition $\langle I, J \rangle$ of the index set 2^{κ} such that $|I| = |J| = 2^{\kappa}$. For every unordered pair $a \in [I]^2$ we shall write $a^+ = \max a$ and $a^- = \min a$, so that $a = \{a^-, a^+\}$.

Let $\{j(a, m) : a \in [I]^2, m < \omega\}$ be pairwise distinct elements of J . For any $a \in [I]^2$ and $m < \omega$ we then define the sets

$$E_{a,m}^0 = D_{j(a,m)}^0 \setminus (D_{a^-}^0 \cap D_{a^+}^0) \text{ and } E_{a,m}^1 = \kappa \setminus E_{a,m}^0.$$

Clearly, then we have

$$E_{a,m}^1 = D_{j(a,m)}^1 \cup (D_{a^-}^0 \cap D_{a^+}^0).$$

In this way we obtained a new family

$$\mathbb{E} = \left\{ \langle E_{a,m}^0, E_{a,m}^1 \rangle : a \in [I]^2, m < \omega \right\}$$

of 2-partitions of κ . We shall show that the space $X_{\mathbb{C} \cup \mathbb{E}}$ satisfies all the requirements of theorem 3.

Claim 3.1. *For any finite function $\eta \in \text{Fn}([I]^2 \times \omega, 2)$ and any ordinal $\alpha \in I$ there is a finite function $\varphi \in \text{Fn}(2^\kappa, 2)$ such that $\alpha \notin \text{dom } \varphi$ and $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$.*

Proof of the Claim. For each $a \in [I]^2$ let us pick $a^* \in a$ with $a^* \neq \alpha$. Then we have

$$\begin{aligned} \mathbb{E}[\eta] &= \bigcap_{\eta(a,m)=0} E_{a,m}^0 \cap \bigcap_{\eta(a,m)=1} E_{a,m}^1 \supset \\ &\supset \bigcap_{\eta(a,m)=0} (D_{j(a,m)}^0 \setminus (D_{a^-}^0 \cap D_{a^+}^0)) \cap \bigcap_{\eta(a,m)=1} D_{j(a,m)}^1 \supset \\ &\supset \bigcap_{\eta(a,m)=0} (D_{j(a,m)}^0 \cap D_{a^*}^1) \cap \bigcap_{\eta(a,m)=1} D_{j(a,m)}^1 = \\ &= \bigcap_{\eta(a,m)=0} D_{a^*}^1 \cap \bigcap_{\langle a,m \rangle \in \text{dom } \eta} D_{j(a,m)}^{\eta(a,m)}. \end{aligned}$$

The expression in the last line above is, however, equal to $\mathbb{D}[\varphi]$ for a suitable $\varphi \in \text{Fn}(2^\kappa, 2)$ because j is an injective map of $[I] \times \omega$ into J and $a^* \neq \alpha$ belongs to $I = \kappa \setminus J$ for all $a \in [I]^2$. \square

Claim 3.2. $\mathbb{C} \cup \mathbb{E}$ is κ -independent, hence $\Delta(X_{\mathbb{C} \cup \mathbb{E}}) = \kappa$.

Proof of the Claim. Let $\varepsilon \in \text{Fn}(2^\kappa, 2)$ and $\eta \in \text{Fn}([I]^2 \times \omega, 2)$ be picked arbitrarily. By Claim 3.1 there is $\varphi \in \text{Fn}(2^\kappa, 2)$ such that $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$. Since $\mathbb{D}[\varphi] \in \mathcal{D}$ we have $|\mathbb{C}[\varepsilon] \cap \mathbb{D}[\varphi]| = \kappa$ because \mathbb{C} satisfies condition (1). Consequently, we have $|\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]| = \kappa$ as well. \square

Claim 3.3. *The family $\{D_\alpha^0 : \alpha \in I\}$ witnesses that $X_{\mathbb{C} \cup \mathbb{E}}$ is almost 2^κ -resolvable.*

Proof of the Claim. First we show that D_α^0 is dense in $X_{\mathbb{C} \cup \mathbb{E}}$ whenever $\alpha \in I$. So fix $\alpha \in I$, moreover let $\varepsilon \in \text{Fn}(2^\kappa, 2)$ and $\eta \in \text{Fn}([I]^2 \times \omega, 2)$. By Claim 3.1 there is $\varphi \in \text{Fn}(2^\kappa, 2)$ such that $\alpha \notin \text{dom } \varphi$ and $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$. Since $\alpha \notin \text{dom } \varphi$ we have $D_\alpha^0 \cap \mathbb{D}[\varphi] \in \mathcal{D}$. Hence, as \mathbb{C} has property (1),

$$\emptyset \neq (D_\alpha^0 \cap \mathbb{D}[\varphi]) \cap \mathbb{C}[\varepsilon] \subset D_\alpha^0 \cap (\mathbb{E}[\eta] \cap \mathbb{C}[\varepsilon])$$

as well. So D_α^0 intersects every basic open subset of $X_{\mathbb{C}\cup\mathbb{E}}$, i. e. D_α^0 is dense in $X_{\mathbb{C}\cup\mathbb{E}}$.

Next we show that $D_\alpha \cap D_\beta$ is nowhere dense in the space $X_{\mathbb{C}\cup\mathbb{E}}$ whenever $a = \{\alpha, \beta\} \in [I]^2$. Indeed, let $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$ be again a basic open set with $\varepsilon \in \text{Fn}(2^\kappa, 2)$ and $\eta \in \text{Fn}([I]^2 \times \omega, 2)$ and let us pick $m < \omega$ such that $\langle a, m \rangle \notin \text{dom } \eta$. Then

$$\eta' = \eta \cup \{\langle \langle a, m \rangle, 0 \rangle\} \in \text{Fn}([I]^2 \times \omega, 2),$$

hence $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta'] \subset \mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$ is a (non-empty) basic open set in the space $X_{\mathbb{C}\cup\mathbb{E}}$. Moreover, $E_{a,m}^0 = D_{j(a,m)}^0 \setminus (D_\alpha^0 \cap D_\beta^0)$ implies

$$(D_\alpha \cap D_\beta) \cap \mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta'] \subset (D_\alpha \cap D_\beta) \cap (D_{j(a,m)}^0 \setminus (D_\alpha^0 \cap D_\beta^0)) = \emptyset,$$

consequently, $D_\alpha \cap D_\beta$ is not dense in $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$. \square

Finally, the following simple claim will complete the proof of our theorem.

Claim 3.4. *The space $X_{\mathbb{C}}$ is ω_1 -irresolvable, that is, not ω_1 -resolvable.*

Proof of the Claim. Assume that $\{F_\zeta : \zeta < \omega_1\}$ is a family of dense subsets of $X_{\mathbb{C}}$. By condition (2) the topology of $X_{\mathbb{C}}$ is \mathcal{D} -forced, so every F_ζ includes a \mathcal{D} -mosaic in $X_{\mathbb{C}}$, consequently for all $\zeta < \omega_1$ there are $\varepsilon_\zeta \in \text{Fn}(2^\kappa, 2)$ and $\phi_\zeta \in \text{Fn}(2^\kappa, 2)$ such that $\mathbb{D}[\phi_\zeta] \cap \mathbb{C}[\varepsilon_\zeta] \subset F_\zeta$. By the well-known Δ -system lemma we may then find $\zeta < \xi < \omega_1$ such that $\varepsilon = \varepsilon_\zeta \cup \varepsilon_\xi \in \text{Fn}(2^\kappa, 2)$ and $\phi = \phi_\zeta \cup \phi_\xi \in \text{Fn}(2^\kappa, 2)$. (Actually, much more is true: there is an uncountable set $S \in [\omega_1]^{\omega_1}$ such that the members of both $\{\varepsilon_\zeta : \zeta \in S\}$ and $\{\phi_\zeta : \zeta \in S\}$ are pairwise compatible.) But then we have

$$F_\zeta \cap F_\xi \supset \mathbb{D}[\phi_\zeta] \cap \mathbb{C}[\varepsilon_\zeta] \cap \mathbb{D}[\phi_\xi] \cap \mathbb{C}[\varepsilon_\xi] = \mathbb{D}[\phi] \cap \mathbb{C}[\phi] \neq \emptyset.$$

\square

To conclude our proof, it suffices to recall the obvious fact that if a topology on a set is λ -resolvable then so is any coarser topology. Hence the ω_1 -irresolvability of $X_{\mathbb{C}}$ implies that of $X_{\mathbb{C}\cup\mathbb{E}}$. \square

Let us point out that as extraresolvability implies almost ω -resolvability that is equivalent to ω -resolvability, any counterexample to problem 1 is automatically an example of an ω -resolvable but not maximally resolvable space, hence it is a solution to the celebrated problem of Ceder and Pearson from [1]. The first Tychonov ZFC examples of such spaces were given in [3] and the spaces constructed in theorem 3 extend the supply of such examples.

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ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS

Email address: `juhasz@renyi.hu`

HEBREW UNIVERSITY

Email address: `shelah@math.huji.ac.il`

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS

Email address: `soukup@renyi.hu`