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SQUARES OF MENGER-BOUNDED GROUPS

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Abstract. Using a portion of the Continuum Hypothesis, we prove that there is a Menger-bounded (also called *o*-bounded) subgroup of the Baire group $\mathbb{Z}^{\mathbb{N}}$, whose square is not Menger-bounded. This settles a major open problem concerning boundedness notions for groups, and implies that Menger-bounded groups need not be Scheepers-bounded.

1. Introduction

Assume that $(G, \cdot)$ is a topological group. For $A, B \subseteq G$, $A \cdot B$ stands for $\{a \cdot b : a \in A,\ b \in B\}$, and $a \cdot B$ stands for $\{a \cdot b : b \in B\}$.

G is *Menger-bounded* (also called *o-bounded*) if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of the unit, there exist finite sets $F_n \subseteq G$, $n \in \mathbb{N}$, such that $G = \bigcup_n F_n \cdot U_n$.

G is *Scheepers-bounded* if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of the unit, there exist finite sets $F_n \subseteq G$, $n \in \mathbb{N}$, such that for each finite set $F \subseteq G$, there is n such that $F \subseteq F_n \cdot U_n$.

A variety of boundedness properties for groups, including the two mentioned ones, were studied extensively in the literature, resulting in an almost complete classification of these notions [15, 8, 9, 11, 4, 16, 2, 1, 12, 5]. Only the following classification problem remained open.

Problem 1. *Is every metrizable Menger-bounded group Scheepers-bounded?*

The notions of Menger-bounded and Scheepers-bounded groups are related in the following elegant manner.

Theorem 2 (Babinkostova-Kočinac-Scheepers [2])**.** *G is Scheepers-bounded if, and only if, G^k is Menger-bounded for all k.*

In light of Theorem 2, Problem 1 asks whether there could be a metrizable group G such that for some k, G^k is Menger-bounded but G^{k+1} is not. We give a negative answer by showing that, assuming

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a portion of the Continuum Hypothesis, there is such an example *for each* k.

Some special hypothesis is necessary in order to prove such a result: Banakh and Zdomskyy [6, 5] proved that consistently, every topological group with Menger-bounded square is Scheepers-bounded.

2. Specializing the question for the Baire group

Subgroups of the *Baire group* $\mathbb{Z}^{\mathbb{N}}$ form a rich source of examples of groups with various boundedness properties [12]. The advantage of working in $\mathbb{Z}^{\mathbb{N}}$ is that the boundedness properties there can be stated in a purely combinatorial manner.

We use mainly self-evident notation. For natural numbers $k < m$, $[k, m) = \{k, k+1, \dots, m-1\}$. For a partial function $f : \mathbb{N} \to \mathbb{Z}$, $|f|$ is the function with the same domain, which satisfies $|f|(n) = |f(n)|$, where in this case $|\cdot|$ denotes the absolute value. For partial functions $f, g : \mathbb{N} \to \mathbb{N}$ with $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, $f \leq g$ means: For each n in the domain of f, $f(n) \leq g(n)$. Similarly, $f \leq k$ means: For each n in the domain of f, $f(n) \leq k$. The quantifiers $(\exists^\infty n)$ and $(\forall^\infty n)$ stand for "there exist infinitely many n" and "for all but finitely many n", respectively.

Theorem 3 ([12])**.** *Assume that G is a subgroup of $\mathbb{Z}^{\mathbb{N}}$. The following conditions are equivalent:*

(1) *G is Menger-bounded.*
(2) *For each increasing $h \in \mathbb{N}^{\mathbb{N}}$, there is $f \in \mathbb{N}^{\mathbb{N}}$ such that:*

$$(\forall g \in G)(\exists^\infty n)\ |g| \restriction [0, h(n)) \leq f(n).$$

The proof of Theorem 2 actually shows that the following holds for each natural number k.

Theorem 4. *G^k is* Menger-bounded *if, and only if, for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of the unit, there exist finite sets $F_n \subseteq G$, $n \in \mathbb{N}$, such that for each $F \subseteq G$ with $|F| = k$, there is n such that $F \subseteq F_n \cdot U_n$.* □

Specializing to $\mathbb{Z}^{\mathbb{N}}$ again and using arguments as in the proof of Theorem 3, we obtain the following.

Theorem 5. *Assume that G is a subgroup of $\mathbb{Z}^{\mathbb{N}}$. The following conditions are equivalent:*

(1) *G^k is Menger-bounded.*
(2) *For each increasing $h \in \mathbb{N}^{\mathbb{N}}$, there is $f \in \mathbb{N}^{\mathbb{N}}$ such that:*

$$(\forall F \in [G]^k)(\exists^\infty n)(\forall g \in F)\ |g| \restriction [0, h(n)) \leq f(n). \quad \square$$

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In light of these results, it is clear that the theorem that we prove in the next section is what we are looking for.

3. The main theorem

The forthcoming Theorem 7 requires a (weak) portion of the Continuum Hypothesis. It is stated in terms of cardinal characteristics of the continuum, see [7] for an introduction. We define the following ad-hoc cardinals.

Definition 6. Fix a partition $\mathcal{P} = \{I_l : l \in \mathbb{N}\}$ of $\mathbb{N}$ such that for each l, there are infinitely many n such that $n, n+1 \in I_l$. For $f \in \mathbb{N}^\mathbb{N}$ and an increasing $h \in \mathbb{N}^\mathbb{N}$, write

$$[f \ll h] = \{n : f(h(n)) < h(n+1)\}.$$

$\mathfrak{d}'(\mathcal{P})$ is the cardinal such that the following are equivalent:

(1) $\kappa < \mathfrak{d}'(\mathcal{P})$;
(2) For each $\mathcal{F} \subseteq \mathbb{N}^\mathbb{N}$ such that $|\mathcal{F}| = \kappa$, there is an increasing $h \in \mathbb{N}^\mathbb{N}$ such that for each $f \in \mathcal{F}$,

$$(\forall l)(\exists^\infty n)\ n, n+1 \in I_l \cap [f \ll h].$$

It is not difficult to show that for each $\mathcal{P}$, $\max\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} \leq \mathfrak{d}'(\mathcal{P}) \leq \mathfrak{d}$, and there are additional bounds on $\mathfrak{d}'(\mathcal{P})$ [14]. (Here $\mathcal{M}$ denotes the ideal of meager subsets of $\mathbb{R}$.)

Theorem 7. *Assume that there is $\mathcal{P}$ such that $\mathfrak{d}'(\mathcal{P}) = \mathfrak{d}$. Then for each k, there is a group $G \leq \mathbb{Z}^\mathbb{N}$ such that G^k is Menger-bounded, but G^{k+1} is not Menger-bounded.*

Proof. Fix a partition $\mathcal{P} = \{I_l : l \in \mathbb{N}\}$ of $\mathbb{N}$ such that for each l, there are infinitely many n such that $n, n+1 \in I_l$, and such that $\mathfrak{d}'(\mathcal{P}) = \mathfrak{d}$.

Enumerate $\mathbb{Z}^{k\times(k+1)}$ as $\{A_n : n \in \mathbb{N}\}$, such that the sequence $\{A_n\}_{n\in I_l}$ is constant for each l. Fix a dominating family of increasing functions $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^\mathbb{N}$. For $v = (v_0, \dots, v_k) \in \mathbb{Z}^{k+1}$, write $\|v\|$ or $\|v_0, \dots, v_k\|$ for $\max\{|v_0|, \dots, |v_k|\}$ (the supremum norm of v).

We carry out a construction by induction on $\alpha < \mathfrak{d}$. Step α: Define functions $\varphi_{\alpha,m} \in \mathbb{N}^\mathbb{N}$, $m \in \mathbb{N}$, by

$$\varphi_{\alpha,m}(n) = \min\{\|v\| : v \in \mathbb{Z}^{k+1},\ \|v\| \geq d_\alpha(n),\ A_m v = \vec{0}\}.$$

Also, define $\varphi_\alpha \in \mathbb{N}^\mathbb{N}$ by

$$\varphi_\alpha(n) = \max\{\varphi_{\alpha,i}(j) : i, j \leq n\}.$$

We define a set $M_\alpha \subseteq \mathbb{Z}^\mathbb{N}$ as follows. Assume, inductively, that for each $\beta < \alpha$, $|M_\beta| \leq \max\{\aleph_0, |\beta|\}$. Let M_α be the smallest set (with respect to inclusion) containing d_α, φ_α and all functions defined in stages $< \alpha$

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(in particular, $\bigcup_{\beta<\alpha} M_\beta \subseteq M_\alpha$), and such that M_α is closed under all operations relevant for the proof.

For example, closing M_α under the following operations suffices:[1]

(a) $f(n) \mapsto c \cdot f(n)$, $c \in \mathbb{N}$;
(b) $g(n) \mapsto \hat{g}(n) = \max\{|g(m)| : m \leq n\}$;
(c) $(f, d) \mapsto f \circ d$ when $d \in M_\alpha \cap \mathbb{N}^\mathbb{N}$;
(d) $(f_0(n), \dots, f_{k-1}(n)) \mapsto \max\{f_0(n), \dots, f_{k-1}(n)\}$;
(e) $(f_0(n), f_1(n), f_2(n), f_3(n)) \mapsto \psi(n)$, where $\psi(n)$ is defined to be

$$\min\{k : (\exists j)\ f_0(j) \leq f_1(j) \text{ and } [f_2(j), f_3(j)) \subseteq [n, k)\},$$

and where $f_2 \in M_\alpha \cap \mathbb{N}^\mathbb{N}$ is increasing, and $f_1 \not\leq^* f_0$.

There are countably many such operations, and $|M_\alpha| \leq \max\{\aleph_0, |\alpha|\} < \mathfrak{d}'(\mathcal{P})$. By the definition of $\mathfrak{d}'(\mathcal{P})$, there is an increasing $h_\alpha \in \mathbb{N}^\mathbb{N}$ such that for each $f \in M_\alpha \cap \mathbb{N}^\mathbb{N}$,

$$(\forall l)(\exists^\infty n)\ n, n+1 \in I_l \cap [f \ll h_\alpha].$$

Define $k+1$ elements $g_0^\alpha, \dots, g_k^\alpha \in \mathbb{Z}^\mathbb{N}$ as follows: For each n, let $v \in \mathbb{Z}^{k+1}$ be a witness for the definition of $\varphi_{\alpha,n}(h_\alpha(n+1))$, and define $(g_0^\alpha(h_\alpha(n)), \dots, g_k^\alpha(h_\alpha(n))) = v$. The remaining values are defined by declaring each g_i^α to be constant on each interval $[h_\alpha(n), h_\alpha(n+1))$.

Take the generated subgroup $G = \langle g_0^\alpha, \dots, g_k^\alpha : \alpha < \mathfrak{d} \rangle$ of $\mathbb{Z}^\mathbb{N}$. We will show that G is as required in the theorem.

G^{k+1} is not Menger-bounded. We use Theorem 5. Take $h(n) = n+1$. Let $f \in \mathbb{N}^\mathbb{N}$. Take $\alpha < \mathfrak{d}$ such that $f \leq^* d_\alpha$. Then $F = \{g_0^\alpha, \dots, g_k^\alpha\} \in [G]^{k+1}$. Let m be large enough, so that $f(m) \leq d_\alpha(m)$. Take n such that $m \in [h_\alpha(n), h_\alpha(n+1))$. Then

$$\begin{aligned} &\|g_0^\alpha(m), \dots, g_k^\alpha(m)\| = \\ &= \ \|g_0^\alpha(h_\alpha(n)), \dots, g_k^\alpha(h_\alpha(n))\| = \varphi_{\alpha,n}(h_\alpha(n+1)) \geq \\ &\geq \ d_\alpha(h_\alpha(n+1)) \geq d_\alpha(m) \geq f(m). \end{aligned}$$

This violates Theorem 5(2) for the power $k+1$.

G^k is Menger-bounded. Let $h \in \mathbb{N}^\mathbb{N}$ be increasing. Take $\delta < \mathfrak{d}$ such that $h \leq^* d_\delta$. It suffices to prove Theorem 5(2) for d_δ instead of h (so that now h is free to denote something else). Abbreviate $d = d_\delta, h = h_\delta$.

Choose an increasing $f \in \mathbb{N}^\mathbb{N}$ dominating all functions $f_c(n) = c \cdot h(n+1)$, $c \in \mathbb{N}$. We will prove that f is as required in Theorem 5(2).

[1]We may, alternatively, use model theory for first-order logic and assume that the sets M_α are elementary submodels of $H(\kappa)$ for a sufficiently large κ.

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Fix $F = \{g_0, \dots, g_{k-1}\} \subseteq G$. Then there are $g'_0, \dots g'_{k-1} \in M_\delta$, $i < k$, $M \in \mathbb{N}$ and $\alpha_1 < \cdots < \alpha_M < \mathfrak{d}$ such that $\delta < \alpha_1$, and matrices $B_1, \dots, B_M \in \mathbb{Z}^{k\times(k+1)}$, such that

$$\begin{pmatrix} g_0 \\ \vdots \\ g_{k-1} \end{pmatrix} = \begin{pmatrix} g'_0 \\ \vdots \\ g'_{k-1} \end{pmatrix} + B_1 \begin{pmatrix} g_0^{\alpha_1} \\ \vdots \\ g_k^{\alpha_1} \end{pmatrix} + \cdots + B_M \begin{pmatrix} g_0^{\alpha_M} \\ \vdots \\ g_k^{\alpha_M} \end{pmatrix}.$$

For each $m \leq M$, let

$$\begin{pmatrix} g_{0,m} \\ \vdots \\ g_{k-1,m} \end{pmatrix} = \begin{pmatrix} g'_0 \\ \vdots \\ g'_{k-1} \end{pmatrix} + B_1 \begin{pmatrix} g_0^{\alpha_1} \\ \vdots \\ g_k^{\alpha_1} \end{pmatrix} + \cdots + B_m \begin{pmatrix} g_0^{\alpha_m} \\ \vdots \\ g_k^{\alpha_m} \end{pmatrix}.$$

We prove, by induction on $m \leq M$, that for an appropriate constant c_m,

$$[\ \|\hat{g}_{0,m} \circ d, \dots, \hat{g}_{k-1,m} \circ d\| \ll c_m \cdot h\ ]$$

is infinite. By the definition of f, this suffices.

$m = 0$: As $g'_0, \dots, g'_{k-1}, d \in M_\delta$, $\|\hat{g}'_0 \circ d, \dots, \hat{g}'_{k-1} \circ d\| \in M_\delta$, and as $h = h_\delta$,

$$[\ \|\hat{g}'_0 \circ d, \dots, \hat{g}'_{k-1} \circ d\| \ll h\ ]$$

is infinite, so that $c_0 = 1$ works.

From $m-1$ to m: Let

$$J_m = [\ \|\hat{g}_{0,m-1} \circ d, \dots, \hat{g}_{k-1,m-1} \circ d\| \ll c_{m-1} \cdot h\ ],$$

and assume that J_m is infinite. We must prove that J_{m+1} is infinite, for an appropriate constant c_m. As $g_{0,m-1}, \dots, g_{k-1,m-1}, d, h \in M_{\alpha_m}$, we have that $\|\hat{g}_{0,m-1} \circ d, \dots, \hat{g}_{k-1,m-1} \circ d\|, c_{m-1} \cdot h, d \circ h \in M_{\alpha_m}$, and thus the (well defined) function

$$\psi_m(n) = \min\{k : (\exists j \in J_m)\ [h(j), d(h(j))) \subseteq [n, k)\}$$

belongs to M_{α_m}. Thus, $\max\{\psi_m, \varphi_{\alpha_m}\} \in M_{\alpha_m}$.

For each $i \leq k$ and each $n > 0$, as $n - 1 \leq h_{\alpha_m}(n)$, we have that

$$|g_i^{\alpha_m}(h_{\alpha_m}(n-1))| \leq \varphi_{\alpha_m, n-1}(h_{\alpha_m}(n)) \leq \varphi_{\alpha_m}(h_{\alpha_m}(n)).$$

As φ_{α_m} is nondecreasing,

$$\|\hat{g}_0^{\alpha_m}(h_{\alpha_m}(n-1)), \dots, \hat{g}_k^{\alpha_m}(h_{\alpha_m}(n-1))\| \leq \varphi_{\alpha_m}(h_{\alpha_m}(n)).$$

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Thus, if l is such that for each $n \in I_l$, $A_n = B_m$, we have that

$$\begin{aligned} I &= \left\{ n : \begin{array}{l} n, n+1 \in I_l, \\ (\exists j \in J_m)\ [h(j), d(h(j))) \subseteq [h_{\alpha_m}(n+1), h_{\alpha_m}(n+2)), \\ \|\hat{g}_0^{\alpha_m}(h_{\alpha_m}(n-1)), \dots, \hat{g}_k^{\alpha_m}(h_{\alpha_m}(n-1))\| < h_{\alpha_m}(n+1) \end{array} \right\} \\ &\supseteq \left\{ n : \begin{array}{l} n, n+1 \in I_l, \\ \psi_m(h_{\alpha_m}(n+1)) < h_{\alpha_m}(n+2), \\ \varphi_{\alpha_m}(h_{\alpha_m}(n)) < h_{\alpha_m}(n+1) \end{array} \right\} \\ &\supseteq \{n : n, n+1 \in I_l \cap [\max\{\psi_m, \varphi_{\alpha_m}\} \ll h_{\alpha_m}]\}. \end{aligned}$$

By the definition of h_{α_m}, the last set is infinite, and therefore so is I.

By the construction,

$$A_n \cdot \begin{pmatrix} g_0^{\alpha_m}(h_{\alpha_m}(n)) \\ \vdots \\ g_k^{\alpha_m}(h_{\alpha_m}(n)) \end{pmatrix} = \vec{0}$$

for all n. Let $n \in I$. Then $n, n+1 \in I_l$ and $A_n = A_{n+1} = B_m$. Thus, for each $i < k$,

$$g_{i,m} \restriction [h_{\alpha_m}(n), h_{\alpha_m}(n+2)) = g_{i,m-1} \restriction [h_{\alpha_m}(n), h_{\alpha_m}(n+2)).$$

Let $j \in J_m$ witness that $n \in I$, and let $p \in [0, d(h(j)))$.

Case 1: $p \in [h_{\alpha_m}(n), d(h(j)))$. By the definition of I,

$$[h_{\alpha_m}(n), d(h(j))) \subseteq [h_{\alpha_m}(n), h_{\alpha_m}(n+2)),$$

and by the equality above (as $j \in J_m$),

$$|g_{i,m}(p)| = |g_{i,m-1}(p)| \le \hat{g}_{i,m-1}(d(h(j))) \le c_{m-1}h(j+1)$$

for all $i < k$.

Case 2: $p \in [0, h_{\alpha_m}(n))$. Let C be the maximal absolute value of a coordinate of B_m. For all $i < k$, by the definition of $g_{i,m}$,

$$|g_{i,m}(p)| \le |g_{i,m-1}(p)| + (k+1)C \cdot \max\{|g_i^{\alpha_m}(p)| : i \le k\}.$$

By the choice of n and j, $p < h_{\alpha_m}(n) \le h(j) \le d(h(j))$, and therefore $|g_{i,m-1}(p)| \le \hat{g}_{i,m-1}(d(h(j))) \le c_{m-1}h(j+1)$. For each $i \le k$,

$$|g_i^{\alpha_m}(p)| \le \hat{g}_i^{\alpha_m}(h_{\alpha_m}(n-1)) \le h_{\alpha_m}(n+1) \le h(j).$$

Thus,

$$\begin{aligned} |g_{i,m}(p)| &\le |g_{i,m-1}(p)| + (k+1)C \cdot \max\{|g_i^{\alpha_m}(p)| : i \le k\} \\ &\le c_{m-1}h(j+1) + (k+1)C \cdot h(j) \\ &\le c_{m-1}h(j+1) + (k+1)C \cdot h(j+1) \\ &= (c_{m-1} + (k+1)C) \cdot h(j+1). \end{aligned}$$

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Take $c_m = c_{m-1} + kC$.

This completes the proof of Theorem 7. □

The problem whether, consistently, every Menger-bounded group is Scheepers-bounded is yet to be addressed.

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Personal appendix: Three fundamental problems

Following are two suggested extensions to F761, and one suggested extension for F762.

Recall that a subgroup G of $\mathbb{Z}^{\mathbb{N}}$ is *Menger-bounded* iff:
For each increasing $h \in \mathbb{N}^{\mathbb{N}}$, there is $f \in \mathbb{N}^{\mathbb{N}}$ such that:

$$(\forall g \in G)(\exists^\infty n)\ |g| \restriction [0, h(n)) \leq f(n).$$

We used a weak but unprovable hypothesis to prove that there is a group $G \leq \mathbb{Z}^{\mathbb{N}}$ such that G is Menger-bounded, but G^2 is not Menger-bounded.

Now assume that we are given more freedom. I expect the following to have a positive answer, and this would solve an important problem of Tkačenko.

Problem F761(A). Are there (in ZFC!) Menger-bounded groups $G, H \leq \mathbb{Z}^{\mathbb{N}}$ such that $G \times H$ is not Menger-bounded.

Definition 8. A subgroup G of $\mathbb{Z}^{\mathbb{N}}$ is *Rothberger-bounded* iff
For each increasing $h \in \mathbb{N}^{\mathbb{N}}$, there is $\varphi : \mathbb{N} \to \mathbb{Z}^{<\aleph_0}$ such that:

$$(\forall g \in G)(\exists n)\ g \restriction [0, h(n)) = \varphi(n).$$

Recall that G^2 is Menger-bounded iff:
For each increasing $h \in \mathbb{N}^{\mathbb{N}}$, there is $f \in \mathbb{N}^{\mathbb{N}}$ such that:

$$(\forall F \in [G]^2)(\exists^\infty n)(\forall g \in F)\ |g| \restriction [0, h(n)) \leq f(n).$$

Problem F761(B). Does CH imply the existence of a group $G \leq \mathbb{Z}^{\mathbb{N}}$ such that G is Rothberger-bounded but G^2 is not Menger-bounded?

Semifilter-trichotomy is the hypothesis equivalent to $\mathfrak{u} < \mathfrak{g}$, which asserts that for each semifilter $\mathcal{F}$ on $\mathbb{N}$ (i.e., $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is nonempty, and for all $A, B \subseteq \mathbb{N}$, $\mathcal{F} \ni A \subseteq^* B \to B \in \mathcal{F}$), there is an increasing sequence h such that $\mathcal{F}/h$ is either the Fréchet filter (all cofinite sets), or an ultrafilter, or $[\mathbb{N}]^{\aleph_0}$.

Problem F762(C). Does semifilter-trichotomy imply that the square of each Menger-bounded subgroup of $\mathbb{Z}^{\mathbb{N}}$ is Menger-bounded?

The question for *larger* powers was settled in the positive by Banakh and Zdomskyy, and independently bey Heike in her work on F762.

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