TIE-POINTS AND FIXED-POINTS IN $\mathbb{N}^*$

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Abstract. A point x is a (bow) tie-point of a space X if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with x in its closure. Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ (e.g. [10, 8]) and in the recent study of (precisely) 2-to-1 maps on $\beta\mathbb{N} \setminus \mathbb{N}$. In these cases the tie-points have been the unique fixed point of an involution on $\beta\mathbb{N} \setminus \mathbb{N}$. This paper is motivated by the search for 2-to-1 maps and obtaining tie-points of strikingly differing characteristics.

1. Introduction

A point x is a tie-point of a space X if there are closed sets A, B of X such that $\{x\} = A \cap B$ and x is an adherent point of each of A and B. We picture (and denote) this as $X = A \underset{x}{\bowtie} B$ where A, B are the closed sets which have a unique common accumulation point x and say that x is a tie-point as witnessed by A, B. Let $A \equiv_x B$ mean that there is a homeomorphism from A to B with x as a fixed point. If $X = A \underset{x}{\bowtie} B$ and $A \equiv_x B$, then there is an involution F of X (i.e. $F^2 = F$) such that $\{x\} = \operatorname{fix}(F)$. In this case we will say that x is a symmetric tie-point of X.

An autohomeomorphism F of $\beta\mathbb{N} \setminus \mathbb{N}$ (or $\mathbb{N}^*$) is said to be *trivial* if there is a bijection f between cofinite subsets of $\mathbb{N}$ such that $F = \beta f \restriction \beta\mathbb{N} \setminus \mathbb{N}$. If F is a trivial autohomeomorphism, then $\operatorname{fix}(F)$ is clopen; so of course $\beta\mathbb{N} \setminus \mathbb{N}$ will have no symmetric tie-points in this case if all autohomeomorphisms are trivial.

If A and B are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \underset{x=y}{\bowtie} B$ denote the quotient space of

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$A \oplus B$ obtained by identifying x and y and let xy denote the collapsed point. Clearly the point xy is a tie-point of this space.

We came to the study of tie-points via the following observation.

Proposition 1.1. *If x, y are symmetric tie-points of $\beta\mathbb{N} \setminus \mathbb{N}$ as witnessed by A, B and A', B' respectively, then there is a 2-to-1 mapping from $\beta\mathbb{N} \setminus \mathbb{N}$ onto the space $A \underset{x=y}{\bowtie} B'$.*

The proposition holds more generally if x and y are fixed points of involutions F, F' respectively. That is, replace A by the quotient space of $\beta\mathbb{N} \setminus \mathbb{N}$ obtained by collapsing all sets $\{z, F(z)\}$ to single points and similary replace B' by the quotient space induced by F'. It is an open problem to determine if 2-to-1 continuous images of $\beta\mathbb{N} \setminus \mathbb{N}$ are homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$ [5]. It is known to be true if CH [3] or PFA [2] holds.

There are many interesting questions that arise naturally when considering the concept of tie-points in $\beta\mathbb{N} \setminus \mathbb{N}$. Given a closed set $A \subset \beta\mathbb{N} \setminus \mathbb{N}$, let $\mathcal{I}_A = \{a \subset \mathbb{N} : a^* \subset A\}$. Given an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, let $\mathcal{I}^\perp = \{b \subset \mathbb{N} : (\forall a \in \mathcal{I})\ a \cap b =^* \emptyset\}$ and $\mathcal{I}^+ = \{d \subset \mathbb{N} : (\forall a \in I)\ d \setminus a \notin \mathcal{I}^\perp\}$. If $\mathcal{J} \subset [\mathbb{N}]^\omega$, let $\mathcal{J}^\downarrow = \bigcup_{J \in \mathcal{J}} \mathcal{P}(J)$. Say that $\mathcal{J} \subset \mathcal{I}$ is unbounded in $\mathcal{I}$ if for each $a \in \mathcal{I}$, there is a $b \in \mathcal{J}$ such that $b \setminus a$ is infinite.

Definition 1.1. If $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$, set $\mathrm{cf}(\mathcal{I})$ to be the cofinality of $\mathcal{I}$; $\mathfrak{b}(\mathcal{I})$ is the minimum cardinality of an unbounded family in $\mathcal{I}$; $\delta(\mathcal{I})$ is the minimum cardinality of a subset $\mathcal{J}$ of $\mathcal{I}$ such that $\mathcal{J}^\downarrow$ is dense in $\mathcal{I}$.

If $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{x}{\bowtie} B$, then $\mathcal{I}_B = \mathcal{I}_A^\perp$ and x is the unique ultrafilter on $\mathbb{N}$ extending $\mathcal{I}_A^+ \cap \mathcal{I}_B^+$. The character of x in $\beta\mathbb{N} \setminus \mathbb{N}$ is equal to the maximum of $\mathrm{cf}(\mathcal{I}_A)$ and $\mathrm{cf}(\mathcal{I}_B)$.

Definition 1.2. Say that a tie-point x has (i) $\mathfrak{b}$-type; (ii) δ-type; respectively (iii) $\mathfrak{b}\delta$-type, (κ, λ) if $\beta\mathbb{N}\setminus\mathbb{N} = A \underset{x}{\bowtie} B$ and (κ, λ) equals: (i) $(\mathfrak{b}(\mathcal{I}_A), \mathfrak{b}(\mathcal{I}_B))$ (ii) $(\delta(\mathcal{I}_A), \delta(\mathcal{I}_B))$; and (iii) each of $(\mathfrak{b}(\mathcal{I}_A), \mathfrak{b}(\mathcal{I}_B))$ and $(\delta(\mathcal{I}_A), \delta(\mathcal{I}_B))$. We will adopt the convention to put the smaller of the pair (κ, λ) in the first coordinate.

Again, it is interesting to note that if x is a tie-point of $\mathfrak{b}$-type (κ, λ), then it is uniquely determined (in $\beta\mathbb{N}\setminus\mathbb{N}$) by λ many subsets of $\mathbb{N}$ since x will be the unique point extending the family $((\mathcal{J}_A)^\downarrow)^+ \cap ((\mathcal{J}_B)^\downarrow)^+$ where $\mathcal{J}_A$ and $\mathcal{J}_B$ are unbounded subfamilies of $\mathcal{I}_A$ and $\mathcal{I}_B$.

Question 1.1. Can there be a tie-point in $\beta\mathbb{N} \setminus \mathbb{N}$ with δ-type (κ, λ) with $\kappa \leq \lambda$ less than the character of the point?

Question 1.2. Can $\beta\mathbb{N} \setminus \mathbb{N}$ have tie-points of δ-type (ω_1, ω_1) and (ω_2, ω_2)?

Proposition 1.2. *If $\beta\mathbb{N} \setminus \mathbb{N}$ has symmetric tie-points of δ-type (κ, κ) and (λ, λ), but no tie-points of δ-type (κ, λ), then $\beta\mathbb{N} \setminus \mathbb{N}$ has a 2-to-1 image which is not homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$.*

One could say that a tie-point x was *radioactive* in X (i.e. $\overset{\triangledown}{\bowtie}$) if $X \setminus \{x\}$ can be similarly split into 3 (or more) relatively clopen sets accumulating to x. This is equivalent to $X = A \underset{x}{\bowtie} B$ such that x is a tie-point in either A or B.

Each point of character ω_1 in $\beta\mathbb{N} \setminus \mathbb{N}$ is a radioactive point (in particular is a tie-point). P-points of character ω_1 are symmetric tie-points of $\mathfrak{b}\delta$-type (ω_1, ω_1), while points of character ω_1 which are not P-points will have $\mathfrak{b}$-type (ω, ω_1) and δ-type (ω_1, ω_1). If there is a tie-point of $\mathfrak{b}$-type (κ, λ), then of course there are (κ, λ)-gaps. If there is a tie-point of δ-type (κ, λ), then $\mathfrak{p} \leq \kappa$.

Proposition 1.3. *If $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{x}{\bowtie} B$, then $\mathfrak{p} \leq \delta(\mathcal{I}_A)$.*

Proof. If $\mathcal{J} \subset \mathcal{I}_A$ has cardinality less than $\mathfrak{p}$, there is, by Solovay's Lemma (and Bell's Theorem) an infinite set $C \subset \mathbb{N}$ such that C and $\mathbb{N} \setminus C$ each meet every infinite set of the form $J \setminus (\bigcup \mathcal{J}')$ where $\{J\} \cup \mathcal{J}' \in [\mathcal{J}]^{<\omega}$. We may assume that $C \notin x$, hence there are $a \in \mathcal{I}_A$ and $b \in \mathcal{I}_B$ such that $C \subset a \cup b$. However no finite union from $\mathcal{J}$ covers a showing that $\mathcal{J}^{\downarrow}$ can not be dense in $\mathcal{I}_A$. $\square$

Although it does not seem to be completely trivial, it can be shown that PFA implies there are no tie-points (the hardest case to eliminate is those of $\mathfrak{b}$-type (ω_1, ω_1))).

Question 1.3. Does $\mathfrak{p} > \omega_1$ imply there are no tie-points of $\mathfrak{b}$-type (ω_1, ω_1)?

Analogous to tie-points, we also define a tie-set: say that $K \subset \beta\mathbb{N} \setminus \mathbb{N}$ is a tie-set if $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{K}{\bowtie} B$ and $K = A \cap B$, $A = \overline{A \setminus K}$, and $B = \overline{B \setminus K}$. Say that K is a symmetric tie-set if there is an involution F such that $K = \mathrm{fix}(F)$ and $F[A] = B$.

Question 1.4. If F is an involution on $\beta\mathbb{N} \setminus \mathbb{N}$ such that $K = \mathrm{fix}(F)$ has empty interior, is K a (symmetric) tie-set?

Question 1.5. Is there some natural restriction on which compact spaces can (or can not) be homeomorphic to the fixed point set of some involution of $\beta\mathbb{N} \setminus \mathbb{N}$?

Again, we note a possible application to 2-to-1 maps.

Proposition 1.4. *Assume that F is an involution of $\beta\mathbb{N} \setminus \mathbb{N}$ with $K = \mathrm{fix}(F) \neq \emptyset$. Further assume that K has a symmetric tie-point x (i.e. $K = A \underset{x}{\bowtie} B$), then $\beta\mathbb{N} \setminus \mathbb{N}$ has a 2-to-1 continuous image which has a symmetric tie-point (and possibly $\beta\mathbb{N} \setminus \mathbb{N}$ does not have such a tie-point).*

Question 1.6. If F is an involution of $\mathbb{N}^*$, is the quotient space $\mathbb{N}^*/F$ (in which each $\{x, F(x)\}$ is collapsed to a single point) a homeomorphic copy of $\beta\mathbb{N} \setminus \mathbb{N}$?

Proposition 1.5 (CH). *If F is an involution of $\beta\mathbb{N} \setminus \mathbb{N}$, then the quotient space $\mathbb{N}^*/F$ is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. If $\mathrm{fix}(F)$ is empty, then $\mathbb{N}^*/F$ is a 2-to-1 image of $\beta\mathbb{N} \setminus \mathbb{N}$, and so is a copy of $\beta\mathbb{N} \setminus \mathbb{N}$. If $\mathrm{fix}(F)$ is not empty, then consider two copies, $(\mathbb{N}^*_1, F_1)$ and $(\mathbb{N}^*_2, F_2)$, of $(\mathbb{N}^*, F)$. The quotient space of $\mathbb{N}^*_1/F_1 \oplus \mathbb{N}^*_2/F_2$ obtained by identifying the two homeomorphic sets $\mathrm{fix}(F_1)$ and $\mathrm{fix}(F_2)$ will be a 2-to-1-image of $\mathbb{N}^*$, hence again a copy of $\mathbb{N}^*$. Since $\mathbb{N}^*_1 \setminus \mathrm{fix}(F_1)$ and $\mathbb{N}^*_2 \setminus \mathrm{fix}(F_2)$ are disjoint and homeomorphic, it follows easily that $\mathrm{fix}(F)$ must be a P-set in $\mathbb{N}^*$. It is trivial to verify that a regular closed set of $\mathbb{N}^*$ with a P-set boundary will be (in a model of CH) a copy of $\mathbb{N}^*$. Therefore the copy of $\mathbb{N}^*_1/F_1$ in this final quotient space is a copy of $\mathbb{N}^*$. □

2. A SPECTRUM OF TIE-SETS

We adapt a method from [1] to produce a model in which there are tie-sets of specified $\mathfrak{b}\delta$-types. We further arrange that these tie-sets will themselves have tie-points but unfortunately we are not able to make the tie-sets symmetric. In the next section we make some progress in involving involutions.

Theorem 2.1. *Assume GCH and that Λ is a set of regular uncountable cardinals such that for each $\lambda \in \Lambda$, T_λ is a $<\lambda$-closed λ^+-Souslin tree. There is a forcing extension in which there is a tie-set K (of $\mathfrak{b}\delta$-type $(\mathfrak{c}, \mathfrak{c})$) and for each $\lambda \in \Lambda$, there is a tie-set K_λ of $\mathfrak{b}\delta$-type (λ^+, λ^+) such that $K \cap K_\lambda$ is a single point which is a tie-point of K_λ. Furthermore, for $\mu \leq \lambda < \mathfrak{c}$, if $\mu \neq \lambda$ or $\lambda \notin \Lambda$, then there is no tie-set of $\mathfrak{b}\delta$-type (μ, λ).*

We will assume that our Souslin trees are well-pruned and are ever ω-ary branching. That is, if T_λ is a λ^+-Souslin tree, we assume that for each $t \in T$, t has exactly ω immediate successors denoted $\{t^\frown\ell : \ell \in \omega\}$ and that $\{s \in T_\lambda : t < s\}$ has cardinality λ^+ (and so has successors on every level). A poset is $<\kappa$-closed if every directed subset of cardinality less than κ has a lower bound. A poset is $<\kappa$-distributive if the intersection of any family of fewer than κ dense open subsets is again dense. For a cardinal μ, let μ^- be the minimum cardinal such that $(\mu^-)^+ \geq \mu$ (i.e. the predecessor if μ is a successor).

The main idea of the construction is nicely illustrated by the following.

Proposition 2.2. *Assume that $\beta\mathbb{N}\backslash\mathbb{N}$ has no tie-sets of $\mathfrak{b}\delta$-type (κ_1,κ_2) for some $\kappa_1 \leq \kappa_2 < \mathfrak{c}$. Also assume that $\lambda^+ < \mathfrak{c}$ is such that λ^+ is distinct from one of κ_1,κ_2 and that T_λ is a λ^+-Souslin tree and $\{(a_t,x_t,b_t) : t \in T_\lambda\} \subset ([\mathbb{N}]^\omega)^3$ satisfy that, for $t < s \in T_\lambda$:*

1. *$\{a_t,x_t,b_t\}$ is a partition of $\mathbb{N}$,*
2. *$x_{t^\frown j} \cap x_{t^\frown \ell} = \emptyset$ for $j < \ell$,*
3. *$x_s \subset^* x_t$, $a_t \subset^* a_s$, and $b_t \subset^* b_s$,*
4. *for each $\ell \in \omega$, $x_{t^\frown \ell+1} \subset^* a_{t^\frown \ell}$ and $x_{t^\frown \ell+2} \subset^* b_{t^\frown \ell}$,*

then if $\rho \in [T_\lambda]^{\lambda^+}$ is a generic branch (i.e. $\rho(\alpha)$ is an element of the α-th level of T_λ for each $\alpha \in \lambda^+$), then $K_\rho = \bigcap_{\alpha\in\lambda^+} x^*_{\rho(\alpha)}$ is a tie-set of $\beta\mathbb{N} \setminus \mathbb{N}$ of $\mathfrak{b}\delta$-type (λ^+,λ^+), and there is no tie-set of $\mathfrak{b}\delta$-type (κ_1,κ_2).

5. *Assume further that $\{(a_\xi,x_\xi,b_\xi) : \xi \in \mathfrak{c}\}$ is a family of partitions of $\mathbb{N}$ such that $\{x_\xi : \xi \in \mathfrak{c}\}$ is a mod finite descending family of subsets of $\mathbb{N}$ such that for each $Y \subset \mathbb{N}$, there is a maximal antichain $A_Y \subset T_\lambda$ and some $\xi \in \mathfrak{c}$ such that for each $t \in A_Y$, $x_t \cap x_\xi$ is a proper subset of either Y or $\mathbb{N}\backslash Y$, then $K = \bigcap_{\xi\in\mathfrak{c}} x^*_\xi$ meets K_ρ in a single point z_λ.*
6. *If we assume further that for each $\xi < \eta < \mathfrak{c}$, $a_\xi \subset^* a_\eta$ and $b_\xi \subset^* b_\eta$, and for each $t \in T_\lambda$, η may be chosen so that x_t meets each of $(a_\eta \setminus a_\xi)$ and $(b_\eta \setminus b_\xi)$, then z_λ is a tie-point of K_ρ.*

Proof. To show that K_ρ is a tie-set it is sufficient to show that $K_\rho \subset \overline{\bigcup_{\alpha\in\lambda^+} a^*_\alpha} \cap \overline{\bigcup_{\alpha\in\lambda^+} b^*_\alpha}$. Since T_λ is a λ^+-Souslin tree, no new subset of λ is added when forcing with T_λ. Of course we use that ρ is T_λ is generic, so assume that $Y \subset \mathbb{N}$ and that some $t \in T_\lambda$ forces that $Y^* \cap K_\rho$ is not empty. We must show that there is some $t < s$ such that s forces that $a_s \cap Y$ and $b_s \cap Y$ are both infinite. However, we know that $x_{t^\frown\ell} \cap Y$ is infinite for each $\ell \in \omega$ since $t^\frown\ell \Vdash_{T_\lambda}$ "$K_\rho \subset x^*_{t^\frown\ell}$". Therefore, by condition 4, for each $\ell \in \omega$, $Y \cap a_{t^\frown\ell}$ and $Y \cap b_{t^\frown\ell}$ are both infinite.

Now let κ_1,κ_2 be regular cardinals at least one of which is distinct from λ^+. Recall that forcing with T_λ preserves cardinals. Assume that in $V[\rho]$, $K \subset \mathbb{N}^*$ and $\mathbb{N}^* = C \bowtie_K D$ with $\mathfrak{b}(\mathcal{I}_C) = \delta(\mathcal{I}_C) = \kappa_1$ and $\mathfrak{b}(\mathcal{I}_D) = \delta(\mathcal{I}_D) = \kappa_2$. In V, let $\{c_\gamma : \gamma \in \kappa_1\}$ be T_λ-names for the increasing cofinal sequence in $\mathcal{I}_C$ and let $\{d_\xi : \xi \in \kappa_2\}$ be T_λ-names for the increasing cofinal sequence in $\mathcal{I}_D$. Again using the fact that T_λ adds no new subsets of $\mathbb{N}$ and the fact that every dense open subset of T_λ will contain an entire level of T_λ, we may choose ordinals $\{\alpha_\gamma : \gamma \in \kappa_1\}$ and $\{\beta_\xi : \xi \in \kappa_2\}$ such that each $t \in T_\lambda$, if t is on level α_γ it will force a value on c_γ and if t is on level β_ξ it will force a value on d_ξ. If $\kappa_1 < \lambda^+$, then $\sup\{\alpha_\gamma : \gamma \in \kappa_1\} < \lambda^+$, hence there are $t \in T_\lambda$ which force a value on each c_γ. If $\lambda^+ < \kappa_2$, then there is some $\beta < \lambda^+$, such that $\{\xi \in \kappa_2 : \beta_\xi \leq \beta\}$ has cardinality κ_2. Therefore there is some $t \in T_\lambda$ such that t forces a value on d_ξ for a cofinal set of $\xi \in \kappa_2$. Of course, if neither κ_1 nor κ_2 is equal to λ^+, then we have a condition that decided cofinal families of each of $\mathcal{I}_C$ and $\mathcal{I}_D$. This implies that $\mathbb{N}^*$ already has tie-sets of $\mathfrak{b}\delta$-type (κ_1, κ_2).

If $\kappa_1 < \kappa_2 = \lambda^+$, then fix $t \in T_\lambda$ deciding $\mathfrak{C} = \{c_\gamma : \gamma \in \kappa_1\}$, and let $\mathfrak{D} = \{d \subset \mathbb{N} : (\exists s > t) s \Vdash_{T_\lambda} \text{“}d^* \subset D\text{”}\}$. It follows easily that $\mathfrak{D} = \mathfrak{C}^\perp$. But also, since forcing with T_λ can not raise $\mathfrak{b}(\mathfrak{D})$ and can not lower $\delta(\mathfrak{D})$, we again have that there are tie-sets of $\mathfrak{b}\delta$-type in V.

The case $\kappa_1 = \lambda^+ < \kappa_2$ is similar.

Now assume we have the family $\{(a_\xi, x_\xi, b_\xi) : \xi \in \mathfrak{c}\}$ as in (5) and (6) and set $K = \bigcap_\xi x_\xi^*$, $A = \{K\} \cup \bigcup\{a_\xi^* : \xi \in \mathfrak{c}\}$, and $B = \{K\} \cup \bigcup\{b_\xi^* : \xi \in \mathfrak{c}\}$. It is routine to see that (5) ensures that the family $\{x_\xi \cap x_{\rho(\alpha)} : \xi \in \mathfrak{c} \text{ and } \alpha \in \lambda^+\}$ generates an ultrafilter when ρ meets each maximal antichain A_Y $(Y \subset \mathbb{N})$. Condition (6) clearly ensures that $A \setminus K$ and $B \setminus K$ each meet $(x_\xi \cap x_{\rho(\alpha)})^*$ for each $\xi \in \mathfrak{c}$ and $\alpha \in \lambda^+$. Thus $A \cap K_\rho$ and $B \cap K_\rho$ witness that z_λ is a tie-point of K_ρ. □

Let θ be a regular cardinal greater than λ^+ for all $\lambda \in \Lambda$. We will need the following well-known Easton lemma (see [4, p234]).

Lemma 2.3. *Let μ be a regular cardinal and assume that P_1 is a poset satisfying the μ-cc. Then any $<\mu$-closed poset P_2 remains $<\mu$-distributive after forcing with P_1. Furthermore any $<\mu$-distributive poset remains $<\mu$-distributive after forcing with a poset of cardinality less than μ.*

Proof. Recall that a poset P is $<\mu$-distributive if forcing with it does not add, for any $\gamma < \mu$, any new γ-sequences of ordinals. Since P_2 is $<\mu$-closed, forcing with P_2 does not add any new antichains to P_1. Therefore it follows that forcing with P_2 preserves that P_1 has the μ-cc and that for every $\gamma < \mu$, each γ-sequence of ordinals in the forcing extension by $P_2 \times P_1$ is really just a P_1-name. Since forcing with $P_1 \times P_2$ is the same as $P_2 \times P_1$, this shows that in the extension by P_1, there are no new P_2-names of γ-sequences of ordinals.

Now suppose that P_2 is μ-distributive and that P_1 has cardinality less than μ. Let $\dot{D}$ be a P_1-name of a dense open subset of P_2. For each $p \in P_1$, let $D_p \subset P_2$ be the set of all q such that some extension of p forces that $q \in \dot{D}$. Since p forces that $\dot{D}$ is dense and that $\dot{D} \subset D_p$, it follows that D_p is dense (and open). Since P_2 is μ-distributive, $\bigcap_{p\in P_1} D_p$ is dense and is clearly going to be a subset of $\dot{D}$. Repeating this argument for at most μ many P_1-names of dense open subsets of P_2 completes the proof. $\square$

We recall the definition of Easton supported product of posets (see [4, p233]).

Definition 2.1. If Λ is a set of cardinals and $\{P_\lambda : \lambda \in \Lambda\}$ is a set of posets, then we will use $\Pi_{\lambda\in\Lambda} P_\lambda$ to denote the collection of partial functions p such that

(1) $\mathrm{dom}(p) \subset \Lambda$,
(2) $|\mathrm{dom}(p) \cap \mu| < \mu$ for all regular cardinals μ,
(3) $p(\lambda) \in P_\lambda$ for all $\lambda \in \mathrm{dom}(p)$.

This collection is a poset when ordered by $q < p$ if $\mathrm{dom}(q) \supset \mathrm{dom}(p)$ and $q(\lambda) \leq p(\lambda)$ for all $\lambda \in \mathrm{dom}(p)$.

Lemma 2.4. *For each cardinal μ, $\Pi_{\lambda\in\Lambda\setminus\mu^+} T_\lambda$ is $<\!\mu^+$-closed and, if μ is regular, $\Pi_{\lambda\in\Lambda\cap\mu} T_\lambda$ has cardinality at most $2^{<\mu} \leq \min(\Lambda \setminus \mu)$.*

Lemma 2.5. *If P is ccc and $G \subset P \times \Pi_{\lambda\in\Lambda} T_\lambda$ is generic, then in $V[G]$, for any μ and any family $\mathcal{A} \subset [\mathbb{N}]^\omega$ with $|\mathcal{A}| = \mu$:*

(1) *if $\mu \leq \omega$, then $\mathcal{A}$ is a member of $V[G \cap P]$;*
(2) *if $\mu = \lambda^+, \lambda \in \Lambda$, then there is an $\mathcal{A}' \subset \mathcal{A}$ of cardinality λ^+ such that $\mathcal{A}'$ is a member of $V[G \cap (P \times T_\lambda)]$;*
(3) *if $\mu^- \notin \Lambda$, then there is an $\mathcal{A}' \subset \mathcal{A}$ of cardinality μ which is a member of $V[G \cap P]$.*

Corollary 2.6. *If P is ccc and $G \subset P \times \Pi_{\lambda\in\Lambda} T_\lambda$ is generic, then for any $\kappa \leq \mu < \mathfrak{c}$ such that either $\kappa \neq \mu$ or $\kappa \notin \{\lambda^+ : \lambda \in \Lambda\}$, if there is a tie-set of $\mathfrak{b}\delta$-type (κ, μ) in $V[G]$, then there is such a tie-set in $V[G \cap P]$.*

Proof. Assume that $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{K}{\bowtie} B$ in $V[G]$ with $\mu = \mathfrak{b}(A)$ and $\lambda = \mathfrak{b}(B)$. Let $\mathcal{J}_A \subset \mathcal{I}_A$ be an increasing mod finite chain, of order type μ, which is dense in $\mathcal{I}_A$. Similarly let $\mathcal{J}_B \subset \mathcal{I}_B$ be such a chain of order type λ. By Lemma 2.5, $\mathcal{J}_A$ and $\mathcal{J}_B$ are subsets of $[\mathbb{N}]^\omega \cap V[G\cap P] = [\mathbb{N}]^\omega$. Choose, if possible $\mu_1 \in \Lambda$ such that $\mu_1^+ = \mu$ and $\lambda_1 \in \Lambda$ such that $\lambda_1^+ = \lambda$. Also by Lemma 2.5, we can, by passing to a subcollection, assume that $\mathcal{J}_A \in V[G\cap(P\times T_{\mu_1})]$ (if there is no μ_1, then let T_{μ_1} denote the trivial order). Similarly, we may assume that $\mathcal{J}_B \in V[G \cap (P \times T_{\lambda_1})]$. Fix a condition $q \in G \subset (P \times \Pi_{\lambda\in\Lambda} T_\lambda)$ which forces that $(\mathcal{J}_A)^\downarrow$ is a $\subset$-dense subset of $\mathcal{I}_A$, that $(\mathcal{J}_B)^\downarrow$ is a $\subset$-dense subset of $\mathcal{I}_B$, and that $(\mathcal{I}_A)^\perp = \mathcal{I}_B$.

Working in the model $V[G \cap P]$ then, there is a family $\{\dot{a}_\alpha : \alpha \in \mu\}$ of T_{μ_1}-names for the members of $\mathcal{J}_A$; and a family $\{\dot{b}_\beta : \beta \in \lambda_1\}$ of T_{λ_1}-names for the members of $\mathcal{J}_B$. Of course if $\mu = \lambda$ and T_{μ_1} is the trivial order, then $\mathcal{J}_A$ and $\mathcal{J}_B$ are already in $V[G \cap P]$ and we have our tie-set in $V[G \cap P]$.

Otherwise, we assume that $\mu_1 < \lambda_1$. Set $\mathcal{A}$ to be the set of all $a \subset \mathbb{N}$ such that there is some $q(\mu_1) \leq t \in T_{\mu_1}$ and $\alpha \in \mu$ such that $t \Vdash_{T_{\mu_1}}$ "$a = \dot{a}_\alpha$". Similarly let $\mathcal{B}$ be the set of all $b \subset \mathbb{N}$ such that there is some $q(\lambda_1) \leq s \in T_{\lambda_1}$ and $\beta \in \lambda$ such that $s \Vdash_{T_{\lambda_1}}$ "$b = \dot{b}_\beta$". It follows from the construction that, in $V[G]$, for any $(a', b') \in \mathcal{J}_A \times \mathcal{J}_B$, there is an $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $a' \subset^* a$ and $b' \subset^* b$. Therefore the ideal generated by $\mathcal{A} \cup \mathcal{B}$ is certainly dense. It remains only to show that $\mathcal{B} \subset (\mathcal{A})^\perp$. Consider any $(a, b) \in \mathcal{A} \times \mathcal{B}$, and choose $(q(\mu_1), q(\lambda_1)) \leq (t, s) \in T_{\mu_1} \times T_{\lambda_1}$ such that $t \Vdash_{T_{\mu_1}}$ "$a \in \mathcal{J}_A$" and $s \Vdash_{T_{\lambda_1}}$ "$b \in \mathcal{J}_B$". It follows that for any condition $\bar{q} \leq q$ with $\bar{q} \in (P \times \Pi_{\lambda\in\Lambda} T_\lambda)$, $\bar{q}(\mu_1) = t$, $\bar{q}(\lambda_1) = s$, we have that

$$\bar{q} \Vdash_{(P \times \Pi_{\lambda\in\Lambda} T_\lambda)} \text{“}a \in \mathcal{J}_A \text{ and } b \in \mathcal{J}_B\text{”} .$$

It is routine now to check that, in $V[G \cap P]$, $\mathcal{A}$ and $\mathcal{B}$ generate ideals that witness that $\bigcap\{(\mathbb{N} \setminus (a \cup b))^* : (a, b) \in \mathcal{A} \times \mathcal{B}\}$ is a tie-set of $\mathfrak{b}\delta$-type (μ, λ). $\square$

Let T be the rooted tree $\{\emptyset\} \cup \bigcup_{\lambda\in\Lambda} T_\lambda$ and we will force an embedding of T into $\mathcal{P}(\mathbb{N})$ mod finite. In fact, we force a structure $\{(a_t, x_t, b_t) : t \in T\}$ satisfying the conditions (1)-(4) of Proposition 2.2.

Definition 2.2. The poset Q_0 is defined as the set of elements $q = (n^q, T^q, f^q)$ where $n^q \in \mathbb{N}$, $T^q \in [T]^{<\omega}$, and $f^q : n^q \times T^q \to \{0, 1, 2\}$. The idea is that x_t will be $\bigcup_{q\in G}\{j \in n^q : f^q(j, t) = 0\}$, a_t will be $\bigcup_{q\in G}\{j \in n^q : f^q(j, t) = 1\}$ and $b_t = \mathbb{N} \setminus (a_t \cup x_t)$. We set $q < p$ if $n^q \geq n^p$, $T^q \supset T^p$, $f^q \supset f^p$ and for $t, s \in T^p$ and $i \in [n^p, n^q)$

(1) if $t < s$ and $f^q(i, t) \in \{1, 2\}$, then $f^q(i, s) = f^q(i, t)$;
(2) if $t < s$ and $f^q(i, s) = 0$, then $f^q(i, t) = 0$;
(3) if $t \perp s$, then $f^q(i, t) + f^q(i, s) > 0$.
(4) if $j \in \{1, 2\}$ and $\{t^\frown \ell, t^\frown(\ell + j)\} \subset T^p$ and $f^q(i, t^\frown(\ell + j)) = 0$, then $f^q(i, t^\frown \ell) = j$.

The next lemma is very routine but we record it for reference.

Lemma 2.7. *The poset Q_0 is ccc and if $G \subset Q_0$ is generic, the family $\mathcal{X}_T = \{(a_t, x_t, b_t) : t \in T\}$ satisfies the conditions of Proposition 2.2.*

We will need some other combinatorial properties of the family $\mathcal{X}_T$.

Definition 2.3. For any $\tilde{T} \in [T]^{<\omega}$, we define the following (Q_0-names).

(1) for $i \in \mathbb{N}$, $[i]_{\tilde{T}} = \{j \in \mathbb{N} : (\forall t \in \tilde{T})\ i \in x_t \text{ iff } j \in x_t\}$,
(2) the collection $\mathrm{fin}(\tilde{T})$ is the set of $[i]_{\tilde{T}}$ which are finite.

We abuse notation and let $\mathrm{fin}(\tilde{T}) \subset n$ abbreviate $\mathrm{fin}(\tilde{T}) \subset \mathcal{P}(n)$.

Lemma 2.8. *For each $q \in Q_0$ and each $\tilde{T} \subset T^q$, $\mathrm{fin}(\tilde{T}) \subset n^q$ and for $i \geq n_q$, $[i]_{\tilde{T}}$ is infinite.*

Definition 2.4. A sequence $\mathcal{S}_W = \{(a_\xi, x_\xi, b_\xi) : \xi \in W\}$ is a tower of T-splitters if for $\xi < \eta \in W$ and $t \in T$:

(1) $\{a_\xi, x_\xi, b_\xi\}$ is a partition of $\mathbb{N}$,
(2) $a_\xi \subset^* a_\eta$, $b_\xi \subset^* b_\eta$,
(3) $x_t \cap x_\xi$ is infinite.

Definition 2.5. If $\mathcal{S}_W$ is a tower of T-splitters and Y is a subset $\mathbb{N}$, then the poset $Q(\mathcal{S}_W, Y)$ is defined as follows. Let E_Y be the (possibly empty) set of minimal elements of T such that there is some finite $H \subset W$ such that $x_t \cap Y \cap \bigcap_{\xi \in H} x_\xi$ is finite. Let $D_Y = E_Y^\perp = \{t \in T : (\forall s \in E_Y)\ t \perp s\}$. A condition $q \in Q(\mathcal{S}_W, Y)$ is a tuple $(n^q, a^q, x^q, b^q, T^q, H^q)$ where

(1) $n^q \in \mathbb{N}$ and $\{a^q, x^q, b^q\}$ is a partition of n^q,
(2) $T^q \in [T]^{<\omega}$ and $H^q \in [W]^{<\omega}$,
(3) $(a_\xi \setminus a_\eta)$, $(b_\xi \setminus b_\eta)$, and $(x_\eta \setminus x_\xi)$ are all contained in n^q for $\xi < \eta \in H^q$.

We define $q < p$ to mean $n^p \leq n^q$, $T^p \subset T^q$, $H^p \subset H^q$, and

(4) for $t \in T^p \cap D_Y$, $x_t \cap (x^q \setminus x^p) \subset Y$,
(5) $x^q \setminus x^p \subset \bigcap_{\xi \in H^p} x_\xi$,
(6) $a^q \setminus a^p$ is disjoint from $b_{\max(H^p)}$,
(7) $b^q \setminus b^p$ is disjoint from $a_{\max(H^p)}$.

Lemma 2.9. *If $W \subset \gamma$, $\mathcal{S}_W$ is a tower of T-splitters, and if G is $Q(\mathcal{S}_W, Y)$-generic, then $\mathcal{S}_W \cup \{(a_\gamma, x_\gamma, b_\gamma)\}$ is also a tower of T-splitters where $a_\gamma = \bigcup\{a_q : q \in G\}$, $x_\gamma = \bigcup\{x_q : q \in G\}$, and $b_\gamma = \bigcup\{b_q : q \in G\}$. In addition, for each $t \in D_Y$, $x_t \cap x_\xi \subset^* Y$ (and $x_t \cap x_\xi \subset^* \mathbb{N} \setminus Y$ for $t \in E_Y$).*

Lemma 2.10. *If W does not have cofinality ω_1, then $Q(\mathcal{S}_W, Y)$ is σ-centered.*

As usual with (ω_1,ω_1)-gaps, $Q(\mathcal{S}_W, Y)$ may not (in general) be ccc if W has a cofinal ω_1 sequence.

Let $0 \notin C \subset \theta$ be cofinal and assume that if $C \cap \gamma$ is cofinal in γ and $\mathrm{cf}(\gamma) = \omega_1$, then $\gamma \in C$.

Definition 2.6. Fix any well-ordering $\prec$ of $H(\theta)$. We define a finite support iteration sequence $\{P_\gamma, \dot{Q}_\gamma : \gamma \in \theta\} \subset H(\theta)$. We abuse notation and use Q_0 rather than $\dot{Q}_0$ from definition 2.2. If $\gamma \notin C$, then let $\dot{Q}_\gamma$ be the $\prec$-least among the list of P_γ-names of ccc posets in $H(\theta) \setminus \{\dot{Q}_\xi : \xi \in \gamma\}$. If $\gamma \in C$, then let $\dot{Y}_\gamma$ be the $\prec$-least P_γ-name of a subset $\mathbb{N}$ which is in $H(\theta) \setminus \{\dot{Y}_\xi : \xi \in C \cap \gamma\}$. Set $\dot{Q}_\gamma$ to be the P_γ name of $Q(\mathcal{S}_{C\cap\gamma}, \dot{Y}_\gamma)$ adding the partition $\{\dot{a}_\gamma, \dot{x}_\gamma, \dot{b}_\gamma\}$ and, where $\mathcal{S}_{C\cap\gamma}$ is the P_γ-name of the T-splitting tower $\{(a_\xi, x_\xi, b_\xi) : \xi \in C \cap \gamma\}$.

We view the members of P_θ as functions p with finite domain (or support) denoted $\mathrm{dom}(p)$.

The main difficulty to the proof of Theorem 2.1 is to prove that the iteration P_θ is ccc. Of course, since it is a finite support iteration, this can be proven by induction at successor ordinals.

Lemma 2.11. *For each $\gamma \in C$ such that $C \cap \gamma$ has cofinality ω_1, $P_{\gamma+1}$ is ccc.*

Proof. We proceed by induction. For each α, define $p \in P^*_\alpha$ if $p \in P_\alpha$ and there is an $n \in \mathbb{N}$ such that

(1) for each $\beta \in \mathrm{dom}(p) \cap C$, with $H^\beta = \mathrm{dom}(p) \cap C \cap \beta$, there are subsets $a^\beta, x^\beta, b^\beta$ of n and $T^\beta \in [T]^{<\omega}$ such that $p \restriction \beta \Vdash_{P_\beta}$ "$p(\beta) = (n, a^\beta, x^\beta, b^\beta, T^\beta, H^\beta)$"

Assume that P^*_β is dense in P_β and let $p \in P_{\beta+1}$. To show that $P^*_{\beta+1}$ is dense in $P_{\beta+1}$ we must find some $p^* \leq p$ in $P^*_{\beta+1}$. If $\beta \notin C$ and $p^* \in P^*_\beta$ is below $p \restriction \beta$, then $p^* \cup \{(\beta, p(\beta)\}$ is the desired element of $P^*_{\beta+1}$. Now assume that $\beta \in C$ and assume that $p \restriction \beta \in P^*_\beta$ and that $p \restriction \beta$ forces that $p(\beta)$ is the tuple $(n_0, a, x, b, \tilde{T}, \tilde{H})$. By an easy density argument, we may assume that $\tilde{H} \subset \mathrm{dom}(p)$. Let n^* be the integer witnessing that $p \restriction \beta \in P^*_\beta$. Let ζ be the maximum element of $\mathrm{dom}(p) \cap C \cap \beta$ and let $p \restriction \zeta \Vdash_{P_\zeta}$ "$p(\zeta) = (n^*, a^\zeta, x^\zeta, b^\zeta, T^\zeta, H^\zeta)$" as per the definition of $P^*_{\zeta+1}$. Notice that since $\tilde{H} \subset H^\zeta$ we have that

$$p \restriction \beta \Vdash_{P_\beta} \text{“}(n^*,\ a^*, x, b^*, T^\zeta \cup \tilde{T}, H^\zeta \cup \{\zeta\}) \leq p(\beta)\text{”}$$

where $a^* = a \cup ([n_0, n^*) \setminus b^\zeta)$ and $b^* = b \cup ([n_0, n^*) \cap b^\zeta)$. Defining $p^* \in P_{\beta+1}$ by $p^* \restriction \beta = p \restriction \beta$ and $p^*(\beta) = (n^*,\ a^*, x, b^*, T^\zeta \cup \tilde{T}, H^\zeta \cup \{\zeta\})$

completes the proof that $P^*_{\beta+1}$ is dense in $P_{\beta+1}$, and by induction, that this holds for $\beta = \gamma$.

Now assume that $\{p_\alpha : \alpha \in \omega_1\} \subset P^*_{\gamma+1}$. By passing to a subcollection, we may assume that

(1) the collection $\{T^{p_\alpha(\gamma)} : \alpha \in \omega_1\}$ forms a Δ-system with root T^*;
(2) the collection $\{\mathrm{dom}(p_\alpha) : \alpha \in \omega_1\}$ also forms a Δ-system with root R;
(3) there is a tuple (n^*, a^*, x^*, b^*) so that for all $\alpha \in \omega_1$, $a^{p_\alpha(\gamma)} = a^*$, $x^{p_\alpha(\gamma)} = x^*$, and $b^{p_\alpha(\gamma)} = b^*$.

Since $C \cap \gamma$ has a cofinal sequence of order type ω_1, there is a $\delta \in \gamma$ such that $R \subset \delta$ and, we may assume, $(\mathrm{dom}(p_\alpha)\setminus\delta) \subset \min(\mathrm{dom}(p_\beta)\setminus\delta)$ for $\alpha < \beta < \omega_1$. Since P_δ is ccc, there is a pair $\alpha < \beta < \omega_1$ such that $p_\alpha \restriction \delta$ is compatible with $p_\beta \restriction \delta$. Define $q \in P_{\gamma+1}$ by

(1) $q \restriction \delta$ is any element of P_δ which is below each of $p_\alpha \restriction \delta$ and $p_\beta \restriction \delta$,
(2) if $\delta \le \xi \in \gamma \cap \mathrm{dom}(p_\alpha)$, then $q(\xi) = p_\alpha(\xi)$,
(3) if $\delta \le \xi \in \mathrm{dom}(p_\beta) \setminus C$, then $q(\xi) = p_\beta(\xi)$,
(4) if $\delta \le \xi \in \mathrm{dom}(p_\beta) \cap C$, then
$$q(\xi) = (n^*, a^{p_\beta(\xi)}, x^{p_\beta(\xi)}, b^{p_\beta(\xi)}, T^{p_\beta(\xi)}, H^{p_\beta(\xi)} \cup H^{p_\alpha(\gamma)}).$$

The main non-trivial fact about q is that it is in $P_{\gamma+1}$ which depends on the fact that, by induction on $\eta \in C \cap \gamma$, $q \restriction \eta$ forces that
$$(a_\eta \setminus a_\xi) \cup (b_\eta \setminus b_\xi) \cup (x_\xi \setminus x_\eta) \subset n^* \text{ for } \xi \in C \cap \eta.$$
It now follows trivially that q is below each of p_α and p_β. $\square$

Proof of Theorem 2.1. This completes the construction of the ccc poset P (P_θ as above). Let $G \subset (P \times \Pi_{\lambda\in\Lambda} T_\lambda)$ be generic. It follows that $V[G \cap P]$ is a model of Martin's Axiom and $\mathfrak{c} = \theta$. Furthermore by applying Lemma 2.4 with $\mu = \omega$ and Lemma 2.3, we have that $P_2 = \Pi_{\lambda\in\Lambda} T_\lambda$ is ω_1-distributive in the model $V[G \cap P]$. Therefore all subsets of $\mathbb{N}$ in the model $V[G]$ are also in the model $V[G \cap P]$.

Fix any $\lambda \in \Lambda$ and let ρ_λ denote the generic branch in T_λ given by G. Let G^λ denote the generic filter on $P \times \Pi\{T_\mu : \lambda \ne \mu \in \Lambda\}$ and work in the model $V[G^\lambda]$. It follows easily by Lemma 2.4 and Lemma 2.3, that T_λ is a λ^+-Souslin tree in this model. Therefore by Proposition 2.2, $K_\lambda = \bigcap_{\alpha<\lambda^+} x^*_{\rho_\lambda(\alpha)}$ is a tie-set of $\mathfrak{b}\delta$-type (λ^+, λ^+) in $V[G]$. By the definition of the iteration in P, it follows that condition (4) of Lemma 2.2 is also satisfied, hence the tie-set $K = \bigcap_{\xi\in C} x^*_\xi$ meets K_λ in a single point z_λ. A simple genericity argument confirms that conditions (5) and (6) of Proposition 2.2 also holds, hence z_λ is a tie-point of K_λ.

It follows from Corollary 2.6 that there are no *unwanted* tie-sets in $\beta\mathbb{N}\setminus\mathbb{N}$ in $V[G]$, at least if there are none in $V[G\cap P]$. Since $\mathfrak{p}=\mathfrak{c}$ in $V[G\cap P]$, it follows from Proposition 1.3 that indeed there are no such tie-sets in $V[G\cap P]$. $\square$

Unfortunately the next result shows that the construction does not provide us with our desired variety of tie-points (even with variations in the definition of the iteration). We do not know if $\mathfrak{b}\delta$-type can be improved to δ-type (or simply exclude tie-points altogether).

Proposition 2.12. *In the model constructed in Theorem 2.1, there are no tie-points with* $\mathfrak{b}\delta$*-type* (κ_1,κ_2) *for any* $\kappa_1\le\kappa_2<\mathfrak{c}$*,*

Proof. Assume that $\beta\mathbb{N}\setminus\mathbb{N}=A\underset{x}{\bowtie}B$ and that $\delta(\mathcal{I}_A)=\kappa_1$ and $\delta(\mathcal{I}_B)=\kappa_2$. It follows from Corollary 2.6 that we can assume that $\kappa_1=\kappa_2=\lambda^+$ for some $\lambda\in\Lambda$. Also, following the proof of Corollary 2.6, there are $P\times T_\lambda$-names $\mathcal{J}_A=\{\tilde{a}_\alpha:\alpha\in\lambda^+\}$ and $P\times T_{\lambda^+}$-names $\mathcal{J}_B=\{\tilde{b}_\beta:\beta\in\lambda^+\}$ such that the valuation of these names by G result in increasing (mod finite) chains in $\mathcal{I}_A$ and $\mathcal{I}_B$ respectively whose downward closures are dense. Passing to $V[G\cap P]$, since T_λ has the θ-cc, there is a Boolean subalgebra $\mathcal{B}\in[\mathcal{P}(\mathbb{N})]^{<\theta}$ such that each $\tilde{a}_\alpha$ and $\tilde{b}_\beta$ is a name of a member of $\mathcal{B}$. Furthermore, there is an infinite $C\subset\mathbb{N}$ such that $C\notin x$ and each of $b\cap C$ and $b\setminus C$ are infinite for all $b\in\mathcal{B}$. Since $C\notin x$, there is a $Y\subset\mathbb{N}$ (in $V[G]$) such that $C\cap Y\in\mathcal{I}_A$ and $C\setminus Y\in\mathcal{I}_B$. Now choose $t_0\in T_\lambda$ which forces this about C and Y. Back in $V[G\cap P]$, set

$$\mathcal{A}=\{b\in\mathcal{B}:(\exists t_1\le t_0\ )\ t_1\Vdash_{T_\lambda}\text{“}b\in\mathcal{J}_A\cup\mathcal{J}_B\text{”}\}\ .$$

Since $V[G\cap P]$ satisfies $\mathfrak{p}=\theta$ and $\mathcal{A}^\downarrow$ is forced by t_0 to be dense in $[\mathbb{N}]^\omega$, there must be a finite subset $\mathcal{A}'$ of $\mathcal{A}$ which covers C. It also follows easily then that there must be some $a,b\in\mathcal{A}'$ and t_1,t_2 each below t_0 such that $t_1\Vdash_{T_{\lambda^+}}$ “$a\in\mathcal{J}_A$”, $t_2\Vdash_{T_{\lambda^+}}$ “$b\in\mathcal{J}_B$”, and $a\cap b$ is infinite. The final contradiction is that we will now have that t_0 fails to force that $C\cap a\subset^* Y$ and $C\cap b\subset^*(\mathbb{N}\setminus Y)$. $\square$

3. T-involutions

In this section we strengthen the result in Theorem 2.1 by making each $K\cap K_\lambda$ a symmetric tie-point in K_λ (at the expense of weakening Martin's Axiom in $V[G\cap P]$). This is progress in producing involutions with some control over the fixed point set but we are still not able to make K the fixed point set of an involution. A poset is said to be σ-linked if there is a countable collection of linked (elements are pairwise compatible) which union to the poset. The statement MA($\sigma-$linked) is, of course, the assertion that Martin's Axiom holds when restricted to σ-linked posets.

Our approach is to replace T-splitting towers by the following notion. If f is a (partial) involution on $\mathbb{N}$, let $\min(f) = \{n \in \mathbb{N} : n < f(n)\}$ and $\max(f) = \{n \in \mathbb{N} : f(n) < n\}$ (hence $\mathrm{dom}(f)$ is partitioned into $\min(f) \cup \mathrm{fix}(f) \cup \max(f)$).

Definition 3.1. A sequence $\mathfrak{T} = \{(A_\xi, f_\xi) : \xi \in W\}$ is a tower of T-involutions if W is a set of ordinals and for $\xi < \nu \in W$ and $t \in T$

(1) $A_\nu \subset^* A_\xi$;
(2) $f_\xi^2 = f_\xi$ and $f_\xi \restriction (\mathbb{N} \setminus \mathrm{fix}(f_\xi)) \subset^* f_\eta$;
(3) $f_\xi[x_t] =^* x_t$ and $\mathrm{fix}(f_\xi) \cap x_t$ is infinite;
(4) $f_\xi([n,m)) = [n,m)$ for $n < m$ both in A_ξ.

Say that $\mathfrak{T}$, a tower of T-involutions, is **full** if $K = K_{\mathfrak{T}} = \bigcap\{\mathrm{fix}(f_\xi)^* : \xi \in W\}$ is a tie-set with $\beta\mathbb{N} \setminus \mathbb{N} = A \underset{K}{\bowtie} B$ where $A = K \cup \bigcup\{\min(f_\xi)^* : \xi \in W\}$ and $B = K \cup \bigcup\{\max(f_\xi)^* : \xi \in W\}$.

If $\mathfrak{T}$ is a tower of T-involutions, then there is a natural involution $F_{\mathcal{T}}$ on $\bigcup_{\xi \in W} (\mathbb{N} \setminus \mathrm{fix}(f_\xi))^*$, but this $F_{\mathcal{T}}$ need not extend to an involution on the closure of the union - even if the tower is full.

In this section we prove the following theorem.

Theorem 3.1. *Assume GCH and that Λ is a set of regular uncountable cardinals such that for each $\lambda \in \Lambda$, T_λ is a $<\!\lambda$-closed λ^+-Souslin tree. Let T denote the tree sum of $\{T_\lambda : \lambda \in \Lambda\}$. There is forcing extension in which there is $\mathfrak{T}$, a full tower of T-involutions, such that the associated tie-set K has $\mathfrak{b}\delta$-type $(\mathfrak{c}, \mathfrak{c})$ and such that for each $\lambda \in \Lambda$, there is a tie-set K_λ of $\mathfrak{b}\delta$-type (λ^+, λ^+) such that $F_{\mathfrak{T}}$ does induce an involution on K_λ with a singleton fixed point set $\{z_\lambda\} = K \cap K_\lambda$. Furthermore, for $\mu \le \lambda < \mathfrak{c}$, if $\mu \neq \lambda$ or $\lambda \notin \Lambda$, then there is no tie-set of $\mathfrak{b}\delta$-type (μ, λ).*

Question 3.1. Can the tower $\mathfrak{T}$ in Theorem 3.1 be constructed so that $F_{\mathfrak{T}}$ extends to an involution of $\beta\mathbb{N} \setminus \mathbb{N}$ with $\mathrm{fix}(F) = K_{\mathfrak{T}}$?

We introduce T-tower extending forcing.

Definition 3.2. If $\mathfrak{T} = \{(A_\xi, f_\xi) : \xi \in W\}$ is a tower of T-involutions and Y is a subset of $\mathbb{N}$, we define the poset $Q = Q(\mathfrak{T}, Y)$ as follows. Let E_Y be the (possibly empty) set of minimal elements of T such that there is some finite $H \subset W$ such that $x_t \cap Y \cap \bigcap_{\xi \in H} \mathrm{fix}(f_\xi)$ is finite. Let $D_Y = E_Y^\perp = \{t \in T : (\forall s \in E_Y)\ t \perp s\}$. A tuple $q \in Q$ if $q = (a^q, f^q, T^q, H^q)$ where:

(1) $H^q \in [W]^{<\omega}$, $T^q \in [T]^{<\omega}$, and $n^q = \max(a^q) \in A_{\alpha^q}$ where $\alpha^q = \max(H^q)$,
(2) f^q is an involution on n^q,
(3) $(A_{\alpha^q} \setminus n^q) \subset A_\xi$ for each $\xi \in H^q$,
(4) $\operatorname{fin}(T^q) \subset n^q$,
(5) $f_\xi \restriction (\mathbb{N} \setminus (\operatorname{fix}(f_\xi) \cup n^q)) \subset f_{\alpha^q}$ for $\xi \in H^q$,
(6) $f_{\alpha^q}[x_t \setminus n^q] = x_t \setminus n^q$ for $t \in T^q$,

We define $p < q$ if $n^p \leq n^q$, and for $t \in T^p$ and $i \in [n^p, n^q)$:

(7) $a^p = a^q \cap n^p$, $T^p \subset T^q$, and $H^p \subset H^q$,
(8) $a^q \setminus a^p \subset A_{\alpha^p}$,
(9) $f_{\alpha^p}(i) \neq i$ implies $f^q(i) = f_{\alpha^p}(i)$,
(10) $f^q([n,m)) = [n,m)$ for $n < m$ both in $a^q \setminus a^p$,
(11) $f^q(x_t \cap [n^p, n^q)) = x_t \cap [n^p, n^q)$,
(12) if $t \in D^p$ and $i \in x_t \cap \operatorname{fix}(f^q)$, then $i \in Y$

It should be clear that the involution f introduced by $Q(\mathfrak{T}, Y)$ satisfies that for each $t \in D_Y$, $\operatorname{fix}(f) \cap x_t \subset^* Y$, and, with the help of the following density argument, that $\mathfrak{T} \cup \{(\gamma, A, f)\}$ is again a tower of T-involutions where A is the infinite set introduced by the first coordinates of the conditions in the generic filter.

Lemma 3.2. *If $W \subset \gamma$, $Y \subset \mathbb{N}$, and $\mathfrak{T} = \{(A_\xi, f_\xi) : \xi \in W\}$ is a tower of T-involutions and $p \in Q(\mathfrak{T}, Y)$, then for any $\tilde{T} \in [T]^{<\omega}$, $\zeta \in W$, and any $m \in \mathbb{N}$, there is a $q < p$ such that $n^q \geq m$, $\zeta \in H^q$, $T^q \supset \tilde{T}$, and $\operatorname{fix}(f^q) \cap (x_t \setminus n^p)$ is not empty for each $t \in T^p$.*

Proof. Let β denote the maximum α^p and ζ and let η denote the minimum. Choose any $n^q \in A_{\alpha^q} \setminus m$ large enough so that

(1) $f_{\alpha^p}[x_t \setminus n^q] = x_t \setminus n^q$ for $t \in \tilde{T}$,
(2) $f_\eta \restriction (\mathbb{N} \setminus (n^q \cup \operatorname{fix}(f_\eta))) \subset f_\beta$,
(3) $A_\beta \setminus A_\eta$ is contained in n^q,
(4) $n^q \cap [i]_{T^p} \cap \operatorname{fix}(f_{\alpha^p})$ is non-empty for each $i \in \mathbb{N}$ such that $[i]_{T^p}$ is in the finite set $\{[i]_{T^p} : i \in \mathbb{N}\} \setminus \operatorname{fin}(T^p)$,
(5) if $i \in x_t \cap n^q \setminus n^p$ for some $t \in D_Y \cap T^p$, then Y meets $[i]_{T^p} \cap n^q \setminus n^p$ in at least two points.

Naturally we also set $H^q = H^p \cup \{\zeta\}$ and $T^q = T^p \cup \tilde{T}$. The choice of n^q is large enough to satisfy (3), (4), (5) and (6) of Definition 3.2. We will set $a^q = a^p \cup \{n^q\}$ ensuring (1) of Definition 3.2. Therefore for any $f^q \supset f^p$ which is an involution on n^q, we will have that $q = (a^q, f^q, T^q, H^q)$ is in the poset. We have to choose f^q more carefully to ensure that $q \leq p$. Let $S = [n^p, n^q) \cap \operatorname{fix}(f_{\alpha^p})$, and $S' = [n^p, n^q) \setminus S$. We choose $\bar{f}$ an involution on S and set $f^q = f^p \cup (f_{\alpha^p} \restriction S') \cup \bar{f}$. We leave it to the reader to check that it suffices to ensure that $\bar{f}$ sends $[i]_{T^p} \cap S$ to itself for each $t \in T^p$ and that $\mathrm{fix}(\bar{f}) \cap x_t \subset Y$ for each $t \in T^p \cap D_Y$. Since the members of $\{[i]_{T^p} \cap S : i \in \mathbb{N}\}$ are pairwise disjoint we can define $\bar{f}$ on each separately.

For each $[i]_{T^p} \cap S$ which has even cardinality, choose two points y_i, z_i from it so that if there is a $p \in D_Y \cap T^p$ such that $[i]_{T^p} \subset x_t$, then $\{y_i, z_i\} \subset Y$. Let $\bar{f}$ be any involution on $[i]_{T^p} \cap S$ so that y_i, z_i are the only fixed points. If $[i]_{T^p} \cap S$ has odd cardinality then choose a point y_i from it so that if $[i]^{T^p}$ is contained in x_t for some $t \in D_y \cap T^p$, then $y_i \in Y \cap [i]_{T^p} \cap S$. Set $\bar{f}(y_i) = y_i$ and choose $\bar{f}$ to be any fixed-point free involution on $[i]^{T^p} \cap S \setminus \{y_i\}$. $\square$

Let P_θ now be the finite support iteration defined as in Definition 2.6 except for two important changes. For $\gamma \in C$, we replace T-splitting towers by the obvious inductive definition of towers of T-involutions when we replace the posets $\dot{Q}(\mathcal{S}_{C\cap\gamma}, \dot{Y}_\gamma)$ by $\dot{Q}(\mathfrak{T}_{C\cap\gamma}, \dot{Y}_\gamma)$. For $\gamma \notin C$ we require that $\Vdash_{P_\gamma}$ "$\dot{Q}_\gamma$ is σ-linked."

Special (parity) properties of the family $\{x_t : t \in T\}$ are needed to ensure that $\Vdash_{P_\gamma}$ "$\dot{Q}(\mathcal{S}_{C\cap\gamma}, \dot{Y}_\gamma)$ is ccc " even for cases when $\mathrm{cf}(\gamma)$ is not ω_1.

The proof of Theorem 3.1 is virtually the same as the proof of Theorem 2.1 (so we skip) once we have established that the iteration is ccc.

Lemma 3.3. *For each $\gamma \in C$, $P_{\gamma+1}$ is ccc.*

Proof. We again define P^*_α to be those $p \in P_\alpha$ for which there is an $n \in \mathbb{N}$ such that for each $\beta \in \mathrm{dom}(p) \cap C$, there are $n \in a^\beta \subset n{+}1$, $f^\beta \in n^n$, $T^\beta \in [T]^{<\omega}$, and $H^\beta = \mathrm{dom}(p) \cap C \cap \beta$ such that $p \restriction \beta \Vdash_{P_\beta}$ "$p(\beta) = (a^\beta, f^\beta, T^\beta, H^\beta)$". However, in this proof we must also make some special assumptions in coordinates other than those in C. For each $\xi \in \gamma \setminus C$, we fix a collection $\{\dot{Q}(\xi, n) : n \in \omega\}$ of P_ξ-names so that

$$1 \Vdash_{P_\xi} \text{“}\dot{Q}_\xi = \bigcup_n \dot{Q}(\xi, n) \text{ and } (\forall n)\ \dot{Q}(\xi, n) \text{ is linked.”}$$

The final restriction on $p \in P^*_\alpha$ is that for each $\xi \in \alpha \setminus C$, there is a $k_\xi \in \omega$ such that $p \restriction \xi \Vdash_{P_\xi}$ "$p(\xi) \in \dot{Q}(\xi, k_\xi)$".

Just as in Lemma 2.11, Lemma 3.2 can be used to show by induction that P^*_α is a dense subset of P_α. This time though, we also demand that $\mathrm{dom}(f^{p(0)}) = n \times T^{p(0)}$ is such that $T^\beta \subset T^{p(0)}$ for all $\beta \in \mathrm{dom}(p) \cap C$ and some extra argument is needed because of needing to decide values in the name $\dot{Y}_\gamma$ as in the proof of Lemma 3.2. Let $p \in P_{\beta+1}$ and assume that P^*_β is dense in P_β. By density, we may assume that $p \restriction$

$\beta \in P^*_\beta$, $H^{p(\beta)} \subset \mathrm{dom}(p)$, $T^{p(\beta)} \subset T^{p(0)}$, and that $p \restriction \beta$ has decided the members of the set $D_{\dot Y_\beta} \cap T^{p(\beta)}$. We can assume further that for each $t \in D_{\dot Y_\beta} \cap T^{p(\beta)}$, $p \restriction \beta$ has forced a value $y_t \in \dot Y_\beta \cap x_t \setminus \bigcup\{x_s : s \in T^p \text{ and } s \not\leq t\}$ such that $y_t > n^{p(\beta)}$. We are using that T is not finitely branching to deduce that if $t \in D_{\dot Y_\beta}$, then $p \restriction \beta \Vdash_{P_\beta}$ "$\dot Y_\beta \cap x_t \setminus \bigcup\{x_s : s \in T^p \text{ and } s \not\leq t\}$ is non-empty" (which follows since $\dot Y_\beta$ must meet x_s for each immediate successor s of t). Choose any m larger than y_t for each $t \in T^{p(\beta)}$. Without loss of generality, we may assume that the integer n^* witnessing that $p \restriction \beta \in P^*_\beta$ is at least as large as m and that $n^* \in \bigcap_{\xi \in H^{p(\beta)}} A_\xi$. Construct $\bar f$ just as in Lemma 3.2, except that this time there is no requirement to actually have fixed points so one member of $\dot Y_\beta$ in each appropriate $[i]_{T^{p(\beta)}}$ is all that is required. Let $\zeta = \max(\mathrm{dom}(p) \cap \beta)$. No new forcing decisions are required of $p \restriction \beta$ in order to construct a suitable $\bar f$, hence this shows that $p \restriction \beta \cup \{(\beta, q)\}$ (where q is constructed below $p(\beta)$ as in Corollary 3.2 in which $H^{p(\zeta)} \cup \{\zeta\}$ is add to H^q) is the desired extension of p which is a member of $P^*_{\beta+1}$.

Now to show that $P_{\gamma+1}$ is ccc, let $\{p_\alpha : \alpha \in \omega_1\} \subset P^*_{\gamma+1}$. Clearly we may assume that the family $\{p_\alpha(0) : \alpha \in \omega_1\}$ are pairwise compatible and that there is a single integer n such that, for each $\alpha \in \omega_1$, $\mathrm{dom}(p_\alpha(0)) = n \times T^\alpha$ for some $T^\alpha \in [T]^{<\omega}$. Also, we may assume that there is some (a, h) such that, for each α,

$$p_\alpha \restriction \gamma \Vdash_{P_\gamma} \text{“}p(\gamma) = (a, h, T^\alpha, H^\alpha)\text{”}$$

where $H^\alpha = \mathrm{dom}(p_\alpha) \cap C \cap \gamma$.

The family $\{\mathrm{dom}(p_\alpha) \cap \gamma : \alpha \in \omega_1\}$ may be assumed to form a Δ-system with root R. For each $\xi \in R$, we may assume that, if $\xi \notin C$, there is a single $k_\xi \in \omega$ such that, for all α, $p_\alpha \restriction \xi \Vdash_{P_\xi}$ "$p_\alpha(\xi) \in \dot Q(\xi, k_\xi)$", and if $\xi \in C$, then there is a single (a_ξ, h_ξ) such that $p_\alpha \restriction \xi \Vdash_{P_\xi}$ "$p_\alpha(\xi) = (a_\xi, h_\xi, T^\alpha, H^\alpha \cap \xi)$". For convenience, for each $\xi \notin C$ let $\dot r_\xi$ be a P_ξ-name of a function from $\omega \times \dot Q^2_\xi$ such that, for each $k \in \omega$,

$$1 \Vdash_{P_\xi} \text{“}\dot r_\xi(k, q, q') \leq q, q' \ (\forall q, q' \in \dot Q(\xi, k))\text{”}.$$

Fix any $\alpha < \beta < \omega_1$ and let $H = H^\alpha \cup H^\beta$. Recall that $p_\alpha(0)$ and $p_\beta(0)$ are compatible. Recursively define a P_ξ-name $q(\xi)$ for $\xi \in$

$\mathrm{dom}(p_\alpha) \cup \mathrm{dom}(p_\beta)$ so that $q \restriction \xi \Vdash_{P_\xi}$

$$\text{“}q(\xi) = \begin{cases} (n, T^\alpha \cup T^\beta, f^{p_\alpha(0)} \cup f^{p_\beta(0)}) & \xi = 0 \\ \dot{r}_\xi(k_\xi, p_\alpha(\xi), p_\beta(\xi)) & \xi \in R \setminus C \\ p_\alpha(\xi) & \xi \in \mathrm{dom}(p_\alpha) \setminus (R \cup C) \text{”}. \\ p_\beta(\xi) & \xi \in \mathrm{dom}(p_\beta) \setminus (R \cup C) \\ (a_\xi, h_\xi, T^\alpha \cup T^\beta, H \cap \xi) & \xi \in C. \end{cases}$$

Now we check that $q \in P_\xi$ by induction on $\xi \in \gamma + 1$.

The first thing to note is that not only is this true for $\xi = 1$, but also that $q(0) \Vdash_{Q_0}$ “$\mathrm{fin}(T^\alpha \cup T^\beta) \subset n$”. Since p_α and p_β are each in $P^*_{\gamma+1}$, this show that condition (4) of Definition 3.2 will hold in all coordinates in C.

We also prove, by induction on ξ, that $q \restriction \xi$ forces that for $\eta < \delta$ both in $H \cap \xi$ and $t \in T^\alpha \cup T^\beta$, $f_\delta[x_t \setminus n] = x_t \setminus n$, $f_\eta \restriction (\mathbb{N} \setminus (\mathrm{fix}(f_\eta) \cup n)) \subset f_\delta$ and $A_\delta \setminus n \subset A_\eta$.

Given $\xi \in H$ and the assumption that $q \restriction \xi \in P_\xi$, and $\alpha = \alpha^{q(\xi)} = \max(H \cap \xi)$, condition (3), (5), and (6) of Definition 3.2 hold by the inductive hypothesis above. It follows then that $q \restriction \xi \Vdash_{P_\xi}$ “$q(\xi) \in \dot{Q}_\xi$”. By the definition of the ordering on $\dot{Q}_\xi$, given that $H \cap \xi = H^{q(\xi)}$ and $T^\alpha \cup T^\beta = T^{q(\xi)}$, it follows that the inductive hypothesis then holds for $\xi + 1$.

It is trivial for $\xi \in \mathrm{dom}(q) \setminus C$, that $q \restriction \xi \in P_\xi$ implies that $q \restriction \xi \Vdash_{P_\xi}$ “$q(\xi) \in \dot{Q}_\xi$”. This completes the proof that $q \in P_{\gamma+1}$, and it is trivial that q is below each of p_α and p_β. □

Remark 1. If we add a trivial tree T_1 to the collection $\{T_\lambda : \lambda \in \Lambda\}$ (i.e. T_1 has only a root), then the root of T has a single extension which is a maximal node t, and with no change to the proof of Theorem 3.1, one obtains that F induces an automorphism on x^*_t with a single fixed point. Therefore, it is consistent (and likely as constructed) that $\beta\mathbb{N} \setminus \mathbb{N}$ will have symmetric tie-points of type $(\mathfrak{c}, \mathfrak{c})$ in the model $V[G \cap P]$ and $V[G]$.

Remark 2. In the proof of Theorem 2.1, it is easy to arrange that each K_λ ($\lambda \in \Lambda$) is also $K_{\mathfrak{T}_\lambda}$ for a (T_λ-generic) full tower, $\mathfrak{T}_\lambda$, of $\mathbb{N}$-involutions. However the generic sets added by the forcing P will prevent this tower of involutions from extending to a full involution.

4. Questions

In this section we list all the questions with their original numbering.

Question 1.1. Can there be a tie-point in $\beta\mathbb{N} \setminus \mathbb{N}$ with δ-type (κ, λ) with $\kappa \leq \lambda$ less than the character of the point?

Question 1.2. Can $\beta\mathbb{N} \setminus \mathbb{N}$ have tie-points of δ-type (ω_1, ω_1) and (ω_2, ω_2)?

Question 1.3. Does $\mathfrak{p} > \omega_1$ imply there are no tie-points of $\mathfrak{b}$-type (ω_1, ω_1)?

Question 1.4. If F is an involution on $\beta\mathbb{N} \setminus \mathbb{N}$ such that $K = \mathrm{fix}(F)$ has empty interior, is K a (symmetric) tie-set?

Question 1.5. Is there some natural restriction on which compact spaces can (or can not) be homeomorphic to the fixed point set of some involution of $\beta\mathbb{N} \setminus \mathbb{N}$?

Question 1.6. If F is an involution of $\mathbb{N}^*$, is the quotient space $\mathbb{N}^*/F$ (in which each $\{x, F(x)\}$ is collapsed to a single point) a homeomorphic copy of $\beta\mathbb{N} \setminus \mathbb{N}$?

Question 3.1. Can the tower $\mathfrak{T}$ in Theorem 3.1 be constructed so that $F_{\mathfrak{T}}$ extends to an involution of $\beta\mathbb{N} \setminus \mathbb{N}$ with $\mathrm{fix}(F) = K_{\mathfrak{T}}$?

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