MORE ON TIE-POINTS AND HOMEOMORPHISM IN $\mathbb{N}^*$

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Abstract. A point x is a (bow) tie-point of a space X if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with x in its closure. We picture (and denote) this as $X = A \underset{x}{\bowtie} B$ where A, B are the closed sets which have a unique common accumulation point x. Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N} = \mathbb{N}^*$ (e.g. [10, 7]) and in the recent study [4, 2] of (precisely) 2-to-1 maps on $\mathbb{N}^*$. In these cases the tie-points have been the unique fixed point of an involution on $\mathbb{N}^*$. One application of the results in this paper is the consistency of there being a 2-to-1 continuous image of $\mathbb{N}^*$ which is not a homeomorph of $\mathbb{N}^*$.

1. Introduction

A point x is a tie-point of a space X if there are closed sets A, B of X such that $\{x\} = A \cap B$ and x is an adherent point of both A and B. We let $X = A \underset{x}{\bowtie} B$ denote this relation and say that x is a tie-point as witnessed by A, B. Let $A \equiv_x B$ mean that there is a homeomorphism from A to B with x as a fixed point. If $X = A \underset{x}{\bowtie} B$ and $A \equiv_x B$, then there is an involution F of X (i.e. $F^2 = F$) such that $\{x\} = \text{fix}(F)$. In this case we will say that x is a symmetric tie-point of X.

An autohomeomorphism F of $\mathbb{N}^*$ is said to be *trivial* if there is a bijection f between cofinite subsets of $\mathbb{N}$ such that $F = \beta f \restriction \mathbb{N}^*$. Since the fixed point set of a trivial autohomeomorphism is clopen, a symmetric tie-point gives rise to a non-trivial autohomeomorphism.

If A and B are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \underset{x=y}{\bowtie} B$ denote the quotient space of

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$A \oplus B$ obtained by identifying x and y and let xy denote the collapsed point. Clearly the point xy is a tie-point of this space.

In this paper we establish the following theorem.

Theorem 1.1. *It is consistent that $\mathbb{N}^*$ has symmetric tie-points x, y as witnessed by A, B and A', B' respectively such that $\mathbb{N}^*$ is not homeomorphic to the space $A \underset{x=y}{\bowtie} A'$*

Corollary 1.1. *It is consistent that there is a 2-to-1 image of $\mathbb{N}^*$ which is not a homeomorph of $\mathbb{N}^*$.*

One can generalize the notion of tie-point and, for a point $x \in \mathbb{N}^*$, consider how many disjoint clopen subsets of $\mathbb{N}^* \setminus \{x\}$ (each accumulating to x) can be found. Let us say that a tie-point x of $\mathbb{N}^*$ satisfies $\tau(x) \geq n$ if $\mathbb{N}^* \setminus \{x\}$ can be partitioned into n many disjoint clopen subsets each accumulating to x. Naturally, we will let $\tau(x) = n$ denote that $\tau(x) \geq n$ and $\tau(x) \not\geq n+1$. Each point x of character ω_1 in $\mathbb{N}^*$ is a symmetric tie-point and satisfies that $\tau(x) \geq n$ for all n. We list several open questions in the final section.

More generally one could study the symmetry group of a point $x \in \mathbb{N}^*$: e.g. set G_x to be the set of autohomeomorphisms F of $\mathbb{N}^*$ that satisfy $\mathrm{fix}(F) = \{x\}$ and two are identified if they are the same on some clopen neighborhood of x.

Theorem 1.2. *It is consistent that $\mathbb{N}^*$ has a tie-point x such that $\tau(x) = 2$ and such that with $\mathbb{N}^* = A \underset{x}{\bowtie} B$, neither A nor B is a homeomorph of $\mathbb{N}^*$. In addition, there are no symmetric tie-points.*

The following partial order $\mathbb{P}_2$, was introduced by Velickovic in [10] to add a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ while doing as little else as possible — at least assuming PFA.

Definition 1.1. The partial order $\mathbb{P}_2$ is defined to consist of all 1-to-1 functions $f : A \to B$ where

- $A \subseteq \omega$ and $B \subseteq \omega$
- for all $i \in \omega$ and $n \in \omega$, $f(i) \in (2^{n+1} \setminus 2^n)$ if and only if $i \in (2^{n+1} \setminus 2^n)$
- $\limsup_{n\to\omega} |(2^{n+1} \setminus 2^n) \setminus A| = \omega$ and hence, by the previous condition, $\limsup_{n\to\omega} |(2^{n+1} \setminus 2^n) \setminus B| = \omega$

The ordering on $\mathbb{P}_2$ is $\subseteq^*$.

We define some trivial generalizations of $\mathbb{P}_2$. We use the notation $\mathbb{P}_2$ to signify that this poset introduces an involution of $\mathbb{N}^*$ because the conditions $g = f \cup f^{-1}$ satisfy that $g^2 = g$. In the definition of $\mathbb{P}_2$ it is possible to suppress mention of A, B (which we do) and to have the poset $\mathbb{P}_2$ consist simply of the functions g (and to treat $A = \min(g) = \{i \in \mathrm{dom}(g) : i < g(i)\}$ and to treat B as $\max(g) = \{i \in \mathrm{dom}(g) : g(i) < i\}$).

Let $\mathbb{P}_1$ denote the poset we get if we omit mention of f but consisting only of disjoint pairs (A, B), satisying the growth condition in Definition 1.1, and extension is coordinatewise mod finite containment. To be consistent with the other two posets, we may instead represent the elements of $\mathbb{P}_1$ as partial functions into 2.

More generally, let $\mathbb{P}_\ell$ be similar to $\mathbb{P}_2$ except that we assume that conditions consist of functions g satisfying that $\{i, g(i), g^2(i), \ldots, g^\ell(i)\}$ has precisely ℓ elements for all $i \in \mathrm{dom}(g)$ (and replace the intervals $2^{n+1} \setminus 2^n$ by $\ell^{n+1} \setminus \ell^n$ in the definition).

The basic properties of $\mathbb{P}_2$ as defined by Velickovic and treated by Shelah and Steprans are also true of $\mathbb{P}_\ell$ for all $\ell \in \mathbb{N}$.

In particular, for example, it is easily seen that

Proposition 1.1. *If $L \subset \mathbb{N}$ and $\mathbb{P}^* = \Pi_{\ell \in L} \mathbb{P}_\ell$ (with full supports) and G is a $\mathbb{P}^*$-generic filter, then in $V[G]$, for each $\ell \in L$, there is a tie-point $x_\ell \in \mathbb{N}^*$ with $\tau(x_\ell) \geq \ell$.*

For the proof of Theorem 1.1 we use $\mathbb{P}_2 \times \mathbb{P}_2$ and for the proof of Theorem 1.2 we use $\mathbb{P}_1$.

An ideal $\mathcal{I}$ on $\mathbb{N}$ is said to be ccc over fin [3], if for each uncountable almost disjoint family, all but countably many of them are in $\mathcal{I}$. An ideal is a P-ideal if it is countably directed closed mod finite.

The following main result is extracted from [6] and [8] which we record without proof.

Lemma 1.1 (PFA)**.** *If $\mathbb{P}^*$ is a finite or countable product (repetitions allowed) of posets from the set $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$ and if G is a $\mathbb{P}^*$-generic filter, then in $V[G]$ every autohomeomorphism of $\mathbb{N}^*$ has the property that the ideal of sets on which it is trivial is a P-ideal which is ccc over fin.*

Corollary 1.2 (PFA)**.** *If $\mathbb{P}^*$ is a finite or countable product of posets from the set $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$, and if G is a $\mathbb{P}^*$-generic filter, then in $V[G]$ if F is an autohomeomorphism of $\mathbb{N}^*$ and $\{Z_\alpha : \alpha \in \omega_2\}$ is an increasing mod finite chain of infinite subsets of $\mathbb{N}$, there is an $\alpha_0 \in \omega_2$ and a collection $\{h_\alpha : \alpha \in \omega_2\}$ of 1-to-1 functions such that $\mathrm{dom}(h_\alpha) = Z_\alpha$ and for all $\beta \in \omega_2$ and $a \subset Z_\beta \setminus Z_{\alpha_0}$, $F[a] =^* h_\beta[a]$.*

Each poset $\mathbb{P}^*$ as above is $\aleph_1$-closed and $\aleph_2$-distributive (see [8, p.4226]). In this paper we will restrict out study to finite products. The following partial order can be used to show that these products are $\aleph_2$-distributive.

Definition 1.2. Let $\mathbb{P}^*$ be a finite product of posets from $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$. Given $\{\vec{f_\xi} : \xi \in \mu\} = \mathfrak{F} \subset \mathbb{P}^*$ (decreasing in the ordering on $\mathbb{P}^*$), define $\mathbb{P}(\mathfrak{F})$ to be the partial order consisting of all $g \in \mathbb{P}^*$ such that there is some $\xi \in \mu$ such that $\vec{g} \equiv^* \vec{f_\xi}$. The ordering on $\mathbb{P}(\mathfrak{F})$ is coordinatewise $\supseteq$ as opposed to ${}^*\supseteq$ in $\mathbb{P}^*$.

Corollary 1.3 (PFA). *If $\mathbb{P}^*$ is a finite or countable product of posets from the set $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$, and if G is a $\mathbb{P}^*$-generic filter, then in $V[G]$ if F is an involution of $\mathbb{N}^*$ with a unique fixed point x, then x is a P_{ω_2}-point and $\mathbb{N}^* = A \underset{x}{\bowtie} B$ for some A, B such that $F[A] = B$.*

Proof. We may assume that F also denotes an arbitrary lifting of F to $[\mathbb{N}]^\omega$ in the sense that for each $Y \subset \mathbb{N}$, $(F[Y])^* = F[Y^*]$. Let $\mathcal{Z}_x = [\mathbb{N}]^\omega \setminus x$ (the dual ideal to x). For each $Z \in \mathcal{Z}_x$, $F[Z]$ is also in $\mathcal{Z}$ and $F[Z \cup F[Z]] =^* Z \cup F[Z]$. So let us now assume that $\mathcal{Z}$ denotes those $Z \in \mathcal{Z}_x$ such that $Z =^* F[Z]$. Given $Z \in \mathcal{Z}$, since $\mathrm{fix}(F) \cap Z^* = \emptyset$, there is a collection $\mathcal{Y} \subset [Z]^\omega$ such that $F[Y] \cap Y =^* \emptyset$ for each $Y \in \mathcal{Y}$, and such that Z^* is covered by $\{Y^* : Y \in \mathcal{Y}\}$. By compactness, we may assume that $\mathcal{Y} = \{Y_0, \ldots, Y_n\}$ is finite. Set $Z_0 = Y_0 \cup F[Y_0]$. By induction, replace Y_k by $Y_k \setminus \bigcup_{j<k} Z_j$ and define $Z_k = Y_k \cup F[Y_k]$. Therefore $Y_Z = \bigcup_k Y_k$ satisfies that $Y_Z \cap F[Y_Z] =^* \emptyset$ and $Z = Y_Z \cup F[Y_Z]$. This shows that for each $Z \in \mathcal{Z}$ there is a partition of $Z = Z^0 \cup Z^1$ such that $F[Z^0] =^* Z^1$. It now follows that x is a P-point, for if $\{Z_n = Z^0_n \cup Z^1_n : n \in \mathbb{N}\} \subset \mathcal{Z}$ are pairwise disjoint, then $x \notin \overline{\bigcup_n Z^*_n}$ since $F[\overline{\bigcup_n (Z^0_n)^*}] = \overline{\bigcup_n (Z^1_n)^*}$ and $\overline{\bigcup_n (Z^0_n)^*}$ is disjoint from $\overline{\bigcup_n (Z^1_n)^*}$.

Now we prove that it is a P_{ω_2}-point. Assume that $\{Z_\alpha : \alpha \in \omega_1\} \subset \mathcal{Z}$ is a mod finite increasing sequence. By Lemma 1.1 (similar to Corollary 1.2) we may assume, by possibly removing some Z_{α_0} from each Z_α, that there is a sequence $\{h_\alpha : \alpha \in \omega_1\}$ of involutions such that h_α induces $F \restriction Z^*_\alpha$. For each $\alpha \in \omega_1$, let $a_\alpha = \min(h_\alpha) = \{i \in \mathrm{dom}(h_\alpha) : i < h_\alpha(i)\}$ and $b_\alpha = Z_\alpha \setminus a_\alpha$. It follows that $F[a_\alpha] =^* b_\alpha$. Since $\mathbb{P}^*$ is $\aleph_2$-distributive, all of these $\aleph_1$-sized sets are in V which is a model of PFA. If x is in the closure of $\bigcup_{\alpha\in\omega_1} Z^*_\alpha$, then x is in the closure of each of $\bigcup_\alpha a^*_\alpha$ and $\bigcup_\alpha b^*_\alpha$. Therefore, it suffices to show that $\mathcal{A} = \{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ can not form a gap in V. As is well-known, if $\mathcal{A}$ does form a gap, there is a ccc poset $Q_{\mathcal{A}}$ which adds an uncountable I such that $\{(a_\alpha, b_\alpha) : \alpha \in I\}$ forms a Hausdorf-gap (i.e. *freezes* the gap). It is easy to prove that if $\mathbb{C}$ is the poset for adding ω_1-many almost disjoint Cohen reals, $\{\dot{C}_\xi : \xi \in \omega_1\}$, then a similar ccc poset $\mathbb{C} * \dot{Q}$ will introduce, for each $\xi \in \omega_1$, an uncountable $I_\xi \subset \omega_1$, such that $\{(\dot{C}_\xi \cap a_\alpha, \dot{C}_\xi \cap b_\alpha) : \alpha \in I_\xi\}$ is a Hausdorff-gap. But now by Lemma

1.1, it follows that there is some $\xi \in \omega_1$ such that $Z = C_\xi \in \mathcal{Z}$, $F \restriction Z^*$ is trivial and for some uncountable $I \subset \omega_1$ $\{(Z \cap a_\alpha, Z \cap b_\alpha) : \alpha \in I\}$ forms a Hausdorff-gap. This however is a contradiction because if h_Z induces $F \restriction Z^*$, then $\min(h_Z) \cap ((Z \cap a_\alpha) \cup (Z \cap b_\alpha))$ is almost equal to a_α for all $\alpha \in \omega_1$, i.e. $\min(h_Z)$ would have to split the Hausdorff-gap. $\square$

The forcing $\mathbb{P}(\mathfrak{F})$ introduces a tuple $\vec{f}$ which satisfies $\vec{f} \leq \vec{f}_\alpha$ for $\vec{f}_\alpha \in \mathfrak{F}$ but for the fact that $\vec{f}$ may not be a member of $\mathbb{P}^*$ simply because the domains of the component functions are too big. There is a σ-centered poset which will choose an appropriate sequence $\vec{f}^*$ of subfunctions of $\vec{f}$ which is a member of $\mathbb{P}^*$ and which is still below each member of $\mathfrak{F}$ (see [6, 2.1]).

A strategic choice of the sequence $\mathfrak{F}$ will ensure that $\mathbb{P}(\mathfrak{F})$ is ccc, but remarkably even more is true. Again we are lifting results from [6, 2.6] and [8, proof of Thm. 3.1]. This is an innovative factoring of Velickovic's original amoeba forcing poset and seems to preserve more properties. Let $\omega_2^{<\omega_1}$ denote the standard collapse which introduces a function from ω_1 onto ω_2.

Lemma 1.2. *Let $\mathbb{P}^*$ be a finite product of posets from $\{\mathbb{P}_\ell : \ell \in \mathbb{N}\}$. In the forcing extension, $V[H]$, by $\omega_2^{<\omega_1}$, there is a descending sequence $\mathfrak{F}$ from $\mathbb{P}^*$ which is $\mathbb{P}^*$-generic over V and, for which, $\mathbb{P}(\mathfrak{F})$ is ccc and ω^ω-bounding.*

It follows also that $\mathbb{P}(\mathfrak{F})$ preserves that $\mathbb{R} \cap V$ is of second category. This was crucial in the proof of Lemma 1.1. We can manage with the ω^ω-bounding property because we are going to use Lemma 1.1. A poset is said to be ω^ω-bounding if every new function in ω^ω is bounded by some ground model function.

The following proposition is probably well-known but we do not have a reference.

Proposition 1.2. *Assume that $\mathbb{Q}$ is a ccc ω^ω-bounding poset and that x is an ultrafilter on $\mathbb{N}$. If G is a $\mathbb{Q}$-generic filter then there is no set $A \subset \mathbb{N}$ such that $A \setminus Y$ is finite for all $Y \in x$.*

Proof. Assume that $\{\dot{a}_n : n \in \omega\}$ are $\mathbb{Q}$-names of integers such that $1 \Vdash_{\mathbb{Q}}$ "$\dot{a}_n \geq n$". Let A denote the $\mathbb{Q}$-name so that $\Vdash_{\mathbb{Q}}$ "$A = \{\dot{a}_n : n \in \omega\}$". Since $\mathbb{Q}$ is ω^ω-bounding, there is some $q \in \mathbb{Q}$ and a sequence $\{n_k : k \in \omega\}$ in V such that $q \Vdash_{\mathbb{Q}}$ "$n_k \leq \dot{a}_i \leq n_{k+2}\ \forall i \in [n_k, n_{k+1})$". There is some $\ell \in 3$ such that $Y = \bigcup_k [n_{3k+\ell}, n_{3k+\ell+1})$ is a member of x. On the other hand, $q \Vdash_{\mathbb{Q}}$ "$A \cap [n_{3k+\ell+1}, n_{3k+\ell+3})$" is not empty for each k. Therefore $q \not\Vdash_{\mathbb{Q}}$ "$A \setminus Y$ is finite". $\square$

Another interesting and useful general lemma is the following.

Lemma 1.3. *Let $\mathcal{F} \subset \mathbb{P}_\ell$ (for any $\ell \in \mathbb{N}$) be generic over V, then for each $\mathbb{P}(\mathfrak{F})$-name $\dot{h} \in \mathbb{N}^{\mathbb{N}}$, either there is an $f \in \mathfrak{F}$ such that $f \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h} \restriction \mathrm{dom}(f) \notin V$", or there is an $f \in \mathfrak{F}$ and an increasing sequence $n_0 < n_1 < \cdots$ of integers such that for each $i \in [n_k, n_{k+1})$ and each $g < f$ such that g forces a value on $\dot{h}(i)$, $f \cup (g \restriction [n_k, n_{k+1}))$ also forces a value on $\dot{h}(i)$.*

Proof. Given any f, perform a standard fusion (see [6, 2.4] or [8, 3.4]) f_k, n_k by picking $L_k \subset [n_{k+1}, n_{k+2})$ (absorbed into $\mathrm{dom}(f_{k+1})$) so that for each partial function s on n_k which extends $f_k \restriction n_k$, if there is some integer $i \geq n_{k+1}$ for which no $<n_k$-preserving extension of $s \cup f_k$ forces a value on $\dot{h}(i)$, then there is such an integer in $\mathrm{dom}(f_{k+1})$. Let $\bar{f}$ be the fusion and note that either $\bar{f}$ forces that $\dot{h} \restriction \mathrm{dom}(\bar{f})$ is not in V, or it forces that our sequence of n_k's does the job. Thus, we have proven that for each f, there is such a $\bar{f}$, hence by genericity, there is such an $\bar{f}$ in $\mathfrak{F}$. □

2. Proof of Theorem 1.1

Theorem 2.1 (PFA). *If G is a generic filter for $\mathbb{P}^* = \mathbb{P}_2 \times \mathbb{P}_2$, then there are symmetric tie-points x, y as witnessed by A, B and C, D respectively such that $\mathbb{N}^*$ is not homeomorphic to the space $A \underset{x=y}{\bowtie} C$*

Assume that $\mathbb{N}^*$ is homeomorphic to $A \underset{x=y}{\bowtie} C$ and that z is the $\mathbb{P}^*$-name of the ultrafilter that is sent (by the assumed homeomorphism) to the point (x, y) in the quotient space $A \underset{x=y}{\bowtie} C$.

Further notation: let $\{a_\alpha : \alpha \in \omega_2\}$ be the $\mathbb{P}_2$-names of the infinite subsets of $\mathbb{N}$ which form the mod-finite increasing chain whose remainders in $\mathbb{N}^*$ cover $A \setminus \{x\}$ and, similarly let $\{c_\alpha : \alpha \in \omega_2\}$ be the $\mathbb{P}_2$-names (second coordinates though) which form the chain in $C \setminus \{y\}$.

If we represent $A \underset{x=y}{\bowtie} C$ as a quotient of $(\mathbb{N} \times 2)^*$, we may assume that F is a $\mathbb{P}^*$-name of a function from $[\mathbb{N}]^\omega$ into $[\mathbb{N} \times 2]^\omega$ such that letting $Z_\alpha = F^{-1}(a_\alpha \times \{0\} \cup c_\alpha \times \{1\})$ for each $\alpha \in \omega_2$, then $\{Z_\alpha : \alpha \in \omega_2\}$ forms the dual ideal to z, and $F : [Z_\alpha]^\omega \to (a_\alpha \times \{0\} \cup c_\alpha \times \{1\})^\omega$ induces the above homeomorphism from Z^*_α onto $(a^*_\alpha \times \{0\}) \cup (c^*_\alpha \times \{1\})$.

By Corollary 1.2, we may assume that for each $\beta \in \omega_2$, there is a bijection h_β between some cofinite subset of Z_β and some cofinite subset of $(a_\beta \times \{0\}) \cup (c_\beta \times \{1\})$ which induces $F \restriction [Z_\beta]^\omega$ (since we can just ignore Z_{α_0} for some fixed α_0). We will use $F \restriction [Z_\beta]^\omega = h_\beta$ to mean that h_β induces $F \restriction [Z_\beta]^\omega$. Note that by the assumptions, for each $\beta \in \omega_2$, there is a $\gamma \in \omega_2$ such that each of $h^{-1}_\gamma(a_\gamma) \setminus Z_\beta$ and $h^{-1}_\gamma(c_\gamma) \setminus Z_\beta$ are infinite.

Let H be a generic filter for $\omega_2^{<\omega_1}$, and assume that $\mathfrak{F} \subset \mathbb{P}^*$ is chosen as in Lemma 1.2. In this model, let us use λ to denote the ω_2 from V. Using the fact that $\mathfrak{F}$ is $\mathbb{P}^*$-generic over V, we may treat all the functions h_α ($\alpha \in \lambda$) as members of V since we can take the valuation of all the $\mathbb{P}^*$-names using $\mathfrak{F}$. Assume that $\dot{h}$ is a $\mathbb{P}(\mathfrak{F})$-name of a finite-to-1 function from $\mathbb{N}$ into $\mathbb{N} \times 2$ satisfying that $h_\alpha \subset^* h$ for all $\alpha \in \lambda$. We show there is no such $\dot{h}$.

Since $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding, there is a increasing sequence of integers $\{n_k : k \in \omega\}$ and an $\vec{f}_0 = (g_0, g_1) \in \mathfrak{F}$ such that

(1) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}(i) \in ([0, n_{k+2}) \times 2)$"
(2) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}^{-1}(\{i\} \times 2) \subset [0, n_{k+2})$"
(3) for each k and each $j \in \{0, 1\}$ there is an m such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \operatorname{dom} g_j$ has at least k elements.

Choose any $(g_0', g_1') = \vec{f}_1 < \vec{f}_0$ such that $\mathbb{N} \setminus \operatorname{dom}(g_0') \subset \bigcup_k [n_{6k+1}, n_{6k+2})$ and $\mathbb{N} \setminus \operatorname{dom}(g_1') \subset \bigcup_k [n_{6k+4}, n_{6k+5})$. Next, choose any $\vec{f}_2 < \vec{f}_1$ and some $\alpha \in \lambda$ such that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\operatorname{dom}(g_0') \subset^* a_\alpha \cup g_0'[a_\alpha]$ and $\operatorname{dom}(g_1') \subset^* c_\alpha \cup g_1'[c_\alpha]$". For each $\gamma \in \lambda$, note that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$a_\gamma \setminus a_\alpha \subset^* \mathbb{N} \setminus \operatorname{dom}(g_0')$" and similarly $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$c_\gamma \setminus c_\alpha \subset^* \mathbb{N} \setminus \operatorname{dom}(g_1')$".

Now consider the two disjoint sets: $Y_0 = \bigcup_k [n_{6k}, n_{6k+3})$ and $Y_1 = \bigcup_k [n_{6k+3}, n_{6k+6})$. Since z is an ultrafilter in this extension, by possibly extending $\vec{f}_2$ even more, we may assume that there is some $j \in \{0, 1\}$ and some $\beta > \alpha$ such that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$Y_j \subset^* Z_\beta$". Without loss of generality (by symmetry) we may assume that $j = 0$. Consider any $\gamma \in \lambda$. Since we are assuming that $h_\gamma \subset^* \dot{h}$, we have that $\vec{f}_2$ forces that $h_\gamma[Z_\gamma \setminus Z_\alpha] =^* \dot{h}[Z_\gamma \setminus Z_\alpha]$. We also have that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathcal{F})}$ "$\dot{h}[Y_0] \,{}^*\!\supset (a_\gamma \setminus a_\alpha) \times \{0\} =^* h_\gamma[Z_\gamma \setminus Z_\alpha] \cap \mathbb{N} \times \{0\}$". Putting this all together, we now have that $\vec{f}_2$ forces that $\dot{h}[Z_\beta]$ almost contains $(a_\gamma \setminus a_\alpha) \times \{0\}$ for all $\gamma \in \lambda$; which clearly contradicts that $\dot{h}[Z_\beta]$ is supposed to be almost equal to $h_\beta[Z_\beta]$.

So now what? Well, let H_2 be a generic filter for $\mathbb{P}(\mathfrak{F})$ and consider the family of functions $\mathcal{H}_\lambda = \{h_\alpha : \alpha \in \lambda\}$ which we know does not have a common finite-to-1 extension.

Before proceeding, we need to show that $\mathcal{H}_\lambda$ does not have any extension h. If $\dot{h}$ is any $\mathbb{P}(\mathfrak{F})$-name of a function for which it is forced that $h_\alpha \subset^* \dot{h}$ for all $\alpha \in \lambda$, then there is some $\ell \in \mathbb{N}$ such that $\dot{Y} = h^{-1}(h(\ell))$ is (forced to be) infinite. It follows easily that $\dot{Y}$ is forced to be almost contained in every member of z. By Lemma 1.2 this cannot happen. Therefore the family $\mathcal{H}_\lambda$ does not have any common extension.

Given such a family as $\mathcal{H}_\lambda$, there is a well-known proper poset Q_1 (see [1, 3.1], [3, 2.2.1], and [10, p9]) which will force an uncountable cofinal $I \subset \lambda$ and a collection of integers $\{k_{\alpha,\beta} : \alpha < \beta \in I\}$ satisfying that $h_\alpha(k_{\alpha,\beta}) \neq h_\beta(k_{\alpha,\beta})$ (and both are defined) for $\alpha < \beta \in I$. So, let $\dot{Q}_1$ be the $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F})$-name of the above mentioned poset. In addition, let $\dot\varphi$ be the $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1$-name of the enumerating function from ω_1 onto I, and let $\dot{k}_{\alpha,\beta}$ (for $\alpha < \beta \in \omega_2$) be the name of the integer $k_{\dot\varphi(\alpha),\dot\varphi(\beta)}$. Thus for each $\alpha < \beta \in \omega_1$, there is a dense set $D(\alpha,\beta) \subset \omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1$ such that for each member p of $D(\alpha,\beta)$, there are functions h_α, h_β in V and sets $Z_\alpha = \mathrm{dom}(h_\alpha), Z_\beta = \mathrm{dom}(h_\beta)$ and integers $k = k(\alpha,\beta) \in Z_\alpha \cap Z_\beta$ such that

$$p \Vdash_{\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1} \text{"} F \restriction [Z_\alpha]^\omega = h_\alpha, F \restriction [Z_\beta]^\omega = h_\beta, h_\alpha(k) \neq h_\beta(k) \text{"}.$$

Finally, let $\dot{Q}_2$ be the $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1$-name of the σ-centered poset which forces an element $\vec{f} \in \mathbb{P}^*$ which is below every member of $\mathfrak{F}$. Again, there is a countable collection of dense subsets of the proper poset $\omega_2^{<\omega} * \mathbb{P}(\mathfrak{F}) * \dot{Q}_1 * \dot{Q}_2$ which determine the values of $\vec{f}$.

Applying PFA to the above proper poset and the family of ω_1 mentioned dense sets, we find there is a sequence $\{h'_\alpha, Z'_\alpha : \alpha \in \omega_1\}$, integers $\{k_{\alpha,\beta} : \alpha < \beta \in \omega_1\}$, and a condition $\vec{f} \in \mathbb{P}^*$ such that, for all $\alpha < \beta$ and $k = k(\alpha,\beta)$,

$$\vec{f} \Vdash_{\mathbb{P}^*} \text{"} F \restriction [Z'_\alpha]^\omega = h'_\alpha, F \restriction [Z'_\beta]^\omega = h'_\beta, h'_\alpha(k) \neq h'_\beta(k) \text{"}.$$

But, we also know that we can choose $\vec{f}$ so that there is some $\lambda \in \omega_2$, and some h_λ, Z_λ such that, for all $\alpha \in \omega_1$, $Z'_\alpha \subset^* Z_\lambda$ and $F \restriction [Z_\lambda]^\omega = h_\lambda$.

It follows of course that for all $\alpha \in \omega_1$, there is some n_α such that $h'_\alpha \restriction [n_\alpha, \omega) \subset h_\lambda$. Let $J \in [\omega_1]^{\omega_1}$, $n \in \omega$, and h' a function with $\mathrm{dom}(h') \subset n$ such that $n_\alpha = n$ and $h'_\alpha \restriction n = h'$ for all $\alpha \in J$. We now have a contradiction since if $\alpha < \beta \in J$ then clearly $k = k(\alpha,\beta) \geq n$ and this contradicts that $h'_\alpha(k)$ and $h'_\beta(k)$ are both supposed to equal $h_\lambda(k)$.

3. proof of Theorem 1.2

Theorem 3.1 (PFA). *If G is a generic filter for $\mathbb{P}_1$, then a tie-point x is introduced such that $\tau(x) = 2$ and with $\mathbb{N}^* = A \underset{x}{\bowtie} B$, neither A nor B is a homeomorph of $\mathbb{N}^*$. In addition, there is no involution F on $\mathbb{N}^*$ which has a unique fixed point, and so, no tie-point is symmetric.*

Assume that V is a model of PFA and that $\mathbb{P} = \mathbb{P}_1$. The elements of $\mathbb{P}$ are partial functions f from $\mathbb{N}$ into 2 which also satisfy that $\limsup_{n\in\mathbb{N}} |2^{n+1} \setminus (2^n \cup \mathrm{dom}(f))| = \infty$. The ordering on $\mathbb{P}$ is that

$f < g$ ($f, g \in \mathbb{P}$) if $g \subset^* f$. For each $f \in \mathbb{P}$, let $a_f = f^{-1}(0)$ and $b_f = f^{-1}(1)$.

Again we assume that $\{a_\alpha : \alpha \in \omega_2\}$ is the sequence of $\mathbb{P}$-names satisfying that $\mathbb{N}^* = A \underset{x}{\bowtie} B$ and $A \setminus \{x\} = \bigcup\{a^*_\alpha : \alpha \in \omega_2\}$. Of course by this we mean that for each $f \in G$, there are $\alpha \in \omega_2$, $a \in [\mathbb{N}]^\omega$, and $f_1 \in G$ such that $a_f \subset^* a \subset a_{f_1}$ and $f_1 \Vdash_{\mathbb{P}}$ "$\check{a} = a_\alpha$".

Next we assume that, if A is homeomorphic to $\mathbb{N}^*$, then F is a $\mathbb{P}$-name of a homeomorphism from $\mathbb{N}^*$ to A and let z denote the point in $\mathbb{N}^*$ which F sends to x. Also, let Z_α be the $\mathbb{P}$-name of $F^{-1}[a_\alpha]$ and recall that $\mathbb{N}^* \setminus \{z\} = \bigcup\{Z^*_\alpha : \alpha \in \omega_2\}$. As above, we may also assume that for each $\alpha \in \omega_2$, there is a $\mathbb{P}$-name of a function h_α with $\mathrm{dom}(h_\alpha) = Z_\alpha$ such that $F \restriction [Z_\alpha]^\omega$ is induced by h_α.

Furthermore if $\tau(x) > 2$, then one of $A \setminus \{x\}$ or $B \setminus \{x\}$ can be partitioned into disjoint clopen non-compact sets. We may assume that it is $A \setminus \{x\}$ which can be so partitioned. Therefore there is some sequence $\{c_\alpha : \alpha \in \omega_2\}$ of $\mathbb{P}$-names such that for each $\alpha < \beta \in \omega_2$, $c_\beta \subset a_\beta$ and $c_\beta \cap a_\alpha =^* c_\alpha$. In addition, for each $\alpha < \omega_2$ there must be a $\beta \in \omega_2$ such that $c_\beta \setminus a_\alpha$ and $a_\beta \setminus (c_\beta \cup a_\alpha)$ are both infinite.

Now assume that H is $\omega_2^{<\omega_1}$-generic and again choose a sequence $\mathfrak{F} \subset \mathbb{P}$ which is V-generic for $\mathbb{P}$ and which forces that $\mathbb{P}(\mathfrak{F})$ is ccc and ω^ω-bounding. For the rest of the proof we work in the model $V[H]$ and we again let λ denote the ordinal ω_2^V.

In the case of $\mathbb{P}_1$ we are able to prove a significant strengthening of Lemma 1.3.

Lemma 3.1. *Assume that $\dot{h}$ is a $\mathbb{P}(\mathfrak{F})$-name of a function from $\mathbb{N}$ to $\mathbb{N}$. Either there is an $f \in \mathfrak{F}$ and such that $f \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h} \restriction \mathrm{dom}(f) \notin V$", or there is an $f \in \mathfrak{F}$ and an increasing sequence $m_1 < m_2 < \cdots$ of integers such that $\mathbb{N} \setminus \mathrm{dom}(f) = \bigcup_k S_k$ where $S_k \subset 2^{m_{k+1}} \setminus 2^{m_k}$ and for each $i \in S_k$ the condition $f \cup \{(i, 0)\}$ forces a value on $\dot{h}(i)$.*

Proof. First we choose $f_0 \in \mathfrak{F}$ and some increasing sequence $n_0 < n_1 < \cdots n_k < \cdots$ as in Lemma 1.3. We may choose, for each k, an m_k such that $n_k \le 2^{m_k} < 2^{m_k+1} \le n_{k+1}$ such that $\limsup_k |2^{m_k+1} \setminus (2^{m_k} \cup \mathrm{dom}(f_0))| = \infty$. For each k, let $S^0_k = 2^{m_k+1} \setminus (2^{m_k} \cup \mathrm{dom}(f_0))$. By re-indexing we may assume that $|S^0_k| \ge k$, and we may arrange that $\mathbb{N} \setminus \mathrm{dom}(f_0)$ is equal to $\bigcup_k S^0_k$ and set $L_0 = \mathbb{N}$. For each $k \in L_0$, let $i^0_k = \min S^0_k$ and choose any $f'_1 < f_0$ such that (by definition of $\mathbb{P}$) $I_0 = \{i^0_k : k \in L_0\} \subset (f'_1)^{-1}(0)$ and (by assumption on $\dot{h}$) f'_1 forces a value on $\dot{h}(i^0_k)$ for each $k \in L_0$. Set $f_1 = f'_1 \restriction (\mathbb{N} \setminus I_0)$ and for each $k \in L_0$, let $S^1_k = S^0_k \setminus (\{i^0_k\} \cup \mathrm{dom}(f_1))$. By further extending f_1 we may also assume that $f_1 \cup \{(i^0_k, 1)\}$ also forces a value on $\dot{h}(i^0_k)$. Choose

$L_1 \subset L_0$ such that $\lim_{k\in L_1} |S^1_k| = \infty$. Notice that each member of i^0_k is the minimum element of S^1_k. Again, we may extend f_1 and assume that $\mathbb{N}\backslash\mathrm{dom}(f_1)$ is equal to $\bigcup_{k\in L_1} S^1_k$. Suppose now we have some infinite L_j, some f_j, and for $k \in L_j$, an increasing sequence $\{i^0_k, i^1_k, \dots, i^{j-1}_k\} \subset S^0_k$. Assume further that

$$S^j_k \cup \{i^\ell_k : \ell < j\} = S^0_k \setminus \mathrm{dom}(f_j)$$

and that $\lim_{k\in L_j} |S^j_k| = \infty$. For each $k \in L_j$, let $i^j_k = \min(S^j_k \setminus \{i^\ell_k : \ell < j\})$. By a simple recursion of length 2^j, there is an $f_{j+1} < f_j$ such that, for each $k \in L_j$, $\{i^\ell_k : \ell \leq j\} \subset S^0_k \setminus \mathrm{dom}(f_{j+1})$ and for each function s from $\{i^\ell_k : \ell \leq j\}$ into 2, the condition $f_{j+1} \cup s$ forces a value on $\dot{h}(i^j_k)$. Again find $L_{j+1} \subset L_j$ so that $\lim_{k\in L_{j+1}} |S^{j+1}_k| = \infty$ (where $S^{j+1}_k = S^0_k \setminus \mathrm{dom}(f_{j+1})$) and extend f_{j+1} so that $\mathbb{N} \setminus \mathrm{dom}(f_{j+1})$ is equal to $\bigcup_{k\in L_{j+1}} S^{j+1}_k$.

We are half-way there. At the end of this fusion, the function $\bar{f} = \bigcup_j f_j$ is a member of $\mathbb{P}$ because for each j and $k \in L_{j+1}$, $2^{m_k+1} \setminus (2^{m_k} \cup \mathrm{dom}(\bar{f})) \supset \{i^0_k, \dots, i^j_k\}$. For each k, let $\bar{S}_k = S^0_k \setminus \mathrm{dom}(\bar{f})$ and, by possibly extending $\bar{f}$, we may again assume that there is some L such that $\lim_{k\in L} |\bar{S}_k| = \infty$ and that, for $k \in L$, $\bar{S}_k = \{i^0, i^1_k, \dots, i^{j_k}_k\}$ for some j_k. What we have proven about $\bar{f}$ is that it satisfies that for each $k \in L$ and each $j < j_k$ and each function s from $\{i^0_k, \dots, i^{j-1}_k\}$ to 2, $\bar{f} \cup s \cup (i^j_k, 0)$ forces a value on $\dot{h}(i^j_k)$.

To finish, simply repeat the process except this time choose maximal values and work down the values in $\bar{S}_k$. Again, by genericity of $\mathfrak{F}$, there must be such a condition as $\bar{f}$ in $\mathfrak{F}$. $\square$

Returning to the proof of Theorem 3.1, we are ready to use Lemma 3.1 to show that forcing with $\mathbb{P}(\mathfrak{F})$ will not introduce undesirable functions h analogous to the argument in Theorem 1.1. Indeed, assume that we are in the case that F is a homeomorphism from $\mathbb{N}^*$ to A as above, and that $\{h_\alpha : \alpha \in \lambda\}$ is the family of functions as above. If we show that $\dot{h}$ does not satisfy that $h_\alpha \subset^* \dot{h}$ for each $\alpha \in \lambda$, then we proceed just as in Theorem 1.1. By Lemma 3.1, we have the condition $f_0 \in \mathfrak{F}$ and the sequence S_k $(k \in \mathbb{N})$ such that $\mathbb{N} \setminus \mathrm{dom}(f_0) = \bigcup_k S_k$ and that for each $i \in \bigcup_k S_k$, $f_0 \cup \{(i, 0)\}$ forces a value (call it $\bar{h}(i)$) on $\dot{h}(i)$. Therefore, $\bar{h}$ is a function with domain $\bigcup_k S_k$ in V. It suffices to find a condition in $\mathbb{P}$ below f_0 which forces that there is some α such that h_α is not extended by $\dot{h}$. It is useful to note that if $Y \subset \bigcup_k S_k$ is such that $\limsup |S_k \setminus Y|$ is infinite, then for any function $g \in 2^Y$, $f_0 \cup g \in \mathbb{P}$.

We first check that $\bar{h}$ is 1-to-1 on a cofinite subset. If not, there is an infinite set of pairs $E_j \subset \bigcup_k \bar{S}_k$, $\bar{h}[E_j]$ is a singleton and such that for each k, $\bar{S}_k \cap \bigcup_j E_j$ has at most two elements. If g is the function with $\mathrm{dom}(g) = \bigcup_j E_j$ which is constantly 0, then $f_0 \cup g$ forces that $\dot{h}$ agrees with $\bar{h}$ on $\mathrm{dom}(g)$ and so is not 1-to-1. On the other hand, this contradicts that there is $f_1 < f_0 \cup g$ such that for some $\alpha \in \omega_2$, a_α almost contains $(f_0 \cup g)^{-1}(0)$ and the 1-to-1 function h_α with domain a_α is supposed to also agree with $\dot{h}$ on $\mathrm{dom}(g)$.

But now that we know that $\bar{h}$ is 1-to-1 we may choose any $f_1 \in \mathfrak{F}$ such that $f_1 < f_0$ and such that there is an $\alpha \in \omega_2$ with $f_0^{-1}(0) \subset a_\alpha \subset f_1^{-1}(0)$, and f_1 has decided the function h_α. Let Y be any infinite subset of $\mathbb{N} \setminus \mathrm{dom}(f_1)$ which meets each $\bar{S}_k$ in at most a single point. If $\bar{h}[Y]$ meets Z_α in an infinite set, then choose $f_2 < f_1$ so that $f_2[Y] = 0$ and there is a $\beta > \alpha$ such that $Y \subset a_\beta$. In this case we will have that f_2 forces that $Y \subset a_\beta \setminus a_\alpha$, $\dot{h} \restriction Y \subset^* h_\beta$, and $h_\beta[Y] \cap h_\beta[a_\alpha]$ is infinite (contradicting that h_β is 1-to-1). Therefore we must have that $\bar{h}[Y]$ is almost disjoint from Z_α. Instead consider $f_2 < f_1$ so that $f_2[Y] = 1$. By extending f_2 we may assume that there is a $\beta < \omega_2$ such that $f_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$Z_\beta \cap \bar{h}[Y]$ is infinite". However, since $f_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$h_\beta \subset^* \dot{h}$", we also have that $f_2 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$h_\beta \restriction (a_\beta \setminus a_\alpha) \subset^* \bar{h}$ and $(a_\beta \setminus a_\alpha) \cap Y =^* \emptyset$" contradicting that $\bar{h}$ is 1-to-1 on $\mathrm{dom}(f_2) \setminus \mathrm{dom}(f_0)$. This finishes the proof that there is no $\mathbb{P}(\mathfrak{F})$ name of a function extending all the h_α's $(\alpha \in \lambda)$ and the proof that F can not exist continues as in Theorem 1.1.

Next assume that we have a family $\{c_\alpha : \alpha \in \lambda\}$ as described above and suppose that $C = \dot{h}^{-1}(0)$ satisfies that (it is forced) $C \cap a_\alpha =^* c_\alpha$ for all $\alpha \in \lambda$. If we can show there is no such $\dot{h}$, then we will know that in the extension obtained by forcing with $\mathbb{P}(\mathcal{F})$, the collection $\{(c_\alpha, (a_\alpha \setminus c_\alpha)) : \alpha \in \lambda\}$ forms an (ω_1, ω_1)-gap and we can use a proper poset Q_1 to "freeze" the gap. Again, meeting ω_1 dense subsets of the iteration $\omega_2^{<\omega_1} * \mathbb{P}(\mathcal{F}) * Q_1 * Q_2$ (where Q_2 is the σ-centered poset as in Theorem 1.1) introduces a condition $f \in \mathbb{P}$ which forces that c_λ will not exist. So, given our name $\dot{h}$, we repeat the steps above up to the point where we have f_0 and the sequence $\{S_k : k \in \mathbb{N}\}$ so that $f_0 \cup \{(i, 0)\}$ forces a value $\bar{h}(i)$ on $\dot{h}(i)$ for each $i \in \bigcup_k S_k$ and $\mathbb{N} \setminus \mathrm{dom}(f_0) = \bigcup_k S_k$. Let $Y = \bar{h}^{-1}(0)$ and $Z = \bar{h}^{-1}(1)$ (of course we may assume that $\bar{h}(i) \in 2$ for all i). Since x is forced to be an ultrafilter, there is an $f_1 < f_0$ such that $\mathrm{dom}(f_1)$ contains one of Y or Z. If $\mathrm{dom}(f_1)$ contains Y, then f_1 forces that $\dot{h}[a_\beta \setminus \mathrm{dom}(f_1)] = 1$ and so $(a_\beta \setminus \mathrm{dom}(f_1)) \subset^* (\mathbb{N} \setminus C)$ for all $\beta \in \omega_2$. While if $\mathrm{dom}(f_1)$ contains

Z, then f_1 forces that $\dot{h}[a_\beta \setminus \mathrm{dom}(f_1)] = 0$, and so $(a_\beta \setminus \mathrm{dom}(f_1)) \subset^* C$ for all $\beta \in \omega_2$. However, taking β so large that each of $c_\beta \setminus \mathrm{dom}(f_1)$ and $(a_\beta \setminus (c_\beta \cup \mathrm{dom}(f_1))$ are infinite shows that no such $\dot{h}$ exists.

Finally we show that there are no involutions on $\mathbb{N}^*$ which have a unique fixed point. Assume that F is such an involution and that z is the unique fixed point of F. Applying Corollaries 1.2 and 1.3, we may assume that $\mathbb{N}^* \setminus \{z\} = \bigcup_{\alpha\in\omega_2} Z^*_\alpha$ and that for each α, $F \restriction Z^*_\alpha$ is induced by an involution h_α.

Again let H be $\omega_2^{<\omega_1}$-generic, $\lambda = \omega_2^V$, and $\mathfrak{F} \subset \mathbb{P}_1$ be $\mathbb{P}_1$-generic over V. Assume that $\dot{h}$ is a $\mathbb{P}(\mathcal{F})$-name of a function from $\mathbb{N}$ into $\mathbb{N}$. It suffices to show that no $f \in \mathfrak{F}$ forces that $\dot{h}$ mod finite extends each h_α $(\alpha \in \lambda)$.

At the risk of being too incomplete, we leave to the reader the fact that Lemma 1.3 can be generalized to show that there is an $f \in \mathfrak{F}$ such that either $f \Vdash_{\mathbb{P}_1}$ "$\dot{h} \restriction Z_\alpha \notin V$", or there is a sequence $\{n_k : k \in \mathbb{N}\}$ as before. This is simply due to the fact that the $\mathbb{P}_1$-name of the ultrafilter x_1 can be replaced by any $\mathbb{P}_1$-name of an ultrafilter on $\mathbb{N}$. Similarly, Lemma 3.1 can be generalized in this setting to establish that there must be an $f \in \mathfrak{F}$ and a sequence of sets $\{m_k, S_k, T_k : k \in K \in [\mathbb{N}]^\omega\}$ with bijections $\psi : S_k \to T_k$ such that $S_k \subset (2^{m_k+1} \setminus 2^{m_k}) \subset [n_k, n_{k+1})$, $T_k \subset [n_k, n_{k+1})$, $\mathbb{N} \setminus \mathrm{dom}(f) \subset \bigcup_k S_K$, and for each k and $i \in S_k$ and $\bar{f} < f$ $\bar{f}$ forces a value on $\dot{h}(\psi(i))$ iff $i \in \mathrm{dom}(\bar{f})$. The difference here is that we may have that $f \Vdash_{\mathbb{P}_1}$ "$\mathrm{dom}(f) \subset Z_\alpha$", but there will be some values of $\dot{h}$ not yet decided since $V[H]$ does not have a function extending all the h_α's. Set $\Psi = \bigcup \psi$ which is a 1-to-1 function.

The contradiction now is that there will be some $f' < f$ such that $f' \Vdash_{\mathbb{P}_1}$ "$\Psi^*(x) \neq z$" (because we know that x is not a tie-point). Therefore we may assume that $\Psi(\mathrm{dom}(f') \cap \mathrm{dom}(\Psi))$ is a member of z and so that $\Psi(\mathrm{dom}(\Psi) \setminus \mathrm{dom}(f'))$ is not a member of z. By assumption, there is some $\bar{f} < f'$ and an $\alpha \in \lambda$ such that $\bar{f} \Vdash_{\mathbb{P}_1}$ "$\Psi(\mathrm{dom}(\Psi) \setminus \mathrm{dom}(f')) \subset Z_\alpha$". However this implies $\bar{f}$ forces that $\dot{h}(\Psi(i)) = h_\alpha(\Psi(i))$ for almost all $i \in \bigcup_k S_k \setminus \mathrm{dom}(\bar{f})$, contradicting that $\bar{f}$ does not force a value on $\bar{h}(\Psi(i))$ for all $i \notin \bar{f}$.

4. QUESTIONS

Question 4.1. Assume PFA. If G is $\mathbb{P}_2$-generic, and $\mathbb{N}^* = A \underset{x}{\bowtie} B$ is the generic tie-point introduced by $\mathbb{P}_2$, is it true that A is not homeomorphic to $\mathbb{N}^*$? Is it true that $\tau(x) = 2$? Is it true that each tie-point is a symmetric tie-point?

Remark 1. The tie-point x_3 introduced by $\mathbb{P}_3$ does not satisfy that $\tau(x_3) = 3$. This can be seen as follows. For each $f \in \mathbb{P}_3$, we can partition $\min(f)$ into $\{i \in \operatorname{dom}(f) : i < f(i) < f^2(i)\}$ and $\{i \in \operatorname{dom}(f) : i < f^2(i) < f(i)\}$.

It seems then that the tie-points x_ℓ introduced by $\mathbb{P}_\ell$ might be better characterized by the property that there is an autohomeomorphism F_ℓ of $\mathbb{N}^*$ satisfying that $\operatorname{fix}(F_\ell) = \{x_\ell\}$, and each $y \in \mathbb{N}^* \setminus \{x\}$ has an orbit of size ℓ.

Remark 2. A small modification to the poset $\mathbb{P}_2$ will result in a tie-point $\mathbb{N}^* = A \underset{x}{\bowtie} B$ such that A (hence the quotient space by the associated involution) is homeomorphic to $\mathbb{N}^*$. The modification is to build into the conditions a map from the pairs $\{i, f(i)\}$ into $\mathbb{N}$. A natural way to do this is the poset $f \in \mathbb{P}_2^+$ if f is a 2-to-1 function such that for each n, f maps $\operatorname{dom}(f) \cap (2^{n+1} \setminus 2^n)$ into $2^n \setminus 2^{n-1}$, and again $\limsup_n |2^{n+1} \setminus (\operatorname{dom}(f) \cup 2^n)| = \infty$. $\mathbb{P}_2^+$ is ordered by almost containment. The generic filter introduces an ω_2-sequence $\{f_\alpha : \alpha \in \omega_2\}$ and two ultrafilters: $x \supset \{\mathbb{N} \setminus \operatorname{dom}(f_\alpha) : \alpha \in \omega_2\}$ and $z \supset \{\mathbb{N} \setminus \operatorname{range}(f_\alpha) : \alpha \in \omega_2\}$. For each α and $a_\alpha = \min(f_\alpha) = \{i \in \operatorname{dom}(f_\alpha) : i = \min(f_\alpha^{-1}(f_\alpha(i))\}$, we set $A = \{x\} \cup \bigcup_\alpha a_\alpha^*$ and $B = \{x\} \cup \bigcup_\alpha (\operatorname{dom}(f_\alpha) \setminus a_\alpha)^*$, and we have that $\mathbb{N}^* = A \underset{x}{\bowtie} B$ is a symmetric tie-point. Finally, we have that $F : A \to \mathbb{N}^*$ defined by $F(x) = z$ and $F \restriction A \setminus \{x\} = \bigcup_\alpha (f_\alpha)^*$ is a homeomorphism.

Question 4.2. Assume PFA. If L is a finite subset of $\mathbb{N}$ and $\mathbb{P}_L = \Pi\{\mathbb{P}_\ell : \ell \in L\}$, is it true that in $V[G]$ that if x is tie-point, then $\tau(x) \in L$; and if $1 \notin L$, then every tie-point is a symmetric tie-point?

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