Correction for "The Amalgamation Spectrum"

John T. Baldwin*
Alexei Kolesnikov†
and Saharon Shelah‡

October 5, 2020

The purpose of this note is to correct an argument in [1]. The argument of Theorem 3.10 relied on the fact that, consistently with ZFC, the partially ordered set $\mathbb{P}$ described in Definition 3.13 satisfies the property that for any collection of fewer than λ dense subsets of $\mathbb{P}$ there is a filter that intersects each subset.

The paper incorrectly claimed in Proposition 3.6 that the assumptions of $(< \mu)$-closure (i.e., that any decreasing sequence of length less than μ has a lower bound) and the strong μ^+-chain condition (described in Definition 3.1(2)) are sufficient to establish the existence of the filter from a version of generalized Martin's axiom (the axiom $\mathrm{Ax}_0(\mu)$).

The axiom $\mathrm{Ax}_0(\mu)$ quoted in the paper requires that the partially ordered set be well-met (i.e., any two compatible elements of $\mathbb{P}$ have a greatest lower bound). In this note, we describe a stronger form $\mathrm{Ax}_{\mu,\omega}$ of generalized Martin's axiom established in [2]. The stronger version ensures the existence of a filter meeting fewer than 2^μ dense subsets of a partially ordered set that is not necessarily well-met. We show that the partial order in [1] satisfies the conditions of the axiom $\mathrm{Ax}_{\mu,\omega}$, thus filling the gap in the main argument of Section 3 of that paper. The authors are grateful to Mirna Dzamonja for pointing out a mistake in the original argument.

The summary of corrections is as follows: Definition 3.1 in [1] should be replaced by Definition 2 below. Fact 3.4 in [1] should be replaced by Theorem 3. Lemma 4 and Claim 6 below together with the statements proved in Claim 3.14 of [1] establish that the partial order satisfies the assumptions of the set-theoretic axiom $\mathrm{Ax}_{\mu,\omega}$.

We start by adapting Definition 1.1 from [2] to our context. In that paper, the property of a partially ordered set described below is denoted $*^\omega_\mu$. To be consistent with [1], we use the ordering convention that "<" means "stronger"; the ordering in [2] is reversed.

Definition 1. Given a forcing notion P with a maximal condition 1_P, we define a game $\mathcal{G}_\mu(P)$. For $n < \omega$, at the nth stage Player I chooses a club $E_n \subset \mu^+$, a function $f_n : \mu^+ \to \mu^+$, and a sequence $\{q^n_i \mid i < \mu^+\} \subset P$. Player II chooses a club $F_n \subset \mu^+$ and a sequence $\{p^n_i \mid i < \mu^+\}$. The selections must satisfy the following rules:

1. $E_0 = \mu^+$, $f_0(i) = 0$ and $q^0_i = 1_P$ for all $i < \mu^+$;
2. $E_n \subset \bigcap_{k<n}(E_k \cap F_k)$; $q^n_i < p^{n-1}_i$ for all $i \in E_n$ such that $\mathrm{cf}(i) = \mu$;

*Partially supported by the Simons Foundation grant G5402.
†partially supported by NSF grant DMS-0901315.
‡The author thank Rutgers University and the Binational Science Foundation for partial support of this research.

3. $f_n(i) < i$ for all $i \in E_n$;
4. $p_i^n < q_i^n$ for all $i \in F_n$ such that $\mathrm{cf}(i) = \mu$.

Player I wins the game if there is a club $E \subset \mu^+$ such that for all $i, j \in E$ with $\mu < i, j$ if $\mathrm{cf}(i) = \mathrm{cf}(j) = \mu$ and $f_n(i) = f_n(j)$ for all $n < \omega$, then the set $\{p_i^n \mid n < \omega\} \cup \{p_j^n \mid n < \omega\}$ has a lower bound.

We say that the partial order P satisfies $(*)_\mu$ if Player I has a winning strategy for game $\mathcal{G}_\mu(P)$.

We formulate a strong form of the generalized Martin's axiom.

Definition 2. $\mathrm{Ax}_{\mu,\omega}$ is the following statement. For every $\mu \leq \kappa < 2^\mu$, for every partially ordered set P which is $(< \mu)$-closed of cardinality less than κ and satisfying $(*)_\mu$, for every collection of fewer than κ dense subsets of P, there exists a filter intersecting all the subsets in the family.

The following is a particular case of Theorem 0.7(3) proved in [2] (here, we take $\varepsilon_\alpha = \omega$ for all α.

Theorem 3. *Let $\{\lambda_\alpha \mid \alpha < \alpha^*\}$ be a sequence of cardinals in V such that*

1. *if λ_α is singular, then $\lambda_{\alpha+1} = \lambda_\alpha^+$;*
2. *if λ_α is regular, then $\lambda_\alpha = \lambda_\alpha^{<\lambda_\alpha}$ and $\lambda_{\alpha+1}$ is a regular cardinal greater than λ_α.*

There is a cardinal and cofinality-preserving forcing extension V^P in which $2^{\lambda_\alpha} = \lambda_{\alpha+1}$ for all $\alpha < \alpha^*$ and for all regular λ_α, $\alpha < \alpha^*$, the axiom $\mathrm{Ax}_{\lambda_\alpha,\omega}$ holds.

The following lemma establishes a sufficient condition for a partially ordered set to satisfy $(*)_\mu$. We then show that the partial order $\mathbb{P}$ defined in [1] satisfies the sufficient condition. This allows us to apply $\mathrm{Ax}_{\mu,\omega}$.

Lemma 4. *Suppose that a partial order P satisfies the following condition: for every two decreasing sequences $\{p_\ell^n \mid n < \omega\}$, $\ell = 1, 2$, such that p_1^n is compatible with p_2^n for all $n < \omega$, the set $\{p_1^n \mid n < \omega\} \cup \{p_2^n \mid n < \omega\}$ has a lower bound.*

If P satisfies the strong μ^+-chain condition, then P satisfies the condition $(*)_\mu$.

Proof. We describe a winning strategy for Player I. The 0-th move of the player is determined by the rules. Suppose that the club subsets E_k, F_k, sequences $\{p_i^k \mid i < \mu^+\}$ and $\{q_i^k \mid i < \mu^+\}$, and functions f_k have been constructed for $k < n$.

For each $i < \mu^+$, choose an element q_i^n so that $q_i^n < p_i^{n-1}$. The strong μ^+-chain condition implies that there is a regressive function f_n and a club set E'_n of μ^+ such that $f_n(i) = f_n(j)$ implies that q_i^n, q_j^n are compatible for all $i, j \in E'_n$ such that $\mathrm{cf}(i) = \mathrm{cf}(j) = \mu$. Let $E_n := E'_n \cap \bigcap_{k<n}(E_k \cap F_k)$.

To show that Player I wins, let $E = \bigcap_{n<\omega} E_n$. For $i, j \in E$, i, j of cofinality μ, if $f_n(i) = f_n(j)$ for all $n < \omega$, then, by the choice of the functions f_n, we have q_i^n is compatible with q_j^n for all $n < \omega$. By the assumption of the lemma, we get that the set $\{q_i^n \mid n < \omega\} \cup \{q_j^n \mid n < \omega\}$ has a lower bound.

It remains to observe that a lower bound for the set $\{q_i^n \mid n < \omega\} \cup \{q_j^n \mid n < \omega\}$ is also a lower bound for the set $\{p_i^n \mid n < \omega\} \cup \{p_j^n \mid n < \omega\}$. □

Note that any well-met ω-closed partial order has to satisfy the condition stated in the above lemma. Therefore, the axiom $\mathrm{Ax}_{\mu,\omega}$ is a strengthening of the axiom $\mathrm{Ax}_0(\mu)$; all the arguments in [1] involving $\mathrm{Ax}_0(\mu)$ and well-met partially ordered sets (Claims 3.7 and 3.8) go through without changes.

Now we show that the partial order $\mathbb{P}$ in [1] satisfies the sufficient condition. We recall the definition of the partially ordered set $\mathbb{P}$.

Definition 5. Let $\mathbb{P}$ be a set of models $M \in \boldsymbol{K}$ such that

1. M has the universe $B \subset \mu$, $|B| < \lambda$;
2. $\{a_l \mid l < k\} \subseteq B$;
3. if $u \subset k$, then $N_u \restriction (B \cap |N_u|)$ is a substructure of M;
4. the rank of $\mathbf{a}$ in M is at least $\beta + 2$;
5. the rank of every $(k+1)$-element indiscernible sequence extending $\mathbf{a}$ in M is at least $\beta + 1$.

The partial order is the reverse $\boldsymbol{K}$-submodel relation.

Claim 6. *For every two decreasing sequences $\{M_\ell^n \mid n < \omega\}$, $\ell = 1, 2$, such that M_1^n is compatible with M_2^n for all $n < \omega$, the set $\{M_1^n \mid n < \omega\} \cup \{M_2^n \mid n < \omega\}$ has a lower bound.*

Proof. For $\ell = 1, 2$ and $n < \omega$, let B_ℓ^n be the universe of the structure M_ℓ^n and let $B_0^n = B_1^n \cap B_2^n$. Let $M_0^n := M_1^n \restriction B_0^n (= M_2^n \restriction B_0^n)$.

It is clear that $M_0^n \subseteq M_0^{n+1}$ for all $n < \omega$. Let $M_\ell := \bigcup_{n<\omega} M_\ell^n$ for $\ell = 0, 1, 2$. We now show that the models M_1, M_2 can be amalgamated over M_0. The amalgam N is the needed lower bound for the sequences $\{M_\ell^n \mid n < \omega\}$, $\ell = 1, 2$.

Without loss of generality, we may assume that $M_\ell = M_0 \cup \{b_\ell\}$ for $\ell = 1, 2$, where $b_1 \neq b_2$. To construct the model N, we amalgamate the special system:

$$\{N_u \restriction (|N_u| \cap |M_0|) \mid u \subset k,\ |u| = k - 1\} \cup \{M_1, M_2\}$$

with the root $M_0 \setminus \{a_0, \ldots, a_{k-1}\}$ and the special sequence $\mathbf{b} = \{a_0, \ldots, a_{k-1}, b_1, b_2\}$.

If the sequence $\mathbf{b}$ is not formally indiscernible, there are no obstacles to amalgamation; otherwise the rank of the subsequence $\{a_0, \ldots, a_{k-1}, b_1\}$ has to be at least $\beta + 1$ by (5) in the definition of $\mathbb{P}$. So the amalgam N exists by the special $(< \lambda, k+2, \beta)$-amalgamation. $\square$

Together with the proof of Claim 3.14 of [1], the above arguments establish that $\mathbb{P}$ satisfies the conditions of the $\mathrm{Ax}_{\mu,\omega}$, thus filling the gap in the proof of Lemma 3.12.

References

[1] John Baldwin, Alexei Kolesnikov, and Saharon Shelah. The amalgamation spectrum. *Journal of Symbolic Logic*, **74**, (2009), 914–927.

[2] Saharon Shelah. Was Sierpiński right IV? Preprint, paper Sh546.