

MAXIMAL FAILURES OF SEQUENCE LOCALITY IN A.E.C.
SH932

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ABSTRACT. We are interested in examples of a.e.c. with amalgamation having some (extreme) behaviour concerning types. Note we deal with \mathfrak{k} being sequence-local, i.e. local for increasing chains of length a regular cardinal (for types, equality of all restrictions imply equality). For any cardinal $\theta \geq \aleph_0$ we construct an a.e.c. with amalgamation \mathfrak{k} with $\text{LST}(\mathfrak{k}) = \theta$, $|\tau_{\mathfrak{R}}| = \theta$ such that $\{\kappa : \kappa \text{ is a regular cardinal and } \mathfrak{R} \text{ is not } (2^\kappa, \kappa)\text{-sequence-local}\}$ is maximal. In fact we have a direct characterization of this class of cardinals: the regular κ such that there is no uniform κ^+ -complete ultrafilter (on any $\lambda > \kappa$). We also prove a similar result to “ $(2^\kappa, \kappa)$ -compact for types”.

Date: December 1, 2017.

1991 Mathematics Subject Classification. Primary: 03C48, 03E17; Secondary: 03E05, 03E50.

Key words and phrases. model theory, abstract elementary classes, types, locality.

The author thanks Alice Leonhardt for the beautiful typing. First typed October 20, 2007.

0. INTRODUCTION

Recall a.e.c. (abstract elementary classes); were introduced in [She87a]; and their (orbital) types defined in [She87b], see on them [She09], [Bal09]. It has seemed to me obvious that even with \mathfrak{k} having amalgamation, those types in general lack the good properties of the classical types in model theory. E.g. “ (λ, κ) -sequence-locality where

Definition 0.1. 1) We say that an a.e.c. \mathfrak{k} is a (λ, κ) -sequence-local (for types) when κ is regular and for every $\leq_{\mathfrak{k}}$ -increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$ of models of cardinality λ and $p, q \in \mathcal{S}(M_\kappa)$ we have $(\forall i < \kappa)(p \upharpoonright M_i = q \upharpoonright M_i) \Rightarrow p = q$. We omit λ when we omit “ $\|M_i\| = \lambda$ ”.

2) We say an a.e.c. \mathfrak{k} is (λ, κ) -local when: $\kappa \geq \text{LST}(\mathfrak{k})$ and if $M \in \mathfrak{k}_\lambda$ and $p_1, p_2 \in \mathcal{S}(M)$ and $N \leq_{\mathfrak{k}} M \wedge \|N\| \leq \kappa \Rightarrow p_1 \upharpoonright N = p_2 \upharpoonright N$ then $p_1 = p_2$.

3) We may replace λ by $\leq \lambda, < \lambda, [\mu, \lambda]$ with the obvious meaning (and allow λ to be infinity).

Of course, being sure is not a substitute for a proof, some examples were provided by Baldwin-Shelah [BS08, §2]. There we give an example of the failure of (λ, κ) -sequence-locality for \mathfrak{k} -types in ZFC for some λ, κ , actually $\kappa = \aleph_0$. This was done by translating our problems to abelian group problems. While those problems seem reasonable by themselves they may hide our real problem.

Here in §1 we get \mathfrak{k} , an a.e.c. with amalgamation with the class $\{\kappa : (< \infty, \kappa)\text{-sequence-localness fail for } \mathfrak{k}\}$ being maximal; what seems to me a major missing point up to it, see Theorem 1.3. Also we deal with “compactness of types” getting unsatisfactory results - classes without amalgamation; in [BS08] this was done only in some universes of set theory but with amalgamation; see §2.

We relay on [BS08] to get that \mathfrak{k} has the JEP and amalgamation.

Question 0.2. Can $\{\kappa : \mathfrak{k} \text{ is } (< \infty, \kappa)\text{-local}\}$ be “wild”? E.g. can it be all odd regular alephs? etc?

Note that for this the present translation theorem of [BS08] is not suitable.

In §2 we deal with sequence-compactness of types.

We thank Will Boney for a correction.

1. AN A.E.C. WITH MAXIMAL FAILURE OF BEING LOCAL

Claim 1.1. *Assume*

- ⊗₁ (a) $\kappa = \text{cf}(\kappa) > \theta \geq \aleph_0$ or just $\kappa = \text{cf}(\kappa) \geq \aleph_0, \theta \geq \aleph_0$
- (b) *there is no uniform θ^+ -complete ultra-filter D on κ*
- (c) τ_θ *is a vocabulary of cardinality θ consisting of θ n -place predicates with each n (and no more say $\{R_{\gamma,n} : \gamma < \theta, n < \omega\}, n = \text{arity}(R_{\gamma,n})$).*

Then

- ⊞ *there are $I_\alpha, M_{\ell,\alpha}, \pi_{\ell,\alpha}$ (for $\ell = 1, 2$ and $\alpha \leq \kappa$), g_α (for $\alpha < \kappa$) satisfying:*
 - (a) I_α , *a set of cardinality θ^κ , is \subseteq -increasing continuous with α*
 - (b) $M_{\ell,\alpha}$, *a τ_θ -model of cardinality $\leq \theta^\kappa$, is increasing continuous with α*
 - (c) $\pi_{\ell,\alpha}$ *is a function from $M_{\ell,\alpha}$ onto I_α , increasing continuous with α*
 - (d) $|\pi_{\ell,\alpha}^{-1}\{t\}| \leq \theta^{\aleph_0}$ *for $t \in I_\alpha, \alpha \leq \kappa$ and $\ell = 1, 2$*
 - (e) *if $t \in I_{\alpha+1} \setminus I_\alpha$ then $\pi_{\ell,\alpha}^{-1}\{t\} \subseteq M_{\ell,\alpha+1} \setminus M_{\ell,\alpha}$*
 - (f) *for $\alpha < \kappa, g_\alpha$ is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting $(\pi_{1,2}, \pi_{2,\kappa})$ which means $a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(g_\alpha(a))$*
 - (g) *for $\alpha = \kappa$ there is no isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ respecting $(\pi_{1,\alpha}, \pi_{2,\alpha})$.*

Proof. Follows from 1.2 which is just a fuller version adding to τ_θ unary function F_c for $c \in G$; this is just a notational change when $\theta^{\aleph_0} = \theta$. Otherwise see (*)₁₂ of the proof of 1.2; anyhow we shall use 1.2. □_{1.1}

Claim 1.2. *Assuming ⊗₁ of 1.1 we have:*

- ⊞ *there are $I_\alpha, M_{\ell,\alpha}, \pi_{\ell,\alpha}$ (for $\ell = 1, 2, \alpha \leq \kappa$) and g_α (for $\alpha < \kappa$) and G such that:*
 - (a) G *is an additive (so abelian) group of cardinality θ^{\aleph_0}*
 - (b) I_α *is a set, increasing continuous with $\alpha, |I_\alpha| = \theta^\kappa$*
 - (c) $M_{\ell,\alpha}$ *is a τ_θ^+ -model, increasing continuous with α , of cardinality θ^κ where $\tau_\theta^+ = \tau_\theta \cup \{F_c : c \in G\}, F_c$ a unary function symbol, τ_θ is from ⊗₁(c) of 1.1*
 - (d) $\pi_{\ell,\alpha}$ *is a function from $M_{\ell,\alpha}$ onto I_α increasing continuous with α*
 - (e) $F_c^{M_{\ell,\alpha}}$ ($c \in G$) *is a permutation of $M_{\ell,\alpha}$, increasing continuous with α*
 - (f) $\pi_{\ell,\alpha}(a) = \pi_{\ell,\alpha}(F_c^{M_{\ell,\alpha}}(a))$
 - (g) $F_{c_1}^{M_{\ell,\alpha}}(F_{c_2}^{M_{\ell,\alpha}}(a)) = F_{c_1+c_2}^{M_{\ell,\alpha}}(a)$
 - (h) $\pi_{\ell,\alpha}(a) = \pi_{\ell,\alpha}(b) \Leftrightarrow \bigvee_{c \in G} F_c^{M_{\ell,\alpha}}(a) = b$
 - (i) *for $\alpha < \kappa, g_\alpha$ is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$ which respects $(\pi_{1,\alpha}, \pi_{2,\alpha})$ which means $a \in M_{1,\alpha} \Rightarrow \pi_{1,\alpha}(a) = \pi_{2,\alpha}(g_\alpha(a))$*
 - (j) *there is no isomorphism from $M_{1,\kappa} \upharpoonright \tau_\theta$ onto $M_{2,\kappa} \upharpoonright \tau_\theta$ respecting $(\pi_{1,\kappa}, \pi_{2,\kappa})$.*

Proof. Let

$$(*)_0 \quad \sigma = \theta^{\aleph_0} \text{ so } \sigma = \sigma^{\aleph_0}$$

- (*)₁ (a) let $G = ([\sigma]^{<\aleph_0}, \Delta)$, i.e., the family of finite subsets of σ with the operation of symmetric difference. This is an abelian group satisfying $\forall x(x+x=0)$
- (b) let $\langle a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < \kappa, u \in G \rangle$ be a sequence without repetitions
- (c) for $\beta \leq \kappa$ let $A_\beta = \{a_{f,\alpha,u} : f \in {}^\kappa\sigma, \alpha < 1 + \beta \text{ and } u \in G\}$
- (d) for $\beta \leq \kappa$ let $I_\beta = ({}^\kappa\sigma) \times (1 + \beta)$
- (e) let $\pi_\beta(a_{f,\alpha,u}) = (f, \alpha)$ when $\alpha < 1 + \beta \leq \kappa$
- (f) for each $\beta < \kappa$ we define a permutation g_β (of order 2) of A_β by $g_\beta(a_{f,\alpha,u}) = a_{f,\alpha,u+G\{f(\beta)\}}$ hence $a \in A_\beta \Rightarrow \pi_\beta(g_\beta(a)) = \pi_\beta(a)$.

Note that

- (*)₂ (a) $|G| = \sigma$
- (b) $\langle A_\beta : \beta \leq \kappa \rangle$ is a \subseteq -increasing continuous, each A_β a set of cardinality $\sigma^\kappa = \theta^\kappa$
- (c) $\langle I_\beta : \beta \leq \kappa \rangle$ is \subseteq -increasing continuous, each I_β of cardinality $\sigma^\kappa = \theta^\kappa$
- (d) π_β is a mapping from A_β onto I_β
- (e) if $t \in I_\alpha \subseteq I_\beta$ then $\pi_\beta^{-1}\{t\} = \pi_\alpha^{-1}\{t\}$ has cardinality $|G| = \sigma$
- (f) if $t \in I_{\alpha+1} \setminus I_\alpha$ then $\pi_{\alpha+1}^{-1}\{t\} \subseteq A_{\alpha+1} \setminus A_\alpha$
- (g) if $\alpha \leq \beta \leq \kappa$ then g_β maps A_α onto itself and $g_\beta \circ g_\beta$ is the identity.

For each $n < \omega$ and $\beta \leq \kappa$ we define equivalence relations $E'_{n,\beta}, E_{n,\beta}$ on ${}^n(A_\beta)$:

- (*)₃ $\bar{a}E'_{n,\beta}\bar{b}$ iff $\pi_\beta(\bar{a}) = \pi_\beta(\bar{b})$ where of course $\pi_\beta(\langle a_\ell : \ell < n \rangle) = \langle \pi_\beta(a_\ell) : \ell < n \rangle$
- (*)₄ $\bar{a}E_{n,\beta}\bar{b}$ iff $\bar{a}E'_{n,\beta}\bar{b}$ and there are $k < \omega$ and $\bar{a}_0, \dots, \bar{a}_k$ such that
- (i) $\bar{a}_\ell \in {}^n(A_\beta)$
- (ii) $\bar{a} = \bar{a}_0$
- (iii) $\bar{b} = \bar{a}_k$
- (iv) for each $\ell < k$ for some $\alpha_1, \alpha_2 < \kappa$ we have $g_{\alpha_2}^{-1}(g_{\alpha_1}(\bar{a}_\ell))$ is well defined and equal to $\bar{a}_{\ell+1}$ or $g_{\alpha_2}(g_{\alpha_1}^{-1}(\bar{a}_\ell))$ is well defined and equal to $\bar{a}_{\ell+1}$.

Note:

- (*)_{4.1} (a) the two possibilities in (*)₄(iv) are one as $g_\alpha^{-1} = g_\alpha$ so the first one is a special case of the second;
- (b) g_α does not preserve $\bar{a}/E_{n,\beta}$!, in fact, $a, g_\alpha(a)$ are never $E_{n,\beta}$ equivalent;
- (c) clearly they are well defined iff $(\forall \ell \leq k)[\bar{a}_\ell \in {}^n(A_{\min\{\alpha_1, \alpha_2\}})]$ because if $\alpha \leq \beta$ then g_β maps A_β onto itself because $g(a_{f_1, \alpha, u_2}) = a_{f_1, \alpha, u_2} \Rightarrow |u_1| + 1 = |u_2| \pmod 2$
- (d) if $\alpha \leq \beta, a \in A_\alpha$, then g_β maps $a/E_{n,\beta}$ onto itself
- (e) if $\alpha, \beta \leq \kappa$, then g_α, g_β commute (on the intersection of their domains, $A_{\min\{\alpha, \beta\}}$).

[Why? E.g. for clause (b) note clause (d).]

Note

- (*)₅ (a) $E'_{n,\beta}, E_{n,\beta}$ are indeed equivalence relations on ${}^n(A_\beta)$
 (b) $E_{\beta,n}$ refine $E'_{\beta,n}$
 (c) if $n < \omega$, $\bar{a} \in {}^n(A_\beta)$ then $\bar{a}/E'_{n,\beta}$ has at most σ members (really exactly two but we shall use only its having $\leq 2^\sigma$ members)
 (d) if $\alpha < \beta \leq \kappa$ then $E'_{n,\beta} \upharpoonright {}^n(A_\alpha) = E'_{n,\alpha}$ and $E_{n,\beta} \upharpoonright {}^n(A_\alpha) = E_{n,\alpha}$
 (read (*)₄(iv) carefully!)
 (e) if $\alpha < \beta \leq \kappa$, $\bar{a} \in {}^n(A_\alpha)$ and $\bar{b} \in \bar{a}/E'_{n,\beta}$ then $\bar{b} \in {}^n(A_\alpha)$
 (f) if $g_\alpha(\bar{a}_\ell) = \bar{b}_\ell$ for $\ell = 1, 2$ then: $\bar{a}_1 E'_{n,\beta} \bar{a}_2$ iff $\bar{b}_1 E'_{n,\beta} \bar{b}_2$.

Now we choose a vocabulary τ_θ^* of cardinality 2^σ (but see (*)₁₂) and for $\alpha \leq \kappa$ we choose a τ_θ^* -model $M_{1,\alpha}$ such that:

- (*)₆ (a) $M_{1,\alpha}$ increasing with α with universe A_α
 (b) assume that \bar{a}, \bar{b} are $E'_{n,\alpha}$ -equivalent (so $\bar{a}, \bar{b} \in {}^n(A_\alpha)$ and $\pi_\alpha(\bar{a}) = \pi_\alpha(\bar{b})$); then \bar{a}, \bar{b} realize the same quantifier free type in $M_{1,\alpha}$ iff $\bar{a} E_{n,\alpha} \bar{b}$
 (c) $\tau_\theta^* = \{E_n : n < \omega\} \cup \{F_c : c \in G\} \cup \{R_e : e \in {}^\sigma\sigma\} \cup \{R_{n,i} : i < \sigma, n < \omega\}$
 where E_n, R_e are two-place predicates, F_c a unary function symbol, $R_{n,i}$ is n -place predicate
 (d) for every function $e \in {}^\sigma\sigma$

$$R_e^{M_{1,\alpha}} = \{(a_{f_1, \beta_1, u_1}, a_{f_2, \beta_2, u_2}) \in A_\alpha \times A_\alpha : f_1 = e \circ f_2 \text{ and}$$

$$\text{if } i < \sigma \text{ then } i \in u_1$$

$$\text{iff } (|\{j \in u_2 : e(j) = i\}| \text{ is odd})\}$$
 recalling $f_\ell \in {}^\kappa\sigma$
 (e) $E_n^{M_{1,\alpha}} = E_{n,\alpha}$ and $F_c^{M_{1,\alpha}} = F_c$ is defined by $F_\ell : A_\alpha \rightarrow A_\alpha$ satisfies
 $F_c(a_{f,\alpha,u}) := a_{f,\alpha,u+Gc}$
 (f) if $\alpha \leq \beta_\ell < \kappa$ for $\ell = 1, 2$ then $g_{\beta_2}^{-1} g_{\beta_1} \upharpoonright A_\alpha$ is an automorphism of $M_{1,\alpha}$.

[Why is this possible? First, we shall show that for each $\alpha < \kappa$, g_α maps $R_e^{M_{1,\alpha}}$ onto itself.

Assume we are given a pair $(a_{f_1, \beta_1, u_1}, a_{f_2, \beta_2, u_2})$ from $A_\alpha \times A_\alpha$ so $\beta_1, \beta_2 < 1 + \alpha$ and $f_1 = e \circ f_2$ so

- (*)_{6.1} $(a_{f_1, \beta_1, u_1}, a_{f_2, \beta_2, u_2}) \in R_e^{M_{1,\alpha}}$ iff $u_1 = \{e(j) : j \in u_2 \text{ and } (\exists^{\text{odd}} \ell \in u_2)(e(\ell) = e(j))\}$.

[Why? Read (*)₆ carefully, in particular note that if $i \notin \{e(j) : j \in u_2\}$ then $i \notin u_1$.]

- (*)_{6.2} $(g_\alpha(a_{f_1, \beta_1, u_1}), g_\alpha(a_{f_2, \beta_2, u_2})) \in R_e^{M_{1,\alpha}}$ iff
 $(a_{f_1, \beta_1, u_1 + \{f_1(\alpha)\}}, a_{f_2, \beta_2, u_2 +_G \{f_2(\alpha)\}}) \in R_e^{M_{1,\alpha}}$ iff
 $u_1 +_G \{f_1(\alpha)\} = \{e(j) : j \in u_2 +_G \{f_2(\alpha)\}\}$ and $(\exists^{\text{odd}} \ell \in (u_2 +_G \{f_2(\alpha)\})(e(\ell) = e(j))\}$.

[Why? Inside $(*)_{6.2}$ the first “iff” holds by the definition of g_α , the second “iff” holds as in $(*)_{6.1}$.]

But $f_1 = e \circ f_2$ hence

$$(*)_{6.3} \quad f_1(\alpha) = e(f_2(\alpha))$$

$$(*)_{6.4} \quad \text{letting } x = f_2(\alpha) < \sigma \text{ we have } u_1 = \{e(j) : j \in u_2 \text{ and } (\exists^{\text{odd}} \iota \in u_2)(e(\iota) = e(j))\} \text{ iff } u_1 +_G \{e(x)\} = \{e(j) : j \in u_2 +_G \{x\} \text{ and } \exists^{\text{odd}} \iota \in (u_2 +_G \{x\})(e(\iota) = e(j))\}.$$

[Why? Check by cases according to whether $x \in u_2$ and whether $e(x) \in u_1$. I.e. by “ G is of order two” it suffices to prove the “only if” so assume the first equality in $(*)_{6.2}$. If $e(x) \notin u_1$, then just add $e(x)$ to both sides. Similarly if $e(x) \in u_1 \cap X \notin u_2$ and if $e(x) \in u_1 \wedge x \in u_2$.]

So together we get equivalence, so the “first” holds.

Second, for defining the $R_{n,i}^{M_{1,\alpha}}$ ’s

- $(*)_{6.5}$ (a) for each N let E_n'' be the following equivalence relation on ${}^n G : \bar{u}_1 E_n'' \bar{u}_2$ iff for some $v \in G$, $|v|$ is even and $\bigwedge_{\ell < n} u_{1,\ell} +_G v = u_{2,\ell}$
- (b) let $\langle \mathcal{X}_{n,i} : i < \sigma \rangle$ list the E_n'' -equivalence classes
- (c) let $R_{n,i}^{M_{1,\alpha}} = \{\bar{a} \in {}^n(A_\alpha) : \text{if } \bar{a} = \langle a_{f_\ell, \alpha_\ell, u_\ell} : \ell < n \rangle, \text{ then } \langle u_\ell : \ell < n \rangle \in \mathcal{X}_{n,i}\}$.

This completed the choice of $M_{1,\alpha}$. Third, g_α preserves “ \bar{a}, \bar{b} are $E_{n,\alpha}$ -equivalent”, “ \bar{a}, \bar{b} are $E'_{n,\alpha}$ -equivalent” and their negations. That is, $\bar{a}, g_\alpha(\bar{a})$ are not $E_{n,\alpha}$ -equivalent, but as $(\forall \beta)(g_\beta = g_\beta^{-1}), \bar{a}, \bar{b}$ being $E_{n,\alpha}$ -equivalent means that there is an even length pass from \bar{a} to \bar{b} , in the graph $\{(\bar{c}, g_\beta(\bar{c})) : \beta \in [\gamma, \kappa) \text{ and } \bar{c} \in {}^n(A_\gamma)\}$ where $\gamma = \min\{\gamma : \bar{a}, \bar{b} \in {}^n(A_\gamma)\}$.

Fourth, no problem in the $M_{1,\alpha}$ ’s are increasing by $(*)_5(d)$, just check that.

Fifth, g_α commutes with $F_c^{M_{1,\alpha}}$ for $c \in G$ because G is an Abelian group.

Sixth, we should check clause $(*)_6(f)$. Now $g_{\beta_2}^{-1} g_{\beta_1} \upharpoonright A_\alpha = (g_{\beta_2} \upharpoonright A_\alpha)(g_{\beta_1} \upharpoonright A_\alpha)$ by $(*)_2(g)$ and it has order 2 because G is of order 2 and it maps $E_n^{M_{1,\alpha}}$ to itself by the “third”, commute with $F_c^{M_{1,\alpha}}$ by the fifth, maps $R_e^{M_{1,\alpha}}$ to itself by the “first”.

Lastly, it maps $R^{M_{1,\alpha}}$ to itself by $(*)_{6.4}$. So we are done proving $(*)_6$.

- $(*)_7$ for $\alpha < \kappa$ let $M_{2,\alpha}$ be the τ_θ^* -model with universe A_α such that g_α is an isomorphism from $M_{1,\alpha}$ onto $M_{2,\alpha}$.

Now we note

$$(*)_8 \quad \text{for } \alpha < \beta < \kappa, M_{2,\alpha} \subseteq M_{2,\beta}.$$

[Why? By the definitions of $M_{1,\gamma}, g_\gamma, E'_{n,\gamma}, E_{n,\gamma}$, in particular, the “first” and “third”, in “why $(*)_6$ ”, fourth, i.e. $(*)_5(d)$.]

$$(*)_9 \quad \text{let } M_{2,\kappa} := \cup\{M_{2,\alpha} : \alpha < \kappa\}, \text{ well defined by } (*)_8$$

$$(*)_{10} \quad \text{let } \pi_{\ell,\beta} = \pi_\beta \text{ for } \ell = 1, 2 \text{ and } \beta < \kappa$$

$$(*)_{11} \quad \text{except clause (j) the demands in the conclusion of } \boxplus \text{ of 1.2 holds easily.}$$

[Why? Just check.]

$$(*)_{12} \quad \text{it is O.K. to use a vocabulary of cardinality } 2^\sigma = 2^{\theta^{\aleph_0}}.$$

[Why? As there is a model M of cardinality 2^σ with $|\tau_M| = \theta$ omitting a quantifier free type p such that $M \subset N \wedge M \equiv N \Rightarrow N$ realizes p . Such M exists as $\sigma = \theta^{\aleph_0}$ and clause (b) of the assumption \otimes_1 of 1.2, 1.1.]

Note

(*)₁₃ if $(a_{f,\alpha,u_1}, a_{f,\alpha,u_2})$ is $E_{2,\alpha}$ -equivalent to $(a_{f,\alpha,v_1}, a_{f,\alpha,v_2})$ then $G \models "u_1 - u_2 = v_1 - v_2"$.

[Why? By induction on the k from (*)₄.]

So to finish we assume toward contradiction

\boxtimes h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ which respects $(\pi_{1,\alpha}, \pi_{2,\alpha})$ for $\alpha < \kappa$.

So trivially

\otimes_1 $h(a_{f,\alpha,u}) \in \{a_{f,\alpha,v} : v \in G\}$ and $\bar{a} \in {}^n(A_\alpha) \Rightarrow h(\bar{a}) \in \bar{a}/E_{n,\alpha} \subseteq \bar{a}/E'_{n,\alpha}$.

[Why? As h respect $(\pi_{1,\kappa}, \pi_{2,\kappa})$ see (*)₁(e) and (*)₁₀ clearly $h(\bar{a}) \in \bar{a}/E'_{n,\alpha}$. But h is an isomorphism from $M_{1,\kappa}$ onto $M_{2,\kappa}$ hence by (*)₆(b) we have $h(\bar{a}) \in (\bar{a}/E_{n,\alpha})$.]

\otimes_2 for $f \in {}^\kappa\sigma$ and $\alpha < \kappa$ let $u_{f,\alpha} \in G$ be the $u \in G$ such that $h(a_{f,\alpha,\emptyset}) = a_{f,\alpha,u}$

\otimes_3 for $f \in {}^\kappa\sigma$, $\alpha < \kappa$ and $v \in G$ we have $h(a_{f,\alpha,v}) = a_{f,\alpha,v+G u_{f,\alpha}}$.

[Why? By \otimes_1 clearly h maps any finite sequence $\bar{b} \in {}^n(A_{1,\kappa})$ to an $E_{n,\alpha}$ -equivalent sequence for each $\alpha < \kappa$. Now apply this to the pair $(a_{f,\alpha,\emptyset}, a_{f,\alpha,u})$ recalling (*)₁₃.]

\otimes_4 we define a partial order \leq on ${}^\kappa\sigma$ as follows:

$f_1 \leq f_2$ iff there is a function $e \in {}^\sigma\sigma$ witnessing it; which means $f_1 = e \circ f_2$

\otimes_5 if $\alpha_1, \alpha_2 < \kappa$ and $f_1 \leq f_2$ (are from ${}^\kappa\sigma$) then $|u_{f_1,\alpha_1}| \leq |u_{f_2,\alpha_2}|$.

[Why? This follows from \otimes_6 below.]

\otimes_6 if $e \in {}^\sigma\sigma$, $f_2 \in {}^\kappa\theta$ and $f_1 = e \circ f_2 \in {}^\kappa\sigma$ and $\alpha_1, \alpha_2 < \kappa$ then

$u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$.

[Why? Choose $\alpha < \kappa$ such that $\alpha > \alpha_1, \alpha > \alpha_2$ so $a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_1,\emptyset} \in M_{\ell,\alpha}$ for $\ell = 1, 2$. Recall that h maps $R_e^{M_{1,\alpha}}$ onto $R_e^{M_{2,\alpha}}$ by \boxtimes and $R_e^{M_{2,\alpha}} = R_e^{M_{1,\alpha}}$ because g_α maps $R_e^{M_{1,\alpha}}$ onto itself (see the proof of (*)₆ above, the "first" in that proof). Now see (*)₆(d), i.e. the definition of $R_e^{M_{1,\alpha}}$, i.e. obviously $(a_{f_1,\alpha_1,\emptyset}, a_{f_2,\alpha_2,\emptyset}) \in R_e^{M_{1,\alpha}}$ so as h is an isomorphism we have $(h(a_{f_1,\alpha_1,\emptyset}), h(a_{f_2,\alpha_2,\emptyset})) \in R_e^{M_{2,\alpha}}$ so by the previous sentence and the definitions of u_{f,α_ℓ} ($\ell = 1, 2$) in \otimes_2 we have $(a_{f_1,\alpha_1,u_{f_1,\alpha_1}}, a_{f_2,\alpha_2,u_{f_2,\alpha_2}}) \in R_e^{M_{1,\alpha}}$ which by the definitions of $R_e^{M_{1,\alpha}}$ in (*)₆(d) implies $u_{f_1,\alpha_1} \subseteq \{e(i) : i \in u_{f_2,\alpha_2}\}$ as promised.]

\otimes_7 (a) $|u_{f,\alpha_1}| = |u_{f,\alpha_2}|$ for $\alpha_1, \alpha_2 < \kappa$, $f \in {}^\kappa\sigma$

(b) $\mathbf{n}(f) = |u_{f,\alpha}|$ is well defined

(c) if $f_1 \leq f_2$ then $\mathbf{n}(f_1) \leq \mathbf{n}(f_2)$.

[Why? For clause (a) use \otimes_6 for the function $e = \text{id}_\sigma$ and $f_1 = f_2 = f$. Clause (b) follows. Clause (c) holds by \otimes_6 equivalently by \otimes_5 .]

- \otimes_8 there are $f_* \in {}^\kappa\sigma$ and $\alpha_* < \kappa$ such that:
- (i) if $f_* \leq f \in {}^\kappa\sigma$ and $\alpha < \kappa$ then $|u_{f_*,\alpha_*}| = |u_{f,\alpha}|$
 - (ii) moreover if $f_* = e \circ f$ where $e \in {}^\sigma\sigma$ and $f \in {}^\kappa\sigma, \alpha < \kappa$ then $e \upharpoonright u_{f,\alpha}$ is one-to-one from $u_{f,\alpha}$ onto $u_{f_*,\alpha}$ so $\mathbf{n}(f_*) = \mathbf{n}(f)$
 - (iii) if $\alpha < \kappa, f_1 = e \circ f_2, f_* = e_1 \circ f_1, f_* = e_2 \circ f_2$ so $e, e_1, e_2 \in {}^\sigma\sigma$, then $e \upharpoonright u_{f_2,\alpha}$ is one-to-one onto $u_{f_1,\alpha}$.

[Why? First note that clause (ii), (iii) follows from clause (i). Second, if clause (i) fails, then we can find a sequence $\langle (f_n, \alpha_n, e_n) : n < \omega \rangle$ such that

- (α) $\alpha_n < \kappa, f_n \in {}^\kappa\sigma$ for $n < \omega$
- (β) $f_n \leq f_{n+1}$ say $f_n = e_n \circ f_{n+1}$ and $e_n \in {}^\sigma\sigma$
- (γ) $(e_n, f_{n+1}, \alpha_{n+1})$ witness that (f_n, α_n) does not satisfy the demand (i) on (f_*, α_*) hence $\mathbf{n}(f_n) < \mathbf{n}(f_{n+1})$.

Let $u_n = u_{f_n, \alpha_n}$ for $n < \omega$. For $n < \omega$ and $i < \sigma$ let $A_{n,i} = \langle \alpha < \kappa : f_n(\alpha) = i \rangle$, so $\langle A_{n,i} : i < \sigma \rangle$ is a partition of κ and $\alpha \in A_{n+1,i} \Rightarrow \alpha \in A_{n,e_n(i)}$. So letting $A_\eta = \bigcap \{A_{n,\eta(n)} : n < \omega\}$ for $\eta \in {}^\omega\sigma$ clearly $\langle A_\eta : \eta \in {}^\sigma\sigma \rangle$ is a partition of κ .

As we have $\sigma = \sigma^{\aleph_0}$ by $(*)_0$, there is a sequence $\langle e^n : n < \omega \rangle$ satisfying $e^n \in {}^\sigma\sigma$ and $f \in {}^\kappa\sigma$ such that $f_n = e^n \circ f$ for each $n < \omega$. So $n < \omega \Rightarrow f_n \leq f$ which by $\otimes_7(c)$ implies $\mathbf{n}(f_n) \leq \mathbf{n}(f)$. As $\langle \mathbf{n}(f_n) : n < \omega \rangle$ is increasing, easily we get a contradiction.]

- \otimes_9 $\mathbf{n}(f_*) > 0$, i.e. $\alpha < \kappa \Rightarrow u_{f_*,\alpha} \neq \emptyset$.

[Why? If $(\forall f \in {}^\kappa\sigma)(\forall \alpha < \kappa)(u_{f,\alpha} = \emptyset)$ then (by \otimes_3) we deduce h is the identity, contradiction. Otherwise assume $u_{f,\alpha} \neq \emptyset$ hence as in the proof of \otimes_8 there is f' such that $f_* \leq f' \wedge f \leq f'$ so by \otimes_5 and \otimes_8 we have $0 < |u_{f,\alpha}| \leq |u_{f',\alpha}| = |u_{f_*,\alpha_*}|$.]

- \otimes_{10} if $f \in {}^\kappa\sigma, \alpha < \kappa$ and $i \in u_{f,\alpha}$ then $\kappa = \sup\{\beta < \kappa : \alpha < \beta \text{ and } f(\beta) = i\}$.

[Why? If not, let $\beta(*) < \kappa$ be $> \sup\{\beta < \kappa : \alpha < \beta, f(\beta) = i\}$ and $> \omega$ and $> \alpha$. Let $Y = \{(a_{f,\alpha,u}, a_{f,\beta(*),u}) : u \in G, i \notin u\}$. Now for every $\beta \in (\beta(*), \kappa)$ the function g_β maps the set Y onto itself (see its definition in $(*)_2(f)$) hence by the definition of $E_{2,\beta(*)+1}$ (in $(*)_4$) it follows that $\bar{a} \in Y \Rightarrow \bar{a}/E_{2,\beta(*)+1} \subseteq Y$ and as h respects $(\pi_{1,\beta(*)+1}, \pi_{2,\beta(*)+1})$ it follows that $h(\bar{a}) \subseteq \bar{a}/E'_{2,\beta(*)+1}$ and so $\kappa > \gamma \geq \beta(*) + 1 \Rightarrow g_\gamma^{-1}(h(\bar{a})) \in \bar{a}/E'_{2,\beta(*)+1}$.

Now for $\bar{a} \in Y$, the pairs $\bar{a}, h(\bar{a})$ realizes the same quantifier free type in $M_{1,\beta(*)+1}, M_{2,\beta(*)+1}$ respectively, hence by the choice of $M_{2,\beta(*)+1}$ the pairs $\bar{a}, g_{\beta(*)+2}^{-1}h(\bar{a})$ realize the same quantifier free type in $M_{1,\alpha}$. By $(*)_6(b)$ recalling $g_{\beta(*)+2}^{-1}(h(\bar{a})) \in \bar{a}/E'_{2,\beta(*)+1}$ this implies that $\bar{a}, g_{\beta(*)+2}^{-1}(h(\bar{a}))$ are $E_{2,\beta(*)+1}$ -equivalent. By the definition of $E_{2,\beta(*)+1}, g_{\beta(*)+2}^{-1}(h(\bar{a}))$ belongs to the closure of $\{\bar{a}\}$ under $\{g_\gamma^{\pm 1} : \gamma \in (\beta(*) ; \kappa)\}$ hence $h(\bar{a})$ belongs to it. But by an earlier sentence Y is closed under those functions so $h(\bar{a}) \in Y$. Similarly $h^{-1}(\bar{a}) \in Y$, hence h maps Y onto itself, recalling \otimes_2 this implies $i \notin u_{f,\alpha}$, contradicting an assumption of \otimes_{10} , so \otimes_{10} holds.]

Now fix f_*, α_* as in \otimes_8 for the rest of the proof, without loss of generality f_* is onto σ and let $u_{f_*, \alpha_*} = \{i_\ell^* : i < \ell(*)\}$ with $\langle i_\ell^* : \ell < \ell(*) \rangle$ increasing for simplicity. Now for every $f \in {}^\kappa \sigma$ such that $f_* \leq f$ and $\alpha < \kappa$ by $\otimes_8(ii), (iii)$ we know that if $e \in {}^\sigma \sigma \wedge f_* = e \circ f$ then e is a one-to-one mapping from $u_{f, \alpha}$ onto u_{f_*, α_*} ; but so $e \upharpoonright u_{f, \alpha}$ is uniquely determined by $(f_*, \alpha_*, f, \alpha)$ so let $i_{f, \alpha, \ell} \in u_{f, \alpha}$ be the unique $i \in u_{f, \alpha}$ such that $e(i) = i_\ell^*$ (equivalently $(\exists \alpha)(f(\alpha) = i \wedge f_*(\alpha) = i_\ell^*)$).

Now if $f_* \leq f \in {}^\kappa \sigma$ and $\alpha_1, \alpha_2 < \kappa$ and we choose $e = \text{id}_\sigma$ so necessarily $f \upharpoonright u_{f, \alpha_1} = e \circ f \upharpoonright u_{f, \alpha_2}$, then $e \upharpoonright \text{Rang}(f \upharpoonright u_{f, \alpha_2})$ map u_{f, α_2} onto u_{f, α_1} but e is the identity so we can write u_f instead of $u_{f, \alpha}$ let $i_{f, \ell} = i_{f, \alpha, \ell}$ for $\ell < \ell(*)$, $\alpha < \kappa$.

Let

$$\mathcal{A} = \{A \subseteq \kappa : \text{for some } f, f_* \leq f \text{ and } \alpha < \kappa \text{ we have } f^{-1}\{i_{f, 0}\} \setminus \alpha \subseteq A\}$$

$$\square_1 \quad \mathcal{A} \subseteq \mathcal{P}(\kappa) \setminus [\kappa]^{< \kappa}.$$

[Why? As κ is regular, this means $A \in \mathcal{A} \Rightarrow A \subseteq \kappa \wedge \text{sup}(A) = \kappa$ which holds by \otimes_{10} .]

$$\square_2 \quad \kappa \in \mathcal{A}.$$

[Why? By the definition of \mathcal{A} .]

$$\square_3 \quad \text{if } A \in \mathcal{A} \text{ and } A \subseteq B \subseteq \kappa \text{ then } B \in \mathcal{A}.$$

[Why? By the definition of \mathcal{A} .]

$$\square_4 \quad \text{if } A_1, A_2 \in \mathcal{A} \text{ then } A =: A_1 \cap A_2 \text{ belongs to } \mathcal{A}.$$

[Why? Let $(f_\ell, e_\ell, \alpha_\ell)$ be such that $f_* = e_\ell \circ f_\ell$ and $f_\ell \in {}^\kappa \sigma$, $\alpha_\ell < \kappa$ and $f_\ell^{-1}\{i_{f_\ell, 0}\} \setminus \alpha_\ell \subseteq A_\ell$ for $\ell = 1, 2$. Let $\text{pr}: \sigma \times \sigma \rightarrow \sigma$ be one-to-one and onto and define $f \in {}^\kappa \sigma$ by $f(\alpha) = \text{pr}(f_1(\alpha), f_2(\alpha))$. Clearly $f_\ell \leq f$ for $\ell = 1, 2$ hence $i_{f, 0}$ is well defined and $i_{f, 0} = \text{pr}(i_{f_1, 0}, i_{f_2, 0})$. Now for every $\alpha < \kappa$, $f(\alpha) = i_{f, 0} \Rightarrow f_1(\alpha) = i_{f_1, 0} \wedge f_2(\alpha) = i_{f_2, 0} \Rightarrow \alpha \in A_1 \wedge \alpha \in A_2 \Rightarrow \alpha \in A_1 \cap A_2 \Rightarrow \alpha \in A$ so $f^{-1}\{i_{f, 0}\} \subseteq A$ hence $A \in \mathcal{A}$.]

$$\square_5 \quad \text{if } A \subseteq \kappa \text{ then } A \in \mathcal{A} \text{ or } \kappa \setminus A \in \mathcal{A}.$$

[Why? Define $f \in {}^\kappa \sigma$:

$$f(\alpha) = \begin{cases} 2f_*(\alpha) & \text{if } \alpha \in A \\ 2f_*(\alpha) + 1 & \text{if } \alpha \in \kappa \setminus A. \end{cases}$$

Let $i = i_{f, 0}$ so by the definition of \mathcal{A} we have $f^{-1}\{i\} = f^{-1}\{i_{f, 0}\} \in \mathcal{A}$. But if i is even then $f^{-1}\{i\} \subseteq A$ and i is odd then $f^{-1}\{i\} \subseteq \kappa \setminus A$ so by \square_3 we are done.]

$$\square_6 \quad \mathcal{A} \text{ is a uniform ultrafilter on } \kappa.$$

[Why? By $\square_1 - \square_5$.]

$$\square_7 \quad \mathcal{A} \text{ is } \sigma^+ \text{-complete.}$$

[Why? Assume $B_\varepsilon \in \mathcal{A}$ for $\varepsilon < \sigma$ and let $B = \bigcap \{B_\varepsilon : \varepsilon < \sigma\}$. Define $A_\varepsilon \subseteq \kappa$ for $\varepsilon < \sigma$ as follows: $A_{1+\varepsilon} = \bigcap_{\zeta < \varepsilon} B_\zeta \setminus B_\zeta$ (so is $\kappa \setminus B_0$ if $\varepsilon = 0$) for $\varepsilon < \sigma$ and

$A_0 = B$. Clearly $\langle A_\varepsilon : \varepsilon < \sigma \rangle$ is a partition of κ , let $f \in {}^\kappa \sigma$ be such that $f \upharpoonright A_\varepsilon$ is constantly ε . Let $f' \in {}^\kappa \theta$ be such that $f \leq f' \wedge f_* \leq f'$. Now $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$ is included in some A_ε . If $\varepsilon = 0$ this exemplifies $\bigcap_{\varepsilon < \sigma} B_\varepsilon \in \mathcal{A}$ as required. If

$\varepsilon = 1 + \zeta < \sigma$, then $(f')^{-1}\{i_{f',0}\} \subseteq A_\varepsilon \subseteq \kappa \setminus B_\varepsilon$, contradiction to \square_6 because $B_\varepsilon \in \mathcal{A}$ and $(f')^{-1}\{i_{f',0}\} \in \mathcal{A}$.]

So by the assumptions of 1.2, that is, $\otimes_1(b)$ of 1.1 we get a contradiction, coming from the assumption “toward contradiction (j) of \boxplus of 1.2 fails”, so it holds and the other clauses were proved so we are done. $\square_{1.2}$

Theorem 1.3. *For every θ there is an $\mathfrak{k} = \mathfrak{k}_\theta^*$ such that*

- \otimes (a) \mathfrak{k} is an a.e.c. with $\text{LST}(\mathfrak{k}) = \theta$, $|\tau_\mathfrak{k}| = \theta$
- (b) \mathfrak{k} has the amalgamation property
- (c) \mathfrak{k} admits intersections (see Definition 1.4 below)
- (d) if κ is a regular cardinal and there is no uniform θ^+ -complete ultrafilter on κ , then: \mathfrak{k} is not $(\leq 2^\kappa, \kappa)$ -sequence-local for types, i.e., we can find an $\leq_\mathfrak{k}$ -increasing continuous sequence $\langle M_i : i \leq \kappa \rangle$ of models and $p \neq q \in \mathcal{S}_\mathfrak{k}(M_\kappa)$ such that $i < \kappa \Rightarrow p \upharpoonright M_i = q \upharpoonright M_i$ and M_κ is of cardinality $\leq 2^\kappa$.

We shall prove 1.3 below. As in [BS08, 1.2, §4] the aim of the definition of “admit intersections” is to ensure types behave reasonably.

Definition 1.4. We say an a.e.c. \mathfrak{k} admits intersections when there is a function $cl_\mathfrak{k}$ such that:

- (a) $cl_\mathfrak{k}(A, M)$ is well defined iff $M \in K_\mathfrak{k}$ and $A \subseteq M$
- (b) $cl_\mathfrak{k}(A, M)$ is preserved under isomorphisms and $\leq_\mathfrak{k}$ -extensions
- (c) for every $M \in K_\mathfrak{k}$ and non-empty $A \subseteq M$ the set $B = cl_\mathfrak{k}(A, M)$ satisfies: $M \upharpoonright B \in K_\mathfrak{k}$, $M \upharpoonright B \leq_\mathfrak{k} M$ and $A \subseteq M_1 \leq_\mathfrak{k} N \wedge M \leq_\mathfrak{k} N \Rightarrow B \subseteq M_1$; we may use $cl_\mathfrak{k}(A, M)$ for $M \upharpoonright cl_\mathfrak{k}(A, M)$.

Claim 1.5. *Assume \mathfrak{k} is an a.e.c. admitting intersections. Then $\text{tp}_\mathfrak{k}(a_1, M, N_1) = \text{tp}_\mathfrak{k}(a_2, M, N_2)$ iff letting $M_\ell = N_\ell \upharpoonright cl_\mathfrak{k}(M \cup \{a_\ell\})$, there is an isomorphism from M_1 onto M_2 over M mapping a_1 to a_2 .*

Proof. Should be clear by the definition. $\square_{1.5}$

Remark 1.6. In Theorem 1.3 we can many times demand $\|M_\kappa\| = \kappa$, e.g., if $(\exists \lambda)(\kappa = 2^\lambda)$.

Note we now show that 1.3 is best possible.

Claim 1.7. *1) If \mathfrak{k} satisfies clause (a) of 1.3, (i.e. \mathfrak{k} is an a.e.c. with LST -number $\leq \theta$ and $|\tau_\mathfrak{k}| \leq \theta$) and κ fails the assumption of clause (d) of 1.3, that is there is a uniform θ^+ -complete ultrafilter on κ , then the conclusion of clause (d) of 1.3 fails, that is \mathfrak{k} is κ -sequence local for types.*

2) If D is a θ^+ -complete ultrafilter on κ and \mathfrak{k} is an a.e.c. with $\text{LST}(\mathfrak{k}) \leq \theta$ then ultraproducts by D preserve “ $M \in \mathfrak{k}$ ”, “ $M \leq_\mathfrak{k} N$ ”, i.e.

☒ if $M_i, N_i (i < \kappa)$ are $\tau(\mathfrak{K})$ -models and $M = \prod_{i < \kappa} M_i/D$ and $N = \prod_{i < \kappa} N_i$ then:

- (a) $M \in K$ if $\{i < \kappa : M_i \in \mathfrak{K}\} \in D$
- (b) $M \leq_{\mathfrak{K}} N$ if $\{i : M_i \leq_{\mathfrak{K}} N_i\} \in D$.

Proof. Note that if D is θ^+ -complete, then it is σ^+ -complete where $\sigma = \theta^{\aleph_0}$ (and much more, it is θ' -complete for the first measurable $\theta' > \theta$).

1) So assume

- ☐ (a) $\langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{K}}$ -increasing
- (b) $M_\kappa = N_0 \leq_{\mathfrak{K}} N_\ell$ for $\ell = 1, 2$
- (c) $p_\ell = \mathbf{tp}_{\mathfrak{K}}(a_\ell, N_0, N_\ell)$ for $\ell = 1, 2$
- (d) $i < \kappa \Rightarrow p_1 \upharpoonright M_i = p_2 \upharpoonright M_i$.

We shall show $p_1 = p_2$, this is enough.

Without loss of generality

- (*)₁ (a) $a_1 = a_2$ call it a
- (b) $\tau_{\mathfrak{K}} \subseteq \mathcal{H}(\theta)$.

By (d) of ☐ we have:

- (d)⁺ for each $i < \kappa$ there are $n_i < \omega$ and $\langle N_{i,m} : n \leq n_i \rangle$ such that
 - (α) $N_{i,0} = N_1$
 - (β) $N_{i,m_i} = N_2$ or just h_i is an isomorphism from N_{i,m_i} onto N_2 such that $h_i \upharpoonright (M_i \cup \{a\})$ is the identity
 - (γ) $a \in N_{i,\ell}$ and $M_i \leq_{\mathfrak{K}} N_{i,\ell}$
 - (δ) if $m < m_i$ then $N_{i,2m+1} \leq_{\mathfrak{K}} N_{i,2m}, N_{i,2m+2}$.

As $\kappa = \text{cf}(\kappa) > \aleph_0$ without loss of generality $i < \kappa \Rightarrow n_i = n_*$. Let χ be such that $\langle M_i : i \leq \kappa \rangle, \langle \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle$ and $\mathfrak{K}_{\text{LST}(\mathfrak{K})}$ all belongs to $\mathcal{H}(\chi)$; concerning $\mathfrak{K}_{\text{LST}(\mathfrak{K})}$ this means τ_χ and $\text{LST}(\mathfrak{K})$ belongs to $\mathcal{H}(\chi)$ hence $\{M \in K_{\mathfrak{K}} : M \in \mathcal{H}(\text{LST}_{\mathfrak{K}}^+)\}$ and $\leq_{\mathfrak{K}} \upharpoonright \mathcal{H}(\text{LST}_{\mathfrak{K}}^+)$ belongs to $\mathcal{H}(\chi)$; those hold by (*)₁(b). Let \mathfrak{B} be the ultrapower $(\mathcal{H}(\chi), \in)^\kappa/D$ and \mathbf{j}_0 the canonical embedding of $(\mathcal{H}(\chi), \in)$ into \mathfrak{B} and let \mathbf{j}_1 be the Mostowski-Collapse of \mathfrak{B} to a transitive set \mathcal{H} and let $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$. So \mathbf{j} is an elementary embedding of $(\mathcal{H}(\chi), \in)$ into (\mathcal{H}, \in) and even an $\mathbb{L}_{\theta^+, \theta^+}$ -elementary one. Recall we are assuming without loss of generality $\tau_{\mathfrak{K}} \subseteq \mathcal{H}(\theta)$ hence $\mathbf{j}(\tau_{\mathfrak{K}}) = \tau_{\mathfrak{K}}$ hence by part (2), \mathbf{j} preserves “ $N \in K_{\mathfrak{K}}$ ”, “ $N^1 \leq_{\mathfrak{K}} N^2$ ”. “ h is an isomorphism from N' onto N'' ”.

So $\mathbf{j}(\langle M_i : i \leq \kappa \rangle)$ has the form $\langle M_i^* : i \leq \mathbf{j}(\kappa) \rangle$ but $\mathbf{j}(\kappa) > \kappa_* := \bigcup_{i < \kappa} \mathbf{j}(i)$ by the uniformity of D and let $\mathbf{j}(\langle \langle N_{i,n} : n \leq n_* \rangle : i < \kappa \rangle) = \langle \langle N_{i,n}^* : n \leq n_* \rangle : i < \mathbf{j}(\kappa) \rangle$ and $\mathbf{j}(\langle h_i : i < \kappa \rangle) = \langle h_i^* : i < \kappa_* \rangle$.

So

- (a) $\mathbf{j} \upharpoonright M_\kappa$ is a $\leq_{\mathfrak{K}}$ -embedding of M_κ into $M_{\kappa_*}^*$ hence even into $M_{\kappa_*}^*$
- (b) $M_{\kappa_*}^* \leq_{\mathfrak{K}} N_{i,n}^*$ and $\mathbf{j}(a) \in N_{i,n}^*$ for $i < \kappa, n \leq n_*$
- (c) $N_{i,0}^* = \mathbf{j}(N_1)$
- (d) h_{κ_*} is an isomorphism from N_{κ_*, n_*} onto $\mathbf{j}(N_2)$
- (e) $N_{\kappa_*, 2m+1}^* \leq_{\mathfrak{K}} N_{\kappa_*, 2m}^*, N_{\kappa_*, 2m+2}^*$ for $2m+1 < n_*$

(f) $\mathbf{j}(a) \in N_{\kappa_*, m}$.

Together we are done.

2) By the representation theorem of a.e.c. [Shea, §1]. □_{1.7}

Proof. Proof of 1.3

Let $\sigma = \theta^{\aleph_0}$. Let $G = ([\sigma]^{<\aleph_0}, \Delta)$ and let $\langle c_i : i < \sigma \rangle$ list the members of G , let $\langle \eta_\alpha : \alpha < \sigma \rangle$ list ${}^\omega\theta$.

Now

□₁ let $B_{\varepsilon, n} \subseteq G$ for $\varepsilon < \theta$ be such that: if $a, b \in G$ then $(\forall \varepsilon < \theta)(\forall n < \omega)(a \in B_{\varepsilon, n} \equiv b \in B_{\varepsilon, n}) \Rightarrow a = b$; moreover, $B_{\varepsilon, n} = \{c_\alpha : \eta_\alpha(n) = \varepsilon\}$.

Let τ have the predicates G, I, J, H (unary), E_1, Q (binary), $R_{n, \alpha}$ (n -place; $n < \omega, \alpha < \theta$), $P_{\varepsilon, n}$ (unary; $\varepsilon < \theta$) and function symbols F_1 (unary), $F_2, \pi, +$ (binary); so $|\tau| = \theta$. We define K as a class of τ -models by:

□₂ $M \in K$ iff (up to isomorphism):

- (a) $\langle G^M, I^M, J^M, H^M \rangle$ is a partition of $|M|$, (recall that they are unary)
- (b) $(G^M, +^M)$ is a subgroup of the group $([\sigma]^{<\aleph_0}, \Delta)$, $P_{\varepsilon, n}^M \subseteq G^M$ for $\varepsilon < \theta$, $\langle P_{\varepsilon, n}^M : \varepsilon < \theta \rangle$ be a partition of M such that $a \neq b \in G^M \Rightarrow (\exists \varepsilon < \theta)(\exists n < \omega)[a \in P_{\varepsilon, n}^M \wedge b \notin P_{\varepsilon, n}^M]$
- (c) $Q^M \subseteq H^M \times J^M$ is such that $(\forall a \in H^M)(\exists^{\leq 1} b)((a, b) \in Q^M)$;
- (d) E_1^M is an equivalence relation on H^M such that: if $(a_1, b) \in Q^M$ and $a_2 \in H^M$ then $a_1 E_1^M a_2 \Leftrightarrow (a_2, b) \in Q^M$
- (e) π^M is a function from H^M into I^M
- (f) $E_2^M = \{(a, b) : a E_1^M b \text{ and } \pi^M(a) = \pi^M(b) \text{ so } a, b \in H\}$
- (g) F_2^M is a partial two-place function such that:
 - (α) $F_2^M(a, b)$ is well defined iff $b \in G^M, a \in H^M$
 - (β) for $a \in H^M$, $\langle F_2^M(a, b) : b \in G^M \rangle$ list $a/E_2^M = \{a' \in H^M : \pi^M(a') = \pi^M(a)\}$ with no repetitions
 - (γ) if $a \in H^M$ and $b, c \in G$ then $F_2^M(a, b +^M c) = F_2^M(F_2^M(a, b), c)$, on the $+$ see clause (b)
 - (δ) $F(a, 0_{G^M}) = a$ for $a \in H^M$
 - (ε) for $n < \omega$ and $\gamma < \theta$ the relation $R_{n, \gamma}^M$ is an n -place relation $\subseteq \cup \{{}^n(a/E_1^M) : a \in H^M\}$.

We define $\leq_{\mathfrak{k}}$ as being a submodel. Easily

□₃ $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is an a.e.c.

For $A \subseteq M \in K$ let

- (a) $cl_M^1(A) =$ the subgroup of $(G^M, +^M)$ generated by $(A \cap G^M) \cup \{b \in G^M : \text{for some } a_1 \neq a_2 \in A \cap H^M \text{ we have } a_1 E_2^M a_2 \text{ and } F_2^M(a_1, b) = a_2\}$
- (b) $cl_M^2(A) = (A \cap I^M) \cup \{\pi^M(a) : a \in A \cap H^M\}$
- (c) $cl_M^3(A) = \{a \in H^M : \text{for some } b \in cl_M^0(A) \text{ and } a'_1 \in A \cap H^M \text{ we have } a = F_2^M(a'_1, b)\}$
- (d) $cl(A, M) = cl_M(A) = M \upharpoonright (\cup \{cl_M^\ell(A) : \ell = 1, 2, 3\})$.

Now this function $c\ell(A, M)$ shows that \mathfrak{k} admits intersections (see Definition 1.4) so

\boxtimes_4 \mathfrak{k} admits closure and $\text{LST}(\mathfrak{k}) + |\tau_{\mathfrak{k}}| = \theta$.

Assume κ is as in clause (d) of 1.3, we use the $M_{\ell, \alpha}$ ($\ell = 1, 2, \alpha \leq \kappa$) constructed in 1.2 (the relevant properties are stated in 1.2). They are not in the right vocabulary so let $M'_{\ell, \alpha}$ be the following τ -model:

- \boxtimes_5 (a) elements $G^{M'_{\ell, \alpha}} = G$
 $I^{M'_{\ell, \alpha}} = I_{\alpha}$
 $J^{M'_{\ell, \alpha}} = \{t_{\ell}^*\}, t_{\ell}^*$ just a new element
 $H^{M'_{\ell, \alpha}} = |M_{\ell, \alpha}|$
 (we assume disjointness)
- (b) $(G^{M'_{\ell, \alpha}}, +^{M'_{\ell, \alpha}})$ is $G = ([\sigma]^{< \aleph_0}, \Delta)$
 $P_{\varepsilon}^{M'_{\ell, \alpha}} \subseteq G^M$ as required in \boxtimes_1 not depending on (ℓ, α)
- (c) $F_1^{M'_{\ell, \alpha}}$ is constantly t_{ℓ}^* on $H^{M'_{\ell, \alpha}}$
- (d) $E_1^{M'_{\ell, \alpha}} = \{(a, b) : F_1^{M'_{\ell, \alpha}}(a) = F_1^{M'_{\ell, \alpha}}(b) \text{ so } a, b \in H^{M'_{\ell, \alpha}}\}$
- (e) $\pi^{M'_{\ell, \alpha}}$ is $\pi_{\ell, \alpha}$ (constructed in 1.2)
- (f) $E_2^{M'_{\ell, \alpha}} = \{(a, b) : aE_1^{M'_{\ell, \alpha}}b \text{ and } \pi^{M'_{\ell, \alpha}}(a) = \pi^{M'_{\ell, \alpha}}(b) \text{ so } a, b \in H^{M'_{\ell, \alpha}}\}$
- (g) $F_2^{M'_{\ell, \alpha}}(a, b) = F_b^{M_{\ell, \alpha}}(b)$ for $a \in H^{M'_{\ell, \alpha}}$
- (h) $R_{\gamma, n}^{M'_{\ell, \alpha}}$ for $n < \omega, \gamma < \sigma$ list the relations of $M_{\ell, \alpha}$.

Let $M'_{0, \alpha} = M'_{\ell, \alpha} \upharpoonright (G^{M'_{\ell, \alpha}} \cup I^{M'_{\ell, \alpha}})$ for $\ell = 1, 2$ and $\alpha \leq \kappa$ (we get the same result).

Note easily

\boxtimes_6 $M_{0, \alpha} \leq_{\mathfrak{k}} M_{\ell, \alpha}, \langle M_{\ell, \alpha} : \alpha \leq \kappa \rangle$ is $\leq_{\mathfrak{k}}$ -increasing (check)

\boxtimes_7 $\text{tp}_{\mathfrak{k}}(t_1^*, M'_{0, \alpha}, M'_{1, \alpha}) = \text{tp}_{\mathfrak{k}}(t_2^*, M'_{0, \alpha}, M'_{2, \alpha})$ for $\alpha < \kappa$.

[Why? By the isomorphism from $M_{1, \alpha}$ onto $M_{2, \alpha}$ respecting $(\pi_{1, \alpha}, \pi_{2, \alpha})$ in 1.1.]

\boxtimes_8 $\text{tp}_{\mathfrak{k}}(t_1^*, M'_{0, \kappa}, M'_{1, \kappa}) \neq \text{tp}_{\mathfrak{k}}(t_2^*, M'_{0, \kappa}, M'_{2, \kappa})$.

[Why? By the non-isomorphism in 1.1; extension will not help.]

Now by the “translation theorem” of [BS08, 4.7] we can find \mathfrak{k}' which has all the needed properties, i.e. also the amalgamation and JEP. $\square_{1.3}$

2. COMPACTNESS OF TYPES IN A.E.C.

Baldwin [Bal09] ask “can we in ZFC prove that some a.e.c. has amalgamation, JEP but fail compactness of types”. The background is that in [BS08] we construct one using diamonds.

To me the question is to show this class can be very large (in ZFC).

Here we omit amalgamation and accomplish both by direct translations of problems of existence of models for theories in $\mathbb{L}_{\kappa^+, \kappa^+}$, first in the propositional logic. So whereas in [BS08] we have an original group G^M , here instead we have a set P^M of propositional “variables” and P^M , set of such sentences (and relations and functions explicating this; so really we use coding but are a little sloppy in stating this obvious translation).

In [BS08] we have I^M , set of indexes, 0 and H , set of Whitehead cases, H_t for $t \in I^M$, here we have I^M , each $t \in I^N$ representing a theory $P_t^M \subseteq P^M$ and in J^M we give each $t \in I^M$ some models $\mathcal{M}_s^M : P^M \rightarrow \{\text{true}, \text{false}\}$. This is set up so that amalgamation holds.

Notation 2.1. In this section types are denoted by \mathbf{p}, \mathbf{q} as p, q are used for propositional variables.

Definition 2.2. 1) We say that an a.e.c. \mathfrak{k} has $(\leq \lambda, \kappa)$ -sequence-compactness (for types) when: if $\langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{k}}$ -increasing continuous and $i < \kappa \Rightarrow \|M_i\| \leq \lambda$ and $\mathbf{p}_i \in \mathcal{S}^{<\omega}(M_i)$ for $i < \kappa$ satisfying $i < j < \kappa \Rightarrow \mathbf{p}_i = \mathbf{p}_j \upharpoonright M_i$ then there is $\mathbf{p}_\kappa \in \mathcal{S}^{<\omega}(M_\kappa)$ such that $i < \kappa \Rightarrow p_\kappa \upharpoonright M_i = p_i$.

2) We define “ $(= \lambda, \kappa)$ -sequence-compactness” similarly. Let (λ, κ) -sequence-compactness mean $(\leq \lambda, \kappa)$ -compactness.

Question 2.3. Can we find an a.e.c. \mathfrak{k} with amalgamation and JEP such that $\{\theta : \mathfrak{k} \text{ have } (\lambda, \theta)\text{-compactness of types for every } \lambda\}$ is complicated, say:

- (a) not an end segment but with “large” members
- (b) any $\{\theta : \theta \text{ satisfies } \psi\}, \psi \in \mathbb{L}_{\kappa^+, \kappa^+}$ (second order).

Definition 2.4. Let $\kappa \geq \aleph_0$, we define $\mathfrak{k} = \mathfrak{k}_\kappa$ as follows:

- (A) the vocabulary $\tau_{\mathfrak{k}}$ consist of $F_i (i \leq \kappa), R_\ell (\ell = 1, 2), P, \Gamma, I, J, c_i (i < \kappa), F_i (i \leq \kappa)$, (pedantically see later),
- (B) the universe of $M \in K_{\mathfrak{k}}$ is the disjoint union of P^M, Γ^M, I^M, J^M so P, Γ, I, J are unary predicates
- (C) (a) P^M a set of propositional variables (i.e. this is how we treat them)
- (b) Γ^M is a set of sentences of one of the forms $\varphi = (p), \varphi = (r \equiv p \wedge q), \varphi = (q \equiv \neg p), \varphi = (q \equiv \bigwedge_{i < \kappa} p_i)$, so $p, q, p_i \in P^M$
but in the last case $\{p_i : i < \kappa\} \subseteq \{c_i^M : i < \kappa\}$ (or code this!)
- (c) for $i < \kappa$ the function $F_i^M : \Gamma^M \rightarrow P^M$ are such that for every $i < \kappa$ and $\varphi \in \Gamma^M$ we have:
 - (α) if $\varphi = (p)$ and $i \leq \kappa$ then $F_{1+i}(\varphi) = p, F_0(\varphi) = c_0$
 - (β) if $\varphi = (r \equiv p \wedge q)$ then $F_i(\varphi)$ is c_1 if $i = 0$, is p if $i = 1$, is q if $r = 2$ is r if $r \geq 3$
 - (γ) if $\varphi = (q \equiv \neg p)$ then $F_i(\varphi)$ is c_2 if $i = 0$, p if $i = 1$, q if $i \geq 2$

- (δ) if $\varphi = (q \equiv \bigwedge_{j < \kappa} p_j)$ then $F_i(\varphi)$ is c_3 if $i = 0$,
 q if $i = 1, p_{2+j}$ if $i = j + 1$
- (d) I a set of theories, i.e. $R_1^M \subseteq \Gamma \times I$ and for $t \in I$ let
 $\Gamma_t^M = \{\psi \in \Gamma^M : \psi R_1^M t\} \subseteq \Gamma^M$
- (e) J is a set of models, i.e. $R_2^M \subseteq (\Gamma \cup P) \times J$ and for $s \in J$ we have
 \mathcal{M}_s^M is the model, i.e. function giving truth values to $p \in P^M$, i.e.
- (α) $\mathcal{M}_s^M(p)$ is true if $p_i R_2^M s$; is false if $\neg p R_2^M s$
- (β) $(\varphi, s) \in R_2^M$ iff computing the truth value of φ in \mathcal{M}_s^M
we get truth
- (f) $F_\kappa^M : J^M \rightarrow I^M$ such that $s \in J^M \Rightarrow \mathcal{M}_s^M$ is a model of $\Gamma_{F_\kappa^M(s)}$
- (g) $(\forall t \in I^M)(\exists s \in J^M)(F_\kappa^M(s) = t)$
- (D) $M \leq_{\mathfrak{k}} N$ iff $M \subseteq N$ are $\tau_{\mathfrak{k}}$ -models from $K_{\mathfrak{k}}$.

Claim 2.5. \mathfrak{k} is an a.e.c., $LST(\mathfrak{k}) = \kappa$.

Proof. Obvious. □_{2.5}

Claim 2.6. \mathfrak{k} has the JEP.

Proof. Just like disjoint unions (also of the relations and functions) except for the individual constants c_i (for $i < \kappa$). □_{2.6}

Claim 2.7. Assume $M_0 \leq_{\mathfrak{k}} M_\ell$ for $\ell = 0, 1$ and $|M_0| = P^{M_0} \cup \Gamma^{M_0} = P^{M_\ell} \cup \Gamma^{M_\ell}$ for $\ell = 1, 2$ and $a_\ell \in I^{M_\ell}$ for $\ell = 1, 2$. Then $\mathfrak{tp}_{\mathfrak{k}}(a_1, M_0, M_1) = \mathfrak{tp}_{\mathfrak{k}}(a_2, M_0, M_2)$ iff $\Gamma_{a_1}^{M_1} = \Gamma_{a_2}^{M_2}$.

Proof. The if direction, \Leftarrow

Let h be a one to one mapping with domain M_1 such that $h \upharpoonright M_0 =$ the identity, $h(a_1) = a_2$ and $h(M_1) \cap M_2 = M_0 \cup \{a_2\}$. Renaming without loss of generality h is the identity. Now define M_3 as $M_1 \cup M_2$, as in 2.6, now $a_1 = a_2$ does not cause trouble because $P^{M_0} = P^{M_\ell}, \Gamma^{M_0} = \Gamma^{M_\ell}$ for $\ell = 1, 2$.

The only if direction, \Rightarrow

Obvious. □_{2.7}

Claim 2.8. Assume λ, θ are such that:

- (a) θ is regular $\leq \lambda$ and $\lambda \geq \kappa$
- (b) $\langle \Gamma_i : i \leq \theta \rangle$ is \subseteq -increasing continuous sequence of sets propositional sentences in $\mathbb{L}_{\kappa^+, \omega}$ such that $[\Gamma_i$ has a model $\Leftrightarrow i < \theta]$
- (c) $|\Gamma_\theta| \leq \lambda$.

Then \mathfrak{k} fail (λ, θ) -sequence-compactness (for types).

Remark 2.9. We may wonder but: for $\theta = \aleph_0$ compactness holds? Yes, but only assuming amalgamation.

Proof. Without loss of generality $|\Gamma_0| = \lambda$. Without loss of generality $\langle p_\varepsilon^* : \varepsilon < \kappa \rangle$ are pairwise distinct propositional variables appearing in Γ_0 (but not necessarily $\in \Gamma_0$) and each $\psi \in \Gamma_i$ is of the form (p) or $r \equiv p \wedge q$ or $r \equiv \neg p$ or $r \equiv \bigwedge_{i < \kappa} p_i$ where $\{p_i : i < \kappa\} \subseteq \{p_\varepsilon^* : \varepsilon < \kappa\}$.

Let P_i be the set of propositional variables appearing in Γ_i without loss of generality $|P_i| = \lambda$.

We choose a model M_i for $i \leq \theta$ such that:

- (a) $|M_i| = P_i \cup \Gamma_i$
- (b) $P^M = P_i$ and $\Gamma^{M_i} = \Gamma_i$
- (c) the natural relations and functions.

Let $\mathcal{M}_i : P_i \rightarrow \{\text{true false}\}$ be a model of Γ_i .

We define a model $N_i \in K_{\mathfrak{k}}$ for $i < \kappa$ (but not for $i = \theta!$)

- (a) $M_i \leq_{\mathfrak{k}} N_i$
- (b) $P^{N_i} = P^{M_i}$
- (c) $\Gamma^{N_i} = \Gamma^{M_i}$
- (d) $I^M = \{t_i\}$
- (e) $J^M = \{s_i\}$
- (f) $F_\kappa^{N_i}(s_i) = t_i$
- (g) $R_1^{N_i} = \Gamma_i \times \{t_i\}$
- (h) $R_2^{N_i}$ is chosen such that $\mathcal{M}_{s_i}^{N_i}$ is \mathcal{M}_i
- (i) $F_i^{N_i}$ ($i < \kappa$) are defined naturally.

Now

$$(*)_1 \quad \mathbf{p}_i = \mathbf{tp}_{\mathfrak{k}}(t_i, M_i, N_i) \in \mathbf{S}^1(M_i).$$

[Why? Trivial.]

$$(*)_2 \quad i < j < \theta \rightarrow \mathbf{p}_i = \mathbf{p}_j \upharpoonright M_j.$$

[Why? Let $N_{i,j} = N_j \upharpoonright (M_j \cup \{s_j, t_j\})$.]

Easily $\mathbf{tp}(t_j, M_i, N_{i,j}) \leq p_j$ and $\mathbf{tp}(t_j, M_i, N_{i,j}) = p_i$ by the claim 2.7 above.]

$$(*)_3 \quad \text{there is no } p \in \mathbf{S}^1(M_\theta) \text{ such that } i < \theta \Rightarrow p \upharpoonright M_i = p_i.$$

Why? We prove more:

- (*)₄ there is no (N, t) such that
 - (a) $M_\kappa \leq_{\mathfrak{k}} N$
 - (b) $t \in I^N$
 - (c) $(\forall \varphi \in \Gamma^{M_\kappa})[\varphi R_1^N t]$.

[Why? As then $\Gamma_\theta = \Gamma^M$ has a model contradiction to an assumption.] □_{2.8}

So e.g.

Conclusion 2.10. If $\theta > \kappa$ is regular with no κ^+ -complete uniform ultrafilter on θ and $\lambda = 2^\theta$, then \mathfrak{k} is not (λ, θ) -sequence-compact.

Remark 2.11. Recall if D is an ultrafilter on θ then $\min\{\sigma' : D \text{ is not } \sigma'\text{-complete}\}$ is \aleph_0 or a measurable cardinality.

Proof. (Well known).

Let M be the model with universe 2^θ , $P_0^M = \theta$ and $R^M \subseteq \theta \times \lambda$ be such that $\{\{\alpha < \lambda : \alpha R^M \beta\} : \beta < \lambda\} = \mathcal{P}(\theta)$, $<^M$ the well ordering of the ordinal on λ the vocabulary has cardinality κ and has elimination of quantifiers and Skolem functions.

Let $\Gamma_i = \text{Th}(M, \beta)_{\beta < \lambda} \cup \{\alpha < c : \alpha < \theta\}$ (c a new individual constant), then $\langle \Gamma_i : i \leq \theta \rangle$ is as¹ required in 2.12 below hence 2.8 apply. $\square_{2.10}$

Conclusion 2.12. In Claim 2.8 if $\lambda = \lambda^\kappa$ then we can allow $\langle \Gamma_i : i \leq \theta \rangle$ to be a sequence of theories in $\mathbb{L}_{\kappa^+, \kappa^+}(\tau)$, τ any vocabulary of cardinality $\leq \lambda$.

Proof. Without loss of generality we can add Skolem functions (each with $\leq \kappa$ places) in particular. So Γ_i becomes universal and adding propositional variables for each quantifier free sentence and writing down the obvious sentences, we get a set of propositional sentences, we get Γ_i as there. $\square_{2.12}$

I think we forgot

Observation 2.13. *If $\lambda \geq \kappa \geq \theta = \text{cf}(\theta)$ then the condition in 2.8 holds.*

Proof. Just let $\Gamma_0 = \{\bigvee_{i < \theta} \neg p_i\}$, $\Gamma_i = \Gamma_0 \cup \{p_j : j < i\}$. \square

Conclusion 2.14. 1) $\mathbf{C}_\kappa = \{\theta : \theta = \text{cf}(\theta) \text{ and for every } \lambda \text{ and a.e.c. } \mathfrak{k} \text{ with } \text{LST}(\mathfrak{k}) \leq \kappa, |\tau_\mathfrak{k}| = \kappa \text{ have } (\lambda, \theta)\text{-compactness of type}\}$ is the class $\{\theta : \theta = \text{cf}(\theta) > \kappa \text{ and there is a uniform } \kappa^+\text{-complete ultrafilter on } \theta\}$.

2) In \mathbf{C}_κ we can replace “every λ ” by $\lambda = 2^\theta + \kappa$.

Proof. Put together 2.10, 2.16. $\square_{2.13}$

Of course, a complimentary result (showing the main claim is best possible) is:

Claim 2.15. *If \mathfrak{k}' is an a.c.c., $\text{LST}(\mathfrak{k}') \leq \kappa$ and on θ there is a uniform κ^+ -complete ultrafilter on θ and θ is regular and λ any cardinality then \mathfrak{k}' has (λ, κ) -compactness of types.*

Proof. Write down a set of sentences on $\mathbb{L}_{\kappa^+, \kappa^+}(\tau_\mathfrak{k}^+)$ expressing the demands.

Let $\langle M_i : i \leq \theta \rangle$ be $<_\mathfrak{k}$ -increasing continuous, $\|M_i\| \leq \lambda$, $p_i = \text{tp}_\mathfrak{k}(a_i, M_i, N_i)$ so $M_i \leq_\mathfrak{k} N_i$ such that $i < j < \theta \Rightarrow p_i = p_j \upharpoonright M_i$. Without loss of generality $\|N_i\| \leq \lambda$.

Let $\langle N_{i,j,\ell} : \ell \leq n_{i,j,\ell} \rangle$, $\pi_{i,1}$ witness $p_i = p_j \upharpoonright M_i$ for $i < j < \theta$ (i.e. $M_i \leq_\mathfrak{k} N_{i,j,\ell}$ (without loss of generality $\|N_{i,j,\ell}\| \leq \lambda$), $N_{i,j,0} = N_i$, $a_i \in N_{i,j,\ell}$, $\bigwedge_{\ell < n_{i,j,\ell}} (N_{i,j,\ell} \leq_\mathfrak{k}$

$N_{i,j,\ell+1} \vee N_{i,j,\ell+1} \leq_\mathfrak{k} N_{i,j,\ell}$ and $\pi_{i,j}$ be an isomorphism from N_j onto $N_{i,j,n_{i,j}}$ over M_i mapping a_j to a_i).

Let $\tau^+ = \tau \cup \{F_{\varepsilon,n} : \varepsilon < \kappa, n < \omega\}$, $\text{arity}(F_{\varepsilon,n}) = n$. Let $\langle M_i^+ : i \leq \theta \rangle$ be \subseteq -increasing, M_i^+ a τ^+ -expansion of M_i such that $u \subseteq M_i^+ \Rightarrow M_i \upharpoonright \text{cl}_{M_i^+}(u) \leq_\mathfrak{k} M_i$. Similarly $\langle N_{i,j,\ell}^{+,\varepsilon} : \ell \leq n_{i,j,\ell} \rangle$; $\varepsilon = 1, \ell$ such that $N_{i,j,\ell}^{+,\varepsilon}$ is a τ^+ -expansion of $N_{i,j,\varepsilon}$ as above such that $(\forall \ell < n_{i,j,\ell})(\exists \varepsilon \in \{1, 2\})(N_{i,j,\ell}^{+,\varepsilon} \subset N_{i,j,\ell+1}^{+,\varepsilon} \vee N_{i,j,\ell+1}^{+,\varepsilon} \subseteq N_{i,j,\ell}^{+,\varepsilon})$.

Now write down a translation of the question, “is there p such that ...” $\square_{2.15}$

¹or directly as Γ_i has Skolem functions

Claim 2.16. *If D is a uniform κ -complete ultrafilter on θ , $\langle M_i : i \leq \theta \rangle$ is $\leq_{\mathfrak{t}}$ -increasing continuous, $p_i \in \mathcal{S}_{\mathfrak{t}}^{\alpha}(M_i)$ as witnessed by (N_i, a_i) for $i < \kappa$, $p_i = p_j \upharpoonright M_i$ for $i < j < \kappa$ as witnessed by $(\pi_i, \langle N_{i,j,\ell} : \ell \leq m_{i,j} \rangle)$ as in the proof above.*

1) *There is $p_{\kappa} \in \mathbf{S}^{\alpha}(M_{\theta})$ such that $i < \theta \Rightarrow p_{\kappa} \upharpoonright M_i$.*

2) *In fact for each $i < \kappa$ let $\mathcal{U}_i \in D$ be such that $i < j \in \mathcal{U}_i \Rightarrow n_{i,j} = n_i^*$. Let $N_{i,\kappa,\ell} = \prod_{j \in \mathcal{U}_i} N_{i,j,\ell}/D$. So $\langle N_{i,\kappa,\ell} : \ell \leq n_{\ell}^* \rangle$ are as above. Let $M = \prod_{i < \kappa} M_i/D$, $\pi_{i,\kappa} =$*

$\prod_{j \in \mathcal{U}_i} \pi_{i,j}/D$, etc.

3. ON SOME STABILITY SPECTRUMS OF AN A.E.C.

Convention 3.1. \mathfrak{k} is an a.e.c. with amalgamation.

Definition 3.2. For $\theta \geq \text{LST}(\mathfrak{k})$. We say \mathfrak{k} is (λ, θ) -stable when $M \in K_\lambda^\mathfrak{k} \Rightarrow |\mathbf{S}(M)/E_M^\theta| \leq \lambda$ where

$$pE_M^\theta q \Leftrightarrow (\forall N)(N \leq_\mathfrak{k} M \wedge \|N\| \leq \theta \Rightarrow p \upharpoonright N = q \upharpoonright \theta).$$

Theorem 3.3. Fixing θ the class $\{\lambda : \mathfrak{k} \text{ is } (\lambda, \theta)\text{-stable}\}$; behave as in [She71].

Remark 3.4. See [Sheb] = [She09, V, §7] or [She00] if not covered.

Definition 3.5. $\kappa_\theta(\mathfrak{k}) := \text{Min}\{\kappa \leq \theta^+ : \text{there is no sequence } \langle M_i : i \leq \kappa \rangle \text{ which is } \leq_\mathfrak{k}\text{-increasing continuous, } \|M_i\| \leq \theta \text{ and } p \in \mathcal{S}(M_\kappa) \text{ such that } p \upharpoonright M_{i+1} \text{ strongly } (\theta)\text{-split over } M_i\}$.

Claim 3.6. 1) If $\lambda > 2^\theta$ and \mathfrak{k} is not (λ, θ) -stable then for some $\kappa \leq \theta^+$ satisfying $\lambda^\kappa > \lambda$ we have $\kappa < \kappa_\theta(\mathfrak{k})$.

2) If $\lambda > \theta, \lambda^\kappa > \lambda$ then $\kappa < \kappa_\theta(\mathfrak{k})$ then \mathfrak{k} not (λ, θ) -stable.

Conclusion 3.7. $(-, \theta)$ -stability spectrum - behave as in [She71].

Discussion 3.8. We can look at $\lambda \in [\theta, 2^\theta)$ using splitting rather than strongly splitting.

It seems to me the main question is

Question 3.9. Assume $(\exists \theta \geq \text{LS}(\mathfrak{k}))(\kappa_\theta(\mathfrak{k}) > \aleph_0)$.

What can you say on $\text{Min}\{\theta : \kappa_\theta(\mathfrak{h}) > \aleph_0, \theta \geq \text{LST}(\mathfrak{k})\}$?

Question 3.10. Assume GCH can we find an a.e.c. \mathfrak{k} such that: $(\forall \theta \geq \text{LST}(\mathfrak{k}))(\kappa_\theta(\mathfrak{k}) = \aleph_0)$ but unstable in every regular $\lambda > \text{LST}(\mathfrak{k})$?

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