# NON FORKING GOOD FRAMES WITHOUT LOCAL CHARACTER

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Abstract. We continue [Sh:h].II, studying stability theory for abstract elementary classes. In [Sh E46], Shelah obtained a non-forking relation for an AEC, $(K, \preceq)$, with $LST$-number at most $\lambda$, which is categorical in $\lambda$ and $\lambda^+$ and has less than $2^{\lambda^+}$ models of cardinality $\lambda^{++}$, but at least one. This non-forking relation satisfies the main properties of the non-forking relation on stable first order theories.

Here we improve this non-forking relation such that it satisfies the local character, too. Therefore it satisfies the main properties of the non-forking relation on superstable first order theories.

Using results of [Sh:h].II, we conclude that the function $\lambda \to I(\lambda, K)$, which assigns to each cardinal $\lambda$, the number of models in K of cardinality $\lambda$, is not arbitrary.

## Contents



## 1. Preliminaries

Familiarity with AEC's is assumed.

*Hypothesis* 1.1.

(1) $(K, \preceq)$ is an AEC.
(2) $\lambda$ is a cardinal.
(3) The Lowenheim Skolem Tarski number of $(K, \preceq)$, $LST(K, \preceq)$, is at most $\lambda$.

**Definition 1.2.** Suppose $M_0 \prec N$ in $K_\lambda$. We say that $N$ is *universal* over $M_0$ if for every $M_1 \succ M_0$, there is an embedding of $M_1$ into $N$ over $M_0$, namely, that fixes $M_0$.

The following proposition is a version of Fodor's Lemma (there is no mathematical reason to choose this version, but we think that it is convenient).

**Proposition 1.3.** *There exist no $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$, $\langle f_\alpha : \alpha \in \lambda^+ \rangle, S$ such that the following conditions are satisfied:*

1. *The sequences $\langle M_\alpha : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha : \alpha \in \lambda^+ \rangle$ are $\preceq$-increasing continuous sequences of models in $K_\lambda$.*
2. *For every $\alpha < \lambda^+$, $f_\alpha : M_\alpha \to N_\alpha$ is a $\preceq$-embedding.*
3. *$\langle f_\alpha : \alpha \in \lambda^+ \rangle$ is an increasing continuous sequence.*
4. *$S$ is a stationary subset of $\lambda^+$.*
5. *For every $\alpha \in S$, there is $a \in M_{\alpha+1} - M_\alpha$ such that $f_{\alpha+1}(a) \in N_\alpha$.*

*Proof.* Suppose there are such sequences. Denote $M = \bigcup \{ f_\alpha[M_\alpha] : \alpha \in \lambda^+ \}$. By clauses (4),(5), $||M|| = K_{\lambda^+}$. $\langle f_\alpha[M_\alpha] : \alpha \in \lambda^+ \rangle$, $\langle N_\alpha \bigcap M : \alpha \in \lambda^+ \rangle$ are filtrations of $M$. So they are equal on a club of $\lambda^+$. Hence there is $\alpha \in S$ such that $f_\alpha[M_\alpha] = N_\alpha \bigcap M$. Hence $f_\alpha[M_\alpha] \subseteq N_\alpha \bigcap f_{\alpha+1}[M_{\alpha+1}] \subseteq N_\alpha \bigcap M = f_\alpha[M_\alpha]$ and so this is a chain of equivalences. Especially $f_{\alpha+1}[M_{\alpha+1}] \bigcap N_\alpha = f_\alpha[M_\alpha]$, in contradiction to condition (5). $\dashv$

## 2. Non-forking frames

The following definition, Definition 2.1 is an axiomatization of the non-forking relation in a superstable first order theory. If we subtract axiom **??**(3)(c), we get the basic properties of the non-forking relation in $(K_\lambda, \preceq \restriction K_\lambda)$ where $(K, \preceq)$ is stable in $\lambda$.

Sometimes we do not find a natural independence relation with respect to all the types. So first we extend the notion of an AEC in $\lambda$ by adding a new function $S^{bs}$ which assigns a collection of basic (because they are basic for our construction) types to each model in $K_\lambda$, and then add an independence relation $\bigcup$ on basic types.

We do not assume *the amalgamation property* in general, but we assume the amalgamation property in $(K_\lambda, \preceq \restriction K_\lambda)$. This is a reasonable assumption, because it is proved in [Sh:h].I, that if an AEC is categorical in $\lambda$ and the amalgamation property fails in $\lambda$, then under a plausible set theoretic assumption, there are $2^{\lambda^+}$ models in $K_{\lambda^+}$.

**Definition 2.1.** $\mathfrak{s} = (K, \preceq, S^{bs}, \bigcup)$ is a *good $\lambda$-frame* if:

1. $(K, \preceq)$ is an AEC in $\lambda$.
2. (a) $(K, \preceq)$ satisfies the joint embedding property.
   (b) $(K, \preceq)$ satisfies the amalgamation property.
   (c) There is no $\preceq$-maximal model in $K$.
3. $S^{bs}$ is a function with domain $K$, which satisfies the following axioms:
   (a) $S^{bs}(M) \subseteq S^{na}(M) = \{ga - tp(a, M, N) : M \prec N \in K,\ a \in N - M\}$.
   (b) $S^{bs}$ respects isomorphisms: if $ga - tp(a, M, N) \in S^{bs}(M)$ and $f : N \to N'$ is an isomorphism, then $ga - tp(f(a), f[M], N') \in S^{bs}(f[M])$.

(c) Density of the basic types: if $M, N \in K_\lambda$ and $M \prec N$, then there is $a \in N - M$ such that $ga - tp(a, M, N) \in S^{bs}(M)$.
(d) Basic stability: for every $M \in K$, the cardinality of $S^{bs}(M)$ is $\leq \lambda$.

(4) the relation $\bigcup$ satisfies the following axioms:
(a) $\bigcup$ is set of quadruples $(M_0, M_1, a, M_3)$ where $M_0, M_1, M_3 \in K$, $a \in M_3 - M_1$ and for $n = 0, 1$ $ga - tp(a, M_n, M_3) \in S^{bs}(M_n)$ and it respects isomorphisms: if $\bigcup(M_0, M_1, a, M_3)$ and $f : M_3 \to M_3'$ is an isomorphism, then $\bigcup(f[M_0], f[M_1], f(a), M_3')$.
(b) Monotonicity: if $M_0 \preceq M_0^* \preceq M_1^* \preceq M_1 \preceq M_3 \preceq M_3^*$, $M_1^* \bigcup \{a\} \subseteq M_3^{**} \preceq M_3^*$, then $\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M_0^*, M_1^*, a, M_3^{**})$. From now on, '$p \in S^{bs}(N)$ does not fork over $M$' will be interpreted as 'for some $a, N^+$ we have $p = ga{-}tp(a, N, N^+)$ and $\bigcup(M, N, a, N^+)$'. See Proposition 2.2.
(c) Local character: for every limit ordinal $\delta < \lambda^+$ if $\langle M_\alpha : \alpha \leq \delta \rangle$ is an increasing continuous sequence of models in $K_\lambda$, and $ga - tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$, then there is $\alpha < \delta$ such that $ga - tp(a, M_\delta, M_{\delta+1})$ does not fork over $M_\alpha$.
(d) Uniqueness of the non-forking extension: if $M, N \in K$, $M \preceq N$, $p, q \in S^{bs}(N)$ do not fork over $M$, and $p \restriction M = q \restriction M$, then $p = q$.
(e) Symmetry: if $M_0, M_1, M_3 \in K_\lambda$, $M_0 \preceq M_1 \preceq M_3$, $a_1 \in M_1$, $ga - tp(a_1, M_0, M_3) \in S^{bs}(M_0)$, and $ga - tp(a_2, M_1, M_3)$ does not fork over $M_0$, *then* there are $M_2, M_3^* \in K_\lambda$ such that $a_2 \in M_2$, $M_0 \preceq M_2 \preceq M_3^*$, $M_3 \preceq M_3^*$, and $ga - tp(a_1, M_2, M_3^*)$ does not fork over $M_0$.
(f) Existence of non-forking extension: if $M, N \in K$, $p \in S^{bs}(M)$ and $M \prec N$, then there is a type $q \in S^{bs}(N)$ such that $q$ does not fork over $M$ and $q \restriction M = p$.
(g) Continuity: let $\delta < \lambda^+$ and $\langle M_\alpha : \alpha \leq \delta \rangle$ be an increasing continuous sequence of models in $K$ and let $p \in S(M_\delta)$. If for every $\alpha \in \delta$, $p \restriction M_\alpha$ does not fork over $M_0$, then $p \in S^{bs}(M_\delta)$ and does not fork over $M_0$.

**Proposition 2.2.** *If $\bigcup(M_0, M_1, a, M_3)$ and the types $ga-\ tp(b, M_1, M_3^*)$, $ga-\ tp(a, M_1, M_3)$ are equal, then we have $\bigcup(M_0, M_1, a, M_3)$.*

*Proof.* Since $ga-\ tp(b, M_1, M_3^*) = ga-\ tp(a, M_1, M_3)$, there is an amalgamation $(id_{M_3}, f, M_3^{**})$ of $M_3$ and $M_3^*$ over $M_1$ with $f(b) = a$. By Definition **??**(3)(b) (monotonicity) $\bigcup(M_0, M_1, a, M_3^{**})$. Using again Definition 2.1(3)(b), we get $\bigcup(M_0, M_1, a, f[M_3^*])$. Therefore by Definition 2.1(3)(a), $\bigcup(M_0, M_1, a, M_3^*)$. $\dashv$

**Definition 2.3.**

(1) $\mathfrak{s} = (K, \preceq, S^{bs}, nf)$ is an *almost* good $\lambda$-frame if $\mathfrak{s}$ satisfies the axioms of a good $\lambda$-frame except maybe local character, but $\mathfrak{s}$ satisfies weak local character.
(2) $\mathfrak{s}$ satisfies *weak local character* when there is a 2-ary relation, $\prec^*$ on $K_\lambda$ which is included in $\prec\restriction K_\lambda$ such that:
  (a) for each $M_0 \in K_\lambda$ there is $M_1 \in K_\lambda$ with $M_0 \prec^* M_1$,
  (b) if $M_0 \prec^* M_1 \preceq M_2 \in K_\lambda$ then $M_0 \prec^* M_2$,
  (c) if $\langle N_\alpha : \alpha < \delta + 1\rangle$ is a $\prec^*$-increasing continuous sequence of models in $K_\lambda$, then for some $a \in N_{\delta+1}$ and some ordinal $\alpha < \delta$, $p =: ga-\ tp(a, N_\delta, N_{\delta+1})$ is a basic type, which does not fork over $N_\alpha$.

In the following definition 'na' means non-algebraic.

**Definition 2.4.** We define a function $S^{na}$ with domain $K_\lambda$ by $S^{na}(M) := \{ga-\ tp(a, M, N) : M \preceq N,\ a \in N - M\}$.

**Definition 2.5.** Let $\mathfrak{s}$ be an almost good $\lambda$-frame. $\mathfrak{s}$ is *full* if $S^{bs} = S^{na}$.

The following theorem says that the stability property in $\lambda$ is satisfied and presents sufficient conditions for a universal model. The stability in $\lambda$ can actually derived from [JrSh 875, Theorem 2.20].

**Theorem 2.6.**

*(1) Suppose:*
  *(a) $\mathfrak{s}$ is an almost good $\lambda$-frame (so indirectly, we assume basic stability).*
  *(b) $\langle M_\alpha : \alpha \leq \lambda\rangle$ is an increasing continuous sequence of models in $K_\lambda$.*
  *(c) $M_{\alpha+1}$ realizes $S^{bs}(M_\alpha)$.*
  *(d) $M_\alpha \prec^* M_{\alpha+1}$.*
  *Then $M_\lambda$ is universal over $M_0$.*
*(2) There is a model in $K_\lambda$ which is universal over $\lambda$.*
*(3) For every $M \in K_\lambda$, $|S(M)| \leq \lambda$.*

*Proof.* Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Why does (1) hold? We have to prove that letting $M_0 \prec N$, $N$ can be embedded in $M_\lambda$ over $M_0$. Toward a contradiction assume that:

(*) There is no an embedding from N into $M_\lambda$ over $M_0$.

Let $cd$ be a bijection from $\lambda \times \lambda$ onto $\lambda$. Now we choose $N_\alpha, A_\alpha, \langle a_{\alpha,\beta} : \beta < \lambda\rangle, f_\alpha$ by induction on $\alpha$ such that:
(1) $N_0 = N,\ f_0 = id_{M_0}$
(2) $\langle N_\alpha : \alpha < \lambda\rangle$ is an increasing continuous sequence of models in $K_\lambda$.
(3) $\langle f_\alpha : \alpha < \lambda\rangle$ is an increasing continuous sequence of functions.
(4) $f_\alpha : M_\alpha \hookrightarrow N_\alpha$ is an embedding.
(5) $N_\alpha = \{a_{\alpha,\beta} : \beta < \lambda\}$.
(6) $A_\alpha = \{cd(\gamma, \beta) : \gamma \preceq \alpha,\ ga-\ tp(a_{\gamma,\beta}, f_\alpha[M_\alpha], N_\alpha) \in S^{bs}(f_\alpha[M_\alpha])\}$.

(7) $a_{\gamma,\beta} \in f_{\alpha+1}[M_{\alpha+1}]$ where $(\gamma, \beta) = cd^{-1}(Min(A_\alpha))$.

*Why can we carry out the induction?* For $\alpha = 0$ or limit, there is no problem. Suppose we have chosen $N_\alpha, A_\alpha, \langle a_{\alpha,\beta} : \beta < \lambda \rangle, f_\alpha$. If $f_\alpha[M_\alpha] = N_\alpha$, then $f_\alpha^{-1} \restriction N_0$ is an embedding over $M_0$, in contradiction to (*). Thus $f_\alpha[M_\alpha] \neq N_\alpha$. Therefore there is a type in $S^{bs}(f_\alpha[M_\alpha])$ which $N_\alpha$ realizes. Hence $A_\alpha \neq \emptyset$. So by the definition of a type, there is no problem to find $N_{\alpha+1}, A_{\alpha+1}, \langle a_{\alpha+1,\beta} : \beta < \lambda \rangle, f_{\alpha+1}$.

*Why is this enough?* Define $N_\lambda := \bigcup\{N_\alpha : \alpha < \lambda\}$, $f_\lambda := \bigcup\{f_\alpha : \alpha < \lambda\}$. By smoothness, $f_\lambda[M_\lambda] \preceq N_\lambda$. But $f_\lambda[M_\lambda] \neq N_\lambda$ (otherwise $f_\lambda^{-1} \restriction N_0$ is an embedding over $M_0$, in contradiction to (*)). So by *weak local character*, there is $c \in N_\lambda - f_\lambda[M_\lambda]$ and there is a $\gamma \in \lambda$ such that $ga-\ tp(c, f_\lambda[M_\lambda], N_\lambda)$ does not fork over $f_\gamma[M_\gamma]$. Without loss of generality, $c \in N_\gamma$, because we can increase $\gamma$. Therefore there is $\beta \in \lambda$ such that $c = a_{\gamma,\beta}$. Hence $ga-\ tp(a_{\gamma,\beta}, f_\gamma[M_\gamma], N_\gamma) \in S^{bs}(f_\gamma[M_\gamma])$. Define an injection $g : [\gamma, \lambda) \to \lambda$ by $g(\alpha)$ :=Min$(A_\alpha)$. For each $\alpha \in [\gamma, \lambda)$, $cd(\gamma, \beta) \in A_\alpha$. So $g(\alpha) < cd(\gamma, \beta)$, (otherwise by (7) $a_{\gamma,\beta} \in f_{\alpha+1}[M_{\alpha+1}] \subset f_\lambda[M_\lambda]$, but $a_{\gamma,\beta} = c \notin f_\lambda[M_\lambda]$), and $g$ is an injection from $[\gamma, \lambda)$ to $cd(\gamma, \beta)$ which is impossible. Thus (*) implies a contradiction. $\dashv$

## 3. Non-forking amalgamation

*Hypothesis* 3.1. $\mathfrak{s}$ is an almost good $\lambda$-frame.

In this section we present a theorem from [JrSh 875], which says that we can derive a non-forking relation on models, from the non-forking relation on elements. First we have to define the conjugation property.

**Definition 3.2.**
(1) Let $p = ga-\ tp(a, M, N)$. Let $f$ be an isomorphism of $M$ (i.e. $f$ is an injection with domain M, and the relations and functions on $f[M]$ are defined such that $f : M \hookrightarrow f[M]$ is an isomorphism). Define $f(p) = ga-\ tp(f(a), f[M], f^+[N])$, where $f^+$ is an extension of $f$ (and the relations and functions on $f^+[N]$ are defined such that $f^+ : N \hookrightarrow f^+[N]$ is an isomorphism).

(2) Let $p_0, p_1$ be types, $n < 2 \to p_n \in S(M_n)$. We say that $p_0, p_1$ are *conjugate* if there is an isomorphism $f : M_0 \hookrightarrow M_1$ such that $f(p_0) = p_1$.

**Claim 3.3.**

*(1) In Definition 3.2, $f(p)$ does not depend on the choice of $f^+$.*
*(2) The conjugation relation is an equivalence relation.*

*Proof.* Easy. $\dashv$

**Definition 3.4.** Let $\mathfrak{s}$ be an almost good $\lambda$-frame. $\mathfrak{s}$ is said to satisfy the *conjugation property*, when: if $p \in S^{bs}(M_1)$ does not fork over $M_0$, then there is an isomorphism $f : M_1 \to M_0$ such that $f(p) = p \restriction M_0$.

**Remark 3.5.** If $\mathfrak{s}$ satisfies the conjugation property, then $K_\lambda$ is categorical.

Now we present the properties that a non-forking relation should satisfy.

**Definition 3.6.** Let $NF \subseteq {}^4K_\lambda$ be a relation. We say $\bigotimes_{NF}$ when the following axioms are satisfied:

(a) If $NF(M_0, M_1, M_2, M_3)$, then $n \in \{1,2\} \rightarrow M_0 \preceq M_n \preceq M_3$ and $M_1 \cap M_2 = M_0$.
(b) The monotonicity axiom: if $NF(M_0, M_1, M_2, M_3)$ *and* $N_0 = M_0, n < 3 \rightarrow N_n \preceq M_n \wedge N_0 \preceq N_n \preceq N_3, (\exists N^*)[M_3 \preceq N^* \wedge N_3 \preceq N^*]$, then $NF(N_0, N_1, N_2, N_3)$.
(c) The existence axiom: for every $N_0, N_1, N_2 \in K_\lambda$, if $l \in \{1,2\} \rightarrow N_0 \preceq N_l$ *and* $N_1 \bigcap N_2 = N_0$, then there is $N_3$ such that $NF(N_0, N_1, N_2, N_3)$.
(d) The uniqueness axiom: suppose for $x = a, b$ we have $NF(N_0, N_1, N_2, N_3^x)$. Then there is a joint embedding of $N^a, N^b$ *over* $N_1 \bigcup N_2$.
(e) The symmetry axiom: $NF(N_0, N_1, N_2, N_3) \leftrightarrow NF(N_0, N_2, N_1, N_3)$.
(f) The long transitivity axiom: for $x = a, b$, let $\langle M_{x,i} : i \preceq \alpha^* \rangle$ be an increasing continuous sequence of models in $K_\lambda$. Suppose $i < \alpha^* \rightarrow NF(M_{a,i}, M_{a,i+1}, M_{b,i}, M_{b,i+1})$. Then $NF(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$.

**Definition 3.7.** Let $NF$ be a relation such that $\bigotimes_{NF}$. We say that $NF$ respects the frame $\mathfrak{s}$ when: if $NF(M_0, M_1, M_2, M_3)$ and $ga-\ tp(a, M_0, M_1) \in S^{bs}(M_0)$, then $ga-\ tp(a, M_2, M_3)$ does not fork over $M_0$.

**Theorem 3.8.** *Suppose:*

*(1) $K$ is categorical in $\lambda$.*
*(2) $\mathfrak{s}$ is an almost good $\lambda$-frame which satisfies the conjugation property.*
*(3) $I(\lambda^{++}, K) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$.*
*(4) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.*
*(5) The ideal $WDmId(\lambda^+)$ is not saturated in $\lambda^{++}$.*

*Then there is a relation $NF$ such that $\bigotimes_{NF}$ and $NF$ respects the frame $\mathfrak{s}$.*

*Proof.* By [JrSh 875]: by Corollary [JrSh 875, 4.18], $K^{3,uq}$ is dense with respect to $\preceq_{bs}$. Hence by Theorem [JrSh 875, 5.15], there is a unique relation, $NF$, with $\bigotimes_{NF}$. Now see Definition [JrSh 875, 5.3]. ⊣

## 4. A full good $\lambda$-frame

*Hypothesis* 4.1. $\mathfrak{s}$ is an almost good $\lambda$-frame which satisfies the conjugation property.

**Definition 4.2.** $nf^{NF} := \{(M_0, M_1, a, M_3) : M_0, M_1, M_3 \in K_\lambda,\ M_0 \preceq M_1 \preceq M_3,\ a \in M_3 - M_1$ and for some $M_2 \in K_\lambda,\ M_0 \preceq M_2,\ a \in M_2 - M_0$ and $NF(M_0, M_1, M_2, M_3)\}$.

The following theorem is similar to Claim [Sh:h, 9.5.2].III.

**Theorem 4.3.** *Let $\mathfrak{s}$ be an almost good $\lambda$-frame which satisfies the conjugation property. Then $\mathfrak{s}^{NF} = (K, \preceq, S^{na}, nf^{NF})$ is a full good $\lambda$-frame.*

*Proof.* We will prove the conditions in Definition **??**:

1. Trivial.

2. (a),(b),(c) are trivial. (d) (basic stability) is satisfied by Theorem 2.6(3).

3. (a) is trivial.

(b) is OK by the monotonicity of $NF$, i.e. Definition 3.6(b).

Axiom (c) (local character) is the heart of the matter. Let $j$ be a limit ordinal, let $\langle N_i : i \preceq j+1\rangle$ be an increasing continuous sequence of models in $K_\lambda$ and let $p =: ga - \; tp(c, N_j, N_{j+1}) \in S^{na}(N_j)$. We have to find $i < j$ such that $p$ does not fork over $N_i$ in the sense of $nf^{NF}$, i.e. $nf^{NF}(N_i, c, N_j, N_{j+1})$. It is enough to find an increasing continuous sequence $\langle M_i : i \preceq j\rangle$ such that for each $i \preceq j$, $N_i \preceq M_i$ and $NF(N_i, N_{i+1}, M_i, M_{i+1})$ (so $NF(N_i, N_j, M_i, M_j)$) and $N_{j+1} \preceq M_j$ (for some $i < j$ $c \in M_i$, so $nf^{NF}(N_i, c, N_j, N_{j+1})$). Without loss of generality, $cf(j) = j$. We try to construct $\langle N_{\alpha,i} : i \preceq j+1\rangle$ by induction on $\alpha \in \lambda^+$, such that:

(1) For each $\alpha \in \lambda^+$, $\langle N_{\alpha,i} : i \preceq j+1\rangle$ is an increasing continuous sequence of models in $K_\lambda$.
(2) For each $i \preceq j$, $\langle N_{\alpha,i} : \alpha < \lambda^+\rangle$ is an $\prec^*$-increasing continuous sequence of models in $K_\lambda$ and $N_{\alpha,j+1} \preceq N_{\alpha+1,j+1}$.
(3) $N_{0,i} = N_i$.
(4) For each $i < j$ and $\alpha < \lambda^+$, we have $NF(N_{\alpha,i}, N_{\alpha,i+1}, N_{\alpha+1,i}, N_{\alpha+1,i+1})$.
(5) For each $\alpha \in S =: \{\delta \in \lambda^+ : cf(\delta) = j\}$, we have $N_{\alpha,j+1} \bigcap N_{\alpha+1,j} \neq N_{\alpha,j}$.

If we succeed, then by clauses (2) and (5), the quadruple

$$\langle N_{\alpha,j} : \alpha < \lambda^+\rangle,\ \langle N_{\alpha,j+1} : \alpha < \lambda^+\rangle,\ \langle id_{N_{\alpha_j}} : \alpha < \lambda^+\rangle,\ S$$

forms a counterexample to Claim 1.3, so it is impossible to carry out this construction.

*Where will we get stuck?* For $\alpha = 0$, we will not get stuck, see item (3).

For $\alpha$ limit, just (1),(2) are relevant, and we just have to take unions and use smoothness.

So we will get stuck at some successor ordinal. Suppose we have defined $\langle N_{\alpha,i} : i \preceq j+1\rangle$. Can we find $\langle N_{\alpha+1,i} : i \preceq j+1\rangle$? If $\alpha \notin S$, then it is easier, so assume $\alpha \in S$. Let $\langle \beta(i) : i \preceq j+1\rangle$ be an increasing continuous sequence of ordinals such that $\beta(j) = \alpha$. If $N_{\alpha,j} = N_{\alpha,j+1}$, then we can define $M_i := N_{\alpha,i}$ and the local character is proved ($N_j \preceq N_{\alpha,j} = M_j$, so see the beginning of the proof). So without loss of generality, $N_{\alpha,j+1} \neq N_{\alpha,j}$.

In the following diagram, the arrows describe the $\prec^*$-increasing continuous sequence $\langle N_{\beta(i),i} : i \preceq j\rangle^\frown\langle N_{\alpha,j+1}\rangle$. A model that appears at the right and above another model is bigger than it.

$$\begin{tikzcd}
N_{\alpha,i^*} & N_{\alpha,i^*+1} & N_{\alpha,i^*+2} & N_{\alpha,j} \arrow[r] & N_{\alpha,j+1} \ni c \\
N_{\beta(i^*+2),i^*} & N_{\beta(i^*+2),i^*+1} & N_{\beta(i^*+2),i^*+2} \arrow[ur] & N_{\beta(i^*+2),j} & N_{\beta(i^*+2),j+1} \\
N_{\beta(i^*+1)+1,i^*} & N_{\beta(i^*+1)+1,i^*+1} & N_{\beta(i^*+1)+1,i^*+2} & N_{\beta(i^*+1)+1,j} & N_{\beta(i^*+1)+1,j+1} \\
N_{\beta(i^*+1),i^*} & N_{\beta(i^*+1),i^*+1} \arrow[uur] & N_{\beta(i^*+1),i^*+2} & N_{\beta(i^*+1),j} & N_{\beta(i^*+1),j+1} \\
N_{\beta(i^*),i^*} \arrow[ur] & N_{\beta(i^*),i^*+1} & N_{\beta(i^*),i^*+2} & N_{\beta(i^*),j} & N_{\beta(i^*),j+1}
\end{tikzcd}$$

By weak local character, there is an element $c$ and an ordinal $i^*$ such that $ga-\ tp(c, N_{\alpha,j}, N_{\alpha,j+1})$ does not fork over $N_{\beta(i^*),i^*}$.

By Definition 2.1(b) (the monotonicity axiom), $ga-\ tp(c, N_{\alpha,j}, N_{\alpha,j+1})$ does not fork over $N_{\alpha,i^*+1}$ and so $ga-\ tp(c, N_{\alpha,i^*+1}, N_{\alpha,j+1}) \in S(N_{\alpha,i^*+1})$. So there is an increasing continuous sequence $\langle N^{temp}_{\alpha+1,i} : i \preceq j\rangle$ such that for $i < j$ we have $NF(N_{\alpha,i}, N_{\alpha,i+1}, N^{temp}_{\alpha+1,i}, N^{temp}_{\alpha+1,i+1})$, and there is $a \in N^{temp}_{\alpha+1,i^*+1}$ such that $ga-\ tp(a, N_{\alpha,i^*+1}, N^{temp}_{\alpha+1,i^*+1}) = ga-\ tp(c, N_{\alpha,i^*+1}, N_{\alpha,j+1})$. [Why? For $i \preceq i^*$ define $N^{temp}_{\alpha+1,i} = N_{\alpha,i}$. Choose $N^{temp}_{\alpha+1,i^*+1}$ which is isomorphic to $N_{\alpha,j+1}$ over $N_{\alpha,i^*+1}$ and $N^{temp}_{\alpha+1,i^*+1} \bigcap N_{\alpha,j+1} = N_{\alpha,i^*+1}$. For $i \in (i^*+1, j]$ choose $N^{temp}_{\alpha+1,i+1}$ such that $NF(N_{\alpha,i}, N_{\alpha,i+1}, N^{temp}_{\alpha+1,i}, N^{temp}_{\alpha+1,i+1})$. If $i$ is limit, then define $N^{temp}_{\alpha+1,i} := \bigcup\{N_{\alpha+1,\varepsilon} : \varepsilon < i\}$]. Now by the long transitivity of $NF$ we have $NF(N_{\alpha,i^*+1}, N_{\alpha,j}, N^{temp}_{\alpha+1,i^*+1}, N^{temp}_{\alpha+1,j})$ and so since $NF$ respects $s$, the type $ga-\ tp(a, N_{\alpha,j}, N^{temp}_{\alpha+1,j})$ does not fork over $N_{\alpha,i^*+1}$. So by Definition 2.1(e), (the uniqueness of the non-forking extension), $ga-\ tp(a, N_{\alpha,j}, N^{temp}_{\alpha+1,j}) = ga-\ tp(c, N_{\alpha,j}, N_{\alpha,j+1})$. Hence by the definition of the equality between types, without loss of generality, there is a model $N_{\alpha+1,j+1}$ such that $N_{\alpha,j+1} \preceq N_{\alpha+1,j+1}$, there is an embedding $f : N^{temp}_{\alpha+1,j} \hookrightarrow N_{\alpha+1,j+1}$ over $N_{\alpha,j}$ and $f(a) = c$. Now for $i \preceq j$ define $N_{\alpha+1} := f[N^{temp}_{\alpha+1,i}]$. Why is (5) satisfied? $c \in N_{\alpha,j+1} \bigcap N_{\alpha+1,i+1} - N_{\alpha,i+1}$. By (4) and the long transitivity of $NF$, we have $NF(N_{\alpha,i+1}, N_{\alpha,j}, N_{\alpha+1,i+1}, N_{\alpha+1,j})$, so $c \notin N_{\alpha,j}$, but since $N_{\alpha+1,i+1} \subset N_{\alpha+1,j}$ we have $c \in N_{\alpha+1,j}$. Hence $c \in N_{\alpha,j+1} \bigcap N_{\alpha+1,j} - N_{\alpha,j}$) Hence we can carry out the construction.

(d) Uniqueness: suppose for $n < 2$, $ga-\ tp(a^n, M_0, M_1^n)$ does not depend on $n$, and $NF(M_0, M_2, M_1^n, M_3^n)$, see the diagram below. We have to prove that $ga-\ tp(a^n, M_2, M_3^n)$ does not depend on $n$. By the definition of the equality between types, there is an amalgamation $f^0, f^1, M_1$ of $M_1^0, M_1^1$ over $M_0$. So there are models $M_3^{n,+}$ and embeddings $f_n^+ : M_3^n \hookrightarrow M_3^{n,+}$, such that for $n < 2$ we have $NF(f_n[M_1^n], f_n^+[M_3^n], M_1, M_3^{n,+})$ and $f_n \subset f_n^+$. Since $M_2 \bigcap M_1^n = M_0$, without loss of generality, $f_n^+ \restriction M_2 = id_{M_2}$ (we can change the names of the elements in $M_2 - M_0$, i.e. $M_2 - M_1^n$). By the long transitivity axiom of $NF$, we have $NF(M_0, M_2, M_1, M_3^{n,+})$. So by the uniqueness of $NF$, there is a joint embedding $g^0, g^1, M_3$ of $M_3^{0,+}, M_3^{1,+}$ over $M_1 \bigcup M_2$. So $g^0 \circ f_0^+, g^1 \circ f_1^+, M_3$ is an amalgamation of $M_3^0, M_3^1$ over $M_2$. Since $a_n \in M_1^n$, $(g^n \circ f_n^+)(a_n) = f_n(a_n)$ and so it does not depend on $n$ (since $f_0, f_1$ are witnesses for $ga-\ tp(a_1, M_0, M_1^n)$ does not depend on $n$). So $ga-\ tp(a^n, M_2, M_3^n)$ does not depend on $n$.

$$\begin{tikzcd}
M_1 \arrow[r] & M_3^{n,+} \arrow[r, "g^n"] & M_3 \\
M_1^n \arrow[u, "f_n"] \arrow[r] & M_3^n \arrow[u, "f_n^+"'] & \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] &
\end{tikzcd}$$

(e) Symmetry: by the symmetry of $NF$, i.e. Definition 3.6(e).
(f) By the corresponding axiom of $NF$, i.e. Definition 3.6(c).
(g) Continuity: it is easy to see that continuity follows by local character, because by definition, $s^{NF}$ is full. $\dashv$

Now we can present the main theorem: we get a good $\lambda$-frame.

**Theorem 4.4.** *Let $(K, \preceq)$ be an AEC such that:*

*(1) $K$ is categorical in $\lambda$,$\lambda^+$ and $1 \leq I(\lambda^{+2}, K) < \mu_{unif}(\lambda^{+2}, 2^{\lambda^+})$.*
*(2) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$, and $WDmId(\lambda^+)$ is not saturated in $\lambda^{+2}$.*

*<u>Then:</u>*
*there is an almost good $\lambda$-frame, $\mathfrak{s}$ with complete... $(K_{\mathfrak{s}}, \preceq_{\mathfrak{s}}) = ((K_\lambda, \preceq)$ and a type is basic if it is minimal. Moreover, if $\mathfrak{s}$ satisfies the conjugation property, then there is a good $\lambda$-frame with $(K_s, \preceq_s) = ((K, \preceq)$.*

**Remark 4.5.** Background on Weak Diamond appears in [DS] and in Chapter 13 of [Gr:book]. Concerning $\mu_{unif}(\mu^+, 2^\mu)$, see the last chapter of [Sh:h], [JrSh 875] or [JrSh 966]. It is "almost $2^{\mu^+}$": $1 < \mu_{unif}(\mu^+, 2^\mu)$, If $\beth_\omega \preceq \mu$, then $\mu_{unif}(\mu^+, 2^\mu) = 2^{\mu^+}$ and in any case it is not clear if $\mu_{unif}(\mu^+, 2^\mu) < 2^{\mu^+}$ is consistent. There are more claims which say that it is a "big cardinal".

*Proof.* By Theorem [Sh E46, 0.2] there is such an almost good frame. So by Theorem 4.3 we have the "moreover". $\dashv$

While in [Sh:h].II we obtained a good $\lambda^+$-frame, here we obtained a $\lambda$-good frame. Why is this important? In Section 1 of [Sh:h].III, Shelah defined weakly dimensionality of a good frame, and proved that it is equal to the categoricity in the successor cardinal. Since here we assume categoricity in $\lambda^+$, the good $\lambda$-frame we obtained here is weakly dimensional.

## 5. The function $\lambda \to I(\lambda, K)$ is not arbitrary

In this section, we prove, under set theoretical assumptions, that there is no AEC, $(K, \preceq)$, which is categorical in $\lambda, \lambda^+ ... \lambda^{+(n-1)}$, but has no model of cardinality $\lambda^{+n}$. The main results of Section 4 enables to prove only a weaker version of this theorem. But we can prove this theorem, using results of [Sh E46] and [Sh:h].II.

By the last section in [Sh:h].II (alternatively, see Corollary [JrSh 875, 12.6]):

**Fact 5.1.** *Suppose:*

(1) $n < \omega$,
(2) $\mathfrak{s} = (K, \preceq, S^{bs}, nf)$ *is a good* $\lambda$*-frame* ,
(3) *For each* $m < n$, $I(\lambda^{+(2+m)}, K) < \mu_{unif}(\lambda^{+(2+m)}, 2^{\lambda^{+(1+m)}})$,
(4) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < ... < 2^{\lambda^{+(1+n)}}$,
(5) *For each* $m < n$, *the ideal* $WDmId(\lambda^{+1+m})$ *is not saturated in* $\lambda^{+(2+m)}$.

Then *there is a model in* $K$ *of cardinality* $\lambda^{+(2+n)}$.

By the following theorem, if $f : card \to card$ is a class function (from the cardinals to the cardinals) with $f(\lambda) = f(\lambda^+) = ...f(\lambda^{+(n-1)}) = 1$ and $f(\lambda^{+n}) = 0$, then under specific set theoretical assumptions (clauses (4),(5), below), $f$ cannot be the spectrum of categoricity of any AEC.

**Theorem 5.2.** *There are no* $K, \preceq, n, \lambda$ *such that*

(1) $n \geq 3$ *is a natural number,*
(2) $(K, \preceq)$ *is an AEC,*
(3) $K$ *is categorical in* $\lambda^{+m}$ *for each* $m < n$, *but* $K_{\lambda^{+n}} = \emptyset$,
(4) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < ... < 2^{\lambda^{+(n-1)}}$,
(5) *For each* $m < n-2$, $WDmId(\lambda^{+1+m})$ *is not saturated in* $\lambda^{+(2+m)}$.

Before we prove Theorem 5.2, we prove a weaker version of it:

**Proposition 5.3.** *The same as Theorem 5.2, but here we assume, in addition, that if* $M_0 \preceq M_1 \preceq M_2$, $a \in M_2 - M_1$ *and* $ga-\ tp(a, M_0, M_2)$ *is minimal, then the types* $ga-\ tp(a, M_1, M_2), ga-\ tp(a, M_0, M_2)$ *are conjugate.*

*Proof.* By Theorem 4.4, there is a good $\lambda$-frame with $(K_s, \preceq_s) = ((K, \preceq)$. Hence by Fact 5.1, there is a model in $K$ of cardinality $\lambda^{+(n)}$. $\dashv$

**Remark 5.4.** To our opinion, by Claim [Sh E46, 7.4](p. 76), it is reasonable to assume that $\mathfrak{s}$ satisfies the conjugation property.

Now we prove Theorem 5.2:

*Proof.* By Theorem [Sh:h, II.3.7](p. 297), there is a good $\lambda^+$-frame, $\mathfrak{s}$ such that its AEC is $(K, \preceq)$. Now use Fact 5.1, where $\lambda^+$ stands for $\lambda$. $\dashv$

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