NONPROPER PRODUCTS

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Abstract. We show that there exist two proper creature forcings having a simple (Borel) definition, whose product is not proper. We also give a new condition ensuring properness of some forcings with norms.

1. Introduction

In Rosłanowski and Shelah [2] a theory of forcings built with the use of norms was developed and a number of conditions to ensure the properness of the resulting forcings was given. However it is not clear how sharp those results really are and this problem was posed in Shelah [4, Question 4.1]. In particular, he asked about the properness of the forcing notion

$$\mathbb{Q} = \{\langle w_n : n<\omega\rangle : w_n \subseteq 2^n, w_n \neq \emptyset \text{ and } \lim_{n\to\omega} |w_n| = \infty\}$$

ordered by $\bar{w} \leq \bar{w}' \Leftrightarrow (\forall n\in\omega)(w'_n \subseteq w_n)$. In the second section we give a general criterion for collapsing the continuum to $\aleph_0$ and then in Corollary 2.8 we apply it to the forcing $\mathbb{Q}$, just showing that it is not proper.

That the property of properness is not productive, i.e. is not preserved under taking products, has been observed by Shelah long ago (see [3, XVII, 2.12]). However, his examples are somewhat artificial and certainly it would be desirable to know of some rich enough subclass of proper forcings that is productively closed. It was a natural conjecture put forth by Zapletal, that the class of definable, say analytic or Borel, proper forcings would have this property. Actually, it was only proved recently by Spinas [5] that finite powers of the Miller rational perfect set forcing and finite powers of the Laver forcing notion are proper. These are two of the most frequently used forcings in the set theory of the reals. However, in this paper we shall show that this phenomenon does not extend to all forcing notions defined in the setting of norms on possibilities. In the fourth section of the paper we give an example of a forcing notion with norms which, by the theory developed in the second section, is not proper and yet it can be decomposed as a product of two proper forcing notions of a very similar type, and both of which have a Borel definition. The properness of the factors is a consequence of a quite general theorem presented in the third section (Theorem 3.3). It occurs that a strong version of halving from [2, Section 2.2] implies the properness of forcing notions of the type $\mathbb{Q}^*_\infty(K,\Sigma)$.

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Notation Most of our notation is standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [1]). However in forcing we keep the convention that *a stronger condition is the larger one.*

In this paper $\mathbf{H}$ will stand for a function with domain ω and such that $(\forall m \in \omega)(2 \leq |\mathbf{H}(m)| < \omega)$. We also assume that $0 \in \mathbf{H}(m)$ (for all $m \in \omega$); if it is not the case then we fix an element of $\mathbf{H}(m)$ and we use it whenever appropriate notions refer to 0.

Creature background: Since our results are stated for creating pairs with several special properties, below we present a somewhat restricted context of the creature forcing, introducing *good creating pairs.*

Definition 1.1. (1) *A creature for* $\mathbf{H}$ is a triple

$$t = (\mathbf{nor}, \mathbf{val}, \mathbf{dis}) = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t])$$

such that $\mathbf{nor} \in \mathbb{R}^{\geq 0}$, $\mathbf{dis} \in \mathcal{H}(\omega_1)$, and for some integers $m^t_{\rm dn} < m^t_{\rm up} < \omega$

$$\emptyset \neq \mathbf{val} \subseteq \{\langle u, v\rangle \in \prod_{i<m^t_{\rm dn}} \mathbf{H}(i) \times \prod_{i<m^t_{\rm up}} \mathbf{H}(i) : u \vartriangleleft v\}.$$

The family of all creatures for $\mathbf{H}$ is denoted by $\mathrm{CR}[\mathbf{H}]$.

(2) Let $K \subseteq \mathrm{CR}[\mathbf{H}]$ and $\Sigma : K \longrightarrow \mathcal{P}(K)$. We say that (K, Σ) is a *good creating pair* for $\mathbf{H}$ whenever the following conditions are satisfied for each $t \in K$.
  (a) [*Fullness*] $\mathrm{dom}(\mathbf{val}[t]) = \prod\limits_{i<m^t_{\rm dn}} \mathbf{H}(i)$.
  (b) $t \in \Sigma(t)$ and if $s \in \Sigma(t)$, then $m^s_{\rm dn} = m^t_{\rm dn}$ and $m^s_{\rm up} = m^t_{\rm up}$.
  (c) [*Transitivity*] If $s \in \Sigma(t)$, then $\Sigma(s) \subseteq \Sigma(t)$.

(3) A good creating pair (K, Σ) is
  - *local* if $m^t_{\rm up} = m^t_{\rm dn} + 1$ for all $t \in K$,
  - *forgetful* if for every $t \in K$, $v \in \prod\limits_{i<m^t_{\rm up}} \mathbf{H}(i)$, and $u \in \prod\limits_{i<m^t_{\rm dn}} \mathbf{H}(i)$ we have

    $$\langle v{\restriction}m^t_{\rm dn}, v\rangle \in \mathbf{val}[t] \qquad \Rightarrow \qquad \langle u, u^\frown v{\restriction}[m^t_{\rm dn}, m^t_{\rm up})\rangle \in \mathbf{val}[t],$$

  - *strongly finitary* if for each $i < \omega$ we have

    $$|\mathbf{H}(i)| < \omega \qquad \text{and} \qquad |\{t \in K : m^t_{\rm dn} = i\}| < \omega.$$

(4) If $t_0, \ldots, t_n \in K$ are such that $m^{t_i}_{\rm up} = m^{t_{i+1}}_{\rm dn}$ (for $i < n$) and $w \in \prod\limits_{i<m^{t_0}_{\rm dn}} \mathbf{H}(i)$, then we let

$$\mathrm{pos}(w, t_0, \ldots, t_n) \stackrel{\rm def}{=} \{v \in \prod_{j<m^{t_n}_{\rm up}} \mathbf{H}(j) : w \vartriangleleft v \ \&\ (\forall i \leq n)(\langle v{\restriction}m^{t_i}_{\rm dn}, v{\restriction}m^{t_i}_{\rm up}\rangle \in \mathbf{val}[t_i])\}.$$

If K is forgetful and $t \in K$, then we also define

$$\mathrm{pos}(t) = \big\{v{\restriction}[m^t_{\rm dn}, m^t_{\rm up}) : \langle v{\restriction}m^t_{\rm dn}, v\rangle \in \mathbf{val}[t]\big\}.$$

Note that if K is forgetful, then to describe a creature in K it is enough to give $\mathrm{pos}(t)$, $\mathbf{nor}[t]$ and $\mathbf{dis}[t]$. This is how our examples will be presented (as they all will be forgetful). Also, if K is additionally local, then we may write $\mathrm{pos}(t) = A$ for some $A \subseteq \mathbf{H}(m^t_{\rm dn})$ with a natural interpretation of this abuse of notation.

Definition 1.2. Let (K,Σ) be a good creating pair for $\mathbf{H}$. We define a forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ as follows.

A condition in $\mathbb{Q}^*_\infty(K,\Sigma)$ is a sequence $p=(w^p,t^p_0,t^p_1,t^p_2,\ldots)$ such that

(a) $t^p_i\in K$ and $m^{t^p_i}_{\rm up}=m^{t^p_{i+1}}_{\rm dn}$ (for $i<\omega$), and
(b) $w\in\prod\limits_{i<m^{t^p_0}_{\rm dn}}\mathbf{H}(i)$ and $\lim\limits_{n\to\infty}\mathbf{nor}[t^p_n]=\infty$.

The relation $\leq$ on $\mathbb{Q}^*_\infty(K,\Sigma)$ is given by: $\quad p\leq q\quad$ if and only if for some $i<\omega$ we have $w^q\in\mathrm{pos}(w^p,t^p_0,\ldots,t^p_{i-1})$ (if $i=0$ this means $w^q=w^p$) and $t^q_n\in\Sigma(t^p_{n+i})$ for all $n<\omega$.

For a condition $p\in\mathbb{Q}^*_\infty(K,\Sigma)$ we let $i(p)=\mathrm{lh}(w^p)$.

2. Collapsing creatures

We will show here that very natural forcing notions of type $\mathbb{Q}^*_\infty(K,\Sigma)$ (for a big local and finitary creating pair (K,Σ)) collapse $\mathfrak{c}$ to $\aleph_0$, in particular answering [4, Question 4.1]. The main ingredient of the proof is similar to the "negative theory" presented in [2, Section 1.4], and Definition 2.1 below should be compared with [2, Definition 1.4.4] (but the two properties are somewhat incomparable).

Definition 2.1. Let $h:\mathbb{R}^{\geq 0}\longrightarrow\mathbb{R}^{\geq 0}$ be a non-decreasing unbounded function and let (K,Σ) be a good creating pair for $\mathbf{H}$. We say that (K,Σ) is *sufficiently h-bad* if there are sequences $\bar m=\langle m_i:i<\omega\rangle$, $\bar A=\langle A_i:i<\omega\rangle$ and $\bar F=\langle F_i:i<\omega\rangle$ such that

(α) $\bar m$ is a strictly increasing sequence of integers, $m_0=0$, and
$$(\forall t\in K)(\exists i<\omega)(m^t_{\rm dn}=m_i\ \&\ m^t_{\rm up}=m_{i+1}),$$
(β) A_i are finite non-empty sets,
(γ) $F_i=(F^0_i,F^1_i):A_i\times\prod\limits_{m<m_{i+1}}\mathbf{H}(m)\longrightarrow A_{i+1}\times 2$,
(δ) if $i<\omega$, $t\in K$, $m^t_{\rm dn}=m_i$ and $\mathbf{nor}[t]>4$, then there is $a\in A_i$ such that for every $x\in A_{i+1}\times 2$, for some $s_x\in\Sigma(t)$ we have $\mathbf{nor}[s_x]\geq\min\{h(\mathbf{nor}[t]),h(i)\}$ and $(\forall u\in\prod\limits_{m<m_i}\mathbf{H}(m))(\forall v\in\mathrm{pos}(u,s_x))(F_i(a,v)=x)$.

Proposition 2.2. *Suppose that $h:\mathbb{R}^{\geq 0}\longrightarrow\mathbb{R}^{\geq 0}$ is a non-decreasing unbounded function, and (K,Σ) is a strongly finitary good creating pair for $\mathbf{H}$. Assume also that (K,Σ) is sufficiently h-bad. Then the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ collapses $\mathfrak{c}$ onto $\aleph_0$.*

Proof. The proof is similar to that of [2, Proposition 1.4.5], but for reader's convenience we present it fully.

Let $\bar m,\bar A$ and $\bar F$ witness that (K,Σ) is sufficiently h-bad. For $i<\omega$ and $a\in A_i$ we define $\mathbb{Q}^*_\infty(K,\Sigma)$–names $\dot\rho_{i,a}$ (for a real in 2^ω) and $\dot\eta_{i,a}$ (for an element of $\prod\limits_{j\geq i}A_j$) as follows:
$$\Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)}\text{“ }\dot\eta_{i,a}(i)=a\ \text{ and }\ \dot\eta_{i,a}(j)=F^0_{j-1}(\dot\eta_{i,a}(j-1),\dot W{\restriction}m_j)\text{ for }j>i\text{ ”},$$
and
$$\Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)}\text{“ }\dot\rho_{i,a}{\restriction}i\equiv 0\ \text{ and }\ \dot\rho_{i,a}(j)=F^1_j(\dot\eta_{i,a}(j),\dot W{\restriction}m_{j+1})\text{ for }j\geq i\text{ ”}.$$

Above, $\dot{W}$ is the canonical name for the generic function in $\prod\limits_{i<\omega} \mathbf{H}(i)$, i.e., $p \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)}$ " $w^p \vartriangleleft \dot{W} \in \prod\limits_{i<\omega} \mathbf{H}(i)$ ". We are going to show that

$$\Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)} \text{ `` } (\forall r \in 2^\omega \cap \mathbf{V})(\exists i < \omega)(\exists a \in A_i)(\forall j \geq i)(\dot{\rho}_{i,a}(j) = r(j)) \text{ ''}.$$

To this end suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $r \in 2^\omega$. Passing to a stronger condition if needed, we may assume that $(\forall j < \omega)(\mathbf{nor}[t^p_j] > 4)$. Let $i < \omega$ be such that $\mathrm{lh}(w^p) = m_i$; then also $m^{t^p_j}_{\mathrm{dn}} = m_{i+j}$ for $j < \omega$ (remember 2.1(α)).

Fix $k < \omega$ for a moment. By downward induction on $j \leq k$ choose $s^k_j \in \Sigma(t^p_j)$ and $a^k_j \in A_{i+j}$ such that

(a) $\mathbf{nor}[s^k_j] \geq \min\{h(\mathbf{nor}[t^p_j]), h(i+j)\}$ for all $j \leq k$,
(b) $(\forall u \in \prod\limits_{m<m_{i+k}} \mathbf{H}(m))(\forall v \in \mathrm{pos}(u, s^k_k))(F^1_{i+k}(a^k_k, v) = r(i+k))$,
(c) for $j < k$:

$$(\forall u \in \prod_{m<m_{i+j}} \mathbf{H}(m))(\forall v \in \mathrm{pos}(u, s^k_j))(F^1_{i+j}(a^k_j, v) = r(i+j) \ \& \ F^0_{i+j}(a^k_j, v) = a^k_{j+1}).$$

(Plainly it is possible by 2.1(δ).)

Since for each $j < \omega$ both $\Sigma(t^p_j)$ and A_{i+j} are finite, we may use König Lemma to pick an increasing sequence $\bar{k} = \langle k(\ell) : \ell < \omega\rangle$ such that

$$a^{k(\ell+1)}_j = a^{k(\ell')}_j \quad \text{and} \quad s^{k(\ell+1)}_j = s^{k(\ell')}_j$$

for $\ell < \ell' < \omega$ and $j \leq k(\ell)$. Put $w^q = w^p$ and $t^q_j = s^{k(j+1)}_j$, $b_j = a^{k(j+1)}_j$ for $j < \omega$. Easily, $q = (w^q, t^q_0, t^q_1, t^q_3, \ldots)$ is a condition in $\mathbb{Q}^*_\infty(K,\Sigma)$ stronger than p. Also, by clause (c) of the choice of s^k_j, we clearly have

$$(\forall j < \omega)(\forall v \in \mathrm{pos}(w^q, t^q_0, \ldots, t^q_j))(F^0_{i+j}(b_j, v) = b_{j+1} \ \& \ F^1_{i+j}(b_j, v) = r(i+j)).$$

Hence

$$q \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)} \text{ `` } (\forall j < \omega)(\dot{\eta}_{i,b_0}(i+j) = b_j \ \& \ \dot{\rho}_{i,b_0}(i+j) = r(i+j)) \text{ ''},$$

finishing the proof. $\square$

Lemma 2.3. *Suppose that positive integers N, M, d satisfy $N \cdot 2^M < d$. Let A, B be finite sets such that $|A| \geq 2^M$ and $|B| \leq N$. Then there is a mapping $F : A \times {}^d M \longrightarrow B$ with the property that:*

($\circledast$) *if $2 \leq \ell \leq M$, $\langle c_i : i < d\rangle \in \prod\limits_{i<d} [M]^\ell$, then there is $a \in A$ such that for every $b \in B$, for some $c^b_i \in [c_i]^{\lfloor \ell/2 \rfloor}$ (for $i < d$) we have*

$$(\forall u \in \prod_{i<d} c^b_i)(F(a,u) = b).$$

Proof. Plainly we may assume that $|A| = 2^M$ and $|B| = N$, and then we may pretend that $A = {}^M 2$ and $B = N$.

For $h \in A = {}^M 2$ and $u \in {}^d M$ we let $F(h,u) < N$ be such that

$$F(h,u) \equiv \sum_{i<d} h(u(i)) \mod N.$$

This defines the function $F : A \times {}^d M \longrightarrow B = N$, and we are going to show that it has the property stated in (⊛). To this end suppose that $2 \le \ell \le M$ and $\langle c_i : i < d \rangle \in \prod_{i<d} [M]^{\ell}$. For each $i < d$ we may choose $h_i \in A$ so that

$$|(h_i)^{-1}[\{0\}] \cap c_i| \ge \lfloor \ell/2 \rfloor \quad \text{and} \quad |(h_i)^{-1}[\{1\}] \cap c_i| \ge \lfloor \ell/2 \rfloor$$

Then, for some $h \in A$ and $I \subseteq d$ we have $|I| \ge d/2^M$ and $h_i = h$ for $i \in I$. For $i \in d \setminus I$ we may pick $c_i^* \in [c_i]^{\lfloor \ell/2 \rfloor}$ and $j_i < 2$ such that $h{\restriction}c_i^* \equiv j_i$.

Now, suppose $b \in B$. Take a set $J \subseteq I$ such that

$$|J| + \sum_{i \in d \setminus I} j_i \equiv b \mod N$$

(possible as $|I| \ge d/2^M > N$). By our choices, we may pick $c_i^b \in [c_i]^{\lfloor \ell/2 \rfloor}$ (for $i \in I$) such that

if $i \in J$, then $h{\restriction}c_i^b \equiv 1$,
if $i \in I \setminus J$, then $h{\restriction}c_i^b \equiv 0$.

For $i \in d \setminus I$ we let $c_i^b = c_i^*$ (selected earlier). It should be clear that then

$$(\forall u \in \prod_{i<d} c_i^b)(F(h,u) = b),$$

as needed. □

Example 2.4. Let $\bar{m} = \langle m_i : i < \omega \rangle$ be an increasing sequence of integers such $m_0 = 0$ and $m_{i+1} - m_i > 4^{i+3}$. Let $h(\ell) = \lfloor \ell/2 \rfloor$ for $\ell < \omega$.

For $j < \omega$ we let $\mathbf{H}^0_{\bar{m}}(j) = i + 2$, where i is such that $m_i \le j < m_{i+1}$. Let $K^0_{\bar{m}}$ consist of all (forgetful) creatures $t \in \mathrm{CR}[\mathbf{H}^0_{\bar{m}}]$ such that

- $\mathbf{dis}[t] = \langle i^t, \langle Z^t_j : m_{i^t} \le j < m_{i^t+1} \rangle \rangle$ for some $i^t < \omega$ and $\emptyset \ne Z^t_j \subseteq \mathbf{H}^0_{\bar{m}}(j)$ (for $m_{i^t} \le j < m_{i^t+1}$),
- $\mathbf{nor}[t] = \min\{|Z^t_j| : m_{i^t} \le j < m_{i^t+1}\}$,
- $\mathrm{pos}(t) = \prod_{j \in [m_{i^t}, m_{i^t+1})} Z^t_j$.

Finally, for $t \in K^0_{\bar{m}}$ we let

$$\Sigma^0_{\bar{m}}(t) = \{s \in K^0_{\bar{m}} : i^t = i^s \ \&\ (\forall j \in [m_{i^t}, m_{i^t+1}))(Z^s_j \subseteq Z^t_j)\}.$$

Then $(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ is a strongly finitary and sufficiently h-bad good creating pair for $\mathbf{H}^0_{\bar{m}}$. Consequently, the forcing notion $\mathbb{Q}^*_\infty(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ collapses $\mathfrak{c}$ onto $\aleph_0$.

Proof. It should be clear that $(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ is a strongly finitary good creating pair for $\mathbf{H}^0_{\bar{m}}$. To show that it is sufficiently h-bad let $A_i = {}^{i+2}2$, $B_i = A_{i+1} \times 2 = {}^{i+3}2 \times 2$ and $M_i = i+2$. Since $|B_i| \cdot 2^{M_i} = 2^{i+4+i+2} < m_{i+1} - m_i \stackrel{\text{def}}{=} d_i$, we may apply Lemma 2.3 for $A = A_i$, $B = B_i$, $M = M_i$ and $d = d_i$ to get functions $F_i : A_i \times {}^{d_i} M_i \longrightarrow B_i$ with the property (⊛) (for those parameters). For $a \in A_i$ and $v \in \prod_{j < m_{i+1}} \mathbf{H}^0_{\bar{m}}(j)$ we interpret $F_i(a, v)$ as $F_i(a, u)$ where $u \in {}^{d_i}(i+1)$ is given by $u(j) = v(m_i + j)$ for $j < d_i$. It is straightforward to show that $\bar{m}$, $\bar{A} = \langle A_i : i < \omega \rangle$ and $\bar{F} = \langle F_i : i < \omega \rangle$ witness that $(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ is h-bad. □

The above example (together with Proposition 2.2) easily gives the answer to [4, Question 4.1]. To show how our problem reduces to this example, let us recall the following.

Definition 2.5 (See [2, Definition 4.2.1])**.** Suppose $0<m<\omega$ and for $i<m$ we have $t_i\in \mathrm{CR}[\mathbf{H}]$ such that $m^{t_i}_{\rm up}\leq m^{t_{i+1}}_{\rm dn}$. Then we define *the sum of the creatures* t_i as a creature $t=\Sigma^{\rm sum}(t_i:i<m)$ such that (if well defined then):

(a) $m^t_{\rm dn}=m^{t_0}_{\rm dn}$, $m^t_{\rm up}=m^{t_{m-1}}_{\rm up}$,
(b) $\mathbf{val}[t]$ is the set of all pairs $\langle h_1,h_2\rangle$ such that:
$\mathrm{lh}(h_1)=m^t_{\rm dn}$, $\mathrm{lh}(h_2)=m^t_{\rm up}$, $h_1\vartriangleleft h_2$,
and $\langle h_2{\restriction} m^{t_i}_{\rm dn}, h_2{\restriction} m^{t_i}_{\rm up}\rangle\in \mathbf{val}[t_i]$ for $i<m$,
and $h_2{\restriction}[m^{t_i}_{\rm up},m^{t_{i+1}}_{\rm dn})$ is identically zero for $i<m-1$,
(c) $\mathbf{nor}[t]=\min\{\mathbf{nor}[t_i]:i<m\}$,
(d) $\mathbf{dis}[t]=\langle t_i:i<m\rangle$.

If for all $i<m-1$ we have $m^{t_i}_{\rm up}=m^{t_{i+1}}_{\rm dn}$, then we call the sum *tight.*

Definition 2.6. Let (K,Σ) be a local good creating pair for $\mathbf{H}$, and let $\bar m=\langle m_i:i<\omega\rangle$ be a strictly increasing sequence with $m_0=0$. We define an $\bar m$–*summarization* $(K^{\bar m},\Sigma^{\bar m},\mathbf{H}^{\bar m})$ *of* $(K,\Sigma,\mathbf{H})$ as follows:

- $\mathbf{H}^{\bar m}(i)=\prod\limits_{m=m_i}^{m_{i+1}-1}\mathbf{H}(m)$,
- $K^{\bar m}$ consists of all tight sums $\Sigma^{\rm sum}(t_\ell:m_i\leq\ell<m_{i+1})$ such that $i<\omega$, $t_\ell\in K$, $m^{t_\ell}_{\rm dn}=\ell$,
- if $t=\Sigma^{\rm sum}(t_\ell:m_i\leq\ell<m_{i+1})\in K^{\bar m}$, then $\Sigma^{\bar m}(t)$ consists of all creatures $s\in K^{\bar m}$ such that $s=\Sigma^{\rm sum}(s_\ell:m_i\leq\ell<m_{i+1})$ for some $s_\ell\in\Sigma(t_\ell)$ (for $\ell=m_i,\ldots,m_{i+1}-1$).

Proposition 2.7. *Assume that (K,Σ) is a local good creating pair for $\mathbf{H}$, $\bar m=\langle m_i:i<\omega\rangle$ is a strictly increasing sequence with $m_0=0$. Then:*

(1) *$(K^{\bar m},\Sigma^{\bar m})$ is a good creating pair for $\mathbf{H}^{\bar m}$;*
(2) *the forcing notion $\mathbb{Q}^*_\infty(K^{\bar m},\Sigma^{\bar m})$ can be embedded as a dense subset of the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ (so the two forcing notions are equivalent).*

Corollary 2.8. *Let $\mathbf{H}:\omega\longrightarrow\omega$ be increasing, $\mathbf{H}(0)\geq 2$, and let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be an unbounded non-decreasing function. We define $(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ as follows: $K^{\mathbf{H}}_g$ consists of all creatures $t\in\mathrm{CR}[\mathbf{H}]$ such that*

- *$\mathbf{dis}[t]=\langle i^t,A^t\rangle$ for some $i^t<\omega$ and $\emptyset\neq A^t\subseteq\mathbf{H}(i^t)$,*
- *$\mathbf{nor}[t]=g(|A^t|)$, $m^t_{\rm dn}=i^t$, $m^t_{\rm up}=i^t+1$ and $\mathrm{pos}(t)=A^t$.*

For $t\in K^{\mathbf{H}}_g$ we let

$$\Sigma^{\mathbf{H}}_g(t)=\{s\in K^{\mathbf{H}}_g:i^t=i^s\ \&\ A^s\subseteq A^t\}.$$

Then $(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ is a local strongly finitary good creating pair for $\mathbf{H}$. The forcing notion $\mathbb{Q}^*_\infty(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ collapses $\mathfrak{c}$ onto $\aleph_0$. In particular, the forcing notion $\mathbb{Q}$ defined in the Introduction is not proper.

Proof. Let $p\in\mathbb{Q}^*_\infty(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$. Plainly, $\lim\limits_{i\to\infty}|A^{t^p_i}|=\infty$, so we may find a condition $q\geq p$ and an increasing sequence $\bar m=\langle m_i:i<\omega\rangle$ such that

- $m_0=0$, $m_1=\mathrm{lh}(w^q)$, $m_{i+1}-m_i>4^{i+3}$,

- if $m_i \le m_{\rm dn}^{t^q_k} < m_{i+1}$, then $|A^{t^q_k}| = i+2$.

Now we define a condition q^* in $\mathbb{Q}^*_\infty((K^{\mathbf{H}}_g)^{\bar m}, (\Sigma^{\mathbf{H}}_g)^{\bar m})$ by

$$w^{q^*} = w^q, \qquad t_i^{q^*} = \Sigma^{\rm sum}(t^q_k : m_{i+1} \le k + m_0 < m_{i+2}) \qquad (\text{for } i < \omega).$$

The forcing notion $\mathbb{Q}^*_\infty(K^{\mathbf{H}}_g, \Sigma^{\mathbf{H}}_g)$ above the condition q is equivalent to the forcing notion $\mathbb{Q}^*_\infty((K^{\mathbf{H}}_g)^{\bar m}, (\Sigma^{\mathbf{H}}_g)^{\bar m})$ above q^*. Plainly, $\mathbb{Q}^*_\infty((K^{\mathbf{H}}_g)^{\bar m}, (\Sigma^{\mathbf{H}}_g)^{\bar m})$ above q^* is isomorphic to $\mathbb{Q}^*_\infty(K^0_{\bar m}, \Sigma^0_{\bar m})$ of Example 2.4 above the minimal condition r with $w^r = w^{q^*}$. The assertion follows now by the last sentence of 2.4. $\square$

Remark 2.9. (1) If, e.g., $g(x) = \log_2(x)$ then the creating pair $(K^{\mathbf{H}}_g, \Sigma^{\mathbf{H}}_g)$ is big (see [2, Definition 2.2.1]), and we may even get "a lot of bigness". Thus the bigness itself is not enough to guarantee properness of the resulting forcing notion.

(2) Forcing notions of the form $\mathbb{Q}^*_\infty(K,\Sigma)$ are special cases of $\mathbb{Q}^*_f(K,\Sigma)$ (see [2, Definition 1.1.10 and Section 2.2]). However if the function f is growing very fast (much faster than $\mathbf{H}$) then our method do no apply. Let us recall that if (K,Σ) is simple, finitary, big and has the Halving Property, and $f : \omega \times \omega \longrightarrow \omega$ is $\mathbf{H}$–fast (see [2, Definition 1.1.12]), then $\mathbb{Q}^*_f(K,\Sigma)$ is proper. Thus one may wonder if we may omit halving - can the forcing notion $\mathbb{Q}^*_f(K^{\mathbf{H}}_g, \Sigma^{\mathbf{H}}_g)$ be proper for $\mathbf{H}$ and f suitably "fast"?

3. Properness from Halving

It was shown in [2, Theorem 2.2.11] that halving and bigness (see [2, Definitions 2.2.1, 2.2.7]) imply properness of the forcings $\mathbb{Q}^*_f(K,\Sigma)$ (for fast f). It occurs that if we have a stronger version of halving, then we may get the properness of $\mathbb{Q}^*_\infty(K,\Sigma)$ even without any bigness assumptions.

Definition 3.1. Let (K,Σ) be a forgetful good creating pair

(1) Let $t \in K$ and $\varepsilon > 0$. We say that a creature $t^* \in \Sigma(t)$ is an ε–**half** of t if the following hold:
   (i) $\mathbf{nor}[t^*] \ge \mathbf{nor}[t] - \varepsilon$, and
   (ii) if $s \in \Sigma(t^*)$ and $\mathbf{nor}[s] > 1$, then we can find $t_0 \in \Sigma(t)$ such that
$$\mathbf{nor}[t_0] \ge \mathbf{nor}[t] - \varepsilon \quad \text{and} \quad \mathrm{pos}(t_0) \subseteq \mathrm{pos}(s).$$

(2) Let $\bar\varepsilon = \langle \varepsilon_i : i < \omega\rangle$ be a sequence of positive real numbers and $\bar m = \langle m_i : i < \omega\rangle$ be a strictly increasing sequence of integers with $m_0 = 0$. We say that the pair (K,Σ) has the $(\bar\varepsilon, \bar m)$*–halving property* if for every $t \in K$ with $m_i \le m^t_{\rm dn}$ and $\mathbf{nor}[t] \ge 2$ there exists an ε_i–**half** of t in $\Sigma(t)$.

Definition 3.2. Let (K,Σ) be a good creating pair. Suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $I \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ is open dense. We say that p *essentially belongs to* I, written $p \in^* I$, if there exists $i_* < \omega$ such that for every $v \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_{i_*-1})$ we have $(v, t^p_{i_*}, t^p_{i_*+1}, t^p_{i_*+2}, \ldots) \in I$

Theorem 3.3. *Let $\bar\varepsilon = \langle \varepsilon_i : i < \omega\rangle$ be a decreasing sequence of positive numbers and $\bar m = \langle m_i : i < \omega\rangle$ be a strictly increasing sequence of integers with $m_0 = 0$.*

Assume that for each $i<\omega$

$$|\prod_{n<m_i}\mathbf{H}(n)|\leq 1/\varepsilon_i.$$

Let (K,Σ) be a good creating pair for $\mathbf{H}$ and suppose that (K,Σ) is local, forgetful and has the $(\bar{\varepsilon},\bar{m})$–halving property. Then the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ is proper.

Proof. The proof will be carried out in a series of claims.

Claim 3.3.1. *Let $a\geq 2$ and $I\subseteq\mathbb{Q}^*_\infty(K,\Sigma)$ be open dense. Furthermore suppose that $p\in\mathbb{Q}^*_\infty(K,\Sigma)$ and $i<\omega$ is such that $\mathbf{nor}[t^p_n]>a$ for every $n\geq m_i-i(p)$. Finally let $v\in\prod\limits_{n<m_i}\mathbf{H}(n)$. Then there exists $q\in\mathbb{Q}^*_\infty(K,\Sigma)$ such that*

(a) *$p\leq q$, $w^p=w^q$ and $t^p_n=t^q_n$ for every $n<m_i-i(p)$;*
(b) *$\mathbf{nor}[t^q_n]\geq a-\varepsilon_i$ for every $n\geq m_i-i(p)$;*
(c) *either, letting $q^{[v]}=(v,t^q_{m_i-i(p)},t^q_{m_i-i(p)+1},t^q_{m_i-i(p)+2},\ldots)$, $q^{[v]}\in^* I$ or else there is no $r\geq q^{[v]}$ such that $r\in I$, $w^r=v$ and $\mathbf{nor}[t^r_n]>1$ for every n.*

Proof of the Claim. We know that (K,Σ) has the $(\bar{\varepsilon},\bar{m})$–halving property and therefore for each $n\geq m_i-i(p)$ we may choose an ε_i–half $t^{q_0}_n\in\Sigma(t^p_n)$ of t^p_n. For $n<m_i-i(p)$ put $t^{q_0}_n=t^p_n$ and let $w^{q_0}=w^p$. This defines a condition $q_0=(w^{q_0},t^{q_0}_0,t^{q_0}_1,t^{q_0}_2,\ldots)\in\mathbb{Q}^*_\infty(K,\Sigma)$. Plainly, (a) and (b) hold for q_0 instead of q. Now if there is no $r\geq q_0^{[v]}$ with $r\in I$, $w^r=v$ and $\mathbf{nor}[t^r_n]>1$ for every $n<\omega$, we can let $q=q_0$. Hence we may assume that such $r=(w^r,t^r_0,t^r_1,t^r_2,\ldots)$ does exist.

Pick $j<\omega$ large enough such that $\mathbf{nor}[t^r_n]\geq a-\varepsilon_i$ for every $n\geq j$. Now we define $q\in\mathbb{Q}^*_\infty(K,\Sigma)$:

- $w^q=w^p$, $t^q_n=t^p_n$ for $n<m_i-i(p)$,
- $t^q_n=t^r_{n-m_i+i(p)}$ for $n\geq m_i-i(p)+j$,
- for $m_i-i(p)\leq n<m_i-i(p)+j$ let $t^q_n\in\Sigma(t^p_n)$ be such that

  $$\mathbf{nor}[t^q_n]\geq\mathbf{nor}[t^p_n]-\varepsilon_i\geq a-\varepsilon_i\quad\text{and}\quad\mathrm{pos}(t^q_n)\subseteq\mathrm{pos}(t^r_{n-m_i+i(p)})$$

  (exists by the halving property).

Clearly $p\leq q$ and (a), (b) hold. Also, for every $u\in\mathrm{pos}(v,t^q_{m_i-i(p)},\ldots,t^q_{m_i-i(p)+j})$ we have $q^{[u]}\geq r$, and hence $q^{[u]}\in I$, as I is open. Consequently, $q^{[v]}\in^* I$. $\square$

Claim 3.3.2. *Let $a\geq 3$ and $I\subseteq\mathbb{Q}^*_\infty(K,\Sigma)$ be open dense. Suppose that $p\in\mathbb{Q}^*_\infty(K,\Sigma)$ and $i<\omega$ is such that $\mathbf{nor}[t^p_n]>a$ for every $n\geq m_i-i(p)$. Then there exists $q\in\mathbb{Q}^*_\infty(K,\Sigma)$ such that*

(a) *$p\leq q$, $w^p=w^q$, and $t^p_n=t^q_n$ for $n<m_i-i(p)$;*
(b) *$\mathbf{nor}[t^q_n]\geq a-1$ for every $n\geq m_i-i(p)$;*
(c) *for every $v\in\prod\limits_{n<m_i}\mathbf{H}(n)$, either $q^{[v]}\in^* I$, or else there is no $r\in I$ such that $r\geq q$, $w^r=v$ and $\mathbf{nor}[t^r_n]>1$ for all n.*

Proof of the Claim. Let $\langle v_l:l<k\rangle$ enumerate $\prod\limits_{n<m_i}\mathbf{H}(n)$, thus $k\leq 1/\varepsilon_i$. Applying Claim 3.3.1 k times, it is straightforward to construct a sequence $\langle q_l:l\leq k\rangle\subseteq\mathbb{Q}^*_\infty(K,\Sigma)$ such that

- $q_0=p$, $q_l\leq q_{l+1}$, $w^{q_l}=w^p$, and $t^{q_l}_n=t^p_n$ for every $n<m_i-i(p)$,
- $\mathbf{nor}[t^{q_l}_n]\geq a-l\cdot\varepsilon_i$ for every $n\geq m_i-i(p)$, and
- $\langle q_l,q_{l+1},v_l,a-l\cdot\varepsilon_i\rangle$ are like $\langle p,q,v,a\rangle$ in Claim 3.3.1.

Then clearly $q = q_k$ is as desired. $\square$

Claim 3.3.3. *Let $a \geq 2$ and $I_j \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ be open dense for $j < \omega$. Suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $\langle i_\ell : \ell < \omega\rangle$ is an increasing sequence in ω such that $\mathbf{nor}[t^p_n] > a + 2\ell + 1$ for every $n \geq m_{i_\ell} - i(p)$, for every ℓ. Then there exists $q \in \mathbb{Q}^*_\infty(K,\Sigma)$ such that*

(a) *$p \leq q$, $w^p = w^q$, and $t^p_n = t^q_n$ for every $n < m_{i_0} - i(p)$;*
(b) *$\mathbf{nor}[t^q_n] \geq a + \ell$ for every $n \geq m_{i_\ell} - i(p)$, for every ℓ;*
(c) *for every $j < \omega$ there are infinitely many $\ell < \omega$ such that for every sequence $v \in \prod\limits_{n<m_{i_\ell}} \mathbf{H}(n)$, if there exists $r \in I_j$ such that $r \geq q$, $w^r = v$ and $\mathbf{nor}[t^r_n] > 1$ for every n, then $q^{[v]} \in^* I_j$.*

Proof of the Claim. Fix a surjection $\pi : \omega \longrightarrow \omega$ such that $\pi^{-1}[\{j\}]$ is infinite for every j. We apply Corollary 3.3.2 infinitely many times in order to construct $\langle p_\ell : \ell < \omega\rangle$ with $p_0 = p$ such that for every ℓ, $\langle p_\ell, p_{\ell+1}, i_\ell, a+\ell+1, I_{\pi(\ell)}\rangle$ is like $\langle p, q, i, a, I\rangle$ there. Then we let q be the natural fusion determined by the p_ℓ. $\square$

Claim 3.3.4. *Suppose that $I_j, j < \omega, p, q$ and $\langle i_\ell : \ell < \omega\rangle$ are like in Claim 3.3.3. Let*
$$I^{[q]}_j = \{q^{[v]} : v \in \prod_{n<m_i} \mathbf{H}(n) \ \&\ i < \omega\} \cap I_j.$$
Then $I^{[q]}_j$ is predense above q.

Proof of the Claim. Suppose $r^* \in \mathbb{Q}^*_\infty(K,\Sigma)$ is stronger than q. Pick a condition $r_0 = (w^{r_0}, t^{r_0}_0, t^{r_0}_1, t^{r_0}_2, \ldots) \geq r^*$ such that $r_0 \in I_j$. By 3.3.3(c) we may find ℓ such that

($*$) $\mathbf{nor}[t^{r_0}_n] > 1$ for every $n \geq m_{i_\ell} - i(r_0)$, and
($**$) for every sequence $v \in \prod\limits_{n<m_{i_\ell}} \mathbf{H}(n)$, if there exists $r \in I_j$ such that $r \geq q$, $w^r = v$ and $\mathbf{nor}[t^r_n] > 1$ for every n, then $q^{[v]} \in^* I_j$.

Choose $v \in \mathrm{pos}(w^{r_0}, t^{r_0}_0, \ldots, t^{r_0}_{m_{i_\ell}-i(r_0)-1})$. Then $r^{[v]}_0 \geq r_0$ so $r^{[v]}_0 \in I_j$ (remember I_j is open) and $r^{[v]}_0 \geq q$. Therefore, by ($**$), we see that $q^{[v]} \in^* I_j$. Clearly $r^{[v]}_0 \geq q^{[v]}$. Hence we may find $u \in \mathrm{pos}(v, t^{r_0}_{m_{i_\ell}-i(r_0)}, \ldots, t^{r_0}_{m_k-i(r_0)-1})$ (for some $k > i_\ell$) such that $q^{[u]} \in I_j$. Then $q^{[u]} \in I^{[q]}_j$ and $r^{[u]}_0$ is a common upper bound to $q^{[u]}$ and r^*. $\square$

As the sets $I^{[q]}_j$ from Claim 3.3.4 are countable, it is now clear how to conclude the properness of $\mathbb{Q}^*_\infty(K,\Sigma)$. $\square$

4. A nonproper product

Here, we will give an example of two proper forcing notions $\mathbb{Q}^*_\infty(K^1,\Sigma^1)$ and $\mathbb{Q}^*_\infty(K^2,\Sigma^2)$ such that their product $\mathbb{Q}^*_\infty(K,\Sigma)$ collapses $\mathfrak{c}$ onto $\aleph_0$.

Throughout this section we write log instead of $\log_2$.

Definition 4.1. Let $x, i \in \mathbb{R}$, $x > 0$, $i \geq 0$, and $k \in \omega \setminus \{0\}$. We let
$$f_k(x,i) = \frac{\log\big(\log(\log(x)) - i\big)}{k}$$
in the case that all three logarithms are well defined and attain a value ≥ 1. In all other cases we define $f_k(x,i) = 1$.

Lemma 4.2. (1) $f_k(\frac{x}{2}, i) \geq f_k(x, i) - \frac{1}{k}$;

(2) *letting* $j = \frac{\log(\log(x))+i}{2}$, *if* $f_k(x,i) \geq 2$ *then* $f_k(x,j) = f_k(x,i) - \frac{1}{k}$;

(3) *letting* j *as in (2), if* $\min\{f_k(x,i), f_k(y,j)\} > 1$ *then* $f_k(y,i) \geq f_k(x,i) - \frac{1}{k}$;

(4) *if* $x \geq 2^{2^{4+i}}$ *and* z *such that* $\log(\log(z)) = \frac{\log(\log(x))+i}{2}$, *then* $f_k(z,i) = f_k(x,i) - \frac{1}{k}$.

Proof. (1) Note that for $x \geq 2$ we have

$$(*) \qquad \log(x-1) \geq \log(x) - 1.$$

Indeed, $x \geq 2$ implies $x - 1 \geq \frac{x}{2}$. Applying log to both sides we get $\log(x-1) \geq \log(\frac{x}{2}) = \log(x) - 1$.

If $x < 2^{2^{2+i}}$, then $\log(\log(\log(x)) - i) < 1$ (if at all defined), and $f_k(x,i) = 1 = f_k(\frac{x}{2}, i)$. So assume $x \geq 2^{2^{2+i}}$. Then $\log(x) \geq 2^{2+i} \geq 2$ and $\log(\log(x)) - i \geq 2$, and hence we may apply $(*)$ with $\log(x)$ and $\log(\log(x)) - i$ and obtain

$$\begin{aligned}&\log\big(\log(\log(\tfrac{x}{2})) - i\big) = \log\big(\log(\log(x)-1) - i\big) \geq \log\big(\log(\log(x)) - 1 - i\big)\\ &\geq \log\big(\log(\log(x)) - i\big) - 1.\end{aligned}$$

By dividing both sides by k we arrive at (1).

(2) Note that $f_k(x,i) \geq 2$ implies $\log(\log(x)) - i \geq 4$ and hence $\log(\log(x)) - j = \frac{\log(\log(x))-i}{2} \geq 2$. Consequently

$$\begin{aligned}f_k(x,j) &= \frac{\log\big(\log(\log(x))-j\big)}{k} = \frac{\log\big((\log(\log(x))-i)/2\big)}{k}\\ &= \frac{\log\big(\log(\log(x))-i\big)-1}{k} = f_k(x,i) - \frac{1}{k}.\end{aligned}$$

(3) By assumption we have $\log(\log(y)) - j \geq 0$. By pluging in the definition of j and adding $j - i$ to both sides we obtain $\log(\log(y)) - i \geq \frac{1}{2}(\log(\log(x)) - i)$ and hence $\log\big(\log(\log(y)) - i\big) \geq \log\big(\log(\log(x)) - i\big) - 1$. After dividing by k we reach at (3).

(4) Note that

$$\begin{aligned}f_k(z,i) &= \tfrac{1}{k}\log\big(\log(\log(z)) - i\big) = \tfrac{1}{k}\log\big((\log(\log(x)) - i)/2\big)\\ &= \tfrac{1}{k}[\log\big(\log(\log(x)) - i\big) - 1] = f_k(x,i) - \tfrac{1}{k}.\end{aligned}$$

□

We are going to modify the example in Corollary 2.8 and Example 2.4.

Let $\overline{m} = \langle m_i : i < \omega\rangle$ be an increasing sequence of integers such that $m_0 = 0$ and $m_{i+1} - m_i > 4^{i+3}$. For $j < \omega$ let $\mathbf{H}(j) = i + 3$, where i is such that $m_i \leq j < m_{i+1}$, and let $g(x) = x$. The local good creating pair $(K^{\mathbf{H}}_g, \Sigma^{\mathbf{H}}_g)$ introduced in 2.8 will be denoted by (K^1, Σ^1). By 2.4 we know that $((K^1)^{\overline{m}}, (\Sigma^1)^{\overline{m}})$ (see 2.6) is sufficiently bad and hence (by 2.8) the forcing $\mathbb{Q}^*_\infty(K^1, \Sigma^1)$ collapses $\mathfrak{c}$ into $\aleph_0$.

Recall that for a creature $t \in K^1$ we have

- $\mathbf{dis}[t] = \langle i^t, A^t\rangle$ for some $i^t < \omega$ and $\emptyset \neq A^t \subseteq \mathbf{H}(i^t)$,
- $\mathbf{nor}[t] = |A^t|$ and $\mathrm{pos}(t) = A^t$.

Let $l_n = |\mathbf{H}(n)|$ and

$$k_n = \lfloor\sqrt{\max\{k \in \omega \setminus \{0\} : f_k(l_n, 0) > 1\}}\rfloor \qquad \text{if } l_n > 16,$$

and $k_n = 1$ if $l_n \leq 16$. Certainly we have $\lim\limits_{n\to\infty} l_n = \infty$ and therefore $\lim\limits_{n\to\infty} k_n = \infty$ as well. Note also that $\lim\limits_{n\to\infty} f_{k_n}(l_n, 0) = \infty$.

Definition 4.3. Let K consist of all creatures $t \in \mathrm{CR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = \langle m^t, A^t, i^t\rangle$ for some $m^t < \omega$ and $\emptyset \neq A^t \subseteq \mathbf{H}(m^t)$, and $i^t \in \omega$, $0 \leq i^t < \log(\log(l_{m^t}))$,
- $\mathbf{nor}[t] = f_{k_{m^t}}(|A^t|, i^t)$, $m^t_{\mathrm{dn}} = m^t$, $m^t_{\mathrm{up}} = m^t + 1$ and $\mathrm{pos}(t) = A^t$.

For $t \in K$ we let

$$\Sigma(t) = \{s \in K : m^s = m^t \ \&\ A^s \subseteq A^t \ \&\ i^s \geq i^t\}.$$

Lemma 4.4. *(K, Σ) is a local forgetful good creating pair for $\mathbf{H}$. The forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ collapses $\mathfrak{c}$ to $\aleph_0$.*

Proof. It is straightforward to check that $(K^{\overline{m}}, \Sigma^{\overline{m}})$ inherits the sufficient badness of $((K^1)^{\overline{m}}, (\Sigma^1)^{\overline{m}})$ (remember 4.2(1)). $\square$

We are now going to define the desired factoring $\mathbb{Q}^*_\infty(K,\Sigma) \simeq \mathbb{P}^0 \times \mathbb{P}^1$ into proper factors $\mathbb{P}^0, \mathbb{P}^1$. For this we recursively define an increasing sequence $\bar{n} = \langle n_i : i < \omega\rangle$ so that $n_0 = 0$ and n_{i+1} is large enough such that

$$k_{n_{i+1}} \geq \prod_{j<n_i} \mathbf{H}(j).$$

We put $U^0 = \bigcup\limits_{i<\omega} [n_{2i}, n_{2i+1})$ and $U^1 = \bigcup\limits_{i<\omega} [n_{2i+1}, n_{2i})$ and we let $\pi^0 : \omega \longrightarrow U^0$ and $\pi^1 : \omega \longrightarrow U^1$ be the increasing enumerations.

Definition 4.5. Let $\ell \in \{0,1\}$. We define $\mathbf{H}^\ell = \mathbf{H} \circ \pi^\ell$ and we introduce K^ℓ, Σ^ℓ as follows.

(1) K^ℓ consist of all creatures $t \in \mathrm{CR}[\mathbf{H}^\ell]$ such that
  - $\mathbf{dis}[t] = \langle m^t, A^t, i^t\rangle$ for some $m^t < \omega$ and $\emptyset \neq A^t \subseteq \mathbf{H}^\ell(m^t)$, and $i^t \in \omega$, $0 \leq i^t < \log(\log(l_n))$, where $n = \pi^\ell(m^t)$,
  - $m^t_{\mathrm{dn}} = m^t$, $m^t_{\mathrm{up}} = m^t + 1$, $\mathrm{pos}(t) = A^t$ and $\mathbf{nor}[t] = f_{k_n}(|A^t|, i^t)$ (where again $n = \pi^\ell(m^t)$).

(2) For $t \in K^\ell$ we let

$$\Sigma^\ell(t) = \{s \in K^\ell : m^s = m^t \ \&\ A^s \subseteq A^t \ \&\ i^s \geq i^t\}.$$

Lemma 4.6. (1) *For $\ell \in \{0,1\}$, (K^ℓ, Σ^ℓ) is a local forgetful good creating pair for $\mathbf{H}$.*

(2) *Let $\bar{m} = \langle m_i : i < \omega\rangle$ and $\bar{\varepsilon} = \langle \varepsilon_i : i < \omega\rangle$ be such that $\pi^0(m_i) = n_{2i}$ and $\varepsilon_i = 1/k_{n_{2i}}$. Then (K^0, Σ^0) has the $(\bar{\varepsilon}, \bar{m})$–halving property.*

(3) *Let $\bar{m} = \langle m_i : i < \omega\rangle$ and $\bar{\varepsilon} = \langle \varepsilon_i : i < \omega\rangle$ be such that $\pi^1(m_i) = n_{2i+1}$ and $\varepsilon_i = 1/k_{n_{2i+1}}$. Then (K^1, Σ^1) has the $(\bar{\varepsilon}, \bar{m})$–halving property.*

Proof. (1) Should be clear.

(2) Let $t \in K^0$, $\mathbf{nor}[t] \geq 2$, $\mathbf{dis}[t] = \langle m, A, i^*\rangle$. Let $n = \pi^0(m) \in [n_{2i}, n_{2i+1})$ (so $m_i \leq m = m^t_{\mathrm{dn}}$). Define $j = \frac{\log(\log(|A|)) + i^*}{2}$ and let z such that $\log(\log(z)) = j$. Certainly $i^* < j < \log(\log(|A|)) \leq \log(\log(l_n))$. Let $t^* \in K^0$ be such that $\mathbf{dis}[t^*] = \langle m, A, j\rangle$. Clearly $t^* \in \Sigma^0(t)$. We are going to argue that t^* is an ε_i–half of t in $\Sigma^0(t)$.

By Lemma 4.2(2), $\mathbf{nor}[t^*] = \mathbf{nor}[t] - \frac{1}{k_n} \geq \mathbf{nor}[t] - \varepsilon_i$. Now let $s \in \Sigma^0(t^*)$ be such that $\mathbf{nor}[s] > 1$. Let $\mathbf{dis}[s] = \langle m, A', i' \rangle$, thus $A' \subseteq A$ and $i' \geq j$. Pick $t_0 \in K^0$ such that $\mathbf{dis}[t_0] = \langle m, A', i^* \rangle$. Then $t_0 \in \Sigma^0(t)$ and $\mathrm{pos}(t_0) = A' = \mathrm{pos}(s)$. Also, $\mathbf{nor}[s] > 1$ implies $\log(\log(|A'|)) > i' \geq j$. By the definition of z we conclude $|A'| > z$. Applying this together with Lemma 4.2(4) we obtain

$$\mathbf{nor}[t_0] = f_{k_n}(|A'|, i^*) \geq f_{k_n}(z, i^*) = f_{k_n}(|A|, i^*) - \frac{1}{k_n} \geq \mathbf{nor}[t] - \varepsilon_i.$$

(3) Like (2) above. □

Corollary 4.7. (1) *The forcing notions $\mathbb{Q}^*_\infty(K^\ell, \Sigma^\ell)$ (for $\ell = 0, 1$) are proper.*

(2) *Let $\mathbb{Q} = \{p \in \mathbb{Q}^*_\infty(K, \Sigma) : i(p) = n_i,\ i < \omega\}$. Then $\mathbb{Q}$ is a dense suborder of $\mathbb{Q}^*_\infty(K, \Sigma)$ and it is isomorphic with a dense suborder of the product $\mathbb{Q}^*_\infty(K^0, \Sigma^0) \times \mathbb{Q}^*_\infty(K^1, \Sigma^1)$. Consequently, the latter forcing collapses $\mathfrak{c}$ to $\aleph_0$.*

Proof. (1) Let $\bar{m}, \bar{\varepsilon}$ be as in 4.6(2). By the choice of $\bar{n}$ we have

$$|\prod_{n<m_i} \mathbf{H}^0(n)| = |\prod\{\mathbf{H}(j) : j \in \bigcup_{\ell<i} [n_{2\ell}, n_{2\ell+1})\}| \leq \prod_{j<n_{2i-1}} \mathbf{H}(i) \leq k_{n_{2i}} = 1/\varepsilon_i.$$

Consequently, Theorem 3.3 and Lemma 4.6(1,2) imply that $\mathbb{Q}^*_\infty(K^0, \Sigma^0)$ is proper. Similarly for $\mathbb{Q}^*_\infty(K^1, \Sigma^1)$

(2) Should be clear. □

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