MODELS OF PA: STANDARD SYTEMS WITHOUT MINIMAL ULTRAFILTERS
SH944

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Abstract. We prove that $\mathbb{N}$, the standard model of arithmetic, has an uncountable elementary extension N such that there is no ultrafilter on the Boolean Algebra of subsets of $\mathbb{N}$ represented in N which is minimal (i.e. as in Rudin-Keisler order for partitions represented in N).

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0. Introduction

Enayat [Ena08], Question III, asked (see Definition 0.4(1)):

Question 0.1. Can we prove in ZFC that there is an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}$ carries no minimal ultrafilter?

He proved the existence of examples, for the stronger notion "2-Ramsey ultrafilter". In [She11] we prove that there is an arithmetically closed Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that any expansion $\mathbb{N}^+$ of $\mathbb{N}$ by any uncountably many members of $\mathbf{B}$ has this property, i.e. the family of definable subsets of $\mathbb{N}^+$ carries no 2-Ramsey ultrafilter.

We deal here with Question 0.1, proving that there is such a family of cardinality $\aleph_1$, this implies the version in the abstract; (since it it well-known that every arithmetically closed family of cardinality at most $\aleph_1$ can be realized as the standard system of some elementary extension of $\mathbb{N}$, as shown by Knight and Nadel [KN82]). We use forcing but the result is proved in ZFC. On other problems from [Ena08] see Enayat-Shelah [ES11] and [She15], [She11].

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Notation 0.2. 1) Let pr:$\omega \times \omega \to \omega$ be the standard pairing function (i.e. $\mathrm{pr}(n,m) = \binom{n+m}{2} + n$, so one to one onto two-place function).
2) Let $\mathcal{A}$ denote a subset of $\mathcal{P}(\omega)$.
3) Let $\mathrm{BA}(\mathcal{A})$ be the Boolean algebra of subsets of ω which $\mathcal{A} \cup [\omega]^{<\aleph_0}$ generates.
4) Let D denote a non-principal ultrafilter on $\mathcal{A}$, meaning that $D \subseteq \mathcal{A}$ and there is a unique non-principal ultrafilter D' on the Boolean algebra $\mathrm{BA}(\mathcal{A})$ satisfying $D = D' \cap \mathcal{A}$, notice that in Definition 0.4 below the distinction between an ultrafilter on $\mathcal{A}$ and on $\mathrm{BA}(\mathcal{A})$ makes a difference.
5) τ denotes a vocabulary extending $\tau_{\mathrm{PA}} = \tau_{\mathbb{N}} = \{0, 1, +, \times, <\}$, usually countable.
6) $\mathrm{PA}(\tau)$ is Peano arithmetic for the vocabulary τ. A model N of $\mathrm{PA}(\tau)$ is called ordinary if $N{\restriction}\tau_{\mathrm{PA}}$ extends $\mathbb{N}$; usually the models will be ordinary.
7) $\varphi(N, \bar{a})$ is $\{b : N \models \varphi[b, \bar{a}]\}$ where $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ and $\bar{a} \in {}^{\ell g(\bar{y})}N$.
8) $\mathrm{Sym}(A)$ is the set (or group) of permutations of N.
9) For sets u, v of ordinals let $\mathrm{OP}_{v,u}$, "the order preserving function from u to v" be defined by: $\mathrm{OP}_{v,u}(\alpha) = \beta$ <u>iff</u> $\beta \in v, \alpha \in u$ and $\mathrm{otp}(v \cap \beta) = \mathrm{otp}(u \cap \alpha)$.
10) We say $u, v \subseteq$ Ord form a Δ-system pair when $\mathrm{otp}(u) = \mathrm{otp}(v)$ and $\mathrm{OP}_{v,u}$ is the identity on $u \cap v$.

Definition 0.3. 1) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\text{ar-cl}(\mathcal{A}) = \{B \subseteq \omega : B$ is first order defined in $(\mathbb{N}, A_1, \ldots, A_n)$ for some $n < \omega$ and $A_1, \ldots, A_n \in \mathcal{A}\}$. This is called the arithmetic closure of $\mathcal{A}$.
2) For a model N of $\mathrm{PA}(\tau)$ let the standard system of N, $\mathrm{SSy}(N)$ be $\{\varphi(M, \bar{a}) \cap \mathbb{N} : \varphi(x, \bar{y}) \in \mathbb{L}(\tau)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M\}$ so $\subseteq \mathcal{P}(\omega)$ for any ordinary model M isomorphic to N, see 0.2(6).

Definition 0.4. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.
0) Let $\mathrm{cd}_0 : \mathcal{H}(\aleph_0) \to \omega$ be one to one, and interpreting $\mathcal{H}(\aleph_0)$ inside $\mathbb{N}$ it is (first order) definable by a bounded formula in $\mathbb{N}$, i.e. $\{\mathrm{cd}_0(x, y) : x \in y \in \mathcal{H}(\aleph_0)\}$ is, and it maps $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$. For $h \in {}^{\omega}\omega$ let $\mathrm{cd}(h) = \{\mathrm{pr}(n, h(n)) : n < \omega\}$, where pr is the standard pairing function of ω, see 0.2(1) and generally for $H \subseteq \mathcal{H}(\aleph_0)$ we let $\mathrm{cd}(H) := \{\mathrm{cd}_0(x) : x \in H\}$; this applies, e.g. to $h \in {}^{[\omega]^k}\omega$.
1) D, an ultrafilter on $\mathcal{A}$, is called minimal <u>when</u>: if $h \in {}^{\omega}\omega$ and $\mathrm{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h{\restriction}X$ is constant or one-to-one.

2) D, an ultrafilter on $\mathcal{A}$, is called Ramsey when: if $k < \omega$ and $h : [\omega]^k \to \{0,1\}$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h{\restriction}[X]^k$ is constant. Similarly k-Ramsey.
3) D, a non-principal ultrafilter on $\mathcal{A}$, is called a Q-point when if $h \in {}^\omega\omega$ is increasing and $\text{cd}(h) \in \mathcal{A}$ then for some increasing sequence $\langle n_i : i < \omega\rangle$ we have $i < \omega \Rightarrow h(2i) \le n_i < h(2i+1)$ and $\{n_i : i < \omega\} \in D$.

Remark 0.5. In [She11] we also use the following notions:
1) D is called 2.5-Ramsey or self-definably closed when: if $\bar{h} = \langle h_i : i < \omega\rangle$ and $h_i \in {}^\omega(i+1)$ and $\text{cd}(\bar{h}) = \{\text{cd}(i, \text{cd}(n, h_i(n)) : i < \omega, n < \omega\}$ belongs to $\mathcal{A}$ then for some $g \in {}^\omega\omega$ we have: $\text{cd}(g) \in \mathcal{A}$ and $(\forall i)[g(i) \le i \wedge \{n < \omega : h_i(n) = g(i)\} \in D]$; this follows from 3-Ramsey and implies 2-Ramsey.
2) D is weakly definably closed when: if $\langle A_i : i < \omega\rangle$ is a sequence of subsets of ω and $\{\text{pr}(n,i) : n \in A_i \text{ and } i < \omega\} \in \mathcal{A}$ then $\{i : A_i \in D\} \in D$, (follows from 2-Ramsey).

Definition 0.6. 1) $\mathbb{L}(\mathbf{Q})$ is first order logic when we add the quantifier $\mathbf{Q}$ where $(\mathbf{Q}x)\varphi$ means that there are uncountable many x's satisfying φ.
2) $\mathbb{L}_{\aleph_1,\aleph_0}(\mathbf{Q})$ is defined parallely.

See on those logics Keisler [Kei71]. We shall use Laver forcing in the proof of Theorem 1.1, so let us define this forcing notion.

Definition 0.7. Let $T \subseteq \{\eta \in {}^{\omega>}\omega : \eta \text{ increasing}\}$ be a subtree. For $a \in T$ let $\text{suc}_T(a) = \{a\hat{\ }\langle i\rangle \in T : i \in \omega\}$. The trunk $\text{tr}(T)$ of T is a maximal element $a \in T$ such that $a \le_T b$ or $b \le_T a$ for every $b \in T$.

Such a tree T will be called a Laver tree iff $s = \text{tr}(T)$ and for every $t \in T$ such that $s \le t$, the set $\text{suc}_T(t)$ is infinite.

We define the forcing notion $\mathbb{Q}$ (= Laver forcing) as follows. A condition $T \in \mathbb{Q}$ is a Laver tree. If $S, T \in \mathbb{Q}$ then $S \le_{\mathbb{Q}} T$ iff $S \supseteq T$. If $\mathbf{G} \subseteq \mathbb{Q}$ is generic, then $\underset{\sim}{\eta}[\mathbf{G}] := \{a \in {}^{\omega>}\omega : \exists T \in \mathbf{G}, a \text{ is the trunk of } T\}$ will be called a Laver real.

Claim 0.8. *If $\boxtimes$ then $\boxplus$ where:*

- $\boxtimes$ (a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha(*), \beta < \alpha(*)\rangle$ *is a CS iteration*
  - (b) $\alpha(*) < \omega_1, k(*) < \omega$ *and* $\beta(k) < \alpha(*)$ *for* $k < k(*)$
  - (c) *each* $\underset{\sim}{\mathbb{Q}}_\alpha$ *is the Laver forcing (in* $\mathbf{V}^{\mathbb{P}_\alpha}$*) and* $\underset{\sim}{\eta}_\alpha$ *its generic*
  - (d) $h \in ({}^\omega\omega)^{\mathbf{V}}$
  - (e) $p \in \mathbb{P}_{\alpha(*)}$
  - (f) $p \Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\underset{\sim}{B}_k \subseteq \omega$ *and* $|\underset{\sim}{B}_k \cap [\underset{\sim}{\eta}_{\beta(k)}(n+1), \underset{\sim}{\eta}_{\beta(k)}(n+2))| \le h(\underset{\sim}{\eta}_{\beta(k)}(n))$ *for every* n *large enough" for* $k < k(*)$
- $\boxplus$ *for some* p_1, p_2 *and* B^*_k *for* $k < k(*)$ *we have*
  - (a) $\mathbb{P}_{\alpha(*)} \models$ "$p \le p_\ell$" *for* $\ell = 1, 2$
  - (b) $B^*_k \subseteq \omega$ *(from* $\mathbf{V}$*)*
  - (c) $p_1 \Vdash$ "$\underset{\sim}{B}_k \subseteq^* B^*_k$"
  - (d) $p_2 \Vdash$ "$\underset{\sim}{B}_k \subseteq^* (\omega \backslash B^*_k)$".

Proof. rm Without loss of generality $\alpha(*) \ge 1$. Clearly letting $\underset{\sim}{B}_* = \cup\{\underset{\sim}{B}_k : k < k(*)\}$ we have

(∗) $p \Vdash_{\mathbb{P}_{\alpha(*)}}$ "for every large enough n the set $\underset{\sim}{B}_* \cap [\underset{\sim}{\eta}_0(n+1), \underset{\sim}{\eta}_0(n+2))$ has $\le \eta_0(n)$ members".

Now by the properties of iterating Laver forcing ([Lav76] or see [She98, Ch.VI]), we have:

(∗) if $\mathbf{G}_1 \subseteq \mathbb{P}_1$ is generic over $\mathbf{V}$ and $\eta = \underset{\sim}{\eta}_0[\mathbf{G}_1]$ then

$\Vdash_{\mathbb{P}_{\alpha(*)}/\mathbf{G}_1}$ " if $\underset{\sim}{B} \subseteq \omega$ and in $\underset{\sim}{B} \cap [\eta(n), \eta(n+1))$ there are $\leq \eta(n))$ elements for every n large enough then for some $B' \in \mathbf{V}[\mathbf{G}_1], B' \subseteq \omega, \underset{\sim}{B} \subseteq B'$ and $B' \cap [\eta(n), \eta(n+1)))$ has $\leq (\eta(n))^n$ members for every n large enough".

Now this applies in particular to $\underset{\sim}{B} = \underset{\sim}{B}_*$ getting $\underset{\sim}{B}'$. Hence without loss of generality $\alpha(*) = 1$ so we can replace $\mathbb{P}_1$ by $\mathbb{Q}_0$, Laver forcing; also for a dense set of $p \in \mathbb{Q}_0$ we have: if $\eta \in p$ is of length $n+1$ so an increasing sequence of natural numbers, then $p^{[\eta]} := \{\nu \in p : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$ forces a value b_η to $\underset{\sim}{B}' \cap [0, \eta(n))$ so necessarily $|b_\eta| \leq \eta(n-1)$ when $n > 1$.

By thinning p, without loss of generality if $\eta \in p$ and $u_\eta = \{n : \eta\hat{\ }\langle n\rangle \in p\}$ is infinite (equivalently is not a singleton) then $\langle b_{\eta\hat{\ }<n>} : n \in u_\eta\rangle$ is a Δ-system.

The rest of the proof should be easy, too. $\square_{0.8}$

1. No minimal ultrafilter on the standard system

Theorem 1.1. *Assume that $\mathbb{N}_*$ is an expansion of $\mathbb{N}$ with countable vocabulary or $\mathbb{N}_*$ is an ordinary model of PA_τ, for some countable $\tau \supseteq \tau_{PA}$ such that $\mathbb{N}_*$ is countable. Then there is M such that*

(a) $\mathbb{N}_* \prec M$
(b) $\|M\| = \aleph_1$
(c) $\mathrm{SSy}(M)$, *the standard system of M, see Definition 0.3, has no minimal ultrafilter on it, see Definition 0.4; moreover*
(d) *there is no Q-point on* $\mathrm{SSy}(M)$
(e) $\mathrm{SSy}(M)$ *is arithmetically closed.*

Proof. Stage A:
Without loss of generality $\mathbb{N}_*$ is the Skolem Hull of $\emptyset$ as we can expand it by $\aleph_0$ individual constants.

We shall choose a sentence $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau^*)$ with $\tau^* \supseteq \tau(\mathbb{N}_*)$ and prove that it has a model, and for every model M^+ of ψ, the model $M^+ {\restriction} \tau(\mathbb{N}_*)$ is as required. By the completeness theorem for $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ it is enough to prove that ψ has a model in some forcing extension; of course it is crucial that ψ can be explicitly defined hence $\in \mathbf{V}$.

Stage B:
Recall $\mathrm{cd} = \mathrm{cd}_0 : \mathcal{H}(\aleph_0) \to \omega$ be one-to-one onto and definable in $\mathbb{N}$ by a bounded formula in the natural sense; see 0.4(0).

Let $\mathbf{V}_0 = \mathbf{V}$ and $\lambda = (2^{\aleph_0})^+$.

Let $\mathbb{R}_0 =$ Levy$(\aleph_1, 2^{\aleph_0})$, let $\mathbf{G}_0 \subseteq \mathbb{R}_0$ be generic over $\mathbf{V}_0$ and let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_0]$, i.e. in $\mathbf{V}_0^{\mathbb{R}_0}$ we have CH.

In $\mathbf{V}_1$ we have $\lambda = \aleph_2$ and let $\mathbb{R}_1$ be $\mathbb{P}_{\omega_2}$ where $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \omega_2, \beta < \omega_2 \rangle$ is a CS iteration, each $\underset{\sim}{\mathbb{Q}}_\alpha$ is a Laver forcing; there are many other possibilities, let $\underset{\sim}{\eta}_\alpha \in {}^\omega\omega$ (increasing) be the $\mathbb{P}_{\alpha+1}$-name of the $\underset{\sim}{\mathbb{Q}}_\alpha$-generic real and $\underset{\sim}{\nu}_\alpha = \langle \mathrm{cd}(\underset{\sim}{\eta}_\alpha {\restriction} n) : n < \omega) \rangle$. Let $\mathbf{G}_1 \subseteq \mathbb{R}_1$ be generic over $\mathbf{V}_1$ and $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}_1]$ and let $\eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}_1], \nu_\alpha = \langle \mathrm{cd}(\eta_\alpha {\restriction} n) : n < \omega \rangle = \underset{\sim}{\nu}_\alpha[\mathbf{G}_1]$.

Let D^2 be a non-principal ultrafilter on ω in the universe $\mathbf{V}_2$.

$\boxplus_1$ In the universe $\mathbf{V}_2$ let $M_1 = \mathbb{N}_*^\omega / D^2$, let $a_\alpha = \eta_\alpha / D^2 \in M_1$

and note

$\boxplus_2$ $\mathrm{SSy}(M_1) = \mathcal{P}(\mathbb{N})^{\mathbf{V}_2}$ hence is arithmetically closed

$\boxplus_3$ let $f_1 \in \mathbf{V}_2$ be the function from $\lambda = \omega_2^{\mathbf{V}_1} = \omega_2^{\mathbf{V}_2}$ into M_1 defined by $f_1(\alpha) = a_\alpha$.

Stage C:
In $\mathbf{V}_1$ (yes, not in $\mathbf{V}_2$) let the forcing notion $\mathbb{R}_2 := \mathbb{P}^+_{\omega_2}$ and the set K be defined as follows (so $\mathbf{B} \in \mathbf{V}_1$ below, which is equivalent to $\mathbf{B} \in \mathbf{V}_0$, similarly for u; so in $\boxplus_4(\alpha)$, $\underset{\sim}{A}$ is a $\mathbb{P}_{\omega_2}$-name):

$\boxplus_4$ (α) $K := \{(\alpha, u, \underset{\sim}{A}) : u \subseteq \lambda$ is countable, $\alpha \in u, \underset{\sim}{A} = \mathbf{B}(\ldots, \underset{\sim}{\eta}_\beta, \ldots)_{\beta \in u}, \mathbf{B}$ a Borel function from ${}^{\text{otp}(u)}({}^\omega\omega)$ to $\mathcal{P}(\omega)$ such that $\Vdash_{\mathbb{P}_{\omega_2}}$ "$\underset{\sim}{A} \cap [\underset{\sim}{\eta}_\alpha(n+1), \underset{\sim}{\eta}_\alpha(n+2))$ has $\leq \underset{\sim}{\eta}_\alpha(n)$ members; moreover $0 = \lim_n(|\underset{\sim}{A} \cap [\underset{\sim}{\eta}_\alpha(n+1), \underset{\sim}{\eta}_\alpha(n+2))/\underset{\sim}{\eta}_\alpha(n)|$"$\}$

(β) $\mathbf{p} \in \mathbb{P}^+_{\omega_2}$ iff

(a) $\mathbf{p} = (p, h) = (p_\mathbf{p}, h_\mathbf{p})$

(b) $p \in \mathbb{P}_{\omega_2}$

(c) h a function from some finite subset $K_\mathbf{p}$ of K to ω_1

(d) if $(\alpha_\ell, u_\ell, \underset{\sim}{A}_\ell) \in K_\mathbf{p}$ for $\ell = 1, 2$ and $h(\alpha_1, u_1, \underset{\sim}{A}_1) = h(\alpha_2, u_2, \underset{\sim}{A}_2)$ and $u_1 \subseteq \alpha_2$ <u>then</u> $p \Vdash_{\mathbb{P}_{\omega_2}}$ "$\underset{\sim}{A}_1 \cap \underset{\sim}{A}_2$ is finite"

(γ) $\mathbb{P}^+_{\omega_2} \models \mathbf{p} \leq \mathbf{q}$ iff:

(a) $\mathbb{P}_{\omega_2} \models p_\mathbf{p} \leq p_\mathbf{q}$

(b) $h_\mathbf{p} \subseteq h_\mathbf{q}$.

Now

$(*)_0$ if $p \in \mathbb{P}_{\omega_2}, \alpha < \omega_2$ and $p \Vdash$ "$\underset{\sim}{A} \subseteq \omega$ satisfies $\underset{\sim}{A} \cap [\underset{\sim}{\eta}_\alpha(n+1), \underset{\sim}{\eta}_\alpha(n+2))$ has $\leq \underset{\sim}{\eta}_\alpha(n)$ members for every n large enough and $0 = \lim\langle|\underset{\sim}{A} \cap [\underset{\sim}{\eta}_\alpha(n+1), \underset{\sim}{\eta}_\alpha(n+2))|/\underset{\sim}{\eta}_\alpha(n) : n < \omega\rangle$" <u>then</u> we can find a triple $(q, u, \underset{\sim}{A}')$ such that:

(α) $\mathbb{P}_{\omega_2} \models$ "$p \leq q$"

(β) $\text{Dom}(q) = u$

(γ) u a countable set of ordinals $< \lambda$ (in $\mathbf{V}_1$ equivalently in $\mathbf{V}_0$)

(δ) $q \Vdash$ "$\underset{\sim}{A} = \underset{\sim}{A}'$"

(ε) $\underset{\sim}{A}' = \mathbf{B}(\ldots, \underset{\sim}{\eta}_{\alpha_i}, \ldots)_{i < \text{otp}(u)}$ where α_i is the i-th member of u, for some Borel function $\mathbf{B}$ from ${}^{\text{otp}(u)}({}^\omega\omega)$ to $\mathcal{P}(\omega)$ so $\mathbf{B} \in \mathbf{V}_1$ equivalently $\mathbf{V}_0$

(ζ) $q(\alpha_i) = \mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{\alpha_j}, \ldots)_{j<i}$ for every $i <$ otp(u) for some Borel fucntion $\mathbf{B}_i$ from ${}^i({}^\omega\omega)$ to Laver forcing, of course, $\mathbf{B}_i$ is from $\mathbf{V}_0$.

[Why? Standard proof.]

$(*)_1$ $\mathbb{P}^+_{\omega_2}$ satisfies the $\aleph_2$-c.c.

[Why? We need a property of the iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ stated in Claim 0.8. In more detail, given a sequence $\langle \mathbf{p}_\alpha : \alpha < \omega_2\rangle$ of members of $\mathbb{P}^+_{\omega_2}$, for each $\alpha < \omega_2$, let $\mathbf{p}_\alpha = (p_\alpha, h_\alpha)$; and without loss of generality for each $(\alpha^*_1, u^*_1, \underset{\sim}{A}^*_1) \in K_{\mathbf{p}_\alpha}$ for some $u^1, \underset{\sim}{A}^1$, the tuple $(p_\alpha, u^1, \underset{\sim}{A}^1)$ is like $(q, u, \underset{\sim}{A}')$ in $(*)_0, (\beta) - (\zeta)$ and $(\alpha, u, \underset{\sim}{A}) \in \text{Dom}(h_\alpha) \Rightarrow u \subseteq \text{Dom}(p_\alpha)$. Letting $u_\alpha = \text{Dom}(p_\alpha)$, we can find a stationary $S \subseteq \{\delta < \omega_2 : \text{cf}(\delta) = \aleph_1\}$ and $p_*, \gamma(*)$ such that:

- $u_\delta \cap \delta = u_*$ for $\delta \in S$ and $u_\alpha \subseteq \delta$ for $\alpha < \delta \in S$
- $p_\delta \restriction \delta \leq p_* \in \mathbb{P}_\delta$ for $\delta \in S$
- without loss of generality $p_\delta \restriction \delta = p_*$ for $\delta \in S$
- $\text{otp}(u_\delta) = \gamma(*)$ for $\delta \in S$

- if $\delta_1, \delta_2 \in S$ then the order preserving function $\mathrm{OP}_{u_{\delta_2}, u_{\delta_1}}$ from u_{δ_1} onto u_{δ_2} maps $\mathbf{p}_{\delta_1}$ to $\mathbf{p}_{\delta_2}$.

Let $\delta(*) = \mathrm{Min}(S)$ and $\mathbf{G}^1_{\delta(*)} \subseteq \mathbb{P}_{\delta(*)}$ be generic over $\mathbf{V}_1$ such that $p_* \in \mathbf{G}^1_{\delta(*)}$. Now we shall apply the conclusion of Claim 0.8 to $\mathbb{P}_{\omega_2}/\mathbf{G}_{\delta(*)}$ and we shall work in $\mathbf{V}[G^1_{\delta(*)}]$.

For $\delta \in S$, let $\alpha_\delta = \mathrm{otp}(u_\delta \backslash \delta_*), \mathbf{h}_\delta$ be the order preserving function from α_δ onto $u_\delta \backslash \delta$ and $(p'_\delta, h'_\delta) \in \mathbb{P}_{\alpha_\delta}$ be such that $\mathbf{h}_\delta$ maps (p'_δ, h'_δ) to (p_δ, h_δ). Clearly $\alpha_\delta, p'_\delta, h'_\delta$ are the same for all $\delta \in S$ so call them $\alpha(*), p', h'$ and applying 0.8 with $p', (\{\alpha, \underset{\sim}{A}$): for some u the tuple $(\alpha, u, \underset{\sim}{A})$ belongs to $\mathrm{Dom}(h)\}$ here stands for $p, \{(\alpha_k, \underset{\sim}{\beta}_k) : k < k(*)\}$ there and get p'_1, p'_2 as there.

Let $\delta_1 < \delta_2$ be from S, let q_{δ_1} be $\mathbf{h}_{\delta_1}(p'_1), q_{\delta_2}$ be $\mathbf{h}_{\delta_2}(p'_2)$. Easily $p_{\delta_\ell} \leq q_{\delta_\ell}$ and $q_{\delta_1} \cup q_{\delta_2}$ is a common upper bound of $p_{\delta_1}, p_{\delta_2}$ in $\mathbb{P}^+_{w_2}/\mathbf{G}^1_{\delta(*)}$.]

$(*)_2$ $\mathbb{P}^+_{\omega_2}$ collapses ω_1 to $\aleph_0$.

[Why? Easy but also we can use $\mathbb{P}^+_{\omega_2} \times \mathrm{Levy}(\aleph_0, \aleph_1)$ instead $\mathbb{P}^+_{\omega_2}$.]

$(*)_3$ the function $p \mapsto (p, \emptyset)$ is a complete embedding of $\mathbb{P}_{\omega_2}$ into $\mathbb{P}^+_{\omega_2}$.

[Why? Should be clear.]

Stage D: Let $\mathbf{G}_2 = \mathbf{G}^+_1 \subseteq \mathbb{P}^+_{\omega_2}$ be generic over $\mathbf{V}_1, \mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ and by $(*)_3$ without loss of generality $\mathbf{G}_1 = \{p : (p, h) \in \mathbf{G}_2\}$. So $\mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ is a generic extension of $\mathbf{V}_2$ and let $f_2 = \cup\{h : (p, h) \in \mathbf{G}_2\}$.

So

$(*)_4$ in $\mathbf{V}_3$ if $f_2(\alpha_1, u_1, \underset{\sim}{A}_1) = f_2(\alpha_2, u_2, \underset{\sim}{A}_2)$ and $u_1 \subseteq \alpha_2$ (hence $\alpha_1 \neq \alpha_2$), then $\underset{\sim}{A}_1[\mathbf{G}_1] \cap \underset{\sim}{A}_2[\mathbf{G}_1]$ is finite.

In $\mathbf{V}_3$ let M_2 be an elementary submodel of $(\mathcal{H}(\beth_\omega), \in, \ldots, \mathbf{V}_\ell \cap \mathcal{H}(\beth_\omega), \ldots)_{\ell=0,1,2}$ of cardinality $\lambda = \aleph_1^{\mathbf{V}_3}$ which includes $\{\alpha : \alpha \leq \lambda\} = \{\alpha : \alpha \leq \omega_1^{\mathbf{V}_3}\}, \{M_1, f_1, f_2, \mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2\}$ and (the universe of) M_1, see $\boxplus_1$ end of stage B, note that $\|M_2\| \subseteq |M_2|$.

Let f_0 be a one-to-one function from M_1 onto M_2, let M_3 be a model such that f_0 is an isomorphism from M_1 onto M_3. Lastly, let M_4 be M_3 expanded by $c_0 = \lambda = \omega_2^{\mathbf{V}_1} = \omega_1^{\mathbf{V}_3}, c_1^{M_4} = \omega_1^{\mathbf{V}}, c_2^{M_4} = M_1, d_{0,\ell}^{M_4} = \mathbf{G}_\ell, d_{1,\ell} = \mathbb{R}_\ell, d^{M_4} = \mathbb{N}_*, \langle d_{2,n}^{M_4} : n < \omega\rangle$ list the members of $\mathbb{N}_*, Q_0^{M_4} = |\mathbb{N}_*|, \in^{M_2} = \in^{\mathbf{V}_3} \restriction |M_2|, F_0^M = f_0, F_1^{M_4} = f_0 \circ f_1$, see end of Stage B, $F_2^{M_4} = f_2, P_\ell^M = \mathbf{V}_\ell \cap M_2$ for $\ell = 0, 1, 2$ (so F_ℓ is a unary function symbol, P_ℓ is a unary predicate) and lastly $<_*^M$, a linear order of $|M_2| = |M_4|$ of order type $\omega_1^{\mathbf{V}_3}$.

We define the sentence ψ: it is the conjunction of the following countable sets and singletons of sentences of $\mathbb{L}_{\aleph_1, \aleph_0}(\mathbf{Q})$ in the vocabulary $\tau(M_4)$ such that $M^+ \models \psi$ iff:

(A) $M^+ \restriction \tau(\mathbb{N}_*)$ is isomorphic to $\mathbb{N}_*$, of cousre, $M^+ \restriction \tau(\mathbb{N}_*)$ has universe $Q_0^{M^+}$

(B) M^+ is uncountable, moreover $M^+ \models (\mathbf{Q}x)$ (x an ordinal $< c_0$)

(C) $<_*^{M^+}$ is a linear order

(D) every proper initial segment by $<_*^{M^+}$ is countable

(E) $(|M^+|, \in^{M^+})$ is a model ZFC$^-$ (even a model of $\mathrm{Th}(\mathcal{H}(\beth_\omega)^{\mathbf{V}_3}, \in)$)

(F) the function $F_1^{M^+} : \{a : M^+ \models$ "a an ordinal $< c_0$"$\} \to M^+$ is one-to-one

(G) $M^+ \models$ "K is as above"

(H) $F_2^{M^+} : K^{M^+} \to \{a : M \models$ "a an ordinal $< c_1$"$\}$ is as above

(I) $M^+ \models$ "for every B we have $B \in \mathcal{P}(\mathbb{N}) \wedge P_2(B)$ iff $B = A \cap \mathbb{N}$ for some definable subset of A in the model c_2".

It is easy to check that

$(*)_5$ $\psi \in \mathbf{V}_0$

$(*)_6$ $M_4 \models \psi$ in $\mathbf{V}_3$.

Hence as the completeness theorem for $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ gives absoluteness

$(*)_7$ ψ has a model in $\mathbf{V} = \mathbf{V}_0$ call it M_5.

By renaming without loss of generality

$(*)_8$ (a) if $M_5 \models$ "a is the n-th natural number" then $a = n$

(b) if $M_5 \models$ "$A \subseteq \omega$" then $A = \{n : M_5 \models$ "$n \in A$"$\}$

(c) if $M_5 \models$ "$b \in {}^\omega\omega$" then $b = \{(n_1, n_2) : M_5 \models f(n_1) = n_2\}$

$(*)_9$ let $N'_* = M_5 {\restriction} \tau(\mathbb{N}_*)$, so isomorphic to N_*, let $N = M_5 {\restriction} \{\in\}$

$(*)_{10}$

(a) let M'_1 be $c_2^{M_5}$ naturally defined

(b) so $M = M'_1$ is a model of $\mathrm{Th}(N'_*) = \mathrm{Th}(N_*), N'_* \prec M'_1$ and $\|M'_1\| = \aleph_1$

(c) let $\mathcal{A}$ be $\mathrm{SSy}(M)$, the standard system of M

Clearly

$(*)_{11}$ (a) $N \models$ "ZC"

(b) M is a model of $\mathrm{Th}(\mathbb{N}_*)$ and $N_* \prec M$

$(*)_{12}$ let $\mathbb{R}'_\ell = d_{1,\ell}^{M_5}$ and $\mathbf{G}'_\ell = d_{2,\ell}^{M_5}$ and let $\mathbf{V}'_\ell = (P_\ell^{M_5}, \in^{M_5})$ for $\ell = 0, 1, 2$.

Stage E:

Clearly M is an uncountable elementary extension of $\mathbb{N}_*$, by clauses (A),(B) of Stage D and without loss of generality $\|M\| = \aleph_1$, so M satisfies clauses (a),(b) of Theorem 1.1. To prove clause (e) recall $\boxplus_2$ and clause (I) above hence $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed; this implies $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{P}(\mathbb{N})$. Also clause (d) implies clause (c), anyhow to prove them, assume toward contradiction that D is an ultrafilter on $\mathcal{A}$ which is minimal or just a Q-point. Let $X = \{a : N \models$ "a is an ordinal $< \omega_1$"$\}$, so X is really an uncountable set. For each $a \in X$ define a sequence $\rho_a \in {}^\omega\omega$ by $\rho_a(n) = k$ iff $M^+ \models$ "$F_1(a)(n) = k$".

Clearly ρ_a is an increasing sequence in ${}^\omega\omega$, hence by the assumption toward contradiction, there is $A_a \in D \subseteq \mathcal{A}$ such that $A_a \cap [\rho_a(n+1), \rho_a(n+2))$ has at most one element (or just $\le \rho_a(n)$ elements) for each $n < \omega$.

So for some element $\underaccent{\sim}{A}_a$ of $N, N \models$ "$\underaccent{\sim}{A}_a$, in $\mathbf{V}'_1$, is a $\mathbb{R}_1$-name of a subset of ω and $\underaccent{\sim}{A}_a[\mathbf{G}'_1] = A_a$".

Clearly $M^+ \models$ "for some countable subset u of $\omega_2^{\mathbf{V}'_1} = \omega_1^{\mathbf{V}'_3}$ from $\mathbf{V}'_1$ and Borel function $\mathbf{B}$ from $\mathbf{V}'_1$ we have $A_a = \mathbf{B}_a(\ldots, \rho_b, \ldots)_{b \in u_a}$ (so some $p \in \mathbf{G}_2^+$ forces $\underaccent{\sim}{A}_a$ satisfies this)". So using $F_2^{M_5}$ there are $a_1 \neq a_2$ from X such that the parallel of clause $(\beta)(d)$ of stage C holds, see clause (G) of stage D, so two members of D are almost disjoint, contradiction. $\square_{1.1}$

Remark 1.2. 1) Note that in 1.1 we can replace $\mathbb{Q}_0$ by any forcing notion similar enough, see [RS99].
2) We can strengthen 1.1 by replacing "Q-point" by a weaker statement.

Similarly we can weaken the demands on how "thin" is $\underset{\sim}{B}$ in 0.8 and in the proof of 1.1.

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