EXAMPLES IN DEPENDENT THEORIES

ITAY KAPLAN AND SAHARON SHELAH

Abstract. In the first part we show a counterexample to a conjecture by Shelah regarding the existence of indiscernible sequences in dependent theories (up to the first inaccessible cardinal). In the second part we discuss generic pairs, and give an example where the pair is not dependent. Then we define the notion of directionality which deals with counting the number of coheirs of a type and we give examples of the different possibilities. Then we discuss non-splintering, an interesting notion that appears in the work of Rami Grossberg, Andrés Villaveces and Monica VanDieren, and we show that it is not trivial (in the sense that it can be different than splitting) whenever the directionality of the theory is not small. In the appendix we study dense types in RCF.

1. Introduction

This paper gives some examples of dependent theories that exemplify certain phenomenons. Recall,

Definition 1.1. A first order theory T is *dependent* (NIP) if it does not have the independence property which means: there are no formula $\varphi(x,y)$ and tuples $\langle a_i, b_s \mid i<\omega, s\subseteq\omega\rangle$ in $\mathfrak{C}$ such that $\varphi(a_i,b_s)$ if and only if $i\in s$.

1.1. **Existence of indiscernibles.** Indiscernible sequences are very important in model theory. Usually one uses Ramsey's theorem to prove their existence. Sometimes we want to have a stronger result. For instance, we may want that any large enough set contains an indiscernible sequence and indeed this was conjectured by Shelah for dependent theories. We will show that at least in some models of ZFC, one cannot hope for such a result to be true.

1.2. **Generic pairs.** In a series of papers ([She, She06, She11, She12a]), Shelah has proved (among other things) that dependent theories give rise to a "generic pair" of models (and in fact this characterizes dependent theories). The natural question is whether the theory of the pair is again dependent. The answer is no. We present an example of an ω-stable theory all of whose generic pairs have the independence property.

The first author's research was partially supported by the SFB 878 grant.

The first author was partially supported by SFB grant 878. The second author would like to thank the Israel Science Foundation for partial support of this research (Grants nos. 710/07 and 1053/11). No. 946 on the second author's list of publications.

1.3. **Directionality.** The directionality of a theory measures the number of finitely satisfiable global extensions of a complete type (these are also called coheirs). We say that a theory has *small* directionality if for every type p over a model M, the number of complete finitely satisfiable (in M) Δ-types which are consistent with p is finite for all finite sets Δ. The theory has *medium* directionality if this number is bounded by $|M|$, and it has *large* directionality if it is not small or medium. We give an equivalent definition (Theorem 4.21 below).

We provide examples of dependent theories of each kind of directionality, and calculate the directionality of some theories, including RCF and ACVF.

1.4. **Splintering.** This section is connected to the work of Rami Grossberg, Andrés Villaveces and Monica VanDieren. In [GVV] they study Shelah's Generic pair conjecture (which is now a theorem) and in their analysis, they came up with the notion of splintering which is similar to splitting. We show that in any dependent theory with medium or large directionality, splintering is different than splitting. We also provide an example of such a theory with small directionality, and prove this cannot happen in the stable realm.

1.5. **Dense types.** In the appendix, we study dense types in RCF. Namely, we show that $\operatorname{ded}\lambda$ — the supremum of the number of cuts of a linear order of size λ — equals the supremum of the number of dense types in a model of RCF of size λ. This is useful for the calculation of the directionality of RCF.

1.6. **Acknowledgment.** We would like to thank the anonymous referee for many useful comments and for suggesting to apply the method used for calculating the directionality of RCF to valued fields (in the previous version it was only shown that certain valued fields are not small). We would also like to thank Marcus Tressl with whom we discussed the directionality of RCF and Pierre Simon and Immanuel Halupczok for discussing valued fields with us.

1.7. **Notation.** When α and β are ordinals, we use left exponentiation ${}^{\beta}\alpha$ to denote the set of functions from β to α, as to not to confuse with ordinal (or cardinal) exponentiation. If there is no room for confusion, and A and B are some sets we use A^B instead. The set $\alpha^{<\beta}$ is the set of sequences (functions) $\bigcup\{{}^{\gamma}\alpha \,|\, \gamma < \beta\}$.

We do not distinguish elements and tuples unless we say so explicitly.

$\mathfrak{C}$ will be the monster model of the theory.

$S_n(A)$ is the set of all complete types in n variables over A. $S_{<\omega}(A)$ is the union $\bigcup_{n<\omega} S_n(A)$. $S(A)$ is the set of all types (perhaps with infinitely mane variables) over A.

For a set of formulas with a partition of variables, $\Delta(x, y)$, $L_\Delta(A)$ is the set of formulas of the form $\varphi(x, a), \neg\varphi(x, a)$ where $\varphi(x, y) \in \Delta$ and $a \in A$. $S_\Delta(A)$ is the set of all complete $L_\Delta(A)$-types. Similarly we may define $\operatorname{tp}_\Delta(b/A)$ as the set of formulas $\varphi(x, a)$ such that $\varphi(x, y) \in \Delta$ and $\mathfrak{C} \models \varphi(b, a)$. For a partial type $p(x)$ over A, $p \restriction \Delta = p \cap L_\Delta(A)$.

Usually we want to consider a set of formulas Δ without specifying a partition of the variables. In this case, for a tuple of variables x, Δ^x is a set of partitioned formulas induced from Δ by partitioning the formulas in Δ to (x, y) in all possible ways. Then $L^x_\Delta(A)$ is just $L_{\Delta^x}(A)$ and $S^x_\Delta(A)$, $p \restriction \Delta^x$ are defined similarly. If x is clear from the context, we omit it. So for instance, when p is a type in x over A, then $p \restriction \Delta$ is the set of all formulas $\varphi(x, a)$ where $\varphi(z, w) \in \Delta$.

2. Few indiscernibles

2.1. Introduction.

Definition 2.1. Let T be a theory. For a cardinal κ, $n \leq \omega$ and an ordinal δ, $\kappa \to (\delta)_{T,n}$ means: for every set $A \subseteq \mathfrak{C}^n$ of size κ, there is a non-constant sequence of elements of A of length δ which is indiscernible.

This definition was suggested by Grossberg and Shelah in [She86, pg. 208, Definition 3.1(2)] with a slightly different form[1].

There it is also conjectured:

Conjecture 2.2. [She86, pg. 209, Conjecture 3.3] *If T is dependent then for every cardinal μ there is some cardinal λ such that $\lambda \to (\mu)_{T,1}$.*

In stable theories this holds: it is known that for any λ satisfying $\lambda = \lambda^{|T|}$, $\lambda^+ \to (\lambda^+)_{T,n}$ (proved by Shelah in [She90], and follows from local character of non-forking). In [She86, pg. 209] it is proved that this conjecture does not hold for simple unstable theories. In [She12b], Shelah proved this conjecture for strongly dependent theories:

Fact 2.3. *If T is strongly dependent (see Definition 2.10 below), then for all $\lambda \geq |T|$, $\beth_{|T|^+}(\lambda) \to (\lambda^+)_{T,n}$ for all $n < \omega$.*

This conjuncture is connected to a result by Shelah and Cohen: in [CS09], they proved that a theory is stable if and only if it can be presented in some sense in a free algebra in a fixed vocabulary but allowing function symbols with infinite arity. If this result could be extended to: a theory is dependent if and only if it can be represented as an algebra with ordering, then this could be used to prove existence of indiscernibles.

In this section, we shall show:

Theorem 2.4. *There is a countable dependent theory T such that if κ is smaller than the first inaccessible cardinal, then for all $n \in \omega$, $\kappa \not\to (\omega)_{T,n}$.*

[1]The definition there is: $\kappa \to (\delta)_{T,n}$ if and only if for each sequence of length κ (of n-tuples), there is an indiscernible sub-sequence of length δ. For us there is no difference because we are dealing with examples where $\kappa \not\to (\mu)_{T,n}$. It is also not hard to see that when δ is an infinite cardinal these two definitions are equivalent.

Thus, Conjecture 2.2 fails in a model of ZFC with no inaccessible cardinals. It appears in a more precise way as Theorem 2.18 below.

An even stronger result can be obtained, namely:

Fact 2.5. [KS10]*For every θ there is a dependent theory T of size θ such that for all κ and δ,* $\kappa \to (\delta)_{T,1}$ if and only if $\kappa \to (\delta)_{\theta}^{<\omega}$.

where:

Definition 2.6. $\kappa \to (\delta)_{\theta}^{<\omega}$ means: for every function $c : [\kappa]^{<\omega} \to \theta$ there is an homogeneous sub-sequence of length δ (i.e., there exists $\langle \alpha_i \,|\, i < \delta \rangle \in {}^{\delta}\kappa$ and $\langle c_n \,|\, n < \omega \rangle \in {}^{\omega}\theta$ such that $c(\alpha_{i_0}, \ldots, \alpha_{i_{n-1}}) = c_n$ for every $i_0 < \cdots < i_{n-1} < \delta$).

By [KS10], whenever $|T| \leq \theta$, $\kappa \to (\delta)_{\theta}^{<\omega}$ always implies that $\kappa \to (\delta)_{T,n}$ for all $n < \omega$, so this is the best result possible. However, the proof of Theorem 2.5 is considerably harder, so it is given in a subsequent work.

The second part of this section is devoted to giving a related example in the field of real numbers. By Fact 2.3, as RCF is strongly dependent, we cannot prove Theorem 2.4 for RCF, but instead we show that the requirement that $n < \omega$ is necessary:

Theorem 2.7. *If κ is smaller than the first strongly inaccessible cardinal, then $\kappa \not\to (\omega)_{RCF,\omega}$.*

This is Theorem 2.23 below.

Notes. It was unknown to us that in 2011 Kudaĭbergenov proved a related result, which refutes a strong version of Conjecture 2.2, namely that $\beth_{\omega+\omega}(\mu + |T|) \to (\mu)_{T,1}$. He proved that for every ordinal α there exists a dependent theory (we have not checked whether it is strongly dependent) T_α such that $|T_\alpha| = |\alpha| + \aleph_0$ and $\beth_\alpha(|T_\alpha|) \not\to (\aleph_0)_{T_\alpha,1}$ and thus seem to indicate that the bound in Fact 2.3 is tight. See [Kud11].

The idea of the construction. The counterexample is a "tree of trees" with functions connecting the different trees. For every η in the tree $2^{<\omega}$ we shall have a predicate P_η and an ordering $<_\eta$ such that $(P_\eta, <_\eta)$ is a dense tree. In addition we shall have functions $G_{\eta,\eta\frown\langle i\rangle} : P_\eta \to P_{\eta\frown\langle i\rangle}$ for $i = 0, 1$. The idea is to prove that $\kappa \not\to (\mu)_{T,1}$ by induction on κ. To use the induction hypothesis, we push the counter examples we already have for smaller κ's to deeper levels in the tree $2^{<\omega}$.

2.2. Preliminaries.

Definition 2.8. We shall need the following fact:

Fact 2.9. [She90, II, 4] *Let T be any theory. Then for all $n < \omega$, T is dependent if and only if $\square_n$ if and only if $\square_1$ where for all $n < \omega$,*

$\square_n$ For every finite set of formulas $\Delta(x, y)$ with $n = \lg(x)$, there is a polynomial f such that for every finite set $A \subseteq M \models T$, $|S_\Delta(A)| \leq f(|A|)$.

Since we also discuss strongly dependent theories, here is the definition:

Definition 2.10. A theory is called *strongly dependent* if there is no sequence of formulas $\langle \varphi_n(x, y_n) \mid n < \omega \rangle$ such that the set $\left\{ \varphi_n(x_\eta, y_{n,k})^{\eta(n)=k} \mid \eta : \omega \to \omega \right\}$ is consistent with the theory (where $\varphi^{\text{True}} = \varphi, \varphi^{\text{False}} = \neg\varphi$).

See [She12b] for further discussion of strongly dependent theories. There it is proved that $\mathrm{Th}(\mathbb{R})$ is strongly dependent, and so is the theory of the p-adics.

2.3. **The example.** Let S_n be the finite binary tree $2^{\leq n}$. On a well ordered tree such as S_n, we define $<_{\text{suc}}$ as follows: $\eta <_{\text{suc}} \nu$ if ν is a successor of η in the tree.

Let L_n be the following language:

$$L_n = \{P_\eta, <_\eta, \wedge_\eta, G_{\eta,\nu} \mid \eta, \nu \in S_n, \eta <_{\text{suc}} \nu\}.$$

Where:

- P_η is a unary predicate; $<_\eta$ is a binary relation symbol; $\wedge_\eta$ is a binary function symbol; $G_{\eta,\nu}$ is a unary function symbol.

Let $T_n^\forall$ be the following theory:

- $P_\eta \cap P_\nu = \emptyset$ for $\eta \neq \nu$.
- $(P_\eta, <_\eta, \wedge_\eta)$ is a tree, where $\wedge_\eta$ is the meet function on P_η, i.e.,

$$x \wedge_\eta y = \max\{z \in P_\eta \mid z \leq_\eta x \,\&\, z \leq_\eta y\}.$$

- $G_{\eta,\nu} : P_\eta \to P_\nu$ and no further restrictions on it.
- In all the axioms above, for elements or pairs outside of the domain of any of the functions $\wedge_\eta$ or $G_{\eta,\nu}$, these functions are the identity on the leftmost coordinate, so for example if $(x, y) \notin P_\eta^2$, then $x \wedge_\eta y = x$.

Thus we have:

Claim 2.11. $T_n^\forall$ is a universal theory.

Claim 2.12. $T_n^\forall$ has the joint embedding property (JEP) and the amalgamation property (AP).

Proof. Easy to see. □

From this we deduce, by e.g., [Hod93, Theorem 7.4.1]:

Corollary 2.13. $T_n^\forall$ *has a model completion,* T_n *which eliminates quantifiers, and moreover: if* $M \models T_{n+1}^\forall$, $M' = M \restriction L_n$ *and* $M' \subseteq N' \models T_n^\forall$ *then* N' *can be enriched to a model* N *of* $T_{n+1}^\forall$ *so that* $M \subseteq N$. *Hence if* M *is an existentially closed model of* $T_{n+1}^\forall$, *then* M' *is an e.c. model of* $T_n^\forall$. *Hence* $T_n \subseteq T_{n+1}$ *(for more see* [Hod93, Theorem 8.2.4]*).*

Proof. The moreover part: for each $\eta \in S_{n+1} \backslash S_n$, we define $P_\eta^N = P_\eta^M$ and in the same way $\wedge_\eta$. The functions $G_{\eta,\nu}^N$ for $\eta \in S_n$ and $\nu \in S_{n+1}$ will be extensions of $G_{\eta,\nu}^M$. $\square$

Now we show that T_n is dependent, but before that, a few easy remarks:

Observation 2.14.

(1) *If* $A \subseteq M \models T_0^\forall$ *is a finite substructure (so just a tree, with no extra structure), then for all* $b \in M$, *the structure generated by* A *and* b *is* $A \cup \{b\} \cup \{\max\{b \wedge a \mid a \in A\}\}$.

(2) *If* $M \models T_n^\forall$ *and* $\eta \in 2^{\leq n}$, *we can define a new structure* $M_\eta \models T_{n-\lg(\eta)}^\forall$ *whose universe is* $\bigcup \left\{P_{\eta\frown\nu}^M \,\middle|\, \nu \in 2^{\leq n-\lg(n)}\right\}$ *by:* $P_\nu^{M_\eta} = P_{\eta\frown\nu}^M$, *and in the same way we interpret every other symbol (for instance,* $G_{\nu_1,\nu_2}^{M_\eta} = G_{\eta\frown\nu_1,\eta\frown\nu_1}^M$*). For every formula* $\varphi(x) \in L_{n-\lg(\eta)}$ *there is a formula* $\varphi'(x) \in L_n$ *such that for all* $a \in M_\eta$, $M \models \varphi'(a)$ *if and only if* $M_\eta \models \varphi(a)$ *(we get* φ' *by concatenating* η *before any symbol).*

(3) *For* M *as before and* $\eta \in 2^{\leq n}$, *for any* $k < \omega$ *there is a bijection between*

$$\left\{p(x_0, \ldots, x_{k-1}) \in S_k^{qf}(M) \mid \forall i < k \left(P_\eta(x_i) \in p\right)\right\}$$

and

$$\left\{p(x_0, \ldots, x_{k-1}) \in S_k^{qf}(M_\eta) \,\middle|\, \forall i < k \left(P_{\langle\rangle}(x_i) \in p\right)\right\}.$$

Proof. (3): The bijection is given by (2). This is well defined, meaning that if $p(x_0, \ldots, x_{k-1})$ is a type over M_η such that $P_{\langle\rangle}(x_i) \in p$ for all $i < k$, then $\{\varphi' \mid \varphi \in p\}$ determines a complete type over M, such that $P_\eta(x_i) \in p$ for all $i < k$. The point is that all atomic formulas over M which mention elements from $M \backslash M_\eta$ or any $\nu \not\geq \eta$ are trivially determined. $\square$

Proposition 2.15. T_n *is dependent.*

Proof. We use Fact 2.9. It is sufficient to find a polynomial $f(x)$ such that for every finite set A, $|S_1(A)| \leq f(|A|)$.

First we note that for a set A, the size of the structure generated by A is bounded by a polynomial in $|A|$: it is generated by applying $\wedge_{\langle\rangle}$ on $P_{\langle\rangle} \cap A$, applying $G_{\langle\rangle,\langle 1\rangle}$ and $G_{\langle\rangle,\langle 0\rangle}$, and then applying $\wedge_{\langle 0\rangle}, \wedge_{\langle 1\rangle}$ and so on. Every step in the process is polynomial, and it ends after n steps.

Hence we can assume that A is a substructure, i.e., $A \models T_n^\forall$.

The proof is by induction on n. To ease notation, we shall omit the subscript η from $<_\eta$ and $\wedge_\eta$.

First we deal with the case $n = 0$. In T_0, $P_{\langle\rangle}$ is a a tree with no extra structure, while outside $P_{\langle\rangle}$ there is no structure at all. The number of types outside $P_{\langle\rangle}$ is bounded by $|A| + 1$ (because there is only one non-algebraic type). In the case that $P_{\langle\rangle}(x) \in p$ for some type p over A, we can characterize p by characterizing the (tree) order-type of $x' := \max\{a \wedge x \,|\, a \in A\}$, i.e., the cut that x' induces on the tree, and by knowing whether $x' = x$ or $x > x'$ (we note that in general, every theory of a tree is dependent by [Par82]).

Now assume that the claim is true for n. Suppose $\eta \in 2^{\leq n+1}$ and $1 \leq \lg(\eta)$. By Observation 2.14(3), there is a bijection between the types $p(x)$ over A where $P_\eta(x) \in p$ and the types $p(x)$ in $T_{n+1-\lg(\eta)}$ over A_η where $P_{\langle\rangle} \in p$. $A_\eta \models T^{\forall}_{n+1-\lg(\eta)}$, and so by the induction hypothesis, the number of types over A_η is bounded by a polynomial in $|A_\eta| \leq |A|$. As the number of types $p(x)$ such that $P_\eta(x) \notin p$ for all η is bounded by $|A|+1$ as in the previous case, we are left with checking the number of types $p(x)$ such that $P_{\langle\rangle}(x) \in p$.

In order to describe p, we first have to describe p restricted to the language $\{<_{\langle\rangle}, \wedge_{\langle\rangle}\}$, and this is polynomially bounded. Let $x' = \max\{a \wedge x \,|\, a \in A\}$. By Observation 2.14(1), if $A \cup \{x\}$ is not closed under $\wedge_{\langle\rangle}$, x' is the only new element in the structure generated by $A \cup \{x\}$ in $P_{\langle\rangle}$. Hence we are left to determine the type of the pairs $\left(G_{\langle\rangle,\langle i\rangle}(x), G_{\langle\rangle,\langle i\rangle}(x')\right)$ over A for $i = 0, 1$ (if x' is not new, then it's enough to determine the type of $G_{\langle\rangle,\langle i\rangle}(x)$). The number of these types is equal to the number of types of pairs in T_n over $A_{\langle i\rangle}$. As T_n is dependent we are done by Fact 2.9. $\square$

Definition 2.16. Let $L = \bigcup_{n<\omega} L_n$, $T = \bigcup_{n<\omega} T_n$ and $T^{\forall} = \bigcup_{n<\omega} T^{\forall}_n$.

We easily have:

Corollary 2.17. T *is complete, it eliminates quantifiers and is dependent.*

We shall prove the following theorem (which implies Theorem 2.4 from the introduction):

Theorem 2.18. *For any two cardinals* $\mu \leq \kappa$ *such that in* $[\mu, \kappa]$ *there are no (uncountable) strongly inaccessible cardinals,* $\kappa \not\to (\mu)_{T,1}$.

We shall prove a slightly stronger statement, by induction on κ:

Proposition 2.19. *Given* μ *and* κ*, such that either* $\kappa < \mu$ *or there are no (uncountable) strongly inaccessible cardinals in* $[\mu, \kappa]$*, there is a model* $M \models T^{\forall}$ *such that* $\left|P^M_{\langle\rangle}\right| \geq \kappa$ *and* $P^M_{\langle\rangle}$ *does not contain a non-constant indiscernible sequence (for quantifier free formulas) of length* μ.

From now on, indiscernible will only mean "indiscernible for quantifier free formulas".

Proof. Fix μ. The proof is by induction on κ. We divide into cases:

Case 1. $\kappa < \mu$. Clear.

Case 2. $\kappa = \mu = \aleph_0$. Denote $\eta_j = \langle 1, \ldots, 1 \rangle$, i.e., the constant sequence of length j and value 1. Find $M \models T^{\forall}$ such that its universe contains a set $\{a_{i,j} \mid i, j < \omega\}$ where $a_{i,j} \neq a_{i',j'}$ for all $(i,j) \neq (i',j')$, $a_{i,j} \in P_{\eta_j}$ and in addition $G_{\eta_j,\eta_{j+1}}(a_{i,j}) = a_{i,j+1}$ if $j < i$ and $G_{\eta_j,\eta_{j+1}}(a_{i,j}) = a_{0,j+1}$ otherwise. We also need that $P^M_{\langle\rangle} = \{a_{i,0} \mid i < \omega\}$. Any model satisfying these properties will do (so no need to specify what the tree structures are). Now, if in $P^M_{\langle\rangle} = \{a_{i,0} \mid i < \omega\}$ there is a non-constant indiscernible sequence, $\langle a_{i_k,0} \mid k < \omega \rangle$, then for $j \geq i_0, i_1$,

$$G_{\eta_j,\eta_{j+1}} \circ \cdots \circ G_{\eta_0,\eta_1}(a_{i_0,0}) = G_{\eta_j,\eta_{j+1}} \circ \cdots \circ G_{\eta_0,\eta_1}(a_{i_1,0}).$$

But for every k such that $i_k > j$, $G_{\eta_j,\eta_{j+1}} \circ \cdots \circ G_{\eta_0,\eta_1}(a_{i_1,0}) \neq G_{\eta_j,\eta_{j+1}} \circ \cdots \circ G_{\eta_0,\eta_1}(a_{i_k,0})$ — contradiction.

Case 3. κ is singular. Suppose $\kappa = \bigcup_{i<\sigma} \lambda_i$ where $\sigma, \lambda_i < \kappa$ for all $i < \sigma$. By the induction hypothesis, for $i < \sigma$ there is a model $M_i \models T^{\forall}$ such that $\left|P^{M_i}_{\langle\rangle}\right| \geq \lambda_i$ and in $P^{M_i}_{\langle\rangle}$ there is no non-constant indiscernible sequence of length μ. Also, there is a model N such that $\left|P^N_{\langle\rangle}\right| \geq \sigma$ and in $P^N_{\langle\rangle}$ there is no non-constant indiscernible sequence of length μ. We may assume that the universes of all these models are pairwise disjoint and disjoint from κ.

Suppose that $\{a_i \mid i < \sigma\} \subseteq P^N_{\langle\rangle}$, and $\left\{b_j \mid \sum_{l<i} \lambda_l \leq j < \lambda_i\right\} \subseteq P^{M_i}_{\langle\rangle}$ witness that $\left|P^N_{\langle\rangle}\right| \geq \sigma$ and $\left|P^{M_i}_{\langle\rangle}\right| \geq \lambda_i \setminus \sum_{l<i} \lambda_l$. Let $\bar{M}$ be a model extending each M_i and containing the disjoint union of the sets $\bigcup_{i<\sigma} M_i$ (exists by JEP).

Define a new model $M \models T^{\forall}$: $\left(P^M_{\langle\rangle}, <_{\langle\rangle}\right) = (\kappa, <)$ (so $\wedge_{\langle\rangle} = \min$); $\left(P^M_{\langle 1\rangle \frown \eta}, <_\eta\right) = \left(P^N_\eta, <_\eta\right)$ and $\left(P^M_{\langle 0\rangle \frown \eta}, <_{\langle 0\rangle,\eta}\right) = \left(P^{\bar{M}}_\eta, <_\eta\right)$. In the same way define $\wedge_\eta$ for all η of length ≥ 1. The functions are also defined in the same way: $G^M_{\langle 1\rangle \frown \eta, \langle 1\rangle \frown \nu} = G^N_{\eta,\nu}$ and $G^M_{\langle 0\rangle \frown \eta, \langle 0\rangle \frown \nu} = G^{\bar{M}}_{\eta,\nu}$. We are left to define $G_{\langle\rangle,\langle 0\rangle}$ and $G_{\langle\rangle,\langle 1\rangle}$. So let: $G_{\langle\rangle,\langle 1\rangle}(\alpha) = a_{\min\{i \mid \alpha < \lambda_i\}}$ and $G_{\langle\rangle,\langle 0\rangle}(\alpha) = b_\alpha$ for all $\alpha < \kappa$.

Note that if I is an indiscernible sequence contained in $P^M_{\langle 1\rangle}$ then I is an indiscernible sequence in N contained in $P^N_{\langle\rangle}$, and the same is true for $P^M_{\langle 0\rangle}$ and $\bar{M}$.

Assume $\langle \alpha_j \mid j < \mu \rangle$ is an indiscernible sequence in $P^M_{\langle\rangle}$. Then $\left\langle G_{\langle\rangle,\langle 1\rangle}(\alpha_j) \mid j < \mu \right\rangle$ is a constant sequence (by the choice of N). So there is $i < \sigma$ such that $\sum_{l<i} \lambda_l \leq \alpha_j < \lambda_i$ for all $j < \mu$. So $\left\langle G_{\langle\rangle,\langle 0\rangle}(\alpha_j) = b_{\alpha_j} \mid j < \mu \right\rangle$ is a constant sequence (it is indiscernible in $P^{\bar{M}}_{\langle\rangle}$ and in fact contained in $P^{M_i}_{\langle\rangle}$), hence $\langle \alpha_j \mid j < \mu \rangle$ is constant, as we wanted.

Case 4. κ is regular uncountable. By the hypothesis of the proposition, κ is not strongly inaccessible, so there is some $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. By the induction hypothesis on λ, there is a model $N \models T^{\forall}$ such that in $P^N_{\langle\rangle}$ there is no non-constant indiscernible sequence of length μ. Let $\{a_i \mid i \leq \lambda\} \subseteq P^N_{\langle\rangle}$ witness that $\left|P^N_{\langle\rangle}\right| \geq \lambda$.

Define $M \models T^{\forall}$ as follows: $P^M_{\langle\rangle} = 2^{\leq\lambda}$ and the ordering is inclusion (equivalently, the ordering is by initial segment). $\wedge_{\langle\rangle}$ is defined naturally: $f \wedge_{\langle\rangle} g = f \restriction \min\{\alpha \,|\, f(\alpha) \neq g(\alpha)\}$.

For all η, let $P^M_{\langle 1\rangle \frown \eta} = P^N_\eta$, and the ordering and the functions are naturally induced from N. The main point is that we set $G_{\langle\rangle,\langle 1\rangle}(f) = a_{\lg(f)}$. Now choose $P^M_{\langle 0\rangle\frown\eta}$, $G_{\langle 0\rangle\frown\eta,\langle 0\rangle\frown\nu}$, etc. arbitrarily, and let $G_{\langle\rangle,\langle 0\rangle}$ be any function.

Suppose that $\langle f_i \,|\, i < \mu\rangle$ is a non-constant indiscernible sequence:

If $f_1 < f_0$ (i.e., $f_1 <_{\langle\rangle} f_0$), we shall have an infinite decreasing sequence in a well-ordered tree — a contradiction.

If $f_0 < f_1$, $\langle f_i \,|\, i < \mu\rangle$ is increasing, so $\left\langle G^M_{\langle\rangle,\langle 1\rangle}(f_i) = a_{\lg(f_i)} \,|\, i < \mu \right\rangle$ is non-constant — contradiction (as it is an indiscernible sequence in M and hence in $P^N_{\langle 0\rangle}$).

Let $h_i = f_0 \wedge f_{i+1}$ for $i < \mu$ (where $\wedge = \wedge_{\langle\rangle}$). This is an indiscernible sequence, and by the same arguments, it cannot increase or decrease, but as $h_i \leq f_0$, and $\left(P_{\langle\rangle}, <_{\langle\rangle}\right)$ is a tree, it follows that h_i is constant.

Assume $f_0 \wedge f_1 < f_1 \wedge f_2$, then $f_{2i} \wedge f_{2i+1} < f_{2(i+1)} \wedge f_{2(i+1)+1}$ for all $i < \mu$, and again $\langle f_{2i} \wedge f_{2i+1} \,|\, i < \mu\rangle$ an increasing indiscernible sequence and we have a contradiction.

By the same reasoning, it cannot be that $f_0 \wedge f_1 > f_1 \wedge f_2$. As $\left(P_{\langle\rangle}, <_{\langle\rangle}\right)$ is a tree, we conclude that $f_0 \wedge f_2 = f_0 \wedge f_1 = f_1 \wedge f_2$. But that is a contradiction (because if $\alpha = \lg(f_0 \wedge f_1)$, then $|\{f_0(\alpha), f_1(\alpha), f_2(\alpha)\}| = 3$).

□

2.4. **In RCF there are few indiscernibles of ω-tuples.** Here we will prove Theorem 2.7. Since RCF is strongly dependent, Fact 2.3 (which discusses finite tuples) holds for it, so we will show that a similar phenomenon as in the previous section holds for ω-tuples in RCF. So assume $\mathfrak{C} \models \mathsf{RCF}$.

Notation 2.20. The set of all open intervals (a, b) (where $a < b$ and $a, b \in \mathfrak{C}$) is denoted by $\mathfrak{I}$.

Definition 2.21. For a cardinal κ, $n \leq \omega$ and an ordinal δ, $\kappa \to (\delta)^{\text{interval}}_n$ means: for every set A of n-tuples of (non-empty, open) intervals (so for each $\bar{I} \in A$, $\bar{I} = \left\langle I^i \,|\, i < n \right\rangle \in \mathfrak{I}^n$) of size κ, there is a sequence $\left\langle \bar{I}_\alpha \,|\, \alpha < \delta \right\rangle \in A^\delta$ of order type δ such that $\bar{I}_\alpha \neq \bar{I}_\beta$ for $\alpha < \beta < \delta$, and there is a sequence $\left\langle \bar{b}_\alpha \,|\, \alpha < \delta \right\rangle$ such that $\bar{b}_\alpha \in \bar{I}_\alpha$ (i.e., $\bar{b}_\alpha = \left\langle b^0_\alpha, \ldots, b^{n-1}_\alpha \right\rangle$ and $b^i_\alpha \in I^i_\alpha$) and such that $\left\langle \bar{b}_\alpha \,|\, \alpha < \delta \right\rangle$ is an indiscernible sequence.

Remark 2.22. Note that:

(1) If $\kappa \to (\delta)^{\text{interval}}_n$ then $\kappa \to (\delta)^{\text{interval}}_m$ for all $m \leq n$.

(2) If $\kappa \not\to (\delta)_n^{\text{interval}}$ then $\kappa \not\to (\delta)_{RCF,n}$ (why? if A witnesses that $\kappa \not\to (\delta)_n^{\text{interval}}$, then for each $\bar{I} \in A$, choose $\bar{b}_{\bar{I}} \in \bar{I}$ (as above) in such a way that $\{\bar{b}_{\bar{I}} \mid \bar{I} \in A\}$ has size κ. By definition this set witnesses $\kappa \not\to (\delta)_{RCF,n}$).

(3) If $\lambda < \kappa$ and $\kappa \not\to (\delta)_n^{\text{interval}}$ then $\lambda \not\to (\delta)_n^{\text{interval}}$.

We shall prove the following theorem (which immediately implies Theorem 2.7):

Theorem 2.23. *For any two cardinals $\mu \le \kappa$ such that in $[\mu, \kappa]$ there are no strongly inaccessible cardinals, $\kappa \not\to (\mu)_\omega^{\text{interval}}$.*

The proof follows from a sequence of claims:

Claim 2.24. If $\kappa < \mu$ then $\kappa \not\to (\mu)_n^{\text{interval}}$ for all $n \le \omega$.

Proof. Obvious. □

Claim 2.25. If $\kappa = \mu = \aleph_0$ then $\kappa \not\to (\mu)_1^{\text{interval}}$.

Proof. For $n < \omega$, let $I_n = (n, n+1)$. □

Claim 2.26. Suppose $\kappa = \sum_{i<\sigma} \lambda_i$ and $n \le \omega$. Then, if $\sigma \not\to (\mu)_n^{\text{interval}}$ and $\lambda_i \not\to (\mu)_n^{\text{interval}}$ then $\kappa \not\to (\mu)_{2+2n}^{\text{interval}}$.

Proof. By assumption, we have a set of intervals $\{\bar{R}_i \mid i < \sigma\}$ that witness $\sigma \not\to (\mu)_n^{\text{interval}}$ and for each $i < \sigma$ we have $\left\{\bar{S}_\beta \,\middle|\, \sum_{j<i} \lambda_j < \beta < \lambda_i\right\}$ that witness $\left|\lambda_i \setminus \sum_{j<i} \lambda_j\right| \not\to (\mu)_n^{\text{interval}}$.

Fix an increasing sequence of elements $\langle b_i \mid i < \sigma\rangle$.

For $\alpha < \kappa$, let $\beta = \beta(\alpha) = \min\{i < \sigma \mid \alpha < \lambda_i\}$ and for $i < 2 + 2n$, define:

- If $i = 0$, let $I_\alpha^i = (b_{2\beta(\alpha)}, b_{2\beta(\alpha)+1})$.
- If $i = 1$, let $I_\alpha^i = (b_{2\beta(\alpha)+1}, b_{2\beta(\alpha)+2})$.
- If $i = 2k+2$, let $I_\alpha^i = R_{\beta(\alpha)}^k$.
- If $i = 2k+3$, let $I_\alpha^i = S_\alpha^k$.

Suppose $\langle \bar{b}_\varepsilon \mid \varepsilon < \mu\rangle$ is an indiscernible sequence such that $\bar{b}_\varepsilon \in \bar{I}_{\alpha_\varepsilon}$ for $\varepsilon < \mu$. Denote $\bar{b}_\varepsilon = \langle b_0^\varepsilon, \ldots, b_{2+2n-1}^\varepsilon\rangle$. Note that $b_1^\varepsilon < b_0^{\varepsilon'}$ if and only if $\beta(\alpha_\varepsilon) < \beta(\alpha_{\varepsilon'})$ (we need two intervals for the "only if" direction).

Hence $\langle \beta(\alpha_\varepsilon) \mid \varepsilon < \mu\rangle$ is increasing or constant. But if it is increasing then we have a contradiction to the choice of $\{\bar{R}_i \mid i < \sigma\}$. So it is constant, and suppose $\beta(\alpha_\varepsilon) = i_0$ for all $\varepsilon < \mu$. But then $\alpha_\varepsilon \in \lambda_{i_0} \setminus \sum_{j<i_0} \lambda_j$ for all $\varepsilon < \mu$ and we get a contradiction to the choice of $\left\langle \bar{S}_\beta \,\middle|\, \sum_{j<i_0} \lambda_j < \beta < \lambda_{i_0}\right\rangle$. □

Claim 2.27. Suppose $\lambda \not\to (\mu)_n^{\text{interval}}$. Then $2^\lambda \not\to (\mu)_{4+2n}^{\text{interval}}$.

Proof. Suppose $\{\bar{I}_\alpha \,|\, \alpha < \lambda\}$ witnesses that $\lambda \not\to (\mu)^{\text{interval}}_n$.

By adding two intervals to each $\bar{I}_\alpha$, we can ensure that it has the extra property that if $\bar{c}_1 \in I_{\alpha_1}$ and $\bar{c}_2 \in \bar{I}_{\alpha_2}$ then $c^1_1 < c^0_2$ if and only if $\alpha_1 < \alpha_2$ (as in the previous claim). By this we have increased the length of $\bar{I}_\alpha$ to $2+n$ (and it is still a witness of $\lambda \not\to (\mu)^{\text{interval}}_n$).

We write $\bar{c}_1 <^* \bar{c}_2$ for $c^1_1 < c^0_2$, but note that it is not really an ordering (it is not transitive in general).

We shall find below a four-place definable function f such that:

- ♡ For every two ordinals, δ, ζ, if $\langle \bar{R}_\alpha \,|\, \alpha < \delta \rangle$ is a sequence of ζ-tuples of intervals, then there exists a set of 2ζ-tuples of intervals, $\left\{\bar{S}_\eta \,\middle|\, \eta \in {}^\delta 2\right\}$ (of size $2^{|\delta|}$) such that for all $i < \zeta$ and $\eta_1 \neq \eta_2$, if $b_1 \in S^{2i}_{\eta_1}, b_2 \in S^{2i+1}_{\eta_1}$ and $b_3 \in S^{2i}_{\eta_2}, b_4 \in S^{2i+1}_{\eta_2}$ then $f(b_1, b_2, b_3, b_4)$ is in $R^i_{\lg(\eta_1 \wedge \eta_2)}$.

Apply ♡ to our situation to get $\left\{\bar{J}_\eta \,\middle|\, \eta \in {}^\lambda 2\right\}$ such that $\bar{J}_\eta = \langle J^i_\eta \,|\, i < 4+2n \rangle$ and for all $k < 2+n$ and $\eta_1 \neq \eta_2$, if $b_1 \in J^{2k}_{\eta_1}, b_2 \in J^{2k+1}_{\eta_1}$ and $b_3 \in J^{2k}_{\eta_2}, b_4 \in J^{2k+1}_{\eta_2}$ then $f(b_1, b_2, b_3, b_4)$ is in $I^k_{\lg(\eta_1 \wedge \eta_2)}$.

This is enough (the reasons are exactly as in the regular case of the proof of Theorem 2.18, but we shall repeat it for clarity):

To simplify notation, we regard f as a function on tuples, so that if $\bar{b}_1 \in \bar{J}_{\eta_1}, \bar{b}_2 \in \bar{J}_{\eta_2}$ then $f(\bar{b}_1, \bar{b}_2)$ is in $\bar{I}_{\lg(\eta_1 \wedge \eta_2)}$ (namely, $f(\bar{b}_1, \bar{b}_2) = \langle a_k \,|\, k < 2+n \rangle$ where $a_k = f\left(b^1_{2k}, b^1_{2k+1}, b^2_{2k}, b^2_{2k+1}\right) \in I^k_{\lg(\eta_1 \wedge \eta_2)}$ for $k < 2+n$).

Suppose $\langle \eta_i \,|\, i < \mu \rangle \subseteq {}^\lambda 2$ is without repetitions and $\langle \bar{b}_{\eta_i} \,|\, i < \mu \rangle$ is an indiscernible sequence such that $\bar{b}_{\eta_i} \in \bar{J}_{\eta_i}$.

Let $h_i = \eta_0 \wedge \eta_{i+1}$ for $i < \mu$. If $\lg(h_i) < \lg(h_j)$ for some $i \neq j$ then $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_{i+1}}) <^* f(\bar{b}_{\eta_0}, \bar{b}_{\eta_{j+1}})$ and so by indiscernibility, $\langle \lg(h_i) \,|\, i < \mu \rangle$ is increasing (it cannot be decreasing), and so $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_{i+1}})$ contradicts our choice of $\langle \bar{I}_\alpha \,|\, \alpha < \lambda \rangle$. Hence (because $h_i \leq \eta_0$) h_i is constant.

Assume $\eta_0 \wedge \eta_1 < \eta_1 \wedge \eta_2$, then $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_1}) <^* f(\bar{b}_{\eta_1}, \bar{b}_{\eta_2})$ so $f(\bar{b}_{\eta_1}, \bar{b}_{\eta_2}) <^* f(\bar{b}_{\eta_2}, \bar{b}_{\eta_3})$, and so $\lg(\eta_0 \wedge \eta_1) < \lg(\eta_1 \wedge \eta_2) < \lg(\eta_2 \wedge \eta_3)$ hence, $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_1}) <^* f(\bar{b}_{\eta_2}, \bar{b}_{\eta_3})$ and it follows that $\langle \lg(\eta_{2i} \wedge \eta_{2i+1}) \,|\, i < \mu \rangle$ is increasing. And this is again a contradiction.

Similarly, it cannot be that $\eta_0 \wedge \eta_1 > \eta_1 \wedge \eta_2$. As both sides are less or equal than η_1, it must be that $\eta_0 \wedge \eta_2 = \eta_0 \wedge \eta_1 = \eta_1 \wedge \eta_2$. But that is impossible (because if $\alpha = \lg(\eta_0 \wedge \eta_1)$, then $|\{\eta_0(\alpha), \eta_1(\alpha), \eta_2(\alpha)\}| = 3$).

Claim. ♡ is true.

Proof. Let $f(x, y, z, w) = (x - z)/(y - w)$ (do not worry about division by 0, we shall explain below).

It is enough, by the definition of ♡, to assume $\zeta = 1$. By compactness, we may assume that δ is finite, and to avoid confusion, denote it by m. So we have a finite set, ${}^m 2$, and a sequence of intervals $\langle R_i \,|\, i < m \rangle$. Each R_i is of the form (a_i, b_i). Let $c_i = (b_i + a_i)/2$. Let

$d \in \mathfrak{C}$ be any element greater than any member of $A := \mathrm{acl}(a_i, b_i \mid i < m)$. For each $\eta \in {}^m 2$, let $a_\eta = \sum_{i<m} \eta(i) c_i d^{m-i}$, and $b_\eta = \sum_{i<m} \eta(i) d^{m-i}$.

Let $S^0_\eta = (a_\eta - 1, a_\eta + 1)$ and $S^1_\eta = (b_\eta - 1, b_\eta + 1)$.

This works:

Assume that $\eta_1 \neq \eta_2$, $b_1 \in S^0_{\eta_1}, b_2 \in S^1_{\eta_1}$ and $b_3 \in S^0_{\eta_2}, b_4 \in S^1_{\eta_2}$.

We have to show $(b_1 - b_3) / (b_2 - b_4) \in R_{\lg(\eta_1 \wedge \eta_2)}$. Denote $k = \lg(\eta_1 \wedge \eta_2)$ (so $k < m$).

$a_{\eta_1} - a_{\eta_2}$ is of the form $\varepsilon c_k d^{m-k} + F(d)$ where $\varepsilon \in \{-1, 1\}$, and $F(d)$ is a polynomial over A of degree $\leq m - k - 1$. $b_{\eta_1} - b_{\eta_2}$ is of the form $\varepsilon d^{m-k} + G(d)$, where ε is the same for both (and G is a polynomial over $\mathbb{Z}$ of degree $\leq m - k - 1$). Now, $b_1 - b_3 \in (a_{\eta_1} - a_{\eta_2} - 2, a_{\eta_1} - a_{\eta_2} + 2)$, and $b_2 - b_4 \in (b_{\eta_1} - b_{\eta_2} - 2, b_{\eta_1} - b_{\eta_2} + 2)$, and hence we know that $b_2 - b_4 \neq 0$. It follows that $(b_1 - b_3) / (b_2 - b_4)$ is inside an interval whose endpoints are $\left\{\left(\varepsilon c_k d^{m-k} + F(d) \pm 2\right) / \left(\varepsilon d^{m-k} + G(d) \pm 2\right)\right\}$. But

$$\left(\varepsilon c_k d^{m-k} + F(d) \pm 2\right) / \left(\varepsilon d^{m-k} + G(d) \pm 2\right) \in R_k$$

by our choice of d, and we are done.

Note that for $\eta_1 \neq \eta_2$, $S^0_{\eta_1} \neq S^0_{\eta_2}$ regardless of the R_i's (which can be a constant interval). $\square$

The proof of Theorem 2.23 now follows by induction on κ: fix μ, and let κ be the first cardinal for which the theorem fails. Then by Claim 2.24, $\kappa \geq \mu$. By Claim 2.25, $\aleph_0 < \kappa$. By Claim 2.26, κ cannot be singular. By Claim 2.27, κ cannot be regular, because if it were, there would be a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$ (because κ is not strongly inaccessible). Note that we did use Claim 2.24 to deal with cases where we couldn't use the induction hypothesis (for example, in the regular case, it might be that $\lambda < \mu$).

Further remarks. Theorem 2.23 can be generalized to allow parameters:

Suppose $\mathfrak{C} \models \mathsf{RCF}$, and $A \subseteq \mathfrak{C}$.

Definition 2.28. $\kappa \to_A (\mu)^{\text{interval}}_\omega$ means the same as in Definition 2.21, but we require that the indiscernible sequence is indiscernible over A.

Then we have:

Theorem 2.29. *For any set of parameters* A *and any two cardinals* μ, κ *such that in* $[\max\{|A|, \mu\}, \kappa]$ *there are no strongly inaccessible cardinals or* $\kappa < \max\{|A|, \mu\}$*,* $\kappa \not\to_A (\mu)^{\text{interval}}_\omega$.

Proof. The proof goes exactly as the proof of Theorem 2.23, but the base case for the induction is different. If $\max\{|A|, \mu\} = \mu$, the proof is exactly the same. Otherwise, we have to deal with the case $\kappa \leq |A|$:

Enumerate $A = \{a_i \mid i < \mu'\}$. Let $\varepsilon \in \mathfrak{C}$ be greater than 0 but smaller than any element in $\mathrm{acl}(A)$. For $i < \mu'$, let $I_i = (a_i, a_i + \varepsilon)$. Then $\{I_i \mid i < \kappa\}$ witnesses $\kappa \not\to_A (\mu)^{\mathrm{interval}}_{\omega}$. □

3. Generic pair

Here we give an example of an ω-stable theory, such that for all weakly generic pairs of structures $M \prec M_1$ (see below for the definition) the theory of the pair (M_1, M) in an extended language where we name M by a predicate has the independence property.

Definition 3.1. A pair (M_1, M) as above is *weakly generic* if for all formula $\varphi(x)$ with parameters from M, if φ has infinitely many solutions in M, then it has a solution in $M_1 \backslash M$.

This definition is induced by the well known "generic pair conjecture" (see [She, She12a]), and it is worth while to give the precise definitions.

Definition 3.2. Assume that $\lambda = \lambda^{<\lambda} > |T|$ (in particular, λ is regular) and that $2^\lambda = \lambda^+$. The *generic pair property* for λ says that there exists a saturated model M of cardinality λ^+, an increasing continuous sequence of models $\langle M_\alpha \mid \alpha < \lambda^+ \rangle$ and a club $E \subseteq \lambda^+$ such that $\bigcup_{\alpha < \lambda^+} M_\alpha = M$ and for all $\alpha < \beta \in E$ of cofinality λ, the pair (M_β, M_α) has the same isomorphism type. We call this pair the *generic pair* of T of size λ.

Proposition 3.3. *Assume that $\lambda = \lambda^{<\lambda} > |T|$ and that $2^\lambda = \lambda^+$. The generic pair property for λ holds iff for every saturated model N of cardinality λ^+ and for every increasing continuous sequence of models $\langle N_\alpha \mid \alpha < \lambda^+ \rangle$ with union N there exists a club $E \subseteq \lambda^+$ such that for all $\alpha < \beta \in E$ of cofinality λ, the pair (N_β, N_α) has the same isomorphism type. Moreover, this type does not depended on the particular choice of N or N_α.*

Proof. Left to right:

Suppose M, $\langle M_\alpha \mid \alpha < \lambda^+ \rangle$ and E witness that T has the generic pair property for λ. If N is another saturated model of size λ^+ and $\langle N_\alpha \mid \alpha < \lambda^+ \rangle$ is as in the Proposition. Then $N \cong M$, so we may assume $N = M$. Let $E_0 = \{\delta < \lambda^+ \mid N_\delta = M_\delta\}$. This is a club of λ^+, and so $E \cap E_0$ is also a club of λ^+ such that (N_β, N_α) has the same isomorphism type for any $\alpha < \beta \in E \cap E_0$ of cofinality λ.

Right to left is clear. □

Justifying definition 3.1 we have:

Claim 3.4. Assume that T has the generic pair property for λ, then every generic pair of size λ is weakly generic.

Proof. Suppose that M, $\langle M_\alpha \mid \alpha < \lambda^+ \rangle$ and E are as in Definition 3.2. Suppose $\alpha, \beta \in E$ and $\alpha < \beta$ are of cofinality λ. We are given a formula $\varphi(x)$ with parameter from M_α, such that

$\aleph_0 \leq |\varphi(M_\alpha)|$. By saturation of M, $\lambda^+ = |\varphi(M)|$. Since $M = \bigcup_{\beta' \in E, \mathrm{cf}(\beta')=\lambda} M_\beta$ there is some $\alpha < \beta' \in E$ of cofinality λ such that $\varphi(M_{\beta'}) \setminus M_\alpha \neq \emptyset$, but as $(M_\beta, M_\alpha) \cong (M_{\beta'}, M_\alpha)$, we are done. $\square$

Proposition 3.3 implies that the generic pair property and the the generic pair are both natural notions. It is important in the study of dependent theories as it lead to the development of a theory of type decomposition in NIP. Using this theory, the second author's [She12a, She] prove that the generic pair property holds for dependent theories and large enough λ's. On the other hand, [She06, She11] prove that if T has IP then it lacks the generic pair property for all large enough λ.

Hence it makes sense to ask whether the theory of the pair is dependent.

The answer is no:

Theorem 3.5. *There exists an ω-stable theory such that for every weakly generic pair of models $M \prec M_1$, the theory of the pair (M_1, M) has the independence property.*

We shall describe this theory:

Let $L = \{P, R, Q_1, Q_2\}$ where R, P are unary predicates and Q_1, Q_2 are binary relations.

Let $\tilde{M}$ be the following structure for L:

(1) The universe is:

$$\tilde{M} = \{u \subseteq \omega \,|\, |u| < \omega\} \cup \{(u, v, i) \,|\, u, v \subseteq \omega, |u| < \omega, |v| < \omega, i < \omega \,\&\, u \subseteq v \Rightarrow i < |v| + 1\}.$$

(2) The predicates are interpreted as follows:
- $P^{\tilde{M}} = \{u \subseteq \omega \,|\, |u| < \aleph_0\}$.
- $R^{\tilde{M}}$ is $\tilde{M} \setminus (P^M)$.
- $Q_1^{\tilde{M}} = \left\{(u, (u, v, i)) \,\middle|\, u \in P^{\tilde{M}}\right\}$.
- $Q_2^{\tilde{M}} = \left\{(v, (u, v, i)) \,\middle|\, v \in P^{\tilde{M}}\right\}$.

Let $T = \mathrm{Th}(\tilde{M})$.

As we shall see in the next claim, T gives rise to the following definition:

Definition 3.6. We call a structure $(B, \cup, \cap, -, \subseteq, 0)$ a *pseudo Boolean algebra* (PBA) when it satisfies all the axioms of a Boolean algebra except:

There is no greatest element 1 (i.e., remove all the axioms concerning it).

Pseudo Boolean algebra can have atoms like in Boolean algebras (nonzero elements that do not contain any smaller nonzero elements).

Definition 3.7. Say that a PBA is of *finite type* if every element is a union of finitely many atoms.

Definition 3.8. For a PBA A, and $C \subseteq A$ a sub-PBA, let $A_C := \left\{a \in A \,\middle|\, \exists c \in C \left(a \subseteq^A c\right)\right\}$, and for a subset $D \subseteq A$, let $at(D)$ be the set of atoms contained in D.

Proposition 3.9. *Every PBA of finite type is isomorphic to $(\mathcal{P}_{<\infty}(\kappa), \cup, \cap, -, \subseteq, \emptyset)$ for some κ where $\mathcal{P}_{<\infty}(\kappa)$ is the set of all finite subsets of κ. Moreover: Assume A, B are PBAs of finite type and $C \subseteq A, B$ is a common sub-PBA. Then, if:*

(1) $|\mathrm{at}(A) \setminus \mathrm{at}(A_C)| = |\mathrm{at}(B) \setminus \mathrm{at}(B_C)|$.

(2) For every $c \in C$, A and B agree on the size of c (the number of atoms it contains).

Then there is an isomorphism of PBAs $f : A \to B$ such that $f \restriction C = \mathrm{id}$.

Proof. The first part follows from the easy observation that in a PBA of finite type, every element has a unique presentation as a union of finitely many atoms. So if A is a PBA, and its set of atoms is $\{a_i \mid i < \kappa\}$, then take a_i to $\{i\}$.

For the moreover part, first we extend id_C to an isomorphism from A_C to B_C: consider all elements in C of minimal size, these are the atoms of C. For each such c, map the set of atoms in A contained in c to the set of atoms in B contained in c. This is well defined and can be extended to all of A_C.

Now, $|at(A) \setminus at(A_C)| = |at(B) \setminus at(B_C)|$, so any bijection between the set of atoms induces an isomorphism. ☐

Claim 3.10. T is ω-stable.

Proof. We prove that an expansion of T to a larger vocabulary is ω-stable, by adding new relations to the language, which are all definable —

$$\{S_n, \subseteq_n, \pi_1, \pi_2, \cap_n, \cup_n, -_n, e \mid n \geq 1\}$$

where S_n is a unary relation defined on P, $\subseteq_n$ is a binary relation defined on P, π_1, π_2 are two unary functions from R to P, $\cap_n, -_n$ are binary functions from S_n to S_n and e is a constant in P. Their interpretation in $\tilde{M}$ are as follows:

- $\pi_1((u, v, i)) = u$, $\pi_2((u, v, i)) = v$.
- For each $1 \leq n < \omega$, $S_n(v) \Leftrightarrow |v| \leq n$.
- For each $1 \leq n$, $u \subseteq_n v$ if and only if $|u| \leq n$, $|v| \leq n$ and $u \subseteq v$.
- $u \cap_n v = u \cap v$ for all $u, v \in S_n$.
- $u -_n v = u \setminus v$ for $v, u \in S_n$.
- $u \cup_n v = u \cup v$ for $u, v \in S_n$.
- $e = \emptyset$.

Note that they are indeed definable:

(1) $\pi_1(x)$ is the unique y such that $Q_1(y, x)$, and similarly π_2 is definable.

(2) Let $E(x,y)$ by an auxiliary equivalence relation defined by $\pi_1(x) = \pi_1(y) \wedge \pi_2(x) = \pi_2(y)$.
(3) e is the unique element $x \in P$ such that there exists exactly one element $z \in R$ such that $\pi_1(z) = x = \pi_2(z)$.
(4) $x \subseteq_n y$ is defined by "$P(x), P(y)$ and the number of elements in the E class of some (equivalently any) element z such that $\pi_1(z) = x$, $\pi_2(z) = y$ is at most $n+1$".
(5) $S_n(x)$ is defined by "$P(x)$ and $e \subseteq_n x$" (In particular, $e \in S_n$ for all n).
(6) $\cap_n$ and $-_n$ are then naturally definable using $\subseteq_n$. For instance $x -_n y = z$ if and only if x,y,z are in S_n, $z \subseteq_n x$ and for each $e \neq w \subseteq_n y$, $w \nsubseteq_n z$.
(7) $x \cup_n y = z$ if and only if $x, y \in S_n$, $z \in S_{2n}$, $x, y \subseteq_{2n} z$ and $z -_{2n} x \subseteq_{2n} y$.

Furthermore, $\subseteq_k \restriction S_n = \subseteq_n$ for $n \leq k$. Hence every model M of T gives rise naturally to an induced PBA: $B^M := \left(\bigcup_n S_n^M, \cup^M, \cap^M, -^M, \subseteq^M, e^M\right)$ where $\cup^M = \bigcup\left\{\cup_n^M \,|\, n < \omega\right\}$, and similarly for $\subseteq^M, -^M$ and $\cap^M$ (see Definition 3.6 above).

Claim. In the extended language, T eliminates quantifiers.

Proof. Suppose $M, N \models T$ are saturated models, $|M| = |N|$ and $A \subseteq M, N$ is a common substructure (where $|A| < |M|$). It is enough to show that we have an isomorphism from M to N fixing A.

By Proposition 3.9, we have an isomorphism f from B^M to B^N preserving $A \cap B^M$ (by saturation and the choice of language, the condition of the proposition are satisfied).

On $P^M \setminus \left(B^M \cup A\right)$ there is no structure and it has the same size as $P^N \setminus \left(B^N \cup A\right)$ (namely $|N|$), so we can extend the isomorphism f to P^M.

We are left with R^M: let $a \in R^M$, and $a_i = \pi_i(a)$ for $i = 1, 2$. We already defined $f(a_1), f(a_2)$. Suppose $a_1 \subseteq_n a_2$ for minimal n. Then there are exactly n elements $z \in R^M$ with $\pi_1(z) = a_1, \pi_2(z) = a_2$. This is true also in R^N, and the number of such z's not in A is the same for both M, N. Hence we can take this E-equivalence class from M to the appropriate class in N.

If not, i.e., $a_1 \nsubseteq_n a_2$ for all n, then there are infinitely many elements z in N and in M with $\pi_1(z) = a_1$, $\pi_2(z) = a_2$, and again we take this E-class in M outside of A to the appropriate E-class in N. □

Now we can conclude the proof by a counting types argument. Let M be a countable model of T. Let $p(x)$ be a non-algebraic type over M. There are some cases:

Case 1. $S_n(x) \in p$ for some n. Then the type is determined by the maximal element c in M such that $c \subseteq_n x$ (this is easy, but also follows from the proof of Proposition 3.9).

Case 2. $S_n(x) \notin p$ for all n but $P(x) \in p$. Then x is already determined — there is nothing more we can say on x.

Case 3. $R(x) \in p$. Then the type of x is determined by the type of $(\pi_1(x), \pi_2(x))$ over M.

So the number of types over M is countable. □

Proposition 3.11. *Every weakly generic pair of models of* T *has the independence property.*

Proof. Suppose (M_1, M) is a weakly generic pair. We think of it as a structure of the language L_Q, where Q is interpreted as M. Consider the formula

$$\varphi(x,y) = P(x) \wedge P(y) \wedge \exists z \notin Q\,(Q_1(x,z) \wedge Q_2(y,z)).$$

This formula has IP: Let $\{a_i \mid i < \omega\} \subseteq M$ be elements from P^M such that $a \in S_1^M$ (as in the language of the proof of Claim 3.10), i.e., they are atoms in the induced PBA, and $a_i \neq a_j$ for $i \neq j$. For any finite $s \subseteq \omega$ of size n, there is an element $b_s \in P^M$ be such that $a_i \subseteq_n^M b_s$ for all $i \in s$. Then for all $i \in \omega$, $\varphi(a_i, b_s)$ if and only if $i \notin s$:

If $\varphi(a_i, b_s)$ there are infinitely many z's in M such that $Q_1(a_i,z) \wedge Q_2(b_s,z)$ (otherwise they would all be in M). This means that $a_i \not\subseteq_n^M b_s$ so $i \notin s$.

For the other direction, the same exact argument works, but this time use the fact that the pair is weakly generic. □

4. Directionality

4.1. Introduction.

Definition 4.1. A global type $p(x) \in S(\mathfrak{C})$ is said to be *finitely satisfiable* in a set A, or a *coheir* over A if for every formula $\varphi(x,y)$, if $\varphi(x,b) \in p$, then for some $a \in A$, $\varphi(a,b)$ holds.

It is well known (see [Adl08]) that a theory T is dependent if and only if given a type $p(x) \in S(M)$ over a model M, the number of complete global types $q \in S(\mathfrak{C})$ that extend p and are finitely satisfiable in M is at most $2^{|M|}$ (while the maximal number is $2^{2^{|M|}}$).

We analyze the behavior of the number of global coheir extensions in a dependent theory and classify theories by what we call directionality:

Say that T has *small* directionality if and only if the number of Δ-coheirs (for a finite set of formulas Δ) that extend a type $p \in S(M)$ is finite. T has *medium* directionality if this number is $|M|$, and it has *large* directionality if it is neither small or medium. In that case we will show that it is at least $\operatorname{ded}|M|$.

We give an equivalent definition in terms of the number of global coheir extensions (see Theorem 4.21).

As far as we know, the first person to give an example of a dependent theory with large directionality was Delon in [Del84].

We give simple combinatorial examples for each of the possible directionalities, and furthermore we show that RCF and some theories of valued fields are large.

We do not always assume that T is dependent in this section.

4.2. **Equivalent definitions of directionality.**

Definition 4.2. For a type $p \in S(A)$, let:

$$\mathrm{uf}(p) = \{q \in S(\mathfrak{C}) \mid q \text{ is a coheir extension of } p \text{ over } A\}.$$

For a partial type $p(x)$ over a set A, and a set of formulas Δ,

$$\mathrm{uf}_\Delta(p) = \{q(x) \in S_\Delta(\mathfrak{C}) \mid q \cup p \text{ is f.s. in } A\}.$$

Note: this definition only makes sense if p is finitely satisfiable in A. The notation uf refers to ultrafilter.

And here is the main definition of this section:

Definition 4.3. Let T be any theory, then:

(1) T is said to have *small* directionality (or just, T is small) if and only if for all finite Δ, $M \models T$ and $p \in S(M)$, $\mathrm{uf}_\Delta(p)$ is finite.
(2) T is said to have *medium* directionality (or just, T is medium) if and only if for every $\lambda \geq |T|$,
$$\lambda = \sup\{|\mathrm{uf}_\Delta(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\}.$$
(3) T is said to have *large* directionality (or just, T is large) if T is neither small nor medium.

Observation 4.4. *If T has the independence property, then it is large. In fact, if $\varphi(x,y)$ has the independence property, then there is a type $p(x)$ over a model M, that has $2^{2^{|M|}}$ many $\{\varphi(x,y)\}$-extensions that are finitely satisfiable in M.*

Proof. We may assume that T has Skolem functions. Let $\lambda \geq |T|$, and let $\bar{a} = \langle a_i \mid i < \lambda \rangle$, $\langle b_s \mid s \subseteq \lambda \rangle$ be such that $\langle a_i \mid i < \lambda \rangle$ is indiscernible and $\varphi(a_i, b_s)$ holds iff $i \in s$. Let M be the Skolem hull of $\bar{a}$. Let $p(x) \in S(M)$ be the limit of $\bar{a}$ in M (so $\psi(x,c) \in p$ iff $\psi(x,c)$ holds for an end segment of $\bar{a}$). Let $P \subseteq \mathcal{P}(\lambda)$ be an independent family of size 2^λ (i.e., such that every finite Boolean combination has size λ). Then for each $D \subseteq P$, $p(x) \cup \left\{\varphi(x,b_s)^{s \in D} \mid s \in P\right\}$ is finitely satisfiable in M. ☐

4.2.1. *Small directionality.* The following construction will be useful (here and in Section 5):

Construction 4.5. *Let T be any complete theory and $M \models T$. Suppose that there is some $p \in S(M)$ and finite Δ such that $\mathrm{uf}_\Delta(p)$ is infinite, and contains $\{q_i \mid i < \omega\}$.*

For all $i < j < \omega$, there is a formula $\varphi_{i,j} \in \Delta$ and $b_{i,j} \in \mathfrak{C}$ such that $\varphi_{i,j}(x, b_{i,j}) \in q_i$, $\neg\varphi_{i,j}(x, b_{i,j}) \in q_j$ (or the other way around). By Ramsey's Theorem we may assume $\varphi_{i,j}$ is constant — $\varphi(x,y)$. Let N be a model containing M and $\{b_{i,j} \mid i < j < \omega\}$.

Suppose $\langle c_i \mid i < \omega \rangle$ *are in* $\mathfrak{C}$ *and* $c_i \models q_i|_N$. *Let* N' *be a model containing* $\{c_i \mid i < \omega\} \cup N$. *Let* $M^* = (N', N, M, Q, \bar{f})$ *where* $Q = \{c_i \mid i < \omega\}$ *and* $\bar{f} : Q^2 \to N$ *is a tuple of functions of length* $\lg(y)$ *defined by* $\bar{f}(c_i, c_j) = \bar{f}(c_j, c_i) = b_{i,j}$ *for* $i < j$.

So if $N \models \mathrm{Th}(M^*)$ *then* $N = (N_0', N_0, M_0, Q_0, \bar{f}_0)$ *and*

- $M_0 \prec N_0 \prec N_0' \models T$,
- $N_0 \cup Q_0 \subseteq N_0'$,
- $\bar{f}_0$ *are functions from* Q_0^2 *to* N_0,
- *For all* $c, d \in Q_0$, $c \equiv_{M_0} d$,
- $\mathrm{tp}_\Delta(c/N_0) \cup \mathrm{tp}(c/M_0)$ *is finitely satisfiable in* M_0 *for all* $c \in Q_0$, *and*
- $\varphi(c, \bar{f}_0(c,d)) \triangle \varphi(d, \bar{f}_0(c,d))$ *(where* $\triangle$ *denotes symmetric difference) holds for all* $c \neq d \in Q_0$.

Claim 4.6. Let T be any theory. Then T is small if and only if for every $M \models T$ and every type $p(x) \in S(M)$, $|\mathrm{uf}(p)| \leq 2^{|T|}$ (here p can also be an infinitary type, but then the bound is $2^{|T| + |\lg(x)|}$).

In addition, if T is not small, then for every $\lambda \geq |T|$, there is a model $M \models T$ of cardinality λ, a type $p \in S(M)$, and a finite set of formulas Δ such that $|\mathrm{uf}_\Delta(p)| \geq \lambda$.

Proof. Assume that T is small. The injective function $\mathrm{uf}(p) \to \prod_{\varphi \in L} \mathrm{uf}_{\{\varphi\}}(p)$ shows that $|\mathrm{uf}(p)| \leq 2^{|T|}$.

Conversely (and the "In addition" part): Assume that there is some p and Δ such that $\mathrm{uf}_\Delta(p)$ is infinite. Use Construction 4.5:

For every $\lambda \geq |T|$ we may find $N \models \mathrm{Th}(M^*)$ of size λ such that $|M_0| = |Q_0| = \lambda$, and we have a model M_0 of T with a type p over it, which has at least λ many Δ-coheirs. □

We conclude this section with a claim on theories with non-small directionality.

Claim 4.7. Suppose T has medium or large directionality. Then there exists some $M \models T$, $p \in S(M)$, $\psi(x,y)$ and $\{c_i \mid i < \omega\} \subseteq \mathfrak{C}$ such that for each $i < \omega$ the set $p(x) \cup \left\{\psi(x, c_j)^{j=i} \mid j < \omega\right\}$ is finitely satisfiable in M.

Proof. We consider the structure M^* introduced in Construction 4.5 and the formula φ chosen there. Find an extension of M^* with an indiscernible sequence $\langle d_i \mid i \in \mathbb{Z} \rangle$ inside Q. Assume without loss that $\varphi(d_0, \bar{f}(d_0, d_1)) \wedge \neg\varphi(d_1, \bar{f}(d_0, d_1))$ holds. This means that $\varphi(d_0, \bar{f}(d_0, d_1)) \wedge \neg\varphi(d_0, \bar{f}(d_{-1}, d_0))$.

We claim that $\varphi(d_i, \bar{f}(d_j, d_{j+1})) \wedge \neg\varphi(d_i, \bar{f}(d_{j-1}, d_j))$ holds if and only if $i = j$:

Suppose this holds but $i \neq j$. If $i > j$ then, since $\neg\varphi(d_1, f(d_0, d_1))$, it must be that $i > j + 1$, but then we have a contradiction to indiscernibility. Similarly, it cannot be that $i < j$. Thus the claim is proved with $\psi(x; y, z) = \varphi(x, y) \wedge \neg\varphi(x, z)$, and $c_i = \langle f(d_i, d_{i+1}), f(d_{i-1}, d_i) \rangle$. □

4.2.2. *Some helpful facts about dependent theories.* Assume T is dependent.

Recall,

Definition 4.8. A global type $p(x)$ is *invariant over* a set A if it does not split over it, namely if whenever b and c have the same type over A, $\varphi(x,b) \in p$ if and only if $\varphi(x,c) \in p$ for every formula $\varphi(x,y)$.

Definition 4.9. Suppose $p(x)$ and $q(y)$ are global A-invariant types. Then $(q \otimes p)(x,y)$ is a global invariant type defined as follows: for any $B \supseteq A$, let $a_B \models p|_B$ and $b_B \models q|_{Ba_B}$, then $p \otimes q = \bigcup_{B \supseteq A} \operatorname{tp}(a_B, b_B/B)$. One can easily check that it is well defined and A-invariant. Let $p^{(n)} = p \otimes p \cdots \otimes p$ where the product is done n times. So $p^{(n)}$ is a type in $(x_0, \ldots, x_{n-1})$, and $p^{(\omega)} = \bigcup_{n<\omega} p^{(n)}$ is a type in $(x_0, \ldots, x_n, \ldots)$. For $n \leq \omega$, $p^{(n)}$ is a type of an A-indiscernible sequence of length n.

Fact 4.10. [HP11, Lemma 2.5] *If T is NIP then for a set A the map $p(x_0) \mapsto p^{(\omega)}(x_0, \ldots)|_A$ from global A-invariant types to ω-types over A is injective.*

In the rest of the section, Δ will always denote a finite set of formulas, closed under negation.

Claim 4.11. For every set $A \subseteq C$, any type $q(x) \in S_\Delta(\mathfrak{C})$ which is finitely satisfiable in A and any choice of a coheir $q' \in S(\mathfrak{C})$ over A which completes q:

- $(a_0, \ldots, a_{n-1}) \models \left(q'^{(n)}|_C\right) \upharpoonright \Delta$ if and only if $a_0 \models q|_C$, $a_1 \models q|_{Ca_0}$, etc.

This enables us to define $q^{(n)}(x_0, \ldots x_{n-1}) \in S_\Delta(\mathfrak{C})$ as $q'^{(n)} \upharpoonright \Delta$.

It follows that $q^{(n)}$ is a type of a Δ-indiscernible sequence of length n.

Proof. The proof is by induction on n:

Right to left: suppose $a_i \models q|_{Ca_0 \cdots a_{i-1}}$ for $i \leq n$, $\varphi(x_0, \ldots, x_n, y) \in \Delta$ and $\varphi(x_0, \ldots, x_n, c) \in q'^{(n+1)}$ for $c \in C$ but $\neg\varphi(a_0, \ldots, a_n, c)$ holds. Then by the choice of a_n, $\neg\varphi(a_0, \ldots, a_{n-1}, x, c) \in q$. Suppose $(b_0, \ldots, b_{n-1}) \models q'^{(n)}|_C$, then $\varphi(b_0, \ldots, b_{n-1}, x, c) \in q$ so there is some $c' \in A$ such that $\varphi(b_0, \ldots, b_{n-1}, c', c) \wedge \neg\varphi(a_0, \ldots, a_{n-1}, c', c)$ holds. But this is a contradiction to the induction hypothesis.

Left to right is similar. □

The following is a local version of Fact 4.10, which will be useful later:

Proposition 4.12. *(T dependent) Suppose Δ is a finite set of formulas, x a finite tuple of variables. Then there exists $n < \omega$ and finite set of formulas Δ_0 such that for every set A, if $q_1(x), q_2(x) \in S(\mathfrak{C})$ are coheirs over A and $\left(q_1^{(n)} \upharpoonright \Delta_0\right)|_A = \left(q_2^{(n)} \upharpoonright \Delta_0\right)|_A$ then $q_1 \upharpoonright \Delta = q_2 \upharpoonright \Delta$.*

Proof. By compactness and NIP,

- there exists some finite set of formulas Δ_0 and some n such that for all $\varphi(x,y) \in \Delta$ and all Δ_0-indiscernible sequences $\langle a_0, \ldots, a_{n_\Delta - 1}\rangle$, there is no c such that $\varphi(a_i, c)$ holds if and only if i is even. We may assume that $\Delta \subseteq \Delta_0$.

By Claim 4.11, we can conclude:

- Suppose that $\left(q_1^{(n)} \restriction \Delta_0\right)|_A = \left(q_2^{(n)} \restriction \Delta_0\right)|_A$, but $q_1 \restriction \Delta \neq q_2 \restriction \Delta$. Then there is some formula $\varphi(x,y) \in \Delta$ and some $c \in \mathfrak{C}$ such that $\varphi(x,c) \in q_1$ and $\neg\varphi(x,c) \in q_2$.

  Since $(q_1 \restriction \Delta_0)^{(n)}|_A = (q_2 \restriction \Delta_0)^{(n)}|_A$, $(q_1 \restriction \Delta_0)^{(m)}|_A = (q_2 \restriction \Delta_0)^{(m)}|_A$ for every $m \leq n$, and it follows by induction on m that the sequence defined by $a_0 \models (q_1 \restriction \Delta_0)|_{Ac}$, $a_1 \models (q_2 \restriction \Delta_0)|_{Aca_0}$, $a_2 \models (q_1 \restriction \Delta_0)|_{Aca_0a_1}$, $\ldots$, $a_{m-1} \models (q_i \restriction \Delta_0)|_{Aca_0\cdots a_{m-2}}$ ($i \in \{1,2\}$) realizes this type. But this entails a contradiction, because $\langle a_0, \ldots, a_{n-1}\rangle$ is a Δ_0 indiscernible sequence (even over A), while $\varphi(a_i, c)$ holds if and only if i is even.

$\square$

Problem 4.13. Does Proposition 4.12 hold for invariant types (not just for coheirs)?

4.2.3. *Large directionality and definability.* Let us recall the definition of $\operatorname{ded}\lambda$.

Definition 4.14. Let $\operatorname{ded}\lambda$ be the supremum of the set:

$$\{|I| \mid I \text{ is a linear order with a dense subset of size } \leq \lambda\}.$$

Fact 4.15. *It is well known that* $\lambda < \operatorname{ded}\lambda \leq (\operatorname{ded}\lambda)^{\aleph_0} \leq 2^\lambda$. *If* $\lambda^{<\lambda} = \lambda$ *then* $\operatorname{ded}\lambda = 2^\lambda$ *so* $\operatorname{ded}\lambda = (\operatorname{ded}\lambda)^{\aleph_0} = 2^\lambda$.

For more, see Section 6 and [CKS12, Section 6].

Definition 4.16. Suppose M is a model and $p \in S(M)$. Let M_p be M enriched with externally definable sets defined over a realization of p. Namely, we enrich the language to a language L_p by adding new relation symbols $\{d_p x\varphi(x,y) \mid \varphi(x,y) \text{ is a formula}\}$ (so d_p is thought of as a quantifier over x), and let M_p be a structure for L_p with universe M where we interpret $d_p x\varphi(x,y)$ as $\{b \in M \mid \varphi(x,b) \in p\}$.

Remark 4.17. Every model $N \models Th(M_p)$ gives rise to a complete L type over N, namely $p^N = \{\varphi(b,x) \mid b \in N, N \models d_p x\varphi(x,b)\}$.

Claim 4.18. Let T be any theory, $M \models T$. Suppose $p \in S(M)$, $q \in \operatorname{uf}(p)$, and $\bar{a} = \langle a_0, a_1, \ldots\rangle \models q^{(\omega)}|_M$. If $\operatorname{tp}(\bar{a}/M)$ is not definable with parameters in M_p, then T is large.

Moreover, in this case

⊗ There exists a finite Δ such that for every $\lambda \geq |T|$,

$$\operatorname{ded}\lambda \leq \sup\{|\operatorname{uf}_\Delta(p)| \mid p \in S(N), |N| = \lambda\}.$$

Proof. We may assume that $|M| = |L|$: let $r = q^{(\omega)}|_M$, and $N \prec M_r$, $|N| = |L|$. Then N gives rise to a complete type $r'(x_0, x_1, \ldots) \in S(N)$. Let $p' = r' \restriction x_0$. It is easy to see that $r' = q'^{(\omega)}|_N$ for some $q' \in \mathrm{uf}(p')$. Also, r' is not definable with parameters in $N_{p'}$.

Let us recall a theorem from [She71a] (we formulate it a bit differently):

Suppose L is a language of cardinality at most λ, P a new predicate (or relation symbol), and S a complete theory in $L(P)$.

Definition. $\mathrm{df}_{\mathrm{iso}}\,\lambda$ is the the supremum of the set of cardinalities:

$$|\{B' \subseteq M \mid (M, B') \cong (M, B)\}|$$

where (M, B) is an $L(P)$ model of S of cardinality λ.

Theorem. [She71a, Hod93, Theorem 12.4.1] *The following are equivalent*[2]*:*

1. *(1) P is not definable with parameters in S, i.e., there is no L-formula $\theta(x, y)$ such that $S \models \exists y \forall x (P(x) \leftrightarrow \theta(x, y))$.*
2. *(2) For every $\lambda \geq |L|$, $\mathrm{df}_{\mathrm{iso}}(\lambda) \geq \mathrm{ded}\,\lambda$.*

Let $n < \omega$ be the integer first such that $\mathrm{tp}(\bar{a} \restriction n/M)$ is not definable with parameters in M_p. So $1 < n$ and $r = \mathrm{tp}(a_0, \ldots, a_{n-2}/M)$ is definable but $\mathrm{tp}(a_0, \ldots, a_{n-1}/M)$ is not.

For a formula $\alpha(x_0, \ldots, x_{n-2}, y)$ let $(d_r\alpha)(y)$ be a formula in $L(M_p)$ defining $\alpha(\bar{a} \restriction n-1, M)$. If $M_p \prec N \models T_p$ then, as in Remark 4.17, there is a complete L type $r^N(x_0, \ldots, x_{n-2})$ over N defined by $\alpha(x_0, \ldots, x_{n-2}, b) \in r^N$ if and only if $N \models (d_r\alpha)(b)$.

There is some formula $\varphi(x_0, \ldots, x_{n-1}, y)$ such that the set $B_0 := \varphi(a_0, \ldots, a_{n-1}, M)$ is not definable with parameters in M_p. Let $S = \mathrm{Th}(M_p, B_0)$ in the language $L(M_p)(P)$ (naming elements from M, so that $N \models S$ implies $M_p \prec N$).

By the theorem cited above, for every $\lambda \geq |L|$ and $\kappa < \mathrm{ded}\,\lambda$, there exists a model $(N_{\lambda,\kappa}, B_{\lambda,\kappa}) = (N, B) \models S$ of cardinality λ such that, letting $\mathcal{B}^N = \{B' \mid (N, B') \cong (N, B)\}$, $\left|\mathcal{B}^N\right| > \kappa$.

Let $\bar{a}^N = \left(a_0^N, \ldots, a_{n-2}^N\right) \models r^N$ and for every $B' \in \mathcal{B}^N$, let

$$q_{B'} = p^N(x) \cup \left\{\varphi\left(\bar{a}^N, x, \bar{b}\right) \,\middle|\, \bar{b} \in B'\right\} \cup \left\{\neg\varphi\left(\bar{a}^N, x, \bar{b}\right) \,\middle|\, \bar{b} \notin B'\right\}.$$

By choice of S, B and $\mathcal{B}$, $q_{B'}$ is finitely satisfiable in N and for $B' \neq B'' \in \mathcal{B}^N$, $q_{B'} \restriction \varphi \neq q_{B''} \restriction \varphi$, so now $N \restriction L$ is a model of T with a type p^N such that $\left|\mathrm{uf}_{\{\varphi\}}(p)\right| > \kappa$. □

If T is small we can say more:

Claim 4.19. Assume T is small, $M \models T$. Suppose $p \in S(M)$, $q \in \mathrm{uf}(p)$, and $\bar{a} = \langle a_0, a_1, \ldots \rangle \models q^{(\omega)}|_M$, then $\mathrm{tp}(\bar{a}/M)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ in M_p.

[2]The original theorem referred to $\mathrm{ded}^*\,\lambda$, which counts the number of branches of the same height in a tree with λ many nodes, but it equals $\mathrm{ded}\,\lambda$, see [CKS12, Section 6] and Fact 6.4.

Proof. By Claim 4.18 it is definable with parameters in M_p. Let $n < \omega$ be minimal such that $q^{(n)}|_M$ is not definable over $\mathrm{acl}^{eq}(\emptyset)$ in M_p. Suppose that for some formula $\varphi(x_0, \ldots, x_{n-1}, y)$, $\mathrm{tp}_\varphi(\bar{a}/M)$ is not definable over $\mathrm{acl}^{eq}(\emptyset)$ in M_p. This means that while the set $\{b \in M \mid \mathfrak{C} \models \varphi(\bar{a}, b)\}$ is definable by $\psi(y, c)$ for some ψ in $L(M_p)$, $c \notin \mathrm{acl}^{eq}(\emptyset)$. We may assume that $c \in M_p^{eq}$ is the code of this set (for every automorphism σ of the monster model $\mathfrak{C}$ of M_p^{eq}, σ fixes $\psi(\mathfrak{C}, c)$ if and only if $\sigma(c) = c$). So in some elementary extension $M_p \prec N$, there are infinitely many conjugates of c over $\mathrm{acl}^{eq}(\emptyset)$, $\{c_i \mid i < \omega\}$, such that $\psi(N, c_i) \neq \psi(N, c_j)$ for $i \neq j$. This implies that $\mathrm{uf}_{\{\varphi\}}(p^N) \geq \aleph_0$, just as in the proof of Claim 4.18. □

Corollary 4.20. *(T dependent) T is large if and only if for every $\lambda \geq |T|$,*

$$\sup\{|\mathrm{uf}_\Delta(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\} = \mathrm{ded}\,\lambda.$$

Proof. Suppose T is large, i.e., for some Δ, M of size $\lambda \geq |T|$ and $p \in S(M)$, $|\mathrm{uf}_\Delta(p)| > \lambda$. By Proposition 4.12, find some Δ_0 and n such that

$$|\mathrm{uf}_\Delta(p)| \leq \left|\left\{\left(q^{(n)} \restriction \Delta_0\right)|_M \mid q \in \mathrm{uf}(p)\right\}\right|.$$

Hence there is some $q \in \mathrm{uf}_\Delta(p)$ such that $q^{(n)} \restriction \Delta_0$ is not definable with parameters over M_p, and we are done by Claim 4.18 (also note that $|S_\Delta(M)| \leq \mathrm{ded}|M|$ for every finite Δ in dependent theories (see e.g., [She71b, Theorem 4.3])). □

4.2.4. *Concluding remarks.*

Theorem 4.21. *For every theory T,*

1. *T is small iff for all $M \models T$, $p(x) \in S(M)$, $|\mathrm{uf}(p)| \leq 2^{|T|+|\lg(x)|}$.*
2. *T is medium iff for all $M \models T$, $p(x) \in S(M)$, $|\mathrm{uf}(p)| \leq |M|^{|T|+|\lg(x)|}$ and T is not small.*

Proof. (1) is Claim 4.6.

(2) Left to right is clear. Conversely, since T is not small, by Claim 4.6, for all $\lambda \geq |T|$,

$$\sup\{|\mathrm{uf}_\Delta(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\} \geq \lambda.$$

On the other hand, if it is strictly greater than λ, then by definition T is large. For λ large enough, $\mathrm{ded}\,\lambda > \lambda = \lambda^{|T|+|\lg(x)|}$ so by Corollary 4.20, we get a contradiction to the right hand side. □

In Section 4.3, we will show that these classes are not empty, and thus:

Corollary 4.22. *For $\lambda \geq |T|$, the cardinality:*

$$\sup\{|\mathrm{uf}_\Delta(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\}$$

has four possibilities: finite / $\aleph_0$; λ; $\mathrm{ded}\,\lambda$; 2^{2^λ}.

This corresponds to small, medium, and large directionality (the last one happens when the theory has the independence property, see Observation 4.4).

Problem 4.23. Suppose T is interpretable in T', and T is large. Does this imply that T' is large or at least not small?

4.3. **Examples of different directionalities.** Here we give examples of the different directionalities.

4.3.1. *Small directionality.*

Example 4.24. $\mathrm{Th}(\mathbb{Q},<)$ has small directionality. In fact, every 1-type over a model M, has at most 2 global coheirs, and in general, a type $p(x_0,\dots,x_{n-1})$ is determined by the order type of $\{x_0,\dots,x_{n-1}\}$ and $p\restriction x_0$, $p\restriction x_1$, etc.

Proposition 4.25. *The theory of dense trees is also small.*

Proof. So here T is the model completion of the theory of trees in the language $\{<,\wedge\}$.

Claim. Let $M\models T$ and $p(x_0,x_1,\dots,x_{n-1})\in S(M)$ be any type. Then $\bigcup_{i,j<n} p\restriction(x_i,x_j)\cup p|_\emptyset\vdash p$.

Proof. Let $\Sigma=\bigcup_{i,j<n} p\restriction(x_i,x_j)\cup p|_\emptyset$. Suppose $(a_0,\dots,a_{n-1})\models\Sigma$. By quantifier elimination, the formulas in p are Boolean combination of formulas of the form $\bigwedge_{k<m} x_{j_k}\wedge a\le\bigwedge_{l<r} x_{r_l}\wedge b$ where $b,a\in M$ and $j_0,\dots,j_{k-1},r_0,\dots,r_{l-1}<n$.

If a,b does not appear, $\bar a$ satisfy this formula because we included $p|_\emptyset$. Consider $\bigwedge_{k<m} x_{j_k}\wedge a$: by assumption we know what is the ordering of $\{x_{j_k}\wedge a\,|\,k<m\}$ (this set is linearly ordered — it is below a). Hence, as $\bar a\models\Sigma$, $\bigwedge_{k<m} a_{j_k}\wedge a$ must be equal to the minimal element in this set, namely $a_{j_k}\wedge a$ for some k, which is determined by Σ. Now $a_{j_k}\wedge a\le\bigwedge_{l<r} a_{r_l}\wedge b$ holds if and only if for each $r<l$, we have $a_{j_k}\wedge a\le a_r$ and $a_{j_k}\wedge a\le b$, both decided in Σ.

Note that we can get rid of $p|_\emptyset$ but we should replace 2-types by 3-types. □

Claim. For any $A\subseteq M\models T$, and $p(x),q(y)\in S(A)$, there are only finitely many complete type $r(x,y)$ that contain both p and q. In fact there is a uniform bound on their number.

Proof. We may assume that A is a substructure. For any $a\in M$, the structure generated by A and a, denoted by $A(a)$, is just $A\cup\{a_A,a\}$ where $a_A=\max\{b\wedge a\,|\,b\in A\}$. Note that a_A need not exist, but if it does, then it is the only new element apart from a (because if $b_1\wedge a<b_2\wedge a$ then $b_1\wedge a=b_1\wedge b_2$).

Now, let $a\models p$ and $b\models q$. Let $B=B(p,q)=\{d\in A\,|\,d\le a\,\&\,d\le b\}$. This set is linearly ordered, and it may have a maximum. If it does, denote it by m. Note that m depends only on p and q.

Now it is easy to show that $\mathrm{tp}(a,a_A,b,b_A,a\wedge b/m)\cup\mathrm{tp}(a/A)\cup\mathrm{tp}(b/A)$ determines $\mathrm{tp}(a,b/A)$ by quantifier elimination. This suffices because the number of types of finite tuples over a finite set is finite. □

Let $M \models T$, $p(\bar{x}) \in S(M)$ and I be the set of all types $r(\bar{x}_0, \bar{x}_1, \ldots)$ over M such that realizations of r are indiscernible sequences of tuples satisfying p.

Let $V = \{(y_0, y_1) \mid y_0, y_1$ are any 2 variables from $\bar{x}_0, \bar{x}_1\}$. By the second claim, for any $(y_0, y_1) \in V$, the set $\{r \upharpoonright (y_0, y_1) \mid r \in I\}$ is finite. By the first claim and indiscernibility, the function taking r to $(r|_\emptyset, \langle r \upharpoonright (y_0, y_1) \mid (y_0, y_1) \in V \rangle)$ is injective. Together, it means that $|I| \leq 2^{\aleph_0 + |\lg(\bar{x})|}$ and we are done by Fact 4.10. □

4.3.2. *Medium directionality.*

Example 4.26. Let $L = \{P, Q, H, <\}$ where P and Q are unary predicates, H is a unary function symbol and $<$ is a binary relation symbol. Let $T^\forall$ be the following theory:

- $P \cap Q = \emptyset$.
- H is a function from P to Q (so $H \upharpoonright Q = \mathrm{id}$).
- $(P, <, \wedge)$ is a tree.

And let T be its model completion (so T eliminates quantifiers). Note that there is no structure on Q. So as in Section 2, T is dependent (this theory is interpretable in the theory there).

Let T' be the restriction of T to the language $L' = L \backslash \{H\}$. The same "moreover" part applies here as in Corollary 2.13, so T' is the model completion of $T^\forall \upharpoonright L'$ and also eliminates quantifiers.

Claim 4.27. T' has small directionality.

Proof. The only difference between T' and dense trees is the new set Q which has no structure. Easily this does not make any difference. □

Proposition 4.28. T *has medium directionality.*

Proof. Let $M \models T$. Let B be a branch in P^M (i.e., a maximal linearly ordered set). Let $p(x) \in S(M)$ be a complete type containing $\Sigma_B := \{b < x \mid b \in B\}$. Note that Σ "almost" isolates p: the only freedom we have, is to determine what is $H(x)$. So suppose $H(x) = m \in p$ for $m \in Q$.

Let $c \models p$ (so $c \notin M$). For each $a \in Q^M$, let $p_a(x) = p \cup \{H(c \wedge x) = a\}$.

Then p_a is finitely satisfiable in M: Suppose $\Gamma \subseteq p_a$ is finite. By quantifier elimination, we may assume that $\Gamma \subseteq \Sigma_B \cup \{H(c \wedge x) = a\} \cup \{H(x) = m\}$. Since B is linearly ordered, we may assume that $\Gamma = \{b < x, H(c \wedge x) = a, H(x) = m\}$ for some $b \in B$. Since T is the model completion of $T^\forall$ which has the amalgamation property, there are two elements $d, e \in M$ such that $b < d, e$, $e \in B$, $d \notin B$, $H(d) = m$ and $H(d \wedge e) = a$. Since $d \wedge c = d \wedge e$, $d \models \Gamma$. We have found $|Q^M|$ coheirs of p, and since M was arbitrary T is not small.

This gives a lower bound on the directionality of T, and we would like to find an upper bound as well. We shall use the same idea as in the proof of Proposition 4.25.

Let $M \models T$, $p(\bar{x}) \in S(M)$ and I be the set of all types $r(\bar{x}_0, \bar{x}_1, \ldots)$ over M such that realizations of r are indiscernible sequences of tuples satisfying p. Let $I' = \{r \restriction L' \mid r \in I\}$. By the proof of Proposition 4.25, $|I'| \leq 2^{\aleph_0 + |\lg(\bar{x})|}$.

Let $V = \{t \mid t \text{ is a term in } L \text{ in the variables } \bar{x}_0, \bar{x}_1, \ldots \text{ over } \emptyset\}$. Suppose $r \in I$, then, as in the proof of Proposition 4.25, for every term t such that $P(t) \in r$, let $t_r = \max\{a \wedge t \mid a \in M\}$ — a term over M (it need not exist). To determine r, it is enough to determine the equations that occur between the images under H of the t_r's and the t's over M. This shows that $|I| \leq |M|^{\aleph_0 + |\lg(\bar{x})|}$. □

4.3.3. *Large directionality.*

Example 4.29. Let $L = \{P, Q, H, <_P, <_Q\}$ where P and Q are unary predicates, H is a unary function symbol and $<_P, <_Q$ are binary relation symbols. Let $T^{\forall}$ be the following theory:

- $P \cap Q = \emptyset$.
- H is a function from P to Q.
- $(P, <_P, \wedge)$ is a tree.
- $(Q, <_Q)$ is a linear order.

Proposition 4.30. T *has large directionality.*

Proof. This is similar to the proof of Proposition 4.28.

Let $M \models T$. Let B be a branch in P^M. Let $p(x) \in S(M)$ be a complete type containing $\Sigma_B := \{b < x \mid b \in B\}$ saying that $H(x) = m$ for some $m \in Q^M$.

Let $c \models p$ (so $c \notin M$). For each cut $I \subseteq Q^M$, let:

$$p_I(x) = p \cup \left\{e < H(c \wedge x) < f \,\middle|\, e \in I, f \in Q^M \backslash I\right\}.$$

Then p_I is finitely satisfiable in M as in the proof of 4.28. So for every cut in Q we found a coheir of p, and since M was arbitrary T is not small nor medium (because for every linear order, we can find a model such that Q contains this order). □

4.3.4. *RCF.* It turns out that even RCF has large directionality, as we shall present now.

Apparently, that RCF was not small was already known and can be deduced from Marcus Tressl's thesis (see [Tre96, 18.13]), but here we give a direct proof that RCF is in fact large and even more.

Definition 4.31. Let $M \models \mathsf{RCF}$. A type $p \in S(M)$ is called *dense* if it is not definable and the differences $b - a$ with $a, b \in M$ and $a < x < b \in p$, are arbitrarily (w.r.t. M) close to 0.

For example, if R is the real closure of $\mathbb{Q}$, then $\operatorname{tp}(\pi/R)$ is dense.

Fact 4.32. *Any real closed field can be embedded into a real closed field of the same cardinality with some dense type.*

Proof. [due to Marcus Tressl] Let R be a real closed field. Let S be the (real closed) field $R\left(\left(t^{\mathbb{Q}}\right)\right)$ of generalized power series over R. Let K be the definable closure of $R(t)$ in S and let p be the 1-type of the formal Taylor series of e^t over K: $\operatorname{tp}\left(1+t^1/1!+t^2/2!+\cdots/K\right)$. Then p is a dense 1-type over K. □

Claim 4.33. Suppose p is dense and q is a definable type over $M \models \mathsf{RCF}$ and both are complete. Then q and p are weakly orthogonal, meaning that $p(x) \cup q(y)$ implies a complete type over M.

Proof. [Remark: this is an easy result that is well known, but for completeness we give a proof.]

Let $\omega \models q, \alpha \models p$.

Note that since p is not definable over M, for every $m \in M$, and even for every $m \in \mathfrak{C}$ such that $\operatorname{tp}(m/M)$ is definable, there is some $\varepsilon_m \in M$ such that $0 < \varepsilon_m < |\alpha - m|$.

Now, suppose that $\varphi(x, y)$ is any formula over M. Then, as q is definable, there is a formula $\psi(x) := (d_q y)\,\varphi(x, y)$ over M that defines $\varphi(M, \omega)$. We claim that $p(x) \cup q(y) \models \varphi(x, y)$ if and only if $p(x) \models (d_q y)\,\varphi(x, y)$.

We know that ψ is equivalent to a finite union of intervals and points from M. We also know that $\varphi(\mathfrak{C}, \omega)$ is such a union, but the types of the end-points over M are definable over M (since we have definable Skolem functions). So denote the set of all these end-points by A. Let $0 < \varepsilon \in M$ be smaller than every ε_m for each $m \in A$. Let $a, b \in M$ such that $a < \alpha < b$ and $b - a < \varepsilon$. Then:

- $\psi(\alpha)$ holds if and only if
- $\psi(m)$ holds for all $m \in M$ such that $a \le m \le b$ if and only if
- $\varphi(m, \omega)$ holds for all $m \in M$ such that $a \le m \le b$ if and only if
- $\varphi(\alpha, \omega)$ holds.

□

We claim that RCF has large directionality. Moreover, we seem to answer an open question raised in [Del84] (at least in some sense, see below), as she says there:

> Mais il laisse ouverte la possibilité que la borne du nombre de cohéritiers soit $\operatorname{ded}|M|$ dans le cas de la propriété de l'ordre et $(\operatorname{ded}|M|)^{(\omega)}$ dans le cas de l'ordre multiple.

So let us make clear what the question means:

Definition 4.34. (Taken from [Kei78, Kei76]) A theory T is said to have the *multiple order property* if there are formulas $\varphi_n(x, y_n)$ for $n < \omega$ such that the following set of formulas is consistent with T:

$$\Gamma = \left\{\varphi_n\left(x_\eta, y_{n,k}\right)^{\eta(k)<n} \mid \eta \in {}^{\omega}\omega\right\}.$$

Remark 4.35. If T is strongly dependent (see 2.10), for example, if $\mathsf{T} = \mathsf{RCF}$, it does not have the multiple order property.

Proof. Suppose T has the multiple order property as witnessed by formulas φ_n. Consider the formulas $\psi_n(x,y,z) = \varphi_n(x,y) \leftrightarrow \varphi_n(x,z)$. It is easy to see that $\{\psi_n \mid n<\omega\}$ exemplify that the theory is not strongly dependent. $\square$

Fact 4.36. [Kei78] *If* T *is countable and has the multiple order property, then for every cardinal* λ*,* $\sup\{S(M) \mid M \models T, |M| = \lambda\} \geq (\mathrm{ded}\,\lambda)^{\omega}$*. If* T *does not have the multiple order property, then* $\sup\{S(M) \mid M \models T, |M| = \lambda\} \leq \mathrm{ded}\,\lambda$.

So the question can be formulated as follows:

- Is there a countable theory without the multiple order property such that for every $\lambda \geq \aleph_0$, $\sup\{|\mathrm{uf}(p)| \mid p \in S_{<\omega}(M), |M| = \lambda\} = (\mathrm{ded}\,\lambda)^{\omega}$ (recall that $S_{<\omega}(M)$ is the set of all finitary types over M).

It is a natural question, because of 2 reasons:

(1) In general, the number of types (in α variables) over a model of size λ in a dependent theory is bounded by $(\mathrm{ded}\,\lambda)^{|T|+|\alpha|+\aleph_0}$ (by [She71b, Theorem 4.3]), so by Fact 4.10 this is an upper bound for $\sup\{|\mathrm{uf}(p)| \mid p \in S(M), |M| = \lambda\}$.
(2) It is very easy to construct an example with the multiple order property that attains this maximum: for example, one can modify example 4.29, and add $\aleph_0$ independent orderings to Q.

Definition 4.37. For $M \models \mathsf{RCF}$, let $S_{\mathrm{dense}}(M)$ be the set of dense complete types over M.

Theorem 4.38. *Suppose* $M \models \mathsf{RCF}$*. Then there is a type* $p \in S_2(M)$ *such that* $|\mathrm{uf}(p)| \geq |S_{\mathrm{dense}}(M)|^{\omega}$.

Proof. We may assume $S_{\mathrm{dense}}(M) \neq \emptyset$. Suppose r_* is a dense type. Let $\alpha \models r_*$, and let $\omega \in \mathfrak{C}$ be an element greater than any element in M. Then $q = \mathrm{tp}(\omega/M)$ is definable and we can apply Claim 4.33. Let $p(x_\omega, x_\alpha) = \mathrm{tp}(\omega, \alpha/M)$.

For every dense type $r(x)$ over M, choose a realization $a_r \in \mathfrak{C}$. For every sequence $\bar{r} = \langle r_i \mid i<\omega\rangle$ of <u>positive</u> dense types over M (i.e., $r_i \models x > 0$), we define a coheir $p_{\bar{r}}$ of p as follows:

Fix $\bar{a} = \langle a_{r_i} \mid i<\omega\rangle$. For every sequence $\bar{b} = \langle b_i \mid i<\omega\rangle \in M^{\omega}$ such that $r_i \models x < b_i$ for all $i<\omega$, and for each $n<\omega$ let $f_n(\bar{a},x) = \alpha + \sum_{i=0}^{n}\left(a_{r_i}/x^{i+1}\right)$ and $g_n\left(\bar{a},\bar{b},x\right) = \alpha + \sum_{i=0}^{n-1}\left(a_{r_i}/x^{i+1}\right) + b_n/x^{n+1}$.

Now, let $p_{\bar{r}}(x_\omega, x_\alpha)$ be:

$$p_{\bar{r}}(x_\omega, x_\alpha) = p(x_\omega, x_\alpha) \cup \left\{f_n(\bar{a}, x_\omega) < x_\alpha < g_n\left(\bar{a},\bar{b},x_\omega\right) \,\middle|\, \bar{b} \text{ as above}, n<\omega\right\}.$$

Claim. $p_{\bar{r}}(x_\omega, x_\alpha)$ (which is over $M \cup \{\alpha\} \cup \{a_{r_i} \mid i < \omega\}$) is finitely satisfiable in M.

Proof. Suppose we are given a finite subset $p_0 \subseteq p_{\bar{r}}(x_\omega, x_\alpha)$, and a finite set of inequalities $S = \left\{f_k(\bar{a}, x_\omega) < x_\alpha < g_k\left(\bar{a}, \bar{b}, x_\omega\right) \middle| k \leq n,\ \bar{b} \in B\right\}$ where B is some finite set of tuples $\langle b_i \mid i \leq n\rangle$ such that $r_i \models x < b_i$ for $i \leq n$.

Let $\bar{b}$ be a tuple $\langle b_i \mid i \leq n\rangle \in M^{n+1}$ such that for $i \leq n$, $r_i \models x < b_i$ and $b_i < b_i'$ for any tuple $\langle b_i' \mid i \leq n\rangle \in B$. We may assume that $p_0 = r_{*,0}(x_\alpha) \cup q_0(x_\omega)$ where $r_{*,0} \subseteq r_*$ and $q_0 \subseteq q$. We may assume in addition that both $r_{*,0}$ and q_0 are intervals over M (i.e., types in the language $\{<\}$). Finally, we may assume that $B = \left\{\bar{b}\right\}$.

We will show:

☺ For all $o \in M$ large enough, there is some $0 < \varepsilon_o \in M$ such that for all $k, l \leq n$, $\varepsilon_o < g_k\left(\bar{a}, \bar{b}, o\right) - f_l(\bar{a}, o)$.

Once ☺ is established, let o be large enough so that it has such an ε_o, o satisfies $q_0(x_\omega)$ and for every $k \leq n$, $f_k(\bar{a}, o), g_k\left(\bar{a}, \bar{b}, o\right) \models r_{*,0}(x_\alpha)$ (so also every element between f_k and g_k). Suppose $l \leq n$ is such that $f_l(\bar{a}, o)$ is maximal and $k \leq n$ is such that $g_k\left(\bar{a}, \bar{b}, o\right)$ is minimal. For $i \leq n$, let $c_i \in M$ be such that $a_{r_i} < c_i$ and $c_i - a_{r_i} < \varepsilon_o \cdot \left(o^{i+1}\right)/(l+2)$ (these exist since the r_i's are dense), and let $\alpha < \alpha_0 \in M$ be such that $\alpha_0 - \alpha < \varepsilon_o/(l+2)$. Let $d = \alpha_0 + \sum_{i=0}^{l}\left(c_i/o^{i+1}\right) \in M$. Then $f_l(\bar{a}, o) < d$ and $d - f_l(\bar{a}, o) = (\alpha_0 - \alpha) + \sum_{i=0}^{l}(c_i - a_{r_i})/o^{i+1} < \varepsilon_o$. So $d < g_k\left(\bar{a}, \bar{b}, o\right)$, and so $(o, d) \models p_0$.

So we only need to show ☺. It is enough to show that for each $k, l \leq n$, for all large enough o, there is some $0 < \varepsilon_{o,k,l} \in M$ such that $\varepsilon_{o,k,l} < g_k\left(\bar{a}, \bar{b}, o\right) - f_l(\bar{a}, o)$. Suppose $k > l$. In that case,

$$g_k\left(\bar{a}, \bar{b}, o\right) - f_l(\bar{a}, o) \geq b_k/o^{k+1} > 0$$

(since the types r_i are positive). Suppose $k \leq l$. So,

$$g_k\left(\bar{a}, \bar{b}, o\right) - f_l(\bar{a}, o) = (b_k - a_{r_k})/o^{k+1} - \left(\sum_{i=k+1}^{l} a_{r_i}/o^{i+1}\right).$$

Since r_k is dense, there is some $0 < \varepsilon \in M$ such that $\varepsilon < b_k - a_{r_k}$. Also, there are some $a_i' \in M$ such that $a_{r_i} < a_i'$. The difference above is greater than:

$$\varepsilon/o^{k+1} - \sum_{i=k+1}^{l} a_i'/o^{i+1} \in M,$$

and for o large enough this number is positive, so let it be $\varepsilon_{o,k,l}$. □

Note that for $\bar{r} \neq \bar{r}'$, $p_{\bar{r}} \cup p_{\bar{r}'}$ is inconsistent. Also, since $r_*, r_* + 1, r_* + 2, \ldots$ are all dense types, $|S_{\text{dense}}(M)| \geq \aleph_0$, so the number of positive dense types over M is equal to the number of all dense types over M. Together, we are done. □

We conclude:

Corollary 4.39. *RCF has large directionality. In addition, RCF does not have the multiple order property but for every* $\lambda \geq \aleph_0$*, with* $\mathrm{cof}\,(\mathrm{ded}\,\lambda) > \omega$*,*

$$\sup\{|\mathrm{uf}\,(p)|\,|\,M \models \mathsf{RCF},\, p \in S_2\,(M)\,,\, |M| = \lambda\} \geq (\mathrm{ded}\,\lambda)^{\omega}\,.$$

Proof. We will use results from Section 6.

By Theorem 6.2, we know that:

$$\sup\{|\mathrm{uf}\,(p)|\,|\,p \in S_2\,(M)\,,|M| = \lambda\} \geq \sup\left\{|S_{\mathrm{dense}}\,(M)|^{\omega}\,|\,|M| = \lambda\right\}.$$

On the other hand, Corollary 6.8 says that:

$$\sup\left\{|S_{\mathrm{dense}}\,(M)|^{\omega}\,|\,|M| = \lambda\right\} = \sup\left\{\left(\lambda^{\langle\mu\rangle_{\mathrm{tr}}}\right)^{\omega}\,|\,\mu \leq \lambda, \mathrm{cof}\,(\mu) = \mu\right\},$$

so this already implies that RCF is large (by Fact 6.4, the right hand side is $\geq \mathrm{ded}\,\lambda$). Corollary 6.10 says that for any cardinal λ, if $\mathrm{cof}\,(\mathrm{ded}\,\lambda) > \aleph_0$, then

$$\sup\left\{\left(\lambda^{\langle\mu\rangle_{\mathrm{tr}}}\right)^{\omega}\,|\,\mu \leq \lambda, \mathrm{cof}\,(\mu) = \mu\right\} = (\mathrm{ded}\,\lambda)^{\omega}\,.$$

Together we are done. □

Remark 4.40. For an easy proof that RCF is large, using the same notation from the proof of Theorem 4.38, for every bounded cut $I \subseteq M$, define:

$$p_I\,(x_\omega, x_\alpha) = r_*\,(x_\alpha) \cup q\,(x_\omega) \cup \{\alpha + a/x_\omega < x_\alpha < \alpha + b/x_\omega\,|\,a \in I, b \notin I\}.$$

Marcus Tressl has pointed out the type $\mathrm{tp}\,(\alpha, \omega)$ to us as a type with infinitely many coheirs (this follows from [Tre96, 18.13]). We thank him for that. This proof that the theory is large is ours.

4.3.5. *Valued fields.* We can combine the techniques of Theorem 4.39 and Example 4.29 in order to prove a similar result for valued fields.

Definition 4.41. The language L of valued fields is the following. It is a 3-sorted language, one sort for the base field K equipped with the ring language $\{0, 1, +, \cdot\}$, another for the valuation group Γ equipped with the ordered abelian groups language $L_\Gamma = \{0, +, <\}$, and another for the residue field k equipped with the ring language L_k. We also have the valuation map $v : K^\times \to \Gamma$ and an angular component map $ac : K \to k$. Recall that an angular component is a function that satisfies $ac\,(0) = 0$ and $ac \restriction K^\times : K^\times \to k^\times$ is a homomorphism such that if $v\,(x) = 0$ then $ac\,(x)$ is the residue of x.

For more on valued fields with angular component, see e.g., [Bél99, Pas89], which also gives us the following fact:

Fact 4.42. [Pas89, Theorem 4.1] *The theory of any Henselian valued field of characteristic* $(0,0)$ *in the language* L *has elimination of field quantifiers: every formula* $\varphi(x_K, x_k, x_\Gamma)$ *(where* x_K*,* x_k *and* x_Γ *are tuples of variables in the base field, the residue field and the valuation group respectively) is equivalent to a Boolean combination of formulas of the form* $\varphi(ac(f_0(x_K)), \ldots, ac(f_{n-1}(x_K)), x_k)$ *and* $\chi(v(g_0(x_K)), \ldots, v(g_{m-1}(x_K)), x_\Gamma)$ *where* φ *is a formula in* L_k*,* χ *is a formula in* L_Γ *and* g_i *and* f_j *are polynomials over the integers.*

Theorem 4.43. *Let* $T = Th(K, \Gamma, k)$ *be any theory of valued fields in* L *which eliminates field quantifiers. Then* T *has large directionality.*

Proof. Let $M_0 = (K_0, \Gamma_0, k_0) \models T$ be a countable model such that Γ_0 contains a copy of the rationals $\{\gamma_q \mid q \in \mathbb{Q}\}$ with the usual order and group structure, so $\gamma_0 = 0_\Gamma$ (by compactness, one only needs to embed a copy of a finitely generated subgroup of $(\mathbb{Q}, +, <)$ in a model of T, but any such subgroup is contained in a subgroup generated by one element, which is isomorphic to $(\mathbb{Z}, +, <)$).

Let S be the tree $2^{\leq\omega}$ and let $S_0 \subseteq S$ be $2^{<\omega}$, so S_0 is countable.

Let $\Sigma \langle x_s \mid s \in S_0 \rangle$ be the following set of formulas with variables in the field sort (over M_0):

$$\left\{ v(x_s - x_t) = \gamma_{\mathrm{lev}(s \wedge t)} \mid s, t \in S_0,\ s \wedge t < s, t \right\} \cup \left\{ v(x_s - x_t) \geq \gamma_{\mathrm{lev}(s)} \mid s, t \in S_0,\ s \leq t \right\}.$$

Then Σ is consistent with M_0: to realize $\Sigma \restriction 2^{<n}$, choose $a_i \in K_0$ with $v(a_i) = \gamma_i$ and let $x_s = \sum_{i < \mathrm{lev}(s)} s(i) a_i$ for $s \in 2^{<n}$. Let $M = (K_1, \Gamma_1, k_1)$ be a countable model containing M_0 and some $\{a_s \mid s \in S_0\}$ realizing Σ.

For each $\eta \in S$ with domain ω (this is a *branch* of S_0), let $p_\eta(x)$ be the following type in the valued field sort:

$$\left\{ v(x - a_s) \geq \gamma_{\mathrm{lev}(s)} \mid s < \eta \right\}.$$

It is consistent since any finite subset if realized by a_t for any $t < \eta$ large enough.

If $\eta_1 \neq \eta_2$ then $p_{\eta_1} \cup p_{\eta_2}$ is inconsistent:

Suppose $s = \eta_1 \wedge \eta_2$ and $s < t < \eta_1$, $s < t' < \eta_2$. If p_{η_2} is consistent with $v(x - a_t) \geq \gamma_{\mathrm{lev}(t)}$, then there is some a such that $v(a - a_t) \geq \gamma_{\mathrm{lev}(t)}$ and $v(a - a_{t'}) \geq \gamma_{\mathrm{lev}(t')}$. So $v(a_t - a_{t'}) \geq \min\left\{\gamma_{\mathrm{lev}(t')}, \gamma_{\mathrm{lev}(t)}\right\}$, but $t \wedge t' = s < t, t'$ and so $v(a_t - a_{t'}) = \gamma_{\mathrm{lev}(s)}$. This is a contradiction since $\gamma_{\mathrm{lev}(t)}, \gamma_{\mathrm{lev}(t')} > \gamma_{\mathrm{lev}(s)}$.

Let Ω be the algebraic closure (as a valued field) of the monster model $\mathfrak{C}$ of M. let $\bar{M} = \left(\bar{K_1}, \bar{\Gamma_1}, \bar{k_1}\right)$ be the algebraic closure of M as a valued field in Ω. Since $\bar{M}$ is countable, there is some branch $\eta \in S$ such that p_η is not realized in $\bar{M}$. Then for every polynomial $f(x)$ over K_1 and every $s < \eta$ large enough, $p_\eta \models ac(f(x)) = ac(f(a_s)) \wedge v(f(x)) = v(f(a_s))$ (decompose f into linear factors $\prod (x - b_i)$. For every large enough s, $v(b_i - a_s) < \gamma_{\mathrm{lev}(s)}$ for all i, and so if $d \models p_\eta$ in $\mathfrak{C}$, then $ac(f(d)) = ac(f(a_s))$ (because $\mathrm{res}\left(\frac{f(d)}{f(a_s)}\right) = 1$ — we do not assume that ac

extends to $\bar{M}$) and $v(f(d)) = v(f(a_s)))$. Since field quantifiers are eliminated in T, this implies that p_η is a complete type. Moreover, we have the following claim:

Claim. For any type $r(y) \in S(M)$ such that y is a tuple of variables in the valuation group sort, p_η and q are weakly orthogonal, meaning that $p_\eta(x) \cup r(y)$ implies a complete type over M.

Proof. By elimination of field quantifiers, we need only to determine whether

$$\chi(v(g_0(x)), \ldots, v(g_{m-1}(x)), y)$$

is in $p_\eta(x) \cup q(y)$ for any formula χ in L_Γ over M and polynomials g_i over M. By the remark above, $p_\eta \models v(g_i(x)) = v(g_i(a_s))$ for any $s < \eta$ large enough and all $i < m$. So $p_\eta \cup r \models \chi$ iff $r \models \chi(v(g_0(a_s)), \ldots, v(g_{m-1}(a_s)), y)$. □

Let $r(y) \in S(M)$ be a type in the valuation group sort which is finitely satisfiable in $\{\gamma_q \mid q \in \mathbb{Q}\}$ and contains $\{y > \gamma_q \mid q \in \mathbb{Q}\}$. By the claim, r and p_η are weakly orthogonal. Fix some $d \models p_\eta$ in $\mathfrak{C}$. For each bounded cut $I \subseteq \mathbb{Q}$, let $p_I(x, y)$ be the following type:

$$p_\eta(x) \cup r(y) \cup \{y + \gamma_q < v(x - d) < y + \gamma_{q'} \mid q \in I,\ q' \notin I\}.$$

Then $p_I(x, y)$ is finitely satisfiable in M:

Suppose we are given finite subsets $p_0 \subseteq p_\eta$ and $r_0 \subseteq r$, $I_0 \subseteq I$ and $I_0' \subseteq \mathbb{Q} \backslash I$. Let $q = \max I_0$ and $q' = \min I_0'$. Note that there is some $s < \eta$ such that for any $a \in K_1$, if $v(a - a_s) \geq \gamma_{\text{lev}(s)}$ then $a \models p_0$. Let $q_0 \in \mathbb{Q}$ be larger than $\text{lev}(s)$, larger than $\text{lev}(s) - q$ and such that $\gamma_{q_0} \models r_0$. Let $q'' \in \mathbb{Q}$ be in the interval $(q_0 + q, q_0 + q')$ and let $b \in K_1$ be such that $v(b) = \gamma_{q''}$. Let $s < t < \eta$ be such that $\text{lev}(t) > q''$ and let $a = a_t + b \in K_1$. Then $p_0(a) \cup r_0(\gamma_{q_0})$ holds, and in addition,

$$v(a - d) = v(a_t - d + b) = v(b) = \gamma_{q''}.$$

Moreover,

$$\gamma_{q_0} + \gamma_q = \gamma_{q_0 + q} < \gamma_{q''} < \gamma_{q_0 + q'} = \gamma_{q_0} + \gamma_{q'}.$$

Obviously, for different cuts I and J, the types p_I and p_J contradict each other.

Together this shows that, letting $p(x, y)$ be the complete type determined by $p_\eta(x) \cup r(y)$, $|\text{uf}_\Delta(p)| \geq 2^{\aleph_0}$ where $\Delta = \{\varphi(x, y; z_0, z_1, z_2)\}$ and:

$$\varphi(x, y; z_0, z_1, z_2) \quad = \quad y + z_1 < v(x - z_0) < y + z_2.$$

So T is large as promised. □

There are other languages of valued fields in addition to the one in Definition 4.41 that would make the proof above work. The only requirements are that we can construct the tree S inside the field, that Γ is a sort and that field quantifiers are eliminated. This can be done in the theory of the p-adics, when we add to the language ac_n for $n < \omega$ as in [Pas90], and also in ACVF —

algebraically closed valued fields (where there is quantifier elimination in any reasonable language, and in fact there is no need for $\mathfrak{ac}$).

Corollary 4.44. *The theory of any Henselian valued field of characteristic* $(0,0)$ *(in the language described in Definition 4.41), ACVF (in any characteristic and any reasonable language with quantifier elimination and a sort for the valuation group), and the theory of* $\mathbb{Q}_p$ *(in the language of Pas with* $\mathfrak{ac}_n$*) are large.*

5. Splintering

This part of the paper is motivated by the work of Rami Grossberg, Andrés Villaveces and Monica VanDieren. In their paper [GVV] they study Shelah's Generic pair conjecture (which is now a theorem — [She, She12a, She11]), and in their analysis they came up with the notion of splintering, a variant of splitting.

Definition 5.1. Let $p \in S(\mathfrak{C})$. Say that p *splinters* over M if there is some $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that

(1) $\sigma(p) \neq p$.
(2) $\sigma(p|_M) = p|_M$.
(3) $\sigma(M) = M$ setwise.

Remark 5.2. [due to Martin Hils] Splitting implies splintering, and if T is stable, then they are equal.

Proof. Suppose $p \in S(\mathfrak{C})$ does not split over M, then, by stability, it is definable over M, and p is the unique non-forking extension of $p|_M$. Then for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, $\sigma(p)$ is the unique non-forking extension of $\sigma(p|_M)$. So if $\sigma(p|_M) = p|_M$, this means that $\sigma(p) = p$ so p does not splinter over M. □

Claim 5.3. Outside of the stable context, splitting $\neq$ splintering.

Proof. Let T be the theory of random graphs in the language $\{I\}$. Let $M \models T$ be countable, and let $a \neq b \in M$ with an automorphism $\sigma \in \mathrm{Aut}(M)$ taking a to b. Let $p(x) \in S(\mathfrak{C})$ say that $x \, I \, c$ for every $c \in M$ and if $c \notin M$ then $x \, I \, c$ if and only if c is connected a and not connected to b. Obviously, p does not split over M. However, let $\sigma' \in \mathrm{Aut}(\mathfrak{C})$ be an extension of σ. Let $c \in \mathfrak{C}$ be such that c is connected to a but not to b. Then $x \, I \, c \in p$ but $x \, I \, c \notin \sigma'(p)$. □

However,

Claim 5.4. If $T = \mathrm{Th}(\mathbb{Q}, <)$, then splitting equals splintering.

Proof. Observe that by quantifier elimination every complete type $r(x_i \,|\, i \in I)$ over a set A is determined by $\mathrm{tp}(\langle x_i \,|\, i \in I \rangle / \emptyset) \cup \bigcup \{\mathrm{tp}(x_i/A) \,|\, i \in I\}$. Assume $q(x_i \,|\, i \in I)$ is a global type that splinters but does not split over a model M. Then it follows that for some $i \in I$, $q \restriction x_i$ splinters, so we may assume $|I| = 1$. Suppose $\sigma \in \mathrm{Aut}(\mathfrak{C})$ is such that $\sigma(M) = M$, $\sigma(q|_M) = q|_M$ and $\sigma(q) \neq q$. Note that $\sigma\left(q^{(\omega)}\right) = \sigma(q)^{(\omega)}$, so by Fact 4.10, $\sigma\left(q^{(\omega)}\right)|_M \neq q^{(\omega)}|_M$. We get a contradiction by quantifier elimination again. □

We shall now generalize Claim 5.3 to every theory with the independence property. In fact, to any theory with large or medium directionality.

Theorem 5.5. *Suppose* T *has medium or large directionality then splitting* $\neq$ *splintering.*

Proof. We know that there is some p and Δ such that $\mathrm{uf}_\Delta(p)$ is infinite. Let us use Construction 4.5:

We may find a saturated model $N = \left(N_0', N_0, M_0, Q_0, \bar{f}_0\right)$ of $Th(M^*)$ of size λ where λ is big enough. Then there is $c \neq d \in Q_0$ such that $\mathrm{tp}(c/\emptyset) = \mathrm{tp}(d/\emptyset)$ in the extended language (with symbols for N_0, M_0, Q_0 and $\bar{f}_0$). So there is an automorphism σ of this structure (in particular of N_0') such that $\sigma(c) = d$. By definition, $\sigma(N_0) = N_0$ and $\sigma(M_0) = M_0$. So $\mathrm{tp}(c/N_0)$ is finitely satisfiable in M_0 and hence does not split over M_0. But it splinters since $\sigma(\mathrm{tp}(c/M_0)) = \mathrm{tp}(d/M_0) = \mathrm{tp}(c/M_0)$ but $\sigma(\mathrm{tp}(c/N_0)) = \mathrm{tp}(d/N_0) \neq \mathrm{tp}(c/N_0)$ as witnessed by φ.

If there are no saturated models, we can take a big enough special model (see [Hod93, Theorem 10.4.4]).

Note that we may also find an example of a type $p \in S(M)$ with a splintering, non-splitting, global extension, with $|M| = |T|$: consider the structure $(N_0', N_0, M_0, \sigma, c, d)$, and find an elementary substructure of size $|T|$. □

Definition 5.6. Let T be a complete theory. We say that $(M, p, \varphi(x; y), A_1, A_2)$ is an *sp-example* for T when:

- $M \models T$; $A_1, A_2 \subseteq M$ are nonempty and disjoint; $p = p(x)$ is a complete type over M, finitely satisfiable in A_1; $\mathrm{Th}(M_p, A_1) = \mathrm{Th}(M_p, A_2)$ (see Definition 4.16); For each pair of finite sets $s_1 \subseteq A_1$ and $s_2 \subseteq A_2$, $M \models \exists y \left(\bigwedge_{a \in s_1} \varphi(a, y) \wedge \bigwedge_{b \in s_2} \neg\varphi(b, y)\right)$.

Proposition 5.7. T *has an sp-example if and only if there is a finitely satisfiable type over a model which splinters over it (in particular, splitting is different than splitting).*

Proof. Suppose $(M, p, \varphi(x, y), A_1, A_2)$ is an sp-example for T. Let M' be the structure (M_p, A_1, A_2) (in the language $L_p \cup \{P_1, P_2\}$ where P_1, P_2 are predicates). Assume $|T| < \mu = \mu^{<\mu}$, and let $N' = (N_q, B_1, B_2)$ be a saturated extension of M' of size μ where $N = N' \restriction L$ and $q = q^N$ is as in Remark 4.17. Since $(N_q, B_1) \equiv (N_q, B_2)$, there is an automorphism σ of N_q, such that σ takes B_1 to B_2 and so $\sigma(q) = q$. Let q' be a global extension of q, finitely satisfiable in B_1 and σ' a global extension of σ.

So q' does not split over N, but it splinters:

Consider the type $\{\varphi(a,y) \mid a \in B_1\} \cup \{\neg\varphi(b,y) \mid b \in B_2\}$. It is finitely satisfiable in N by choice of φ. Let $c \in \mathfrak{C}$ satisfy this type. Then $\varphi(x,c) \in q'$ but $\varphi(x,c) \notin \sigma(q')$ (because $\sigma(q')$ is finitely satisfiable in B_2).

If we do not assume the existence of such a μ, we can use special models.

Now suppose that splitting is different than splintering, as witnessed by some global type p that splinters over a model M but is finitely satisfiable in it. Then there is some automorphism σ of $\mathfrak{C}$ that witnesses it. There is a formula $\varphi(x,y)$ and $a \in \mathfrak{C}$ such that $\varphi(x,a) \in p, \neg\varphi(x,a) \in \sigma(p)$. Let $B_1 = \left\{m \in M \,\middle|\, \varphi(m,a) \wedge \neg\varphi\left(m, \sigma^{-1}(a)\right)\right\}$, $B_2 = \sigma(B_1)$. It is easy to check that $(M, p, \varphi, B_1, B_2)$ is an sp-example □

The following theorem answers the natural question:

Theorem 5.8. *There is a theory with small directionality in which splitting $\neq$ splintering.*

Proof. Let $L = \{R\}$ where R is a ternary relation symbol. Let $M_0 = \langle \mathbb{Q}, < \rangle$ and define $R(x,y,z)$ by $x < y < z$ or $z < y < x$, i.e., y is between x and z. Let $T = Th(M_0 \restriction L)$.

Claim. T has small directionality.

Proof. Suppose $M \models T$. Let (a,b) denote $\{c \mid R(a,c,b)\}$.

Then, for any choice of a pair of distinct elements a, b there is a unique enrichment of M to a model M' of $Th(\mathbb{Q}, <)$ such that R is defined as above and $a < b$:

For $w \neq z$, $w < z$ if and only if

$(a,b) \cap (a,z) \neq \emptyset$ and $((a,w) \subseteq (a,z)$ or $(a,w) \cap (a,z) = \emptyset)$ or

$(a,b) \cap (a,z) = \emptyset$ and $(z,b) \subseteq (w,b)$.

From this observation, it follows that there is a unique completion of any type $p \in S(M)$ to a type $p' \in S(M')$. So if Δ is a finite set of L formulas and $\mathrm{uf}_\Delta(p)$ is infinite, then $\mathrm{uf}_\Delta(p')$ is also infinite — contradiction to Example 4.24. □

Claim. T has an sp-example

Proof. Let $M = M_0 \restriction L$. Let $p(x) = \mathrm{tp}(\pi/M)$. Let $A_1 = \{x \in \mathbb{Q} \mid x > \pi\}$ and $A_2 = \mathbb{Q} \backslash A_1$, and let $\varphi(x; y_1, y_2) = R(y_1, x, y_2)$. We claim that $(M, p, \varphi(x, y_1, y_2), A_1, A_2)$ is an sp-example:

First, let M' be the reduct of $(\mathbb{Q} \cup \{\pi\}, <)$ to L. There is some $\sigma \in \mathrm{Aut}(M'/\pi)$ such that $\sigma(A_1) = A_2$. Hence $(M_p, A_1) \cong (M_p, A_2)$. Also, since $\mathrm{tp}(\pi/M_0)$ (in $\{<\}$) is finitely satisfiable in both A_1 and A_2 (by quantifier elimination), p is finitely satisfiable in both A_1 and A_2. Finally, for finite $s_1 \subseteq A_1$ and $s_2 \subseteq A_2$, there exists $c_1, c_2 \in \mathbb{Q}$ such that $R(c_1, a, c_2)$ for all $a \in s_1$ and $\neg R(c_1, b, c_2)$ for all $b \in s_2$. □

□

6. Appendix: dense types in RCF

Definition 6.1. For $M \models \mathsf{RCF}$, let $S_{\text{dense}}(M)$ be the set of dense complete types over M (see Definition 4.31).

Here we will prove the following theorem:

Theorem 6.2. $\operatorname{ded}\lambda = \sup\{|S_{\text{dense}}(M)| \mid M \models \mathsf{RCF}, |M| = \lambda\}$.

For the proof we will need some definitions and facts:

Definition 6.3. (1) By a *tree* we mean a partial order $(T, <)$ such that for every $a \in T$, $T_{<a} = \{x \in T \mid x < a\}$ is well ordered. For $a \in T$, the order type of $T_{<a}$ is a's *level*. By a *branch* in T we mean a maximally linearly ordered subset of T. Its length is its order type.

(2) For two cardinals λ and μ, let $\lambda^{\langle\mu\rangle_{\text{tr}}}$ be:

$$\sup\{\kappa \mid \text{there is some tree } T \text{ with } \lambda \text{ many nodes and } \kappa \text{ branches of length } \mu\}.$$

Fact 6.4. *(See* [Bau76, Theorem 2.1(a)]*) The following cardinalities are the same:*

(1) $\operatorname{ded}\lambda$.

(2) $\sup\{\kappa \mid$ *there is a regular* μ *and a tree* T *with* κ *branches of length* μ *and* $|T| \leq \lambda\}$.

(3) $\sup\left\{\lambda^{\langle\mu\rangle_{\text{tr}}} \mid \mu \leq \lambda \text{ is regular}\right\}$.

It is somewhat easier to consider trees which are sub-trees of $\lambda^{<\mu}$ (with the usual "first-segment" order) for some λ, μ. Given any tree T, and any cardinal μ, suppose we are interested in computing the number of branches of length μ. For this we may assume that the level of each element in T is $< \mu$. Suppose $|T| = \lambda$, so we may assume that its universe is λ. Let T' be $\{T_{<a} \mid a \in T\}$. This is easily seen to be a tree with the inclusion ordering, and moreover it is isomorphic to a complete sub-tree T'' of $\lambda^{<\mu}$ (in the sense that if $\eta \in T''$ and ν is an initial segment of η, then $\nu \in T''$): if $\operatorname{lev}(a) = \alpha$, map $T_{<a}$ to $\eta : \alpha \to \lambda$ where $\eta(\beta)$ is the β'th element in $T_{<a}$. If $B \subseteq T$ is a branch of length μ, let $B' = \{T_{<a} \mid a \in B\}$. Then B' is also a branch of length μ, and in addition if $B_1 \neq B_2$ are branches of T, then $B_1' \neq B_2'$ in T'. This shows that T' (so also T'') has at least as many branches as T, and so in calculating $\operatorname{ded}\lambda$ we can add to our list of cardinalities from Fact 6.4:

(4) $\sup\{\kappa \mid$ there is a regular μ and a tree $T \subseteq \lambda^{<\mu}$ with κ branches of length μ and $|T| \leq \lambda\}$.

Theorem 6.2 follows from:

Proposition 6.5. *For every tree* $T \subseteq \lambda^{<\mu}$ *of size* λ*, there is a model* $M \models \mathsf{RCF}$ *of size* λ *such that* $|S_{\text{dense}}(M)|$ *is at least the number of branches in* T *of length* μ.

Proof. We may assume that $\mu \leq \lambda$. For $i < \mu$, let $T_i = T \cap {}^{i}\lambda$, $T_{<i} = T \cap \lambda^{<i}$. By induction on $i < \mu$ we construct a sequence of models $\bar{M} = \langle M_i \mid i < \mu\rangle$ and $\langle a_\eta, b_\eta \mid \eta \in T_{<i}\rangle$ such that:

$\bar{M}$ is an $\prec$-increasing continuous sequence of models of RCF; For all $\eta \in T_{<i}$, $a_\eta, b_\eta \in M_{\lg(\eta)+1}$; $a_\eta < b_\eta$; $b_\eta - a_\eta < c$ for all $0 < c \in M_{\lg(\eta)}$; If $\alpha < \beta < \lambda$ and $\eta \frown \langle \alpha \rangle, \eta \frown \langle \beta \rangle \in T_{<i}$ then $b_{\eta \frown \langle \alpha \rangle} < a_{\eta \frown \langle \beta \rangle}$; For $\nu < \eta$, $a_\nu < a_\eta < b_\eta < b_\nu$.

The construction:

Let M_0 be any model of size λ.

For i limit, let $M_i = \bigcup_{j<i} M_j$ (there are no new (a_η, b_η)'s).

For $i = j+1$ for j a successor, let M_i be a model of size λ containing M_j and an increasing sequence $\langle c_\alpha \,|\, \alpha < \lambda \rangle$ such that $0 < c_\alpha < d$ for all $0 < d \in M_j$. For $\eta \in T_{j-1}$, if $\eta \frown \langle \alpha \rangle \in T$, let $a_{\eta \frown \langle \alpha \rangle} = a_\eta + c_{2\alpha}$ and $b_{\eta \frown \langle \alpha \rangle} = a_\eta + c_{2\alpha+1}$ (note that $b_{\eta \frown \langle \alpha \rangle} < b_\eta$).

For $i = j+1$ for j limit (or $j = 0$), let M_i be model of size λ containing M_j and a_η, b_η for $\eta \in T_j$ where $a_{\eta \restriction j'} < a_\eta < b_\eta < b_{\eta \restriction j'}$ for all $j' < j$ (so for $j = 0$ this just means $a_{\langle\rangle} < b_{\langle\rangle}$) and $b_\eta - a_\eta < d$ for all $d \in M_j$.

Finally, we let $M = \bigcup_{i<\mu} M_i$. For each branch $\eta \in {}^{\mu}\lambda$ of T, let $p_\eta = \{a_{\eta \restriction i} < x < b_{\eta \restriction i} \,|\, i < \mu\}$. This is easily seen to be a dense type. Also, it is very easy to see that $p_\eta \neq p_{\eta'}$ for $\eta \neq \eta'$. □

Remark 6.6. Note that this proof only used the fact that the order is dense, and so this holds in any densely ordered abelian group.

Next we will show that Proposition 6.5 is "as good as it gets".

Proposition 6.7. *If* $M \models \mathsf{RCF}$, $|M| = \lambda$, *and* $\mu = \mathrm{cof}\,(M, <)$, *then* $|S_{\mathrm{dense}}\,(M)| \leq \lambda^{\langle \mu \rangle_{\mathrm{tr}}}$.

Proof. We shall construct a tree of size λ with $|S_{\mathrm{dense}}\,(M)|$ branches of length μ.

Let $\langle d_i \,|\, i < \mu \rangle$ be an increasing cofinal sequence of positive elements in M. Let $<^*$ be a well ordering on M^2. We define a sequence of pairs $\langle (a_{i,p}, b_{i,p}) \,|\, i < \mu, p \in S_{\mathrm{dense}}\,(M) \rangle$ by induction on $i < \mu$ such that:

$(a_{i,p}, b_{i,p}) \in M^2$ is the $<^*$-first pair such that $p\,(x) \models a_{i,p} < x < b_{i,p}$, $b_{i,p} - a_{i,p} < 1/d_i$ and for $j < i$, $a_{j,p} < a_{i,p}$, $b_{i,p} < b_{j,p}$.

Claim. $(a_{i,p}, b_{i,p})$ exist for all $p \in S_{\mathrm{dense}}\,(M)$ and $i < \mu$.

Proof. Fix some $p \in S_{\mathrm{dense}}\,(M)$. Suppose $i < \mu$ is the first such that $(a_{i,p}, b_{i,p})$ do not exist. For $j < i$, let $0 < c_j \in M$ be such that $p\,(x) \models x + c_j < b_{j,p}$ and $p\,(x) \models a_{j,p} + c_j < x$ (exists since p is dense, since otherwise it would be definable). Since the cofinality of M is μ, there must be some $e \in M$ such that $e > d_i$ and $e > 1/c_j$ for all $j < i$. Since p is dense there must be some $a, b \in M$ such that $p\,(x) \models a < x < b$ and $b - a < 1/e$. By choice of e for all $j < i$, $a_{j,p} < a$, $b < b_{j,p}$ and $b - a < 1/d_i$. □

For $i < \mu$, let:

$$T_i = \left\{ \eta : i \to M^2 \,|\, \exists p \in S_{\mathrm{dense}}\,(M)\, \forall j < i\, [\eta\,(j) = (a_{j,p}, b_{j,p})] \right\}.$$

Claim. If $\eta \in T_i$ then $\eta \restriction j \in T_j$ for all $j < i$.

Claim. $|T_i| \leq \lambda$.

Proof. By the first claim, if $\eta \in T_i$ then it can be extended to some ν in T_{i+1}. So it is enough to show that $|T_{i+1}| \leq \lambda$. For that it is enough to show that the map $\eta \mapsto \eta(i)$ from T_{i+1} to M^2 is injective. But this follows from definition of $(a_{i,p}, b_{i,p})$. □

Let $T = \bigcup_{i<\mu} T_i$. Then T a tree, and for each dense type $p \in S_{\text{dense}}(M)$, we can find a branch $\eta_p : \mu \to M^2$ defined by $\eta(i) = (a_{i,p}, b_{i,p})$. The following claim finishes the proof:

Claim. For $p_1 \neq p_2$, $\eta_{p_1} \neq \eta_{p_2}$.

Proof. Suppose $p_1(x) \models x < b$ and $p_2(x) \models b < x$, and let $0 < e \in M$ be such that $p_1(x) \models x + e < b$ and $p_2(x) \models b + e < x$ (exists since p_1 and p_2 are not definable). For some $i < \mu$, $d_i > 1/e$. Then it follows that $\eta_{p_1}(i) \neq \eta_{p_2}(i)$. □

□

Corollary 6.8. *The following equality holds for all cardinals* $\lambda \geq \aleph_0$*:*

$$\sup\left\{\left|S_{\text{dense}}(M)^{\omega}\right| \mid M \models \mathsf{RCF}, |M| = \lambda\right\} = \sup\left\{\left(\lambda^{\langle\mu\rangle_{\text{tr}}}\right)^{\omega} \mid \mu \leq \lambda, \operatorname{cof}(\mu) = \mu\right\}.$$

Proof. The inequality $\leq$ follows immediately from Proposition 6.7. For $\geq$ we will show that for every regular $\mu \leq \lambda$, $\left(\lambda^{\langle\mu\rangle_{\text{tr}}}\right)^{\omega} \leq \sup\left\{\left|S_{\text{dense}}(M)^{\omega}\right| \mid |M| = \lambda\right\}$.

Suppose $\lambda^{\langle\mu\rangle_{\text{tr}}}$ is attained, i.e., there is a tree of size λ with $\lambda^{\langle\mu\rangle_{\text{tr}}}$ branches of length μ. Then by Proposition 6.5, for some model $M \models \mathsf{RCF}$ of size λ, $|S_{\text{dense}}(M)| \geq \lambda^{\langle\mu\rangle_{\text{tr}}}$, so in that case we are done.

Suppose $\lambda^{\langle\mu\rangle_{\text{tr}}}$ is not attained. In that case $\operatorname{cof}\left(\lambda^{\langle\mu\rangle_{\text{tr}}}\right) > \lambda$. Indeed, if not, then $\lambda^{\langle\mu\rangle_{\text{tr}}} = \bigcup_{i<\lambda} \sigma_i$ for some cardinals $\sigma_i < \lambda^{\langle\mu\rangle_{\text{tr}}}$. For each $i < \lambda$, there is a tree T_i of size λ with more than σ_i branches of length μ. Let T be the disjoint union of T_i for $i < \lambda$. Then T is a tree of size λ, with at least $\lambda^{\langle\mu\rangle_{\text{tr}}}$ branches of length μ — contradiction. In particular, $\operatorname{cof}\left(\lambda^{\langle\mu\rangle_{\text{tr}}}\right) > \omega$, so every function $f : \omega \to \lambda^{\langle\mu\rangle_{\text{tr}}}$ is bounded, and hence:

$$\left(\lambda^{\langle\mu\rangle_{\text{tr}}}\right)^{\omega} = \sup\left\{\kappa^{\omega} \,\middle|\, \kappa < \lambda^{\langle\mu\rangle_{\text{tr}}}\right\}.$$

So it is enough to show that for each $\kappa < \lambda^{\langle\mu\rangle_{\text{tr}}}$, there is a model M of size λ with more than κ dense types, which follows from Proposition 6.5. □

Example 6.9. In [CKS12, Section 6] it is shown that it is consistent with ZFC that there is an uncountable cardinal λ such that:

(1) $\operatorname{cof}(\operatorname{ded}\lambda) = \operatorname{cof}(\lambda) = \aleph_0$, so $(\operatorname{ded}\lambda)^{\omega} > \operatorname{ded}\lambda$.
(2) For all regular cardinals $\mu < \lambda$, $\lambda^{\mu} \leq \operatorname{ded}\lambda$.

So in this case,

$$\sup\left\{\left(\lambda^{\langle\mu\rangle_{\mathrm{tr}}}\right)^{\omega} \mid \mu \leq \lambda, \operatorname{cof}(\mu)=\mu\right\} = \operatorname{ded}\lambda.$$

However,

Corollary 6.10. *For any cardinal* λ*, if* $\operatorname{cof}(\operatorname{ded}\lambda) > \aleph_0$*, then*

$$\sup\left\{\left(\lambda^{\langle\mu\rangle_{\mathrm{tr}}}\right)^{\omega} \mid \mu \leq \lambda, \operatorname{cof}(\mu)=\mu\right\} = (\operatorname{ded}\lambda)^{\omega}.$$

Proof. $\leq$ is clear. For $\geq$, we use a similar argument as in the proof of Corollary 6.8. Since $(\operatorname{ded}\lambda)^{\omega} = \bigcup\{\kappa^{\omega} \mid \kappa < \operatorname{ded}\lambda\}$, we only have to show that every $\kappa < \operatorname{ded}\lambda$, $\kappa < \lambda^{\langle\mu\rangle_{\mathrm{tr}}}$ for some regular $\mu \leq \lambda$. But that already follows from Fact 6.4. □

References

[Adl08] Hans Adler. An introduction to theories without the independence property. *Archive of Mathematical Logic*, 2008. accepted.

[Bau76] James E. Baumgartner. Almost-disjoint sets, the dense set problem and the partition calculus. *Ann. Math. Logic*, 9(4):401–439, 1976.

[Bél99] Luc Bélair. Types dans les corps valués munis d'applications coefficients. *Illinois J. Math.*, 43(2):410–425, 1999.

[CKS12] Artem Chernikov, Itay Kaplan, and Saharon Shelah. On non-forking spectra. 2012. arXiv:1205.3101, 1007 in Shelah's archiv.

[CS09] Moran Cohen and Saharon Shelah. Stable theories and representation over sets, 2009. arXiv:0906.3050.

[Del84] Françoise Delon. Espaces ultramétriques. *Journal of Symbolic Logic*, 49:405–424, 1984.

[GVV] Rami Grossberg, Andrés Villaveces, and Monica VanDieren. Existence of a model of cardinality λ^{++}. *work in progress*.

[Hod93] Wilfrid Hodges. *Model Theory*, volume 42 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, Great Britain, 1993.

[HP11] Ehud Hrushovski and Anand Pillay. On NIP and invariant measures. *J. Eur. Math. Soc. (JEMS)*, 13(4):1005–1061, 2011.

[Kei76] H. Jerome Keisler. Six classes of theories. *J. Austral. Math. Soc. Ser. A*, 21(3):257–266, 1976.

[Kei78] H. Jerome Keisler. The stability function of a theory. *J. Symbolic Logic*, 43(3):481–486, 1978.

[KS10] Itay Kaplan and Saharon Shelah. A dependent theory with few indiscernibles. 2010. arXiv:1010.0388.

[Kud11] K. Zh. Kudaĭbergenov. Independence property of first-order theories and indiscernible sequences. *Mat. Tr.*, 14(1):126–140, 2011.

[Par82] Michel Parigot. Théories d'arbres. *J. Symbolic Logic*, 47(4):841–853 (1983), 1982.

[Pas89] Johan Pas. Uniform p-adic cell decomposition and local zeta functions. *J. Reine Angew. Math.*, 399:137–172, 1989.

[Pas90] Johan Pas. Cell decomposition and local zeta functions in a tower of unramified extensions of a p-adic field. *Proc. London Math. Soc. (3)*, 60(1):37–67, 1990.

[She] Saharon Shelah. Dependent theories and the generic pair conjecture. *Communications in Contemporary Mathematics*, submitted. math.LO/0702292.

[She71a] S. Shelah. Remark to "local definability theory" of Reyes. *Ann. Math. Logic*, 2(4):441–447, 1970/1971.

[She71b] Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory. *Ann. Math. Logic*, 3(3):271–362, 1971.

[She86] Saharon Shelah. *Around classification theory of models*, volume 1182 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[She90] Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, second edition, 1990.

[She06] Saharon Shelah. Dependent T and existence of limit models. 2006. arXiv:math/0609636.

[She11] Saharon Shelah. No limit model in inaccessibles. In *Models, logics, and higher-dimensional categories*, volume 53 of *CRM Proc. Lecture Notes*, pages 277–290. Amer. Math. Soc., Providence, RI, 2011.

[She12a] Saharon Shelah. Dependent dreams: recounting types. 2012. arXiv:1202.5795.

[She12b] Saharon Shelah. Strongly dependent theories. *Israel Journal of Mathematics*, 2012. accepted, arXiv:math/0504197.

[Tre96] Marcus Tressl. *Dedekind cuts in polynomially bounded, O-minimal expansions of real closed Fields*. PhD thesis, Universität Regensburg, 1996. http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/papers/thesis.pdf.