DEPENDENT DREAMS: RECOUNTING TYPES
SH950

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Abstract. We investigate the class of models of a general dependent theory. We continue [She15] in particular investigating the so called "decomposition of types"; our thesis is that what holds for stable theory and for $\mathrm{Th}(\mathbb{Q},<)$ hold for dependent theories. Another way to say this is: we have to look at small enough neighborhood and use reasonably definable types to analyze general types; also we presently concentrate on complete types over saturated models (and sometimes just quite saturated models). We now mention the main results understandable without reading the paper. First, a parallel to the "stability spectrum", we consider the "(problem of) recounting of types", that is assume $\lambda = \lambda^{<\lambda}$ is large enough, M a saturated model of T of cardinality λ, let $\mathfrak{S}_{\mathrm{aut}}(M)$ be the set of complete types over M up to being conjugate, i.e. we identify p, q when some automorphism of M maps p to q. Whereas for independent T usually the number is 2^λ, for dependent T the number is $\leq \lambda$ moreover it is $\leq |\alpha|^{|T|}$ when $\lambda = \aleph_\alpha$ and λ is not too small, see §(5B). Second, for stable T, recall that a model is κ-saturated iff it is $\aleph_\varepsilon$-saturated and every infinite indiscernible set (of elements) of cardinality $< \kappa$ is not $\subseteq$-maximal. We prove here an analog in §(7B). Third, if M is saturated and $p \in \mathbf{S}(M)$ then p is the average of an indiscernible sequence of length $\|M\|$ inside the model, see §(6A). Fourth, we prove a (weak) relative of the existence of indiscernibles, see §(4A). Lastly, the so-called generic pair conjecture was proved in [She15] for κ measurable, here it is essentially proved, i.e. for $\kappa = \kappa^{<\kappa} > |T| + \beth_\omega$, see §(7A).

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Anotated Content

§ 0. Introduction

§ 0(A). **What is done here.**

This is a step in trying to understand a dependent elementary class Mod_T. Our approach is:

Thesis 0.1. 1) It is fruitful to prove that questions on (first order complete) T and a cardinal does not depend too much on the cardinal, by finding syntactical equivalent condition; this suggests it is an interesting dividing line.
2) We should first analyze saturated models (then quite saturated models and only then general models).
3) In particular we should first try to understand complete types over saturated models, etc.

More specifically:

Thesis 0.2. For $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ we shall try to analyze $p \in \mathbf{S}^\varepsilon(M)$ by types of two simple kinds:

Kind A: $\mathrm{Av}(D, M), D$ an ultrafilter on ${}^\varepsilon B$ for some $B \subseteq M$ of cardinality $< \mu$ (μ a fix cardinal $\ll \|M\|$).

Kind B: $\mathrm{Av}(\mathbf{I}, M)$ where $\mathbf{I} = \langle \bar{a}_\alpha : \alpha < \lambda \rangle$ an indiscernible sequence (of ε-tuples) inside M.

Remark 0.3. For stable T, if M is $|T|^+$-saturated then every $p \in \mathbf{S}(M)$ is $\mathrm{Av}(\mathbf{I}, M)$ for some indiscernible sequence (so set) $\mathbf{I}$ of cardinality $\aleph_0$, so it falls under both kinds.

Consider a fixed complete first order theory T which is dependent. The problem we try to address here is analyzing a complete type over a saturated model, say $p \in \mathbf{S}^{<\theta^+}(M)$ where $\theta \geq |T|$. The reader may wonder why not $p \in \mathbf{S}^{<\omega}(M)$? The reason is that anyhow we are driven to consider infinitely many variables.

Trying to analyze $p \in \mathbf{S}^\theta(M), M \in \mathrm{EC}_{\kappa,\kappa}(T)$, clearly whatever occurs for some stable theories may appear, so in the analysis we allow types definable over small sets (though presently not stable types, just definable in a weak sense) where any fix bound will be O.K. but as it happens here "small sets" mean a set of cardinality say $< \beth_\omega + |T|^+$.

Also in dense linear order there are cuts defined say by a sequence of elements of length any regular $\sigma < \kappa$ (e.g. $p(x) \in \mathbf{S}(M)$ say that x induces a cut of M whose lower half has cofinality σ), we cannot avoid this so we allow types gotten as averages of indiscernible sequences of length σ. Note that types related to large cofinalities are not covered by Kind A, just as in [She14a, §1], where the cuts with both cofinalities maximal are fine - there expanding by them preserve saturation.

An approximation to analyzing p is $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}$; a characteristic case is $\kappa = \kappa^{<\kappa}$ large enough, $\theta = |T| = \aleph_0, \mu = \beth_\omega$ (actually we use $\bar{\mu}$ but ignore it in the introduction). Now, see Definition 2.2, such $\mathbf{x}$ consist of the model $M = M_\mathbf{x}$, which is κ-saturated (and in general may have larger cardinality), the sequence $\bar{d} = \bar{d}_\mathbf{x}$ realizing a complete type p over M which we are trying to analyze, $\bar{c} = (\ldots {}^\frown \bar{c}_i {}^\frown \ldots)_{i \in v(\mathbf{x})}$ an initial segment of the analysis where $v(\mathbf{x})$ is an ordinal $< \theta^+$ or just a linear order of cardinality $\leq \theta$. This means that for each $i \in v(\mathbf{x})$ one of the following two cases occurs, letting $r_{\mathbf{x},i} = \mathrm{tp}(\bar{c}_i, M_\mathbf{x} + \Sigma\{\bar{c}_j : j < i\})$.

In the first case, formally $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$, the type $r_{\mathbf{x},i}$ does not split over some $B_{\mathbf{x},i} \in [M_{\mathbf{x}}]^{<\mu}$ (or even is finitely satisfiable in it). So this type is in a suitable sense definable over some small set as in the stable case, so is the "stable part" called "Kind A" above.

In the second case, formally $i \in u_{\mathbf{x}}$ the type $r_{\mathbf{x},i}$ is the average of an indiscernible sequence $\mathbf{I}_{\mathbf{x},i} = \langle \bar{a}_{\mathbf{x},i,\alpha} : \alpha < \kappa_i \rangle$ where $\kappa_i = \mathrm{cf}(\kappa_i) \in [\mu, \kappa)$.

In [She15] some relatives were used but there $\mu = \kappa$ hence $B^+_{\mathbf{x}} = \cup\{\mathbf{I}_{\mathbf{x},i} : i \in u_{\mathbf{x}}\} \cup \bigcup\{B_{\mathbf{x},i} : i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}\}$ here corresponds to $B_{\mathbf{x}}$ there, so there the analysis is by information of size just smaller than κ, whereas here it is by $\leq \theta$ indiscernible sequences of length a regular cardinal + information of bounded size, i.e. $< \mu$, a major difference.

How does such $\mathbf{x}$ help? For each $i \in v_{\mathbf{x}}$ we define when $\mathbf{x}$ is active in i; it is the parallel of forking, i.e. of "$\mathrm{tp}(\bar{d}_{\mathbf{x}}, M_{\mathbf{x}} + \Sigma\{\bar{c}_j : j \leq i\})$ forks over $M_{\mathbf{x}} + \Sigma\{\bar{c}_j : j < i\}$", this cannot occur θ^+ times so there is $\mathbf{y}$ above $\mathbf{x}$ maximal in this sense; i.e. we cannot increase $v_{\mathbf{x}}$ having a "new" activity but not changing $M_{\mathbf{x}}, \bar{d}_{\mathbf{x}}, \bar{c}_i (i \in v_{\mathbf{x}})$ but possibly increasing $v_{\mathbf{x}}$. Moreover, see 2.14(2) we have further versions, local and/or less demanding, but we skip this in the introduction. The class of maximal such $\mathbf{y}$'s is called $\mathrm{qK}'_{\kappa,\mu,\theta}$, see Definition 2.11(1); for them we can prove:

(∗) if $A \subseteq M_{\mathbf{y}}, |A| < \mu$ <u>then</u> some $\bar{e} \in {}^{\theta}(M_{\mathbf{x}})$ solve $(\mathbf{x}, A)$ which means that $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{e}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A)$ and even uniformly, which is expressed by "according to $\bar{\psi}$".

This is the parallel of: if M is a dense linear form, $p \in \mathbf{S}(M), \mathscr{C} = (C_1, C_2)$ the cut of the linear order M which $p(x)$ induces and it has both cofinalities $\geq \mu$ and $A \subseteq M, |A| < \mu$ then we can choose $a \in C_1, b \in C_2$ such that $(a,b)_M \cap A = \emptyset$ hence $(a < x < b) \in p$ and $(a < x < b) \vdash p(x) \restriction A$.

All this seems to support:

Thesis 0.4. 1) The theory of dependent elementary classes is the combination of what occurs in stable classes and in the theory of dense linear orders.
2) We analyze general types by decompositions to three kinds: one are finitely satisfiable in a small set (or just does not split over a small set), second are averages of indiscernible sequences, third, are like branches of trees (include cuts of a linear order) any "bounded" subset are implied by a very small subset.

But we really gain understanding by the density of $\mathrm{tK}_{\kappa,\mu,\theta} \subseteq \mathrm{pK}_{\kappa,\mu,\theta}$ for some pair (κ, μ), (to cover all relevant cases better use $\mathrm{vK}^{\otimes}$, see §3). That is for $\bar{d} \in {}^{\theta}\mathfrak{C}$, we can find $\mathbf{x} \in \mathrm{tK}_{\kappa,\mu,\theta}$ such that $\bar{d} \triangleleft \bar{d}_{\mathbf{x}}, M = M_{\mathbf{x}}$ and for every $A \subseteq M$ of cardinality $< \kappa$ we can find $(\bar{c}', \bar{d}')$ in M realizing the same type as $(\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}})$ over M and $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{c}' + d') \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A + \bar{c}'_{\mathbf{x}} + \bar{d}')$, even uniformly and fixing the type of $\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}} {}^\frown \bar{c}' {}^\frown \bar{d}'$. In a stronger sense the type of $\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}}$ over M really combine parts definable over a small set and one like a (partial) order.

Another thesis is (see [She09, §1])

Thesis 0.5. In dependent (elementary) classes the family of outside definable sets ($\mathrm{Def}_{<\alpha}(M)$, see Definition 1.19) replace the family of inside definable sets for stable classes.

* * *

This work may be continued [S⁺a] and as said above it continues [She15] though does not depend on it. More specifically, how are [She15] and the present work related?

In both cases decomposition ($\mathrm{pK}_{\kappa,\mu,\theta}$ here, K^1 there) are central and $\mathrm{qK}', \mathrm{qK}$ here[1] are parallel to mxK there and also $\le_1, \le_2$ are similar here and there. In both cases the model $M_{\mathbf{x}}$ is κ-saturated and $\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}$ are of cardinality $\le \theta$ (normally $\kappa > \theta \ge |T|$). But here we use $|B_{\mathbf{x},i}| < \mu$ and allow $\mu \ll \kappa$ rather than $|B_{\mathbf{x},i}| < \kappa$, and instead use indiscernible sequences $\mathbf{I}_{\mathbf{x},i}$ for some i's. Hence $B^+_{\mathbf{x}}$ here stands for $B_{\mathbf{x}}$ there, both have cardinality $< \kappa$, but there $B_{\mathbf{x}}$ is any set, here without loss of generality $\mathbf{x}$ is smooth so $B^+_{\mathbf{x}}$ is a so called $(\bar{\mu}, \theta)$-set, essentially a set of cardinality $< \mu$ plus $\le \theta$ mutually indiscernible sequences of $(\le \theta)$-tuples. Such sets have some affinity to stable $\mathfrak{C}$, e.g. $|\mathbf{S}(B^+_{\mathbf{x}})| \le 2^{<\mu} + |B^+_{\mathbf{x}}|^{|T|}$.

Also $\mathrm{tK}_{\kappa,\mu,\theta}$ here is related to strict decompositions in [She15]. But in [She15] we get existence assuming only κ is a measurable cardinal so a quite large cardinal, so cannot prove in ZFC that it exists; whereas here this is proved for every large enough regular cardinal provably in ZFC, and the bound is small (at least for my taste), $\beth_\omega$, well $+|T|$, of course.

All this is a good point in favor of large cardinals by the criterion (first suggested by Gödel): we can first prove things assuming them, this helps us to find the way to really sort out things.

§ 0(B). **From Higher Perspective: The Test.**

What questions do we address here?

Question 0.6. <u>The serious/dull question</u> 1) Is the equation dependent/stable = groups/Abelian groups true?

That is, is dependence a better dividing line than stable (among say elementary classes), but we have been (and are) just too dim to see it?
2) The use of cardinals ($> \aleph_0$) in model theory: has it passed its time OR is it the key to dependent classes and will continue to be central.

Alas, most (relevant) people already know the answers, unfortunately not all of them know the same answer.

In more serious mode, we suggest here to put dependent theories to "end of first level examination". Trying to be objective we ask: do we have a good analog to what is in the first paper on stable T, [She69b] (and [She71]), essentially equivalently at the time of stability being three years old.

So here is the test composed of four questions (as presented in a lecture in MAMLS, Fall 2008 Meeting in honor of Gregory Cherlin) and a fifth question (as urged by the audience):

Question 0.7. <u>Question/Test</u> Find parallels of (1)-(4) and answer (5) for dependent T.
1)The stablility spectrum Theorem (for stable theory T on a model of cardinality λ there are $\le \lambda$ completer 1-types).
2) Strong partition theorems, i.e. existence of indiscernibles: for stable T , if $a_\alpha \in \mathfrak{C}$ for $\alpha < \lambda^+$ are given, $\lambda = \lambda^{|T|}$ then for some unbounded, even stationary subset S of λ^+ the sequence $\langle a_\alpha : \alpha \in S \rangle$ is indiscernible.

[1] In the context of [She15], i.e. $\mu_0 = \kappa$ essentially we get $\mathrm{qK}' = \mathrm{qK}$, see 2.15(3),(5)

3) "Understanding" complete types over models and indiscernible sequences (for stable T, the finite equivalence relation theorem which was somewhat later).
4) Characterize saturated models by indiscernible sequences, (for stable T, M is κ saturated iff it is $\aleph_\epsilon$-saturated and every infinite indiscernible set of cardinality $< \kappa$).
5) The generic pair conjecture, a major question from [She15] and more generally the existence of (λ, κ)-limit models ($\kappa = 2$ is the generic pair case).

We did not mention two problems having been answered earlier: majority on indiscernibles (see §(1C)) and definability of types (as we may consider the following theorem as an answer: expansion by outside definable sets preserved the theory of the model being dependent, by [She09, §1]).

We will present the questions in §1 and present solutions to (1),(4) and the first part of (5) in §5,§7. Unfortunately we do not solve the original interpretation of questions (2),(3) as we hoped, but, not surprisingly, we think we have excellent excuses. Now the answer to the parallel of (3) we considered, i.e. "no case of high directionality" that is bounding the number of ultrafilters D on M such that $\mathrm{Av}(D, M) = p$, has already been known to be false for many years, proved by Delon.

As for the existence of indiscernibles, i.e. 0.7(2) and actually also 0.7(3), subsequently Kaplan-Shelah [KS14b], proved that the premature assertion in the Rutgers lecture is false, (and nothing can be saved by Kaplan-Shelah [KS14a]). This is the negative half of the excuse, i.e. this version cannot be proved being false.

However on the positive side, we believe we have reasonable substitutes, i.e. reasonable parallels of parts (2),(3) of 0.7 for dependent T.

For part (3):

$\boxplus_1$ if $M \in \mathrm{EM}_{\kappa,\kappa}(T)$ and $p \in \mathbf{S}(M)$ then p is the average of an indiscernible sequence in M of length κ, see 6.2, (more in §(6A) and the results of §(6B)).

About the existence of indiscernibles, i.e. part (2) of 0.7, by §6 we have

$\boxplus_2$ existence for T with low or medium directionality (introduced in §(1B)).

Probably this is not convincing: but a true answer for 0.7(2) is another relative (or you may say a weak version) of the existence of indiscernibles

$\boxplus_3$ if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite and $a_{\alpha,n} \in \mathfrak{C}$ for $\alpha < \kappa, n < n(*) < \omega$ then we can find stationary $\mathscr{S}_n \subseteq \kappa$ for $n < n(*)$ such that: for $\bar\alpha \in \prod\limits_{\ell<n} \mathscr{S}_\ell$, the Δ-type of $\langle a_{\alpha_0}, \ldots, a_{\alpha_{n(*)-1}} \rangle$ depends just on the truth values of $\alpha_{\ell(1)} < \alpha_{\ell(2)}$ for $\ell(1), \ell(2) < n(*)$.

This holds by 4.6, (note that we can apply it for any permutation of $\{\langle 0, \ldots, n(*)-1\rangle\}$ and the formulation here is simpler because we use the club filter on χ, i.e. use diagonal intersection of clubs). Note that for T any completion of Peano arithmetic (or any 2-independent T) this holds only for (some) large cardinal.

There has been work on dependent theories in the previous century, see e.g. in the introductions of [She04, §1], [She09, §0], [She15, §0]; there was much activity in the first decade of the present century, but in different directions; on indiscernibility see §(1C) here.

§ 0(C). **Basic Definitions.**

We assume basic knowledge in model theory.

Convention 0.8. 1) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model of the complete first order T.
2) The vocabulary of T is τ_T.
3) $\mathbb{L}(\tau)$ is the set of first order formulas in the vocabulary τ.

Definition 0.9. 1) Let $\mathrm{EC}_{\lambda,\kappa}(T)$ be the class of κ-saturated models of T of cardinality λ; if $\kappa = 1$ this means that we omit the κ-saturation; we may omit κ when $\kappa = \lambda$.
2) Let $\bar{x}_{\bar{a}} = \langle x_{a_t} : t \in \mathrm{Dom}(\bar{a})\rangle$ where $\bar{a} \in {}^I\mathfrak{C}$ for some index set $I = \mathrm{Dom}(\bar{a})$, usually I an ordinal. Let $\bar{x}_{[\alpha]} = \langle x_\beta : \beta < \alpha\rangle$, similarly $\bar{x}_{[u]}$ for u a set or linear order. Generally we allow infinite sequence of variables but the formulas are finitary so only finitely many variables are mentioned.
2A) Let $\bar{x}'_{\bar{a}} = \langle x'_{a_t} : t \in \mathrm{Dom}(\bar{a})\rangle$, etc.; note $\bar{x}_{\bar{a}\restriction u} = \bar{x}_{\bar{a}}\restriction u$.
2B) If $\eta \in {}^I\mathrm{Dom}(\bar{a})$ then: $\bar{x}_{\bar{a},\eta} = \langle x_{a_{\eta(s)}} : s \in I\rangle$ and $\bar{a}_\eta = \langle a_{\eta(s)} : s \in \mathrm{Dom}(\eta)\rangle$; see 5.22.
2C) Let $\ell g(\bar{a}) = \mathrm{Dom}(\bar{a})$. Note $\ell g(\bar{x}_{\bar{a}}) = \mathrm{Dom}(\bar{x}_{\bar{a}})$ and $\ell g(\bar{x}_{[u]}) = u$.
3) Let $\varphi(\bar{x})$ be the pair $(\varphi, \bar{x})$, where

- φ is a first order formula (in $\mathbb{L}(\tau_T), T$ the first order theory understood from the content
- $\bar{x}$ is a sequence without repetition of variables, including all the variables occuring in φ freely.

We normally use $\varphi(\bar{x}, \bar{y})$ as a different object than $\varphi(\bar{x}\restriction u, \bar{y}\restriction v)$ and φ may stand for such object, e.g. $\langle \psi_\varphi(\bar{y}, \bar{z}) : \varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)\rangle$. This is ambiguous in principle but clear in practice. See more in Definition 1.2(4).
4) We may use $A + B$ instead of $A \cup B$ and $\sum\limits_{t\in I} A_t$ for $\cup\{A_t : t \in I\}$.

Observation 0.10. *The number of formulas $\varphi(\bar{x}_{\bar{c}}, x_{\bar{d}}) \in \mathbb{L}(\tau_T)$ is $|T| + |\ell g(\bar{c})| + |\ell g(\bar{d})|$ so $\geq \aleph_0$ and maybe $> |T|$.*

Definition 0.11. 1) For $M \prec \mathfrak{C}$ and $B \subseteq \mathfrak{C}$ let $M_{[B]}$ be M expanded by relations definable in $\mathfrak{C}$ with parameters from B, as in [She09, §1].
2) Similarly $M_{[p(\bar{x})]}$ for $p(\bar{x}) \in \mathbf{S}^\varepsilon(M)$ is M expanded by $R_{\varphi(\bar{x}_{[n]})} \equiv \{\bar{a} \in {}^nM : M \models \varphi[\bar{a}]\}$ for $\varphi(\bar{x}_{[n]}) \in \mathbb{L}(\tau_T)$.

Convention 0.12. E.g. saying "$\bar{c}\char`^\bar{d}$ realizes $\mathrm{tp}(\bar{c}_{\mathbf{x}}\char`^\bar{d}_{\mathbf{x}}, A)$" we may forget to say $\ell g(\bar{c}) = \ell g(\bar{c}_{\mathbf{x}}), \ell g(\bar{d}) = \ell g(\bar{d}_{\mathbf{x}})$.

Notation 0.13. 1) $\mathrm{tp}_\varphi(\bar{d}, \bar{c} + A)$ for $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y})$ is $\{\varphi(\bar{x}_{\bar{d}}, \bar{c}, \bar{a}) : \bar{a} \in {}^{\ell g(\bar{y})}A$ and $\mathfrak{C} \models \varphi[\bar{d}, \bar{c}, \bar{a}]\}$.
2) Similarly $\mathrm{tp}_\Delta(\bar{d}, \bar{c} + A)$ where $\Delta \subseteq \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \mathbb{L}(\tau_T)\}$, members of Δ not of this form are ignored.
3) $\mathrm{tp}_{\pm\varphi}$ means $\mathrm{tp}_{\{\varphi,\neg\varphi\}}$.
4) Let $\Gamma_{[\zeta]} = \{\varphi : \varphi = \varphi(\bar{x}_{[\zeta]}, \bar{y}) \in \mathbb{L}(\tau_T)\}$; similarly $\Gamma_{[\zeta],n} = \{\varphi : \varphi = \varphi(\bar{x}^0_{[\zeta]}, \dots, \bar{x}^{n-1}_{[\zeta]}, \bar{y})\}$.
5) Let $(\forall^\kappa t \in I)\vartheta(t)$ means: for all but $< \kappa$ members $t \in I$ we have $\vartheta(t)$ (but may use $(\forall^\infty n)$ instead $(\forall^{\aleph_0} n \in \mathbb{N})\vartheta(n)$). Similarly $(\exists^\kappa t \in I)$ means: there are $\geq \kappa$ members t of I such that $\vartheta(t)$.

Definition 0.14. 1) We say that a model M is a κ-sequence homogeneous <u>when</u> : if f is a partial one-to-one function from M to M of cardinality $< \kappa$, i.e. $|\mathrm{Dom}(f)| < \kappa$ and f is elementary in M then: for every $a \in M$ for some $b \in M$ the function $f' = f \cup \{\langle a, b\rangle\}$ is elementary in M, where
1A) We say the function f is elementary in M when: $\mathrm{Dom}(f) \subseteq M, \mathrm{Rang}(f)$ and if $M \models \varphi[a_0, \ldots]$ and $a_0, \ldots \in \mathrm{Dom}(f)$ then $M \models \varphi[f(a_0), \ldots]$.
2) We say that a model M is strongly κ-sequence homogeneous <u>when</u> : if f is as in part (1) then f can be extended to an automorphism of M.
3) We say that a model M is strongly κ-saturated <u>when</u> M is κ-saturated and strongly κ-sequence homogeneous.

Convention 0.15. 1) Generally (i.e. from §2 on if not said otherwise) in this work, I vary on K_{lin}, the class of linear orders which are endless.

§ 1. Presenting questions, definitions and facts

We here recall and make some definitions and questions related to the family of dependent theories and say some easy things to clarify, mostly those questions are dealt with later in this work.

§ 1(A). Recounting types.

We define the new version of the number of types, i.e. up to automorphisms, considering saturated model and generalizations. We then have a "first look at them". First, about the function $f^{\rm aut}_T$, counting the types up to automorphisms, see Definition 1.1:

$\boxplus$ (a) if T is stable, the function $f^{\rm aut}_T(\lambda)$ is constant, $\le 2^{|T|}$

(b) if T is countable then

- (α) the constant value belongs to $\{2,3,\ldots\} \cup \{\aleph_0, 2^{\aleph_0}\}$, see 1.3(1),(2)
- (β) every one of the values occurs even for superstable T, see 1.2

(c) if T is $\aleph_0$-stable then except $2^{\aleph_0}$ every one of the values (listed in $(b)(\beta)$) occurs

$(d)(\alpha)$ if T is independent then $f^{\rm aut}_T(\lambda) = 2^\lambda$ when $(\exists\mu)(\lambda = \lambda^{<\lambda} = 2^\mu > |T|)$, see 1.4

- (β) if T is independent, $\lambda = \lambda^{<\lambda} > |T|$ but not as in (α) then still $f^{\rm aut}_T(\lambda) \ge \lambda$

(e) if T is dependent and unstable then $f^{\rm aut}_T(\aleph_\zeta) \ge |\zeta + 1|$, see 1.3(4),(5).

This explains that the problem is about dependent (unstable) T. Note that the case of independent T and strongly inaccessible $\lambda > |T|$ is not resolved here, see on it [S⁺b].

The rest of this subsection is devoted to looking at relatives of $f^{\rm aut}_T$ motivated by a desire not to use instances of G.C.H.

Definition 1.1. 1) Let $\mathbf{C} := \{\lambda : \lambda = \lambda^{<\lambda}\}$ and $\mathbf{C}_{>\mu} = \mathbf{C}(>\mu)$ be $\mathbf{C} \backslash \mu^+$.
2) For T a complete first order theory and $\theta \ge 1$ we define the function $f^{\rm aut}_{T,\theta} : \mathbf{C} \to$ Card by $f^{\rm aut}_{T,\theta}(\lambda) = |\mathfrak{S}^\theta_{\rm aut}(M_\lambda)|$ for $M_\lambda \in {\rm EC}_{\lambda,\lambda}(T)$, i.e. a saturated model of T of cardinality λ, where
3) $\mathfrak{S}^\theta_{\rm aut}(M) = (\mathbf{S}^\theta(M)/\equiv_{\rm aut})$ where $\equiv_{\rm aut}$ or more[2] fully $\equiv^{\rm aut}_M$ is the following equivalence relation: $p, q \in \mathbf{S}^\theta(M)$ are $\equiv_{\rm aut}$- equivalent iff they are conjugate, i.e. there is an automorphism of M mapping p to q.
4) If we omit θ we mean $\theta = 1$, if we write "$<\aleph_0$" we mean "for any finite $n > 0$".

Example 1.2. 1) Assume $T = {\rm Th}(\mathbb{Q}, <)$, the theory of dense linear orders with neither first nor last element. Then $f^{\rm aut}_T(\aleph_0)$ is equal to 6, yes, six.
2) If $T = {\rm Th}(\mathbb{C})$, or T is the theory of some algbraically closed field of characteristic p, p prime or zero, then $f^{\rm aut}_T(\lambda) = \aleph_0$, for $\lambda \ge \aleph_0$.
3) In part (1), in general, $f^{\rm aut}_T(\aleph_\alpha) = 6 + 2|\alpha|$ for $\aleph_\alpha \in \mathbf{C}$.

[2] We can define also when $p_\ell \in \mathbf{S}^\theta(M_\ell)$ are equivalent = conjugate for $\ell = 1,2$ as in [Shea] which deal in a non-first order but for a stable context.

4) Let $\tau = \{P_i : i < \alpha\}, P_i$ a unary predicate and T says that each P_i is infinite, they are pairwise disjoint, and if α is finite then $\{x : \bigwedge\limits_{i<\alpha} \neg P_i(x)\}$ is infinite. Then T is stable (even totally transcendental so $\aleph_0$-stable if α is countable) and $f^{\rm aut}_T(\lambda) = 2(|\alpha+1|)$ for $\lambda \geq \aleph_0 + |\alpha|$. If α is finite > 0 and $\beta \leq \alpha$ and above we demand P_ℓ is a singleton when $\ell < \beta$, infinite when $\ell \geq \beta$ then we get $f^{\rm aut}_\tau(\lambda) = 2|\alpha - \beta| + |\beta| + 2$.
5) Let $T = {\rm Th}(M)$ where $M = ({}^\omega 2, P^M_n)_{n<\omega}$ and $P^M_n = \{\eta \in {}^\omega 2 : \eta(n) = 1\}$ for $n \in \mathbb{N}$. Then T is countable superstable and $f^{\rm aut}_T(\lambda) = 2^{\aleph_0}$ for $\lambda \geq 2^{\aleph_0}$.
6) Let $T = {\rm Th}({}^\omega\omega, E_n)_{n<\omega}$ where $E_n = \{(\eta,\nu) : \eta,\nu \in {}^\omega\omega$ and $\eta{\restriction}n = \nu{\restriction}n\}$. So T is countable, stable not superstable and $f^{\rm aut}_T(\lambda) = 2, f^{\rm aut}_{T,2}(\lambda) = \aleph_0$ for every $\lambda = \lambda^{\aleph_0}$; noting that T has no saturated models of cardinality λ when $\aleph_0 < \lambda < \lambda^{\aleph_0}$.

Observation 1.3. *1) If T is stable, then $f^{\rm aut}_T(\lambda)$ is constant and is $\leq 2^{|T|}$ for every $\lambda \in \mathbf{C}_{>|T|}$ (or just T has a saturated model of cardinality λ, e.g. $\lambda = \lambda^{|T|}$). Similarly $f^{\rm aut}_{T,\theta}(\lambda) \leq 2^{|T|+\theta}$ and is constant.*
2) If T is countable and stable and e.g. $\lambda = \lambda^{\aleph_0}$ then $f^{\rm aut}_T(\lambda)$, either is constantly some $\theta \in [2,\aleph_0]$ or is constantly $2^{\aleph_0}$.
3) If T is $\aleph_0$-stable then $f^{\rm aut}_T(\lambda) \leq \aleph_0$.
4) If T is unstable and is dependent, then $f^{\rm aut}_T(\aleph_\zeta) \geq |\zeta+1|$ for $\aleph_\zeta \in \mathbf{C}$ which is $> |T|$.
5) If T is independent, $\lambda > |T|$ is inaccessible then $f^{\rm aut}_T(\lambda) \geq \lambda$.

Proof. 1) Assume M is saturated of cardinality $> |T|$ or just a strongly $|T|^+$-sequence homogeneous (see Definition 0.14). Every $p \in \mathbf{S}^m(M)$ is definable, in fact there is a sequence $\langle \psi_\varphi(\bar y, \bar z_\varphi) : \varphi = \varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T)\rangle$ so $\bar x = \bar x_{[m]}$, not depending on M such that for every $p \in \mathbf{S}^m(M)$ there is a sequence $\bar c^p := \langle \bar c^p_\varphi : \varphi = \varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T)\rangle$, of sequences from M such that $\ell g(\bar c^p_\varphi) = \ell g(\bar z_\varphi)$ and $\varphi(\bar x, \bar b) \in p$ iff $\bar b \in {}^{\ell g(\bar y)}M$ and $M \models \psi_\varphi[\bar b, \bar c^p_\varphi]$, see [She78, Ch.II]. Now the number of complete types of sequences of the form $\langle \bar c_\varphi : \varphi = \varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T)\rangle$ in M with $\bar c^p_\varphi \in {}^{\ell g(\bar z_\varphi)}M$ is $\leq 2^{|T|}$. But M is strongly $|T|^+$-sequence homogeneous, see Definition 0.14(3), so this piece of information suffices, that is, if $p, q \in \mathbf{S}^m(M)$ and ${\rm tp}(\bar c^p, \emptyset, M) = {\rm tp}(\bar c^q, \emptyset, M)$ then there is an automorphism f of M which maps $\bar c^p$ to $\bar c^q$ hence f maps p to q. Of course, this works for $\mathfrak{S}^\zeta_{\rm aut}(M)$ too, only the bound is $2^{|\zeta|+|T|}$, so for $\zeta \geq |T|$ we moreover get equality.
2) As in part (1), but this constant value is the number of equivalence classes of some Borel relation hence by a theorem of Silver is $\leq \aleph_0$ or is $2^{\aleph_0}$, see e.g. [HS82], [She84]. Note that the value is always ≥ 2 as the type ${\rm tp}(a, M)$ for $a \in M$ is not conjugate to the type ${\rm tp}(b, M, N)$ when $M \prec N, b \in N \backslash M$.
3) By the proof of part (1) and the definition of being $\aleph_0$-stable.
4) Recall T has the strict order property (by [She90, Ch.II]) hence some formula $\varphi(x, \bar y_n)$ has the strict order property. We fix such φ; and any $M \in {\rm EC}_{\aleph_\zeta,\aleph_\zeta}(T)$ for any regular $\kappa \leq \aleph_\zeta$ we can find an indiscernible sequence $\mathbf{I}_\kappa = \langle \langle b_{\kappa,\alpha}\rangle\hat{\ }\bar a_{\kappa,\alpha} : \alpha < \kappa\rangle$ in M such that:

$(*)$ (a) $\mathfrak{C} \models \varphi[b_{\kappa,\beta}, \bar a_{\kappa,\alpha}]$ iff $\alpha < \beta$
(b) $\varphi(x, \bar a_{\kappa,\alpha}) \vdash \varphi(x, \bar a_{\kappa,\beta})$ if $\alpha < \beta$.

Let $p_\kappa = {\rm Av}(\langle b_{\kappa,\alpha} : \alpha < \kappa\rangle, M)$, so it is enough to prove that for regular $\kappa_1 \neq \kappa_2$, the types $p_{\kappa_1}, p_{\kappa_2}$ are not conjugate. For this it is enough to prove $p_{\kappa_1} \neq p_{\kappa_2}$ (as the assumptions in the choice of $\mathbf{I}_\kappa, p_\kappa$ are preserved by automorphisms of M).

Toward contradiction assume $p_{\kappa_1} = p = p_{\kappa_2}$ and without loss of generality $\kappa_1 < \kappa_2$. For $\ell = 1,2$ we have $\alpha < \kappa_\ell \Rightarrow (\forall^{\kappa_\ell}\beta < \kappa_\ell)\varphi(b_{\kappa_\ell,\beta},\bar a_{\kappa_\ell,\alpha}) \Rightarrow \varphi(x,\bar a_{\kappa_\ell,\alpha}) \in p \Rightarrow (\forall^{\kappa_{3-\ell}}\beta < \kappa_{3-\ell})\varphi(b_{\kappa_{3-\ell},\beta},\bar a_{\kappa_\ell,\alpha})$; so applying this to $3-\ell$ we also have $\alpha < \kappa_{3-\ell} \Rightarrow (\forall^{\kappa_\ell}\beta < \kappa_\ell)[\varphi(b_{\kappa_\ell,\beta},\bar a_{\kappa_{3-\ell},\alpha})]$. So as $\kappa_1 < \kappa_2 = \mathrm{cf}(\kappa_2)$ necessarily there is a co-bounded $u \subseteq \kappa_2$ such that $\alpha < \kappa_1 \wedge \beta \in u \Rightarrow M \models \varphi[b_{\kappa_2,\beta},\bar a_{\kappa_1,\alpha}]$. Renaming, without loss of generality $u = \kappa_2$.

First, assume $\kappa_2 < \aleph_\zeta$. Let $q(\bar y) = \{\neg\varphi(b_{\kappa_1,i},\bar y) : i < \kappa_1\} \cup \{\varphi(b_{\kappa_2,j},\bar y) : j < \kappa_2\}$. If $\bar a \in {}^{\ell g(\bar y)}M$ realizes $q(\bar y)$ we get $\neg\varphi(x,\bar a) \in p_{\kappa_1} \wedge \varphi(x,\bar a) \in p_{\kappa_2}$, contradiction.

But if $r \subseteq q(\bar y)$ is finite and $i_* = \sup\{i : b_{\kappa_1,i}$ appear in $r\}$ then $\bar a_{\kappa_1,i_*+1}$ realizes r so $q(\bar y)$ is a type in M but we are assuming $|q| = \kappa_2 < \aleph_\zeta$ and M is saturated so q is realized in M, contradiction.

Second, assume $\kappa_2 = \aleph_\zeta$; we could have chosen p_{κ_2} using a linear order $I = I'_2 + I''_2$, isomorphic to $(\kappa_2 + \kappa_2^*)$ such that $I'_2 = \{s_\alpha : \alpha < \kappa\}, I''_\alpha = \{t_\alpha : \alpha < \kappa\}$ and $\alpha < \beta < \kappa_2 \Rightarrow s_\alpha <_I s_\beta <_I t_\beta <_I t_\alpha$.

We choose $\langle b_s, \bar a_s : s \in I\rangle$ in M such that $M \models \varphi[b_t,\bar a_s]$ if $s <_I t$. Also without loss of generality for every $A \subseteq M$ of cardinality $< \aleph_\zeta$ for some $\alpha < \aleph_\zeta$ the set $\langle \bar b_s \char`^ \bar a_s : s \in \{s_\beta, t_\beta\} : \beta \in (\alpha,\aleph_\zeta)\rangle$ is indiscernible over A.

Lastly, without loss of generality $\beta < \kappa_2 \Rightarrow b_{s_\beta} = b_{\aleph_\zeta,\beta}$ so $p_{\kappa_2}(\lambda) = \{\psi(x,\bar c) : \bar c \subseteq M$ and $M \models \psi[b_{s_\alpha},\bar c]$ for every $\alpha < \kappa_2$ large enough$\}$. Now for any $\alpha < \kappa_2$ we have $(\varphi(x,\bar a_{s_\alpha}) \wedge \neg\varphi(x,\bar a_{t_\alpha})) \in p_{\kappa_2}$ hence for some $\gamma(\alpha) = \gamma_\alpha < \kappa_1$ we have $\mathfrak{C} \models \varphi[b_{\kappa_1,\gamma(\alpha)},\bar a_{s_\alpha}] \wedge \neg\varphi[b_{\kappa_1,\gamma(\alpha)},\bar a_{t_\alpha}]$ so for some $\gamma < \kappa_1$ the set $u = \{\beta < \kappa_2 : \gamma(\beta) = \gamma\}$ is unbounded in $\kappa_2 = \aleph_\zeta$. So choose above $A = \{b_{\kappa_1,\gamma}\}$ and get a contradiction.

5) See more in [S$^+$b], still we state 1.4 below. $\square_{1.3}$

Observation 1.4. *Assume T is independent, then:*
$f^{\mathrm{aut}}_T(\lambda) = 2^\lambda$ *for* $\lambda = 2^\mu \in \mathbf{C}_{>|T|}$

Proof. Because there are $M_0 \in \mathrm{EC}_{\lambda,1}(T)$ such that $A \subseteq M_0, |A| = \mu$ such that $\mathscr{P} = \{p \in \mathbf{S}(M) : p$ finitely satisfiable in $A\}$ has cardinality 2^λ, but $\mathscr{P}_q = \{p \in \mathscr{P} : p$ conjugate to $q\}$ has cardinality $\le \lambda^\mu = \lambda$ for each $q \in \mathscr{P}$. $\square_{1.4}$

Dealing with saturated models, for unstable T, force us to have the suitable cardinality with $(\kappa = \kappa^{<\kappa})$! so our restriction to such cardinals is natural, that is recall

Claim 1.5. *If $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ but T is unstable and $\kappa > \aleph_0$ then $\kappa = \kappa^{<\kappa}$.*

Proof. By [She90, Ch.III]. $\square_{1.5}$

Our aim is

Conjecture 1.6. 1) If T is dependent, then $f^{\mathrm{aut}}_T(\aleph_\alpha) \le |\alpha|^{2^{|T|}}$ for $\aleph_\alpha \in \mathbf{C}$.
2) If T is dependent unstable, then for some $\kappa^+(T) \le |T|^+$ we have $f_T(\aleph_\alpha) = |\alpha|^{<\kappa^+(|T|)}$ when $\aleph_\alpha \in \mathbf{C}$ is large enough (see [She90, Ch.III] on number of independent orders).

Discussion 1.7. 1) During a try to improve [She15], raising this Conjecture changes my outlook and leads to this work.
2) We may like to eliminate the use of G.C.H. or weak relatives, though 1.5 show this is not straight. We may consider the following relatives, $f^{\mathrm{wat}}_{T,\theta}(-)$ and $f^{\mathrm{vwa}}_{T,\theta}(-)$, those are not further dealt with in this work, i.e. after §(1A).

Definition 1.8. 1) For $\lambda \geq |T|$ let $f^{\rm wat}_{T,\theta}(\lambda) = \min\{\mu$: for every $M \prec \mathfrak{C}$ of cardinality λ there is $N \prec \mathfrak{C}$ of cardinality λ extending M such that $|\mathfrak{S}_{\rm aut}(N)| \leq \mu\}$.
2) Let $f^{\rm vwa}_{T,\theta}(\lambda) = \min\{\mu$: for every $M \prec \mathfrak{C}$ of cardinality λ there is $N \prec \mathfrak{C}$ of cardinality λ extending M and function $g : \mathbf{S}(M) \to \mathbf{S}(N)$ such that $p \in \mathbf{S}(M) \Rightarrow g(p){\restriction}M = p$ and $|\{g(p)/\equiv^{\rm aut}_N : p \in \mathbf{S}(M)\}| \leq \mu\}$; so $f^{\rm vwa}_\lambda(T) \leq f^{\rm wat}_T(\lambda)$.
3) Omitting θ means $\theta = 1$, writing "$< \theta$" means we use $\mathbf{S}^{<\theta}(-)$.

Discussion 1.9. Let us consider $T = T_{\rm ord} := {\rm Th}(\mathbb{Q}, <)$, we concentrate on $f^{\rm vwt}_T(\lambda)$, the case $f^{\rm wat}_\lambda(T)$ can be analyzed similarly. For any λ letting $\Theta^{\rm tr}_\lambda = \{\kappa : \kappa = {\rm cf}(\kappa) \leq \lambda$ and $\lambda^{<\kappa>_{\rm tr}} > \lambda\}$, see Definition 1.10 below, so for some $M \in {\rm EC}_{\lambda,1}(T)$ for each $\kappa \in \Theta^{\rm tr}_\lambda$ it has a set κ of $> \lambda$ cuts of cofinality (κ, κ). Now if we consider $N, M \prec N \in {\rm EC}_{\lambda,1}(T)$, some of these will not be filled, hence $f_T(\lambda) \geq |\Theta^{\rm tr}_\lambda|$.

Concerning the size of $\Theta^{\rm tr}_\lambda$ note that by Easton forcing (using a not necessarily increasing function f from RCard to Car), if $\mu = \min\{\mu : 2^\mu \geq \lambda\}$ then $\Theta^{\rm tr}_\lambda \cap [\mu, \lambda)$ is quite arbitrary. However, by pcf theorems $\Theta^{\rm tr}_\lambda \cap \mu$ is quite small, that is, if $\chi \leq \mu$ is strong limit, then $\Theta^{\rm tr}_\lambda \cap \chi$ is a bounded subset of χ, see [She00b], [She06] and maybe even is provably always finite.

Given $M \in {\rm EC}_{\lambda,1}(T)$ there is $N \in {\rm EC}_{\lambda,1}(T)$ extending it which is strongly $\aleph_0$-saturated (equivalently, 2-transitive), filling as many cuts as we can. Now all the cuts of N of cofinality $(\aleph_0, \aleph_0)$ are conjugate; also the types corresponding to cuts $\mathscr{C}$ with cofinality $(\kappa^1_{\mathscr{C}}, \kappa^2_{\mathscr{C}})$ such that $\kappa^1_{\mathscr{C}} \neq \kappa^2_{\mathscr{C}} \vee \kappa^1_{\mathscr{C}} = \kappa^2_{\mathscr{C}} \notin \Theta^{\rm tr}_\lambda \backslash \{\aleph_0\}$ are easy to handle; because their number is $\leq \lambda$, and we fill the cut $\mathscr{C}$ by $J_{\mathscr{C}}$ such that $J_{\mathscr{C}}$ has both cofinalities $\aleph_0$ as well as treating increasing sequences leading to the cuts from both sides; in fact we can choose N such that this occurs to any cut of M filled by some member of $N \backslash M$.

But when $\kappa^1_{\mathscr{C}} = \kappa^2_{\mathscr{C}} \notin \Theta^{\rm tr}_\lambda$ call it $\kappa_{\mathscr{C}}$ and it $\in \Theta^{\rm tr}_\lambda \backslash \{\aleph_0\}$ it is not immediately clear whether all such cuts can be treated to ensure uniqueness up to conjugacy.

Let $\langle (a_{\mathscr{C},i}, b_{\mathscr{C},i}) : i < \kappa_{\mathscr{C}} \rangle$ be a decreasing sequence of intervals converging to the cut $\mathscr{C}$; now the isomorphism type of $\mathscr{C}$ can be handled <u>when</u>:

$\boxplus_M$ the following set contains a club of $\kappa_{\mathscr{C}}, \{i < \kappa_{\mathscr{C}}$: the cut of M with lower half $\{a : \bigvee\limits_{j<i} a <_M a_{\mathscr{C},j}\}$ is filled in N and the cut of M with upper half $\{b : \bigvee\limits_{j<i} b_{\mathscr{C},j} < b\}$ is filled in $N\}$.

Now as classically known we can find a tree $\mathscr{T}$ of cardinality λ with $\leq \lambda$ levels and $\leq \lambda$ nodes, with nodes intervals of I and cuts correspond to branches. So clearly we can ensure $\boxplus_M$ and this is clearly enough. So we can understand $f^{\rm vwt}_\lambda(T)$ for $T = {\rm Th}(\mathbb{Q}, <)$. We may formalize 1.9 as a claim in 1.11. (Note that computing $f^{\rm wat}_{T,\theta}(\lambda), f^{\rm vwt}_{T,\theta}(\lambda)$ for $\theta > 1$ is easy from the case $\theta = 1$. We use $\alpha(*) \geq \omega$ below to simplify.

Definition 1.10. $\lambda^{<\theta>_{\rm tr}} = \sup\{|\lim_\theta(\mathscr{T})| : \mathscr{T} \subseteq {}^{\theta>}\lambda$ is closed under initial segments and has cardinality $\leq \lambda\}$ where $\lim_\theta(\mathscr{T}) = \{\eta \in {}^\theta\lambda : \eta{\restriction}i \in \mathscr{T}$ for every $i < \theta\}$.

Claim 1.11. *Let $T = T_{\rm ord} := {\rm Th}(\mathbb{Q}, <)$. For any cardinal $\lambda = \aleph_{\alpha(*)} \geq \aleph_\omega$ we have $f^{\rm wat}_T(\lambda) = |\alpha(*)|, f^{\rm vwt}_T(\lambda) = |\Theta| + 1 = |\Theta|$ where $\Theta = \Theta^{\rm tr}_\lambda := \{\theta : \theta = {\rm cf}(\theta) \leq \lambda$ and $\lambda^{<\theta>_{\rm tr}} > \lambda\}$.*

Proof. Let $M \in {\rm EC}_{\lambda,1}(T)$ be given, without loss of generality M is such that:

$(*)$ for every $\theta \in \mathrm{Reg} \cap \lambda^+$, in M

- (a) there is an increasing sequence $\langle a^1_{\theta,\alpha} : \alpha \le \theta \rangle$
- (b) there is an decreasing sequence $\langle a^2_{\theta,\alpha} : \alpha \le \theta \rangle$
- (c) if $\theta \in \Theta$ there is a tree $\mathscr{T}_\theta \subseteq {}^{\theta>}\lambda$ exemplifying $\lambda^{<\theta>_{\rm tr}} > \lambda$ and sequence $\langle b_{\theta,\eta}, c_{\theta,\eta} : \eta \in \mathscr{T}_\theta \rangle$ of members of M such that $\nu \triangleleft \eta \in \mathscr{T}_\theta \Rightarrow b_{\theta,\nu} <_M b_{\theta,\eta} <_M c_{\theta,\eta} <_M c_{\theta,\nu}$ and $[\eta\hat{}\langle\alpha\rangle, \eta\hat{}\langle\beta\rangle \in \mathscr{T}_\theta, \alpha < \beta \Rightarrow c_{\theta,\eta\hat{}\langle\alpha\rangle} <_M b_{\theta,\eta\hat{}\langle\beta\rangle}]$.

Assume $M \prec N \in \mathrm{EC}_{\lambda,1}(T)$ and let N^+ be such that $N \prec N^+$ and N^+ is λ^+-saturated. For $\ell \le 4$ choose $d_\ell \in N^+$ be such that: $d_0 \in M, (\forall a \in N)(d_1 < a < d_2), (\forall a \in N)(a < d_0 \to a < d_3 < d_0), (\forall a \in N)(d_0 < a \to d_0 < d_4 < a)$. For $\theta \in \Theta$, let $\eta = \eta_\theta \in \lim_\theta(\mathscr{T}_\theta)$ and $p_\theta = \{b_{\theta,\eta\restriction i} < x < b_{\theta,\eta\restriction i} : i < \theta\}$ be such that p_θ is omitted by N, exists by cardinality consideration; and so p_θ has unique extension p^+_θ in $\mathbf{S}(N)$ and let $e^0_\theta \in N^+$ realize it. For $\theta \in \mathrm{Reg} \cap \lambda^+$ let $e^1_\theta \in N^+$ be such that $\alpha < \theta \Rightarrow a^1_{\theta,\alpha} < e^1_\alpha$ and $(\forall a \in N)(\bigwedge\limits_{\alpha<\theta} a^1_{\theta,\alpha} < a \Rightarrow e^1_\alpha < a)$. Let $e^2_\alpha \in N^+$ be such that $\alpha < \theta \Rightarrow e^2_\theta < a^2_{\theta,\alpha}$ and $(\forall a \in N)[\bigwedge\limits_{\alpha<\theta} a^2_\theta < a^2_{\theta,\alpha} \to a < e^2_\theta]$. (So the most "economical" way is to have $a^1_{\theta,\alpha} = b_{\theta,\eta\restriction\alpha}, a^2_{\theta,\alpha} = c_{\theta,\eta\restriction\alpha}$ and $e^0_\theta = e^1_\theta = e^2_\theta, \theta \in \Theta \Rightarrow e^0_\theta = e^1_\theta$.)

Now we prove the four needed inequalities

$\boxplus_1$ $f^{\rm vwt}_T(\lambda) \ge |\Theta| + 1$.

Why? It suffices to prove that for any $\mathbf{f} : \mathbf{S}(M) \to \mathbf{S}(N)$ such that $p \in \mathbf{S}(M) \Rightarrow (f(p))\restriction M = p$ we have $|\{\mathbf{f}(p)/\equiv_{\rm aut} : p \in \mathbf{S}(M)\}| \ge |\Theta| + 1$. The types $p_0 = \mathrm{tp}(d_0, M)$ and $p_\theta = \mathrm{tp}(e^0_\theta, M)$ for $\theta \in \Theta$ have unique extensions in $\mathbf{S}(N)$ and clearly $\mathbf{f}(p_0), \mathbf{f}(p_\theta), \theta \in \Theta$ are pairwise non-conjugate.

$\boxplus_2$ $f^{\rm wat}_T(\lambda) \ge |\mathrm{Reg} \cap \lambda^+| + 5$.

Why? It suffices to prove that $\mathbf{S}(N)/\equiv_{\rm aut}$ has cardinality $\ge |\mathrm{Reg} \cap \lambda^+| + 5$. Now the types $\mathrm{tp}(d_0, N), \mathrm{tp}(d_1, N), \mathrm{tp}(d_2, N), \mathrm{tp}(d_3, N), \mathrm{tp}(d_4, N)$ and $\mathrm{tp}(e^1_\theta, N)$ for $\theta \in \mathrm{Reg} \cap \lambda^+$ are pairwise non-conjugate.

$\boxplus_3$ $f^{\rm vwt}_T(\lambda) \le |\Theta| + 1$.

Why? It suffices to show that we can choose a model N_* such that $M \prec N_* \in \mathrm{EC}_{\lambda,1}(T)$ and a function $\mathbf{f} : \mathbf{S}(M) \to \mathbf{S}(N_*)$ such that $p \in \mathbf{S}(M) \Rightarrow \mathbf{f}(p)\restriction M = p$ and $\{\mathbf{f}(p)/\equiv_{\rm aut} : p \in \mathbf{S}(M)\}$ has cardinality $\le |\Theta| + 1$. Note that

$(*)_1$ $\sigma := \min(\Theta)$ is equal to $\min\{\partial : \lambda^\partial > \lambda\}$.

Now choose N_* such that

$(*)_2$ (a) $N \prec N_* \in \mathrm{EC}_{\lambda,1}(T)$

(b) if $d \in \mathfrak{C} \backslash M$ and $(\theta^-_{M,d}, \theta^+_{M,d}) := (\mathrm{cf}\{a \in M : a < d\}, <_M)$, $\mathrm{cf}(\{a \in M : d \in a\}, >_M)) \notin \{(\theta, \theta) : \theta \in \Theta\}$ <u>then</u> the type $\mathrm{tp}(d, M)$ is realized in N_*

(c) N_* is σ-saturated

- $(c)^+$ moreover N_* is strongly σ-saturated (i.e. every partial automorphism of cardinality $< \sigma$ can be extended to an automorphism)
- (d) $\mathscr{T} \subseteq {}^{\lambda>}2$ is a tree with $\le \lambda$ nodes (and $\le \lambda$ levels) and $\bar{a} = \langle a_\eta : \eta \in \mathscr{T}\rangle$ list the members of M with no repetitions such that for $\eta \in \mathscr{T}$ we have $\alpha < \beta < \ell g(\eta) \Rightarrow (a_{\eta\restriction\alpha} < a_\eta \equiv a_{\eta\restriction\alpha} < \bar{a}_{\eta\restriction\beta})$ and $\alpha < \ell g(\eta) \to (a_{\eta\restriction\alpha} < a_\eta \equiv \eta(\alpha) = 1)$
- (e) if $\eta \in \mathscr{T}$ <u>then</u> for some $e^0_\eta, e^1_\eta \in N_* \backslash M$ we have $\{a \in M : a < e^0_\eta\} = \{a \in M : (\exists \alpha < \ell g(\eta)[\eta(\alpha) = 1 \wedge \alpha < \ell g(\eta), a \le a_{\eta\restriction\alpha}]\}$ and $\{a \in M : e^1_\eta < a\} = \{a \in M : (\exists \alpha < \ell g(\eta)[\eta(\alpha) = 0 \wedge a_{\eta\restriction\alpha} \le a]\}$
- (f) if for $\ell = 1,2$ we have $c_\ell < d_\ell$ are both from $N_* \backslash M$ and we let $A_\ell = \{b \in N_*$: if $a \in M$ <u>then</u> $a < c_\ell \Rightarrow a < b$ and $d_\ell < a \Rightarrow b < a\}$ then there is an automorphism of N_* mapping A_1 onto A_2.

Why is this possible: for (c) as $\lambda = \lambda^{<\sigma}$, for (b) as $\{\mathrm{tp}(d, M) : d \in \mathfrak{C}$ and $\theta^-_{M,d} \ne \theta^+_{M,d}$ are infinite$\}$ has $\le \lambda$ members and $\{\mathrm{tp}(d, M) : d \in \mathfrak{C}$ and $\aleph_0 \le \theta^-_{M,d} = \theta^+_{M,d} \notin \Theta\}$ has $\le \lambda$ members by the definition of Θ (and the well known old equivalence of trees and number of cuts); lastly $\{\mathrm{tp}(d, M) : d \in \mathfrak{C}$ and $\theta^-_{M,d} \in \{0, 1\}$ or $\theta^-_{M,d} \in \{0, 1\}\}$ has $\le \lambda$ members trivially. Also clauses (d),(e),(f) are straight.

Now we define $\mathbf{f}$, so let $p \in \mathbf{S}(M)$; the proof is divided to two tasks. <u>First</u>, if some $d = d_p \in N_*$ realize p, then let $\mathbf{f}(p) = \mathrm{tp}(d_p, N_*)$ so by clause $(c)^+$ clearly $\mathbf{f}(p), p_0 = \mathrm{tp}(d_0, N_*)$ are conjugate. <u>Second</u>, if $p \in \mathbf{S}(M)$ is not realized in N_* then by clause (b) there are $\theta \in \Theta$ and $<_M$-increasing $\langle d^-_{p,i} : i < \theta\rangle$ and $<_M$-decreasing $\langle d^+_{p,i} : i < \theta\}$ such that $d^-_{p,i} <_N d^+_{p,i}$ for $i < \theta$ and p include $p' = \{d^-_{p,i} < x < d^+_{p,i} : i < \theta\}$ which N_* omits hence p has unique extension $\mathbf{f}(p)$ in $\mathbf{S}(N_*)$.

Let $\bar{d}_{p,i} = a_{\eta_{p,i}}$ for $i < \theta$, now

- $(*)_3$ without loss of generality $\ell g(\eta_{p,i})$ is constant or is increasing
- $(*)_4$ if $i_0 < i_1 < i_2 < \theta$ then $\ell g(\eta_{p,i_0} \cap \eta_{p,i_1}) \ge \ell g(\eta_{p,i_0} \cap \eta_{p,i_2})$.

[Why? Check separately when $\eta_{p,i_0} \triangleleft \eta_{p,i_1}$ and when not.]

- $(*)_5$ without loss of generality if $i_0 < i_1 < \theta$ then $\ell g(\eta_{p,i_0} \cap \eta_{p,i_1}) = \ell g(\eta_{p,i_0} \cap \eta_{p,i_0+1})$
- $(*)_6$ without loss of generality either
  - (a) $\langle \eta_{p,i} : i < \theta\rangle$ is $\triangleleft$-increasing and $\eta_{p,i+1}(\ell g(\eta_{p,i})) = 1$

    or
  - (b) $\ell g(\eta_{p,i}) > \alpha_i := \ell g(\eta_{p,i+1} \cap \eta_{p,i})$ so $\eta_{p,i}(\alpha_i) = 0$ for every i.

But if $(*)(b)$ holds we can use $\eta'_{p,i} = \eta_{p,i}\restriction\alpha_i$, so without loss of generality $(*)_6(a)$ holds so

- $(*)_7$ for some $\eta \in c\ell(\mathscr{T})\backslash\mathscr{T}$ we have $\mathrm{cf}(\ell g(\eta)) = \theta, \langle \alpha_i : i < \theta\rangle$ is increasing with limit $\ell g(\eta), \eta_{p,i} = \eta\restriction\alpha_i, \eta(\alpha_i) = 1$.

Similarly without loss of generality

- $(*)_8$ for some $\nu \in c\ell(\mathscr{T})\backslash\mathscr{T}, \mathrm{cf}(\ell g(\nu)) = \theta, \langle \beta_i : i < \theta\rangle$ is increasing with limit $\ell g(\nu), d_{p,i} = \nu\restriction\beta_i, \eta(\beta_i) = 0$.

But for each limit $\delta < \theta$ the types $\{d^-_{p,i} < x < a : i < \delta$ and $a \in M$ and $j < \delta \Rightarrow d^-_{p,j} < a\}, \{a < x < d^+_{p,j} : i < \delta, a \in M$ and $j < \delta \Rightarrow a < d^+_{p,j}\}$ are realized by clauses (d),(e). Now easily $\mathbf{f}(p)$, $\mathrm{tp}(e^0_\theta, N)$ are conjugate by some $g \in \mathrm{aut}(N)$ such that $g(b_{\theta,i}) = d^-_{p,i}, g(c_{\theta,i}) = d^+_{p,i}$, because we can choose it in each relevant convex set by clause (f).

$\boxplus_4$ $f^{\mathrm{wat}}_T(\lambda) \le |\alpha(*) + 6|$.

It is simpler when $\alpha(*) \ge \omega$ and the proof is similar to the proof of $\boxplus_3$ but use $\prec$-increasing continuous $\langle N^*_\varepsilon : \varepsilon \le \sigma\rangle, N^*_0 = N$, etc. $\square_{1.11}$

Question 1.12. 1) For T countable, dependent and unstable, is $f^{\mathrm{vwt}}_\lambda(T)$ essentially equal to $f^{\mathrm{vwt}}_{\mathrm{Th}(\mathbb{Q},<)}(\lambda)$? at least can we understand it (and $f^{\mathrm{wat}}_T(\lambda)$)?
2) What can we say on $f^{\mathrm{vwt}}_T(\lambda), f^{\mathrm{wat}}_T(\lambda)$ for independent T?, see below.

Discussion 1.13. 1) Concerning Part (2) of 1.12, it is easy to note: if T is independent and $|T| \le \mu < \lambda \le 2^\mu < 2^\lambda$ and $\mathrm{cf}([2^\mu]^\lambda, \subseteq) > 2^\mu$ hold, e.g. if $\mathrm{cf}(2^\mu) \le \lambda$, then $f^{\mathrm{wat}}_T(\lambda) = 2^\lambda$; see more in Kojman-Shelah [KS92], [Shea, 4.7].

Let $\langle u_0 : i < \lambda\rangle$ be a sequence of subsets of μ which is independent and $\varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T)$ is an independent. Let $M \in \mathrm{EC}_{\lambda,1}(T)$ be such that $\bar a_\alpha \in \mu, \bar b_i \in {}^{\ell g(\bar y)}M$ for $\alpha < \mu, i < \lambda$ be such that $M \models \varphi[\bar a_\alpha, \bar b_i]$ iff $\alpha \in u_i$. It suffices to prove that $|\mathscr{S}(M)/\equiv^{\mathrm{aut}}_M| \ge 2^\lambda$ and for $s \subseteq \lambda$ let $P_s \in \mathbf{S}(M)$ be finitely satisfiable in $A = \{a_\alpha : \alpha < \mu\}$ and such that $\varphi(x, b_i) \in p_\alpha$ <u>iff</u> $i \in s$.

So it suffices to prove that for $s_* \subseteq \lambda$ the set $\mathscr{S} = \{s \subseteq \lambda : p_s, p_{s_*}$ are conjugate$\}$ has cardinality $\le \lambda^\mu = 2^\mu$. For $s \in \mathscr{S}$ let $g_s \in \mathrm{aut}(M)$ map p_{s_*} to p_s. So if $|\mathscr{S}| > 2^\mu$ then for some $g : A \to M$ the set $\mathscr{S}_g = \{s \subseteq \lambda : g_s{\restriction}A = g\}$ has cardinality $> 2^\mu$. We continue as there.
2) For independent T the situation concerning $f^{\mathrm{vwt}}_{T,\theta}(-)$ is very different than for $f^{\mathrm{aut}}_{T,\theta}(-)$. Why? By the following.

Claim 1.14. *1) If $\lambda = \lambda^{<\lambda} > \theta + |T| + \aleph_0$ and T is a complete first order theory, <u>then</u> $f^{\mathrm{vwa}}_{T,\theta}(\lambda) \le 2^\theta$.*
2) Moreover for every $M \in \mathrm{EC}_{\lambda,1}(T)$ there is an elementary extension $M^+ \in \mathrm{EC}_{\lambda,1}(T)$ such that:

$(*)_{M_1,M_2}$ *if $p \in \mathbf{S}^\theta(M)$ then for some $q = q_p \in \mathbf{S}^\theta(M^+)$ extending p the model $M^+_{[q]}$ is saturated, see Definition 0.11.*

Proof. 1) By (2).
2) Let N be such that $M \prec N$ and every $p \in \mathbf{S}^\theta(M)$ is realized by $\bar a_p \in {}^\theta N$.

For $\alpha < \lambda$ let D_α be a regular ultrafilter on $I_\alpha = |\alpha| + \aleph_0$. Now we choose (N_α, M_α) by induction on $\alpha \le \lambda$ such that

- (a) (N_β, M_β) is elementarily equivalent to (N, M) (where (N_β, M_β) is the $(\tau_T \cup \{P\})$-model expanding N_β by $P^{(N_\beta,M_\beta)} = |M_\beta|$, so P is a new unary predicate)
- (b) $(N_0, M_0) = (N, M)$
- (c) the sequence $\langle (N_\beta, M_\beta) : \beta \le \alpha\rangle$ is $\prec$-increasing continuous
- (d) if $\alpha = \beta+1$ then there is an isomorphism $\mathbf{j}^+_\beta$ from (N_α, M_α) onto $(N_\beta, M_\beta)^{I_\beta}/D_\beta$ extending the canonical embedding $\mathbf{j}_\beta$ from(N_β, M_β) into $(N_\beta, M_\beta)^{\lambda_\beta}/D_\beta$, i.e. for $a \in N_\beta, \mathbf{j}_\beta(a) = f_{a,\beta}/D_\beta$ where $f_{a,\beta} : \lambda_\beta \to N_\beta$ is constantly a.

There is no problem to carry the definition and $M^+ := M_\lambda$ is as required. That is, we can prove by induction on α that $\|M_\alpha\| = \lambda$: if $\alpha = 0$ by clause (b) if $\alpha = \beta + 1$ as $\lambda \le \lambda^{I_\beta}/D_\alpha \le \lambda^{|I_\beta|} \le \lambda^{<\lambda} = \lambda$ and for α limit by the induction hypothesis. Also, as D_β is a regular ultrafilter, clearly M_λ is saturated hence $M_\lambda \in \mathrm{EC}_{\lambda,\lambda}(T)$. Similarly (N_λ, M_λ) is λ-saturated hence if $\bar{a} \in {}^{\lambda>}(N_\lambda)$ then $(M_\lambda)_{[\bar{a}]}$ is saturated. We choose $M^+ = M_\lambda$ so indeed $M \prec M^+ \in \mathrm{EC}_{\lambda,\lambda}(T)$.

Now for every $p \in \mathbf{S}^\theta(M)$ recall that $\bar{a}_p \in {}^\theta N \subseteq {}^\theta(N_\lambda)$ realizes p, so let $q_p(\bar{x}_{[\bar{a}]}) = \mathrm{tp}(\bar{a}_p, M_\lambda)$, so we are done. $\square_{1.14}$

Claim 1.15. *1) Assume T is a complete first order theory and λ is strong limit singular of cofinality $\kappa, \lambda > \theta, \lambda > |T| + \aleph_0$. Then $f^{\mathrm{vwa}}_{T,\theta}(\lambda) \le 2^\theta$.*
2) Like 1.14(2) replacing "saturated" by "special", see [CK73].

Proof. 1) By part (2).
2) Similar to the proof of 1.14, but we elaborate. Now the definition of "special" says that there is $\bar{M} = \langle M^*_i : i < \kappa\rangle$ which is a $\prec$-increasing continuous sequence of models (of T) with union M such that M_{i+1} is $\|M_i\|^+$-saturated and $i < \kappa \Rightarrow \|M^*_i\| < \lambda$. Let $\langle \lambda_i : i < \kappa\rangle$ be an increasing sequence of regular cardinals with limit λ. We choose $N, \langle \bar{a}_p : p \in \mathbf{S}^\theta(M)\rangle$ and $\langle D_\alpha : \alpha < \lambda\rangle$ as in the proof of 1.14. We now choose $(N_\alpha, \bar{M}_\alpha)$ by induction on $\alpha < \lambda$ such that:

$\boxplus$ $(a)(\alpha)$ $\bar{M}_\alpha = \langle M_{\alpha,i} : i \le \kappa\rangle$ is $\prec$-increasing continuous
- (β) $(N_\alpha, M_{\alpha,i})$ is elementarily equivalent to (N, M^*_i) for $i < \kappa$ such that $\lambda_i \ge \alpha$ so $M_{\alpha,i} \prec N_\alpha$

$(b)(\alpha)$ $N_0 = N$
- (β) $M_{0,i} = M^*_i$ for $i < \kappa$
- (γ) $M_{0,\kappa} = M$

(c) $\langle (N_\beta, M_{\beta,i}) : \beta \le \alpha\rangle$ is $\prec$-increasing continuous

(d) if $\alpha = \beta + 1$ then
- (α) there is an isomorphism $\mathbf{j}^+_\beta$ from N_α onto $N_\beta^{I_\beta}/D_\beta$ extending the canonical embedding of N_β into $N_\beta^{I_\beta}/D_\beta$
- (β) if $\beta < \lambda_i$ then $\mathbf{j}^+_\beta$ maps $M_{\alpha,i}$ onto $M_{\beta,i}^{I_\beta}/D_\beta$
- (γ) if $\beta \ge \lambda_i$ then $M_{\alpha,i} = M_{\beta,i}$.

In the end $\langle M_{\lambda_i,\lambda_i} : i < \kappa\rangle$ witness that $M^+ := \cup\{M_{\lambda_i,\lambda_i} : i < \lambda\}$ is special; moreover, if $\bar{a} \in {}^\theta N$ then $q_p := \mathrm{tp}(\bar{a}_p, M^+, N_\lambda)$ is as promised. $\square_{1.15}$

If you do not like the use of instances of GCH, i.e. $\kappa = \kappa^{<\kappa}$, but like to stick to essentially the same property, we can reformulate it.

Definition 1.16. Let $f^{\mathrm{aut},*}_{T,\theta}(\lambda)$, for λ regular be the minimal μ such that for any λ-saturated $M \prec \mathfrak{C}$, e.g. of cardinality $2^{<\lambda}$ we can find a subset $\mathbf{P}$ of $\mathbf{S}^\theta(M)$ of cardinality $\le \mu$ satisfying that:

$(*)$ for any $p_1(\bar{x}_{[\theta]}) \in \mathbf{S}^\theta(M)$ there is $p_2(\bar{x}_{[\theta]}) \in \mathbf{P}$ such that letting $\bar{a}_\ell = \langle a_{\ell,i} : i < \theta\rangle$ realizes $p_\ell(\bar{x}_{[\theta]})$ in $\mathfrak{C}$ for $\ell = 1, 2$ we have
- $\odot$ in the E.F. (i.e. Ehrenfeucht-Fräïssé) game of length λ for the pair $(M_{[\bar{a}_1]}, M_{[\bar{a}_2]})$ the ISO player has a winning strategy.

Discussion 1.17. Concerning $f^{\mathrm{aut},*}_{T,\theta}(-)$.
1) The positive result, i.e. upper bound for dependent T (see end of §4) still holds as well as the negative ones.
2) The negative results for independent T holds.
3) The question is closed to the one on "what occurs in $\mathbf{V}^{\mathrm{Levy}(\lambda,\chi)}$ for some χ".

Question 1.18. Generalize to any dependent T the theorem: a linear order of cardinality λ has $\leq \lambda$ cuts of different lower cofinality and upper cofinality.

§ 1(B). On the outside definable sets and uf(p).

Definition 1.19. 1) Let $\mathrm{Def}^\alpha_\Delta(M) = \{\varphi(M,\bar c) : \varphi(\bar x,\bar y) \in \Delta, \ell g(\bar x) = \alpha$ and $\bar c \in {}^{\ell g(\bar y)}\mathfrak{C}\}$ and $\mathrm{Def}_\alpha(M) = \mathrm{Def}^\alpha_{\mathbb{L}(\tau_T)}(M)$ where see below; of course instead $\mathfrak{C}$ we can use any $\|M\|^+$-saturated elementary extension of M.
2) $\varphi(M,\bar c) = \{\bar b : \bar b \in {}^{\ell g(\bar x)}M$ and $\mathfrak{C} \models \varphi[\bar b,\bar c]\}$ where $\varphi = \varphi(\bar x,\bar y)$.
2A) We say $\mathbf{I} \subseteq {}^\alpha M$ is outside definable when it belongs to $\in \mathrm{Def}_\alpha(M)$.
3) If $p(\bar x) \in \mathbf{S}^\alpha(M)$ let $\mathrm{uf}(p) = \{D : D$ an ultrafilter on the Boolean Algebra $\mathrm{Def}_\alpha(M)$ containing $\{\varphi(M,\bar a) : \varphi(\bar x,\bar a) \in p\}\}$.
3A) If $p \in \mathbf{S}^\alpha(M)$ and $\Delta \subseteq \{\varphi(\bar x_\alpha,\bar y) : \bar y \in \{\bar y_{[n]} : n < \omega\}, \varphi \in \mathbb{L}(\tau_T)\}$ <u>then</u> let $\mathrm{uf}_\Delta(p) = \{D \cap \mathrm{Def}^\alpha_\Delta(M) : D \in \mathrm{uf}(p)\}$. If $\Delta = \{\varphi\}$ we may write φ.
4) We say p has super multiplicity 1 <u>when</u> $|\mathrm{uf}(p)| = 1$.
5) If $q(\bar x,\bar y) = \mathrm{tp}(\bar a{}^\frown\bar b, M)$ and $\mathrm{p}(\bar x) = \mathrm{tp}(\bar a, M)$ then $\pi = \pi_{p(\bar x),q(\bar x,\bar y)}$ is the function from $\mathrm{uf}(q)$ onto $\mathrm{uf}(p)$, we call it the projection, such that if $D \in \mathrm{uf}(q)$ and $M \subseteq A \subset \mathfrak{C}$ and $\bar a'{}^\frown\bar b'$ realizes $\mathrm{Av}(D,A)$ then $\bar a'$ realizes $\mathrm{Av}(\pi(D),A)$, see 1.20(1) below.
6) We say $\mathbf{I} = \langle \bar a_t : t \in I\rangle$ is an indiscernible sequence based on $p \in \mathbf{S}^\alpha(M)$ <u>when</u> (I is a linear order and) $\mathbf{I}$ is based on some $D \in \mathrm{uf}(p)$ which means that: for each $t \in I, \mathrm{tp}(\bar a_t, M \cup \{\bar a_s : s \text{ satisfies } t <_I s\} \cup M)$ is $\mathrm{Av}(D, \{\bar a_s : s \text{ satisfies } t <_I s\} \cup M)$. Similarly for $p \in \mathbf{S}^m(A)$ which is finitely satisfiable in M and $\mathbf{I}$ is based on (D,A).
7) Assume $p \in \mathbf{S}^\alpha(M)$ and $D \in \mathrm{uf}(p)$, let $\mathrm{Dom}(D) = |M|$, (we can replace it by an set). We say $\bar a$ realizes $\mathrm{tp}(D,A)$ <u>when</u> there is $\langle \bar a_n : n < \omega\rangle$, as in part (6), i.e. such that $\bar a_n$ realizes $\mathrm{Av}(D, \{\bar a_\ell : \ell \in (n,\omega)\} \cup \bar a \cup \mathrm{Dom}(D))$ for $n < \omega$ and $\langle \bar a\rangle{}^\frown\langle \bar a_n : n < \omega\rangle$ is an indiscernible sequence over A.
8) Above we say "realizes $\mathrm{tp}_\Delta(D,A)$", when in the end $\langle \bar a\rangle{}^\frown\langle \bar a_n : n < \omega\rangle$ is demanded only to be an Δ-indiscernible over A.
9) For D as above let $\lambda(D) = \min\{\sigma$: for some $A \subseteq \mathrm{Dom}(D)$ of cardinality σ, no $\bar a \in {}^\gamma(\mathrm{Dom}(D))$ realizes $(A,D)\}$.
10) $\lambda_\Delta(D)$ is defined similarly restricting ourselves to Δ.
11) $\lambda_{\mathrm{loc}}(D) = \sup\{\lambda_\Delta(D) : \Delta \subseteq \mathbb{L}(\tau_T) \text{ finite}\}$.

Claim 1.20. *1) For $M, \bar a, \bar b, p(\bar x), q(\bar x,\bar y)$ as in Definition 1.19(5), the function $\pi_{p(\bar x),q(\bar x,\bar y))} : \mathrm{uf}(q) \to \mathrm{uf}(p)$ is well defined.*
2) Moreover it is onto.

Proof. 1) Should be clear.
2) So assume $D_1 \in \mathrm{uf}(p(\bar x))$. It suffices to prove that

- $(*)$ the family $\mathscr{X}_1 \cup \mathscr{X}_2$ can be extended to an ultrafilter on ${}^\alpha M$ where
  - (a) $\mathscr{X}_1 := \{X'_1$: for some $X_1 \in D$ we have $X'_1 = \{\bar a'{}^\frown\bar b' : \bar a' \in {}^{\ell g(\bar x)}M, \bar b' \in {}^{\ell g(\bar y)}M$ and $\bar a' \in X_1\}\}$ and

(b) $\mathscr{X}_2 := \{\{\bar a' \hat{}\, \bar b' : \bar a' \in {}^{\ell g(\bar x)}M, \bar b' \in {}^{\ell g(\bar y)}M$ and $M \models \varphi[\bar a', \bar b', \bar c]\} : \mathfrak{C} \models \varphi[\bar a, \bar b, \bar c]$ and $\bar c \subseteq M, \varphi = \varphi(\bar x, \bar y, \bar z) \in \mathbb{L}(\tau_T)\}$.

As each of the two families in the union is closed under (finite) intersection, it suffices to prove:

$\odot$ assume $\varphi = \varphi(\bar x, \bar z_1), \bar c_1 \in {}^{\ell g(\bar z_1)}\mathfrak{C}$ and $X_1 := \varphi(M, \bar c_1) \in D_1$ define $X_1' \in \mathscr{X}_1$ as in $(*)(a)$ and $\psi = \psi(\bar x, \bar y, \bar z_2), \bar c_2 \in {}^{\ell g(\bar z_2)}M$ such that $\mathfrak{C} \models \psi[\bar a, \bar b, \bar c]$, defines $X_2 \in \mathscr{X}_2$ as in $(*)(b)$ <u>then</u> we can find $\bar a', \bar b'$ in M such that $\mathfrak{C} \models \varphi[\bar a', \bar c_1] \wedge \psi[\bar a', \bar b', \bar c_2]$.

To prove $\odot$ note that the set $Y_1 := \{\bar a' \in {}^{\ell g(\bar x)}M : M \models (\exists \bar y)\psi(\bar a', \bar y, \bar c_2)\}$ belongs to D_1 because $D_1 \in \mathrm{uf}(p), p = \mathrm{tp}(\bar a, M)$ and $\mathfrak{C} \models (\exists \bar y)\psi[\bar a, y, \bar c_2]$. Hence $X_1 \cap Y_1 \in D_1$ and choose $\bar a' \in X_1 \cap Y_1$. As $\bar a' \in Y_1$ there is $\bar b' \in {}^{\ell g(\bar y)}M$ such that $M \models \psi[\bar a', \bar b', \bar c_2]$ and as $\bar a' \in X_1$ we have $M \models \varphi[\bar a', \bar c_2]$. Together $\bar a' \hat{}\, \bar b'$ is as required in $\odot$. $\square_{1.20}$

Claim 1.21. *We assume (needed really just in parts (0),(2),(4), that T is dependent.*

0) If $\mathbf{I}$ is an infinite indiscernible set, <u>then</u> $\mathbf{I}$ sits stably, see 1.36(2), (so every $p \in \mathbf{S}^{<\omega}(\cup \mathbf{I})$ is definable).

1) If $D \in \mathrm{uf}(p(\bar x)), p(\bar x) \in \mathbf{S}^\alpha(M)$ and I is a linear order <u>then</u> there is an indiscernible sequence $\langle \bar a_t : t \in I\rangle$ over M based on D, see Definition 1.19(6). We can replace M by a set A.

2) In part (1) if $\mathbf{I}_\ell = \langle \bar a^\ell_t : t \in I\rangle$ is an indiscernible sequence based on D, I is a linear order with no first element and $\bar a^\ell_t$ realizes $\mathrm{Av}(D, \cup\{a^k_s : s$ satisfies $t <_I s$ and $k = 1,2\})$ <u>then</u> $\mathbf{I}^*_1, \mathbf{I}^*_2$, i.e. $\mathbf{I}_1, \mathbf{I}_2$ inverted are equivalent, see 1.36(5).

3) In Definition 1.19(7), it is equivalent "for every infinite linear order I there is an indiscernible sequence $\langle \bar a_t : t \in I\rangle$ over M based on D".

4) Assume $D_\ell \in \mathrm{uf}_\gamma(M)$ and $\langle \bar a^\ell_n : n < \omega\rangle$ is an indiscernible sequence based on D_ℓ, see Definition 1.19(6) for $\ell = 1,2$, <u>then</u> $D_1 = D_2$ iff $\mathrm{tp}(\langle \bar a^1_n : n < \omega\rangle, M) = \mathrm{tp}(\langle a^2_n : n < \omega\rangle, M)$.

4A) Assume for transparency $\gamma < \omega$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite. <u>Then</u> for some $n_\Delta < \omega$ for every $D_1, D_2, M, \bar a^\ell_n$ as in part (4) we have: $D_1 \cap \mathrm{Def}^\gamma_\Delta(M) = D_2 \cap \mathrm{Def}^\gamma_\Delta(M)$ <u>iff</u> $\mathrm{tp}_\Delta(\langle \bar a^1_n : n < n_\Delta\rangle, M) = \mathrm{tp}_\Delta(\langle \bar a^2_n : n < n_\Delta\rangle, M)$.

5) If $\zeta < \theta^+, M$ is κ-saturated, $\mathrm{cf}(\kappa) > 2^\theta$ and $p \in \mathbf{S}^\zeta(M)$ <u>then</u> for some $\mathbf{u}, A_*$ we have (we write $\mathbf{u} = \mathbf{u}_\kappa(p)$):

(a) *$\mathbf{u}$ is a non-empty subset of $\mathrm{uf}(p)$, see 1.19(8)*
(b) *if $D \in \mathbf{u}$ and $A \subseteq M$ has cardinality $< \kappa$ <u>then</u> some $\bar a \in {}^\zeta M$ realizes $\mathrm{tp}(D, A)$*
(c) *$A_* \subseteq M$ and $|A_*| < \kappa$*
(d) *if $\bar a \in {}^\zeta M$ realizes $p{\restriction}A$ then for some $D \in \mathbf{u}, \bar a$ realizes $\mathrm{tp}(D, A_*)$, see Definition 1.19(8)*
(e) *if $D \in \mathrm{uf}(p) \backslash \mathbf{u}$, <u>then</u> no $\bar a \in {}^\zeta M$ realizes $\mathrm{tp}(D, A)$.*

Proof. Parts (0),(2),(4),(4A) and (5) by [She04], the others are obvious. $\square_{1.21}$

Observation 1.22. *Assume $p \in \mathbf{S}^\gamma(M)$ and $|T| + \gamma < \theta^+$. If $\mathrm{uf}(p)$ has cardinality $> 2^\theta$ <u>then</u> for some $\varphi = \varphi(\bar x_\gamma, \bar y)$, also $\mathrm{uf}_{\varphi(\bar x_\gamma, \bar y)}(p)$ has cardinality $> 2^\theta$. In fact $|\mathrm{uf}(p)| \le \Pi\{\mathrm{uf}_{\{\varphi\}}(p) : \varphi = \varphi(\bar x_{[\gamma]}) \in \mathbb{L}(\tau_T)\}$.*

Proof. Obvious. $\square_{1.22}$

Recall question [She04, 6.1].

Definition 1.23. 1) T has bounded directionality <u>when</u>: if $p \in \mathbf{S}^\alpha(M)$, then the set $\mathrm{uf}(p) = \{D : D$ an ultrafilter on $\mathrm{Def}^\alpha(M)$ such that $\mathrm{Av}(D, {}^\alpha M) = p\}$ has cardinality $\leq 2^{|T|+|\alpha|}$.
1A) We define "finite directionality" similarly when we consider only $p \in \mathbf{S}^{<\omega}(M)$.
1B) We define "unary directionality" similarly when we consider only $p \in \mathbf{S}(M)$.
2) We say T has medium directionality <u>when</u> for every $p \in \mathbf{S}^\alpha(M)$, the set $\mathrm{uf}(p)$ has cardinality $\leq \|M\|^{|\alpha|+|T|}$, but T does not have bounded directionality.
3) We say that T has large directionality <u>when</u> it neither has bounded directionality nor medium directionality.

Claim 1.24. *1) T has bounded directionality <u>iff</u> $\mathrm{uf}_\Delta(p)$ is finite whenever $p \in \mathbf{S}^\varepsilon(M), \Delta \subseteq \Gamma_\varepsilon$ is finite <u>iff</u> for some $\lambda \geq |T|$ we have $M \in \mathrm{EC}_{\lambda,1}(T) \wedge \Delta \subseteq \mathbb{L}(\tau_T)$ finite $\wedge p \in \mathbf{S}^{<\omega}(M) \Rightarrow |\mathrm{uf}_\Delta(p)| < \lambda$.*
2) If T has medium directionality <u>iff</u> for every $\lambda \geq |T|$ we have $\lambda = \sup\{|\mathrm{uf}_\Delta(p)| : p \in \mathbf{S}^{<\omega}(M)$ and $M \in \mathrm{EC}_{\lambda,1}(T)$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite$\}$.
3) If T has large directionality <u>iff</u> for every $\lambda > |T|$ we have $\sup\{\lambda^{<\theta>_{\mathrm{tr}}} : \theta \leq \lambda$ regular$\} = \sup\{|\mathrm{uf}_\Delta(p)| : p \in \mathbf{S}^{<\omega}(M), M \in \mathrm{EC}_{\lambda,1}(T)$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite$\}$.
4) If T has medium or bounded directionality, $M \prec \mathfrak{C}, p = \mathrm{tp}(\bar{a}, M) \in \mathbf{S}^\varepsilon(M)$ and $D \in \mathrm{uf}(p)$ and $\langle \bar{a}_n : n < \omega\rangle$ is an indiscernible sequence based on D <u>then</u> for every $\varphi = \varphi(\bar{x}^0_{[\varepsilon]}, \ldots, \bar{x}^{n-1}_{[\varepsilon]}; \bar{y})$ the type $p = \mathrm{tp}_\varphi(\bar{a}_0 \hat{} \ldots \hat{} \bar{a}_{n-1}, M)$ is definable with parameters in the model $M_{[\bar{a}]}$.
5) If T has bounded directionality <u>then</u> in part (4) the type p_φ is definable almost on $\emptyset$ in the model $M_{[\bar{a}]}$, i.e. in $M^{\mathrm{eq}}_{[\bar{a}]}$ it is definable with parameters.

Proof. 1) Clearly the second implies the first which implies the third.

Lastly, assume the second fails and we shall prove the third fails. So we are assuming $p \in \mathbf{S}^m(M)$ and $\mathrm{uf}_\Delta(p)$ infinite, Δ finite. let $\langle D_n : n < \omega\rangle$ be a member of $\mathrm{uf}(p)$ such that $\langle \mathrm{Av}_\Delta(D_n, \mathfrak{C}) : n < \omega\rangle$ are pairwise distinct. For $n < m < \omega$ as $\mathrm{Av}(D_n, \mathfrak{C}) \neq \mathrm{Av}(D_m, \mathfrak{C})$ we can choose let $\varphi_{n,m}(\bar{x}, \bar{b}_{n,m}) \in \mathrm{Av}(D_n, \bar{b}_{n,m})$ such that $\neg\varphi_{n,m}(\bar{x}, \bar{b}_{n,m}) \in \mathrm{Av}(D_m, \bar{b}_{n,n})$.

Let $M = M_0 \prec M_1 \prec M_2 \cup \{\bar{b}_{m,n} : m < n\} \subseteq N_1, \bar{c}_n \subseteq M_2$ realizes $\mathrm{Av}(D_n, N)$ hence it realizes $p \in \mathbf{S}^m(M)$. Let $M^+ = (N, P^{N^+}_1, Q^{N^+}_1, <^{N^+}), P^{N^+}_1 = |M|, Q^N_1 = \{\bar{b}_{m,m} : m < n < \omega\}, P^{N^+}_2 = |N_1|, Q_2 = \{\bar{c}_n : n < \omega\}$.

Let N^{++}_0 be a μ^+-saturated model of $\mathrm{Th}(M^+)$. Without loss of generality $N_2 = N^+ \restriction \tau_T \prec \mathfrak{C}$, let $N_0 = \mathfrak{C} \restriction P^{N^{++}}_1, N_1 = \mathfrak{C} \restriction P^{N^{++}}_2$.

Let $p' = \mathrm{tp}(\bar{c}, M')$ for every $\bar{c} \in Q^{N^{++}}_2$. For every $\bar{c} \in Q^{N^{++}}_2$ let $q_{\bar{c}} = \mathrm{tp}(\bar{c}, N'_1)$, this type is finitely satisfiable in M' (by $N^{++} \equiv M^+$) hence for some ultrafilter D on M' we have $q_{\bar{c}} = \mathrm{Av}(D_{\bar{c}}, N'_1)$. Now for any $\bar{c}_1 \neq \bar{c}_2$ in $Q^{N^{++}}_2$ we have $D_{\bar{c}_1} \neq D_{\bar{c}_2}$ so for some $\bar{b} \in Q^{\mathrm{pos}}_1, \varphi(\bar{x}_1, \bar{b}) \equiv \neg\varphi(\bar{c}_2, \bar{b})$ hence for some $\mathbf{t}$ we hvae $\varphi(\bar{x}, \bar{b})^{\mathbf{t}} \in q_{\bar{c}_1}, \neg\varphi(\bar{x}, \bar{b})^{\mathbf{t}} \in q_{\bar{c}_2}$.

So $\langle D_{\bar{c}} \cap \mathrm{def}_\Delta(M_0) : \bar{c} \in Q^{N^{++}}_0\rangle$ is a sequence of pairwise distinct members of $\mathrm{uf}_\Delta(p')$. As $|Q^{N^{++}}_2| \geq \mu^+$ we are done.
2),4) See Kaplan-Shelah [KS14b] using [She78] and 1.21(4).
5) Obvious. $\square_{1.24}$

Remark 1.25. Can define $\mathrm{uf}_\Delta(p) = \{D \cap \mathrm{def}_\Delta(M) : D \in \mathrm{uf}(M)$ and $\mathrm{Av}(D, M) \supseteq p{\restriction}\Delta\}$, no difference in the proof.

Question 1.26. If $M_{[p(\bar{x})]} \prec N_{[q(\bar{x})]}$ how are $\mathrm{uf}(p(\bar{x})), \mathrm{uf}(q(\bar{x}))$ related?

Question 1.27. Can we prove a substitute? We do not deal with it presently. E.g. we may consider $\mathrm{uf}(\mathbf{I}), \mathbf{I}$ a k-end-homogeneous sequence, see below and §(5B).

Definition 1.28. For a sequence $\mathbf{I} = \langle \bar{a}_s : s \in I\rangle$ of ζ-tuples, (I a linear order) let
1) $\mathbb{B}(\mathbf{I}) = \{J \subseteq I$: for some $\varphi(\bar{x}_{[\zeta]}, \bar{b})$ for every $t \in I$ we have $\varphi[\bar{a}_t, \bar{b}] \Leftrightarrow t \in J\}$.
2) $\mathrm{uf}(\mathbf{I})$ is the set of ultrafilters D on $\mathbb{B}(\mathbf{I})$ containing all co-bounded subsets, so interesting only when $\mathbf{I}$ has no last element.
3) For $\Delta \subseteq \Gamma_\zeta = \{\varphi : \varphi = \varphi(\bar{x}_{[\zeta]}, \bar{y}) \in \mathbb{L}(T)\}$ let $\mathbb{B}_\Delta(\mathbf{I}) = \{J \subseteq I$: for some $\varphi(\bar{x}_{[\zeta]}, \bar{y}) \in \Delta$ and $\bar{b} \in {}^{\ell g(\bar{y})}\mathfrak{C}$ for every $t \in I$ we have $t \in J \Rightarrow \varphi[\bar{a}_t, \bar{b}]\}$.
4) $\mathrm{uf}_\Delta(\mathbf{I}) = \{D \cap \mathbb{B}_\Delta(\mathbf{I}) : D \in \mathrm{uf}(\mathbf{I})\}$.

Probably we may do better.

Question 1.29. Does the directionality of T essentially determine when $\lambda \to_T (\kappa)_\sigma$? See on the directionality, see 1.24 and on the arrow see 1.42 and on $\lambda \to_T (\kappa)_\sigma$ for dependent T see [KS14b].

We have divided the family of dependent unstable T's to three.

Claim 1.30. *1) Every dependent T satisfies exactly one of the following possibilities: stable, unstable (dependent with) bounded directionality, unstable dependent with medium directionality and unstable dependent with large directionality.*
2) Each of those classes is non-empty.

Remark 1.31. 1) Delon, see Poizat [Poi00], gives an example of a dependent T with $|\mathrm{uf}(p)| > \|M\|$, in the present terminology this means a dependent T with large directionality.

Proof. 1) By 1.24.
2) See Kaplan-Shelah [KS14b]. $\square_{1.30}$

Question 1.32. In the definition of medium/large directionality, can we use $p \in \mathbf{S}(M)$?

§ 1(C). Indiscernibles.

Definition 1.33. 1) For an index model I and model M we say $\mathbf{I} = \langle \bar{a}_\eta : \eta \in I\rangle$ is (Δ, n)-indiscernible in M when: $\bar{a}_\eta$ is a sequence from M of length depending only $\mathrm{tp}_{\mathrm{qf}}(\eta, \emptyset, I)$ and such that if the sequences $\bar{\eta}_1 = \langle \eta^1_\ell : \ell < n\rangle, \bar{\eta}_2 = \langle \eta^2_\ell : \ell < n\rangle$ realize the same quantifier free type in I then $\bar{a}_{\bar{\eta}_1}, \bar{a}_{\bar{\eta}_2}$ realize the same Δ-type in M where:
1A) For $\bar{\eta} = \langle \eta_\ell : \ell < n\rangle$ we let $\bar{a}_{\bar{\eta}} := \bar{a}_{\eta_0}\hat{}\ \ldots\hat{}\ \bar{a}_{\eta_{n-1}}$.
2) If $\Delta = \mathbb{L}(\tau_M)$ we may omit it; if $M \prec \mathfrak{C} = \mathfrak{C}_T$ we may omit M, we may write "$< n$" instead n and omit n meaning all n's.
3) Note: saying $\mathbf{I}$ is $\{\varphi(\bar{x}_0, \ldots, \bar{x}_{n-1})\}$-indiscernible in M mean that we consider only $\bar{a}_{\eta_0}\hat{}\ \ldots\hat{}\ \bar{a}_{\eta_{n-1}}, \ell g(\eta_\ell) = \ell g(\bar{x}_\ell)$, so do not allow to divide the variables differently.

4) $\langle \bar{a}_\eta : \eta \in I\rangle$ is continuously indiscernible in M <u>when</u>, say[3] $\ell g(\bar{a}_\eta) = \zeta$ for every $\eta \in I$ and for any formula $\varphi(\bar{x}_0, \dots, \bar{x}_{n-1}) \in \Gamma_{(\zeta)_n}$, with $\ell g(\bar{x}_\ell) = \zeta$ for $\ell < n$ see 0.13, there is a quantifier free formula $\vartheta(y_0, \dots, y_{n-1}) \in \mathbb{L}(\tau_I)$ such that for every $\eta_0, \dots, \eta_{n-1} \in I$ we have $M \models \varphi[\bar{a}_{\eta_0}, \dots, \bar{a}_{\eta_{n-1}}]$ <u>iff</u> $I \models \vartheta[\eta_0, \dots, \eta_{n-1}]$.
5) We add "over B" when we use the expansion $(M, b)_{b \in B}$

Claim 1.34. *Let T be dependent.*
1) Assume

- (a) *Δ is a finite set of formulas*
- (b) *M a model of T and $A \subseteq M$*
- (c) *I is a linear order*
- (d) *$\mathbf{I} = \langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [I]^\ell\rangle$ is indiscernible[4] over A*
- (e) *$\bar{c} \in {}^{\omega >}M$.*

<u>Then</u> there is a finite subset J of I or of the completion $\mathrm{comp}(I)$ *of I such that $\langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [I]^\ell\rangle$ is Δ-indiscernible over $A \cup \bar{d}$ above J.*
2) Moreover, there is a bound on $|J|$ which depend just on $\Delta, \langle k_\ell : \ell < n\rangle$ (and T), and so it is enough that $\mathbf{I}$ is Δ_1-indiscernible for appropriate finite Δ_1.
3) So for every $C \subseteq \mathfrak{C}$ there is $J \subseteq \mathrm{comp}(I)$ of cardinality $\le |C| + |T|$ such that $\mathbf{I}$ is indiscernible above J over $A \cup C$.
4) Let $I \in K_{p,\sigma}$, see Definition 1.39(1) below. If σ is finite then parts (1),(2) holds. Part (3) holds when we demand J to be just of cardinality $\le |C| + |T| + \sigma$.

Proof. See [She04, §3]. $\square_{1.34}$

More generally

Definition 1.35. Let $\mathfrak{k} = (K, \le_{\mathfrak{k}}) = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ be an a.e.c. class of index models; normally $\le_{\mathfrak{k}}$ is $\subseteq$, then we may write K.
1) We say that the theory T has the $\mathfrak{k} - \theta$-indiscernibility property <u>when</u>: if $I \in K$ (see below) and the sequence $\mathbf{I} = \langle \bar{a}_\eta : \eta \in I\rangle$ is indiscernible over A in $\mathfrak{C}_T$ and $\bar{b} \in {}^{\omega >}\mathfrak{C}$ <u>then</u> there is $I_* \in K$ $\le_{\mathfrak{k}}$-extending I and subset J of I_* of cardinality $< \theta$ such that: if $\eta_\ell = \langle \eta_{\ell,m} : m < n\rangle \in {}^n I$ for $\ell = 1, 2$ realizes the same quantifier free type over J in I_* <u>then</u>, the sequences $\bar{a}_{\bar{\eta}_\ell} := \bar{a}_{\eta_{\ell,0}} \hat{} \dots \hat{} \bar{a}_{\eta_{\ell,n-1}}$ for $\ell = 1, 2$ realize the same type over A.
2) Writing "$\mathfrak{k} - (< \theta, n_*)$-indiscernible property" means that above $n \le n_*$.
3) Writing "$\mathfrak{k} - (< \theta, \Delta, n_*)$-indiscernible property" means that above we restrict ourselves to the Δ-type, i.e. which means that $\Delta \subseteq \{\varphi(\bar{x}_0; \bar{x}_1; \dots; \bar{x}_{n-1}; \bar{y}) : \varphi(\bar{x}_0; \dots; \bar{x}_{n-1}; \bar{y}) \in \mathbb{L}(\tau_T); \bar{y} \text{ finite}\}$ and $\bar{\eta}_1, \bar{\eta}_2 \in {}^{(}\mathfrak{C}_T)$ we use only $\varphi(\bar{x}_0, \dots, \bar{x}_{n-1}; \bar{y}) \in \Delta$ such that $\ell g(\bar{y}) = \ell g(\bar{b})$ and $\ell g(\bar{x}_m) = \ell g(\bar{a}_{\eta_{1,m}}) = \ell g(\bar{a}_{\eta_{2,m}})$.
4) Writing "$\mathfrak{k}$ − local − $(< \theta)$-indiscernible property" means that "$\mathfrak{k} - (\theta, \Delta, n)$-indiscernible property", and for every finite Δ.
5) Omitting θ means $\aleph_0$ for the local case, $|T|^+$ for the other case; and instead "$< \theta^+$" we may write θ.
6) We say $I \in K$ is full[5] <u>when</u> for every $J \in K$ which $\le_{\mathfrak{k}}$-extends I, every quantifier free type (in finitely many variables) realized in J is realized in I.

[3]for transparency
[4]so $[I]^{\le n}$ is defined as an index model naturally
[5]it is many reasonable to restrict ourselves to full I

7) We say $I \in K$ is locally full <u>when</u> we replace above type by a formula.

Definition 1.36. 1) An indiscernible sequence $\langle \bar{a}_s : s \in I\rangle$ in $\mathfrak{C}_T$ is dependent (in $\mathfrak{C}_T$) <u>when</u> for every $\bar{b} \in {}^{\omega>}\mathfrak{C}$ it satisfies the conclusion of 1.35 for $\kappa = |T|^+ +$ (the number of quantifier free $(<\omega, \tau_I)$-types realized in I).
1A) Above "κ-dependent" means we use κ.
2) If $I \in K_{\rm set}$, see 1.39 below, an indiscernible set $\mathbf{I} = \{\bar{a}_t : t \in I\}$ in $\mathfrak{C}$ is stable or sit stably <u>when</u> it satisfies the conclusion of 1.35(1).
2A) Above we say κ-stably when we use κ, superstably when $\kappa = \aleph_0$.
2B) An infinite indiscernible sequence $\mathbf{I} = \langle \bar{a}_t : t \in I\rangle$ of ζ-tuples is dependent when for every $\varphi = \varphi(\bar{x}_{[\zeta]}, \bar{y})$ and $\bar{b} \in {}^{\ell g(\bar{y})}\mathfrak{C}$ there is a convex equivalent relation E on I with finitely many equivalence classes such that $sEt \Rightarrow \mathfrak{C} \models \varphi[\bar{a}_s, \bar{b}] \equiv \varphi[\bar{a}_t, \bar{b}]$.
3) For indiscernible $\mathbf{I} = \langle \bar{a}_t : t \in I\rangle \subseteq {}^{\zeta}\mathfrak{C}$ as in part (2) and $A_* \subseteq \mathfrak{C}$ let $\mathrm{Av}(\mathbf{I}, A) = \{\varphi(\bar{x}_{[\zeta]}, \bar{a}) : \bar{b} \in {}^{\omega>}A$ and $\mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}]$ holds for all but $< \aleph_0$ elements $t \in I\}$.
4) For endless $I \in K_{\rm lin}$, see 1.39, indiscernible sequence $\mathbf{I} = \langle \bar{a}_t : t \in I\rangle \subseteq {}^{\zeta}\mathfrak{C}$ and set A let $\mathrm{Av}(\mathbf{I}, A) = \{\varphi(\bar{x}_{[\zeta]}, \bar{b}) : \bar{b} \in {}^{\omega>}M$ and $\mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}]$ for every $<_I$-large enough $t \in I\}$.
5) We call the infinite indiscernible sequences $\mathbf{I}, \mathbf{J}$ equivalent when $\mathrm{Av}(\mathbf{I}, A) = \mathrm{Av}(\mathbf{J}, A)$ for every A.
6) Given endless indiscernible sequences $\mathbf{I}_\ell = \langle \bar{a}^\ell_t : t \in I_\ell\rangle$ for $\ell = 1, 2$, we say $\mathbf{I}_1, \mathbf{I}_2$ are immediate neighbours when $\mathbf{I}_\ell + \mathbf{I}_{3-\ell}$ naturally defined is an indiscernible sequence for some $\ell \in \{1, 2\}$. They are n-neighbours when there are $\mathbf{J}^*_0, \ldots, \mathbf{J}^*_n$ such that $\mathbf{J}_k, \mathbf{J}_{k+1}$ are endless indiscernible sequences which are immediate neighbours for $k = 0, \ldots, n-1$ and $\mathbf{I}_1 = \mathbf{J}_0, \mathbf{I}_2 = \mathbf{J}_n$. Let being neighbours mean n-neighbours for some n.

Discussion 1.37. <u>Historical review for §(1C)</u>:

Of course, Eherenfeucht-Mostowski [EM56] use indiscernibles, i.e. their models were generated by a sequence of indiscernibles. Morley [Mor65] prove that for $\aleph_0$-stable T: when $\lambda = \mu$ is regular $\lambda \rightarrow_T (\lambda)_1$ which mean for any $a_\alpha \in \mathfrak{C}_T(\alpha < \lambda)$ for some $\mathscr{U} \subseteq \lambda$, of cardinality μ the sequence $\langle a_\alpha : \alpha \in \mathscr{U}\rangle$ is an indiscernible set, using $\varphi(x, \bar{b})$ of minimal rank such that $(\exists^\lambda \alpha)(\varphi[a_\alpha, \bar{b}])$, see Definition 1.42. The author [She69a],[She90, III], got a parallel result for stable theory using e.g. Fodor lemma, as minimality does not work, when e.g. $\alpha < \lambda \Rightarrow |\alpha|^{|T|} < \lambda$.

Also for stable T:

- (a) if $\langle \bar{a}_\alpha : \alpha \in I\rangle$ is an infinite indiscernible set, I a set, i.e. with equality only
  - (α) $\varphi(\bar{x}, \bar{b})$ can divide it only to finite/co-finite sets, so we have average
  - (β) for some $u \subseteq \lambda, |u| < \lambda, \langle a_\alpha : \alpha \in I \backslash u\rangle$ is indiscernible over $\bar{b} \cup \{\bar{a}_\alpha : \alpha \in u\} \cup b$.

On general models see [Sheb, §5]. Grossberg and the author suggest to classify first order T by $\lambda \rightarrow_T (\mu)_1$, see 1.42(2) this remains untraceable, see [She00a, §2]. We can consider parallel to Erdös-Rado, see Definition 1.42(3). This is proved for stable T (and more general context) in [Shec, §1], e.g.

- (b) $[\lambda]^{\leq 2} \rightarrow_T ([\mu]^{\leq 2})_\theta$ when $\lambda = (2^\mu)^+$ and $\mu \geq 2^{|T|}$, see 1.42(4).

For dependent T, the parallel to $(a)(\alpha)$ above is in [She90, Ch.II,4.13,pg.77] or [She04, 3.2(1)] the parallel to (β), is in Baldwin-Bendikt [BB00] (not seeing it is also [She04, 3.2(3)]). For I being essentially $I^{\leq n}$ the parallel of $(a)(\beta)$ in [She04, 3.4=1.34] here. Here we state also another generalization using end-homogeneity.

In [She14b, §3] some advance was made for strongly stable theories, $\beth_\delta \to_T (\lambda^+)_1$ when $\delta = (2^{|T|+\lambda})^+$, also in [She14b] it was suggested to look at infinite sequences having better prospects for dichotomies and "T is n-dependent", see more [She17, §2].

Question 1.38. Is the combination reasonable?

Definition 1.39. 1) Let $K_{p,\sigma}$ be the class of $I = (I, <_I, P^I_i)_{i<\sigma}$ where $<_I$ is a linear order of I and $\langle P^I_i : i < \sigma\rangle$ a partition of I, (as in many cases we disabuse our notation not distinguished the (index) model and its universe).
1A) $K_{q,\sigma}$ is the class of $I = (I, <_I, P^I_i)_{i<\sigma}$ where $<_I$ is a linear order of I and P_i a unary predicate. If $\sigma = 0$ we may omit it and so if I is endless this means $I \in K_{\rm lin}$.
2) For $I \in K_{p,\sigma}$, let $E_I = \{(s,t) : s,t \in P^I_i$ for some $i < \sigma\}$; so E_I an equivalence relation on I with $\leq \sigma$ equivalence classes.
3) Let $\mathfrak{k}_e$ be the class of $(I, <, E)$ where $<$ is a linear order on J and E an equivalence relation on I.
4) $K_{{\rm set},\sigma}$ is the class of $(I, P^I_i)_{i<\sigma}$ where $\langle P^I_i : i < \sigma\rangle$ a partition, if $\sigma = 1$ we may omit σ and P^I_0.

Remark 1.40. So by 1.34(1) this case is covered, i.e. if T is dependent <u>then</u> it has the $K_{p,\sigma}$-indiscernibility property.

Observation 1.41. *1) If T is independent <u>then</u> the conclusion of 1.40 fails.*
2) But there is T which is unstable, but have the $K_{\rm set} - \aleph_0$-indiscernible property, e.g. any expansion of the theory of linear order.
3) If T is a dependent theory, <u>then</u> it has the $K_{\rm set}$-indiscernible property (see [She04, §1]*).*
4) Trivially T has the $K_{\rm set}$-indiscernible property <u>iff</u> for every n, every infinite indiscernible set $\mathbf{I} = \{\bar a_\alpha : \alpha < \lambda\}$ of n-tuples in $\mathfrak{C}_T$ is stable (in $\mathfrak{C}_T$, see Definition 1.36(2)).

Definition 1.42. 1) For a linear order I, we say that $\langle \bar a_t : t \in I\rangle$ is an n-end-homogeneous over A <u>when</u> if $m \leq n$ and $t(0,\ell) <_I t(1,\ell) <_I \ldots <_I t(m-1,\ell)$ for $\ell = 1,2$ then the sequence $\bar a_{t(0,1)}{}^\frown \ldots {}^\frown \bar a_{t(m-1,1)}$ and $\bar a_{t(0,2)}{}^\frown \ldots {}^\frown \bar a_{t(m-1,2)}$ realize the same type over $\cup\{\bar a_t : t <_I t(0,1)$ and $t <_I t(0,2)\} \cup A$.
1A) Replacing n by "$< n$" has the obvious meaning (and allow $m = \omega$), $u \in [\lambda]^{<n}$.
2) Let $\lambda \to_T (\gamma)_\sigma$ means that: if $\bar a_\alpha \in {}^\sigma\mathfrak{C}$ for $\alpha < \lambda$ and $\langle \bar a_\alpha : \alpha < \lambda\rangle$ is end-homogeneous <u>then</u> for some $u \subseteq \lambda$, of order type γ the sequence $\langle \bar a_\alpha : \alpha \in u\rangle$ is indiscernible.
3) Let $\lambda \to_T (\gamma)^n_\sigma$ when: if $\bar a_u \in {}^\sigma\mathfrak{C}$ for $u \in [\lambda]^n$ <u>then</u> for some $\mathscr{U} \subseteq \lambda$ of order type γ the sequence $\langle \bar a_n : u \in [\mathscr{U}]^n\rangle$ is $(< n)$-indiscernible. Similarly with $< n$ instead of n.
4) Fix $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ an a.e.c. of index models. Then $I \to_{\mathfrak{k},T} (J)_\theta$ for $I, J \in K_{\mathfrak{k}}$ is defined naturally.

Question 1.43. Find reasonable sufficient conditions on T for the following: for every σ, γ the cardinality $\min\{\lambda : \lambda \to_T (\gamma)_\sigma\}$ is quite small or at least $< \min\{\lambda : \lambda \to (\gamma)^{<\omega}_{\sigma_1}$ where $\sigma_1 = 2^{|T|+\sigma}\}$. (Of course, Erdös-Rado theorem gives lower bounds, see [EHMR84].)

We may consider

Question 1.44. <u>The Strong Indiscernibility Question</u>
1) Give sufficient conditions on T for the following; where $|T| \le \theta$ and $\kappa = \mathrm{cf}(\kappa) > 2^\theta$ (or just large enough). For some $k_1 < \omega, T$ has the strong k_1-indiscernibility existence property for (κ, θ), meaning: if $\gamma(*) < \theta^+$ and $\bar a_\alpha \in {}^{\gamma(*)}\mathfrak{C}$ for $\alpha < \kappa$ and $\mathbf{I} = \langle \bar a_\alpha : \alpha < \kappa \rangle$ is k-end-homogeneous <u>then</u> for some unbounded $\mathscr{U} \subseteq \kappa$ the sequence $\langle \bar a_\alpha : \alpha \in \mathscr{U} \rangle$ is indiscernible.
2) Similarly for "T has the k-strong$^+$ indiscernibility existence property for κ" which means that above $\mathbf{I}$ is mod clubs locally indiscernible.

Discussion 1.45. 1) We will be glad even for weaker versions, anything better than Erdös cardinal.
2) If T is ω-independent we are no better off than in set theory (because we allow ω-tuples).
3) Independent theories can satisfy strong versions of 1.44, see example below.

Definition 1.46. Assume κ is regular uncountable.

We say $\mathbf{I} = \langle \bar a_\alpha : \alpha < \kappa \rangle$ is mod clubs locally indiscernible <u>when</u> for some club E of κ and $I \in K_{q,|T|}$ expanding $(\kappa, <)$ the sequence $\langle \bar a_\alpha : \alpha \in I{\restriction}E \rangle$ is locally indiscernible, see 1.33(4), this means that for every finite Δ there is a finite $\tau_\Delta \subseteq \tau_I$ such that $\langle \bar a_\alpha : \alpha \in (I{\restriction}E{\restriction}\tau_\Delta) \rangle$ is Δ-indiscernible.

Similarly n-indiscernible, n-end-homogeneous.

Recall ([She17, §2])

Definition 1.47. 1) We say T is 2-independent or $2 \times$ independent <u>when</u> , we can find an independent sequence of formulas of the form $\langle \varphi(\bar x, \bar b_n, \bar c_m) : n, m < \omega \rangle$ in $\mathfrak{C} = \mathfrak{C}_T$ or just in some model of T.
2) "T is 2-dependent" (or dependent/2) means the negation of 2-independent (see [She14b, §5 (H)]).
3) We say $\varphi(\bar x, \bar y_0, \dots, \bar y_{n-1})$ is n-independent (for T) <u>when</u> in $\mathfrak{C}_T$ we can, for each $\lambda < \bar\kappa$, find $\bar a^\ell_\alpha \in {}^{\ell g(\bar y_\ell)}(\mathfrak{C}_T)$ for $\alpha < \lambda, \ell < n$ such that the sequence $\langle \varphi(\bar x, \bar a^0_{\eta(0)}, \dots, \bar a^{n-1}_{\eta(n-1)}) : \eta \in {}^n\lambda \rangle$ is an independent sequence of formulas.
4) T is n-independent <u>when</u> some formula $\varphi(\bar x, \bar y_0, \bar y_1, \dots, \bar y_{n-1})$ is n-independent.
5) T is n-dependent (or dependent/n) <u>when</u> it is not n-independent.

Example 1.48. 1) For a first order T which is 3-independent assuming $\lambda = (2^\mu)^+$ we can find $n < \omega$ and $\bar d_\alpha \in {}^n\mathfrak{C}$ for $\alpha \le \lambda$ such that $\langle \bar d_\alpha : \alpha \le \lambda \rangle$ is one-end-homogeneous, equivalently $\mathrm{tp}(\bar d_\alpha, \cup\{\bar d_\beta : \beta < \alpha\})$ is increasing, but for no unbounded $\mathscr{U} \subseteq \lambda$ and even no $\mathscr{U}$ of cardinality μ^+ is $\langle \bar d_\alpha : \alpha \in \mathscr{U} \rangle$ an indiscernible sequence.
2) For a first order T which is $(k+2)$-independent and $\lambda = (2^\mu)^+$ we can find $n < \omega$ and $\bar d_\alpha \in {}^n\mathfrak{C}$ for $\alpha \le \lambda$ such that $\langle \bar d_\alpha : \alpha \le n \rangle$ is k-end-homogeneous for no $u \subseteq \lambda$ of cardinality μ^+ is $\langle \bar d_\alpha : \alpha \in \mathscr{U} \rangle$ an indiscernible sequence.

Example 1.49. T_{rd}, the theory of random graphs has the strongly one-indiscernibility property.

Definition 1.50. We say T has bounded/medium/large k-directionality <u>when</u> : if $\mathbf{I} = \langle \bar a_\alpha : \alpha < \delta \rangle$ has a k-type-increasing (= k-end-homogeneous) <u>then</u> $\mathrm{uf}(\mathbf{I})$ is defined as in Definition 1.23, replacing $\|M\|$ by $|\delta|$.

Remark 1.51. We may consider replacing well orderings by other classes of index models.

Question 1.52. Can we answer 1.43 or 1.44 when T has bounded or at least medium k-directionality for some k.

Question 1.53. 1) Can we characterize $\mathbf{U}_{T,m,\Delta} = \{(n_1, n_2) : n_1 \to (n_2)_{T,\Delta,m}\}$? for finite Δ, m when T dependent?
2) Similarly for T k-dependent?
3) If T is k-dependent is there k_1 such that: if for T is k-dependent, $m < \omega, \Delta \subseteq \mathbb{L}(\tau_T)$ is finite, then $\beth_{k_1}(n) \to (n)_{T,\Delta,m}$ for every n large enough?
4) As in (2) for $k = 1$? (i.e. T dependent).

Question 1.54. Assume $\Delta = \mathbb{L}(\tau_T)$ or Δ-finite, $p = p(\bar{x})$ a (Δ, m)-type over $A, \ell g(\bar{x}) < \theta$ and every subset of p of cardinality $< \kappa$ is realized in M. Can we find $q \in \mathbf{S}^m_\Delta(A)$ extending $p(\bar{x})$ such that every subset of q of cardinality $< \kappa$ is realized in M?

Conjecture 1.55. Assume M is a saturated model of a cardinality $\kappa > |T|$ of a dependent complete T.
1) If $p \in \mathbf{S}(M)$ then there is an indiscernible sequence $\mathbf{I} = \langle a_\alpha : \alpha < \kappa \rangle$ in M such that $p = \mathrm{Av}(I, M)$.
2) Similarly for $p \in \mathbf{S}^\theta(M)$ where $\theta < \kappa$.
See more on this in §6.

§ 1(D). **Limit Models and Generic Pairs.**

We shall address in this paper also the following conjectures.

Conjecture 1.56. We can characterize "M is κ-saturated" parallely to stable T, e.g. M is a κ-saturated model of T iff it is $|T|^+$-saturated and every indiscernible sequence $\langle a_\alpha : \alpha < \delta \rangle$ of elements in M of length $\delta < \kappa$ can be continued and similarly for cuts.

Conjecture 1.57. The Generic Pair Conjecture
Assume [6] $\lambda = \lambda^{<\lambda} > |T|, 2^\lambda = \lambda^+, M_\alpha \in \mathrm{EC}_{\lambda,1}(T)$ is $\prec$-increasing continuous for $\alpha < \lambda^+$ with $\cup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_{\lambda^+,\lambda^+}(T)$, i.e. being saturated. Then T is dependent iff for some club E of λ^+ for all pairs $\alpha < \beta < \lambda^+$ from E of cofinality $\lambda, (M_\beta, M_\alpha)$ has the same isomorphism type.

Remark 1.58. We proved in [She15] the "structure" side, i.e. the implication $\Rightarrow$ in 1.57 when $\lambda = \kappa$ is measurable, on the non-structure side of 1.57, 1.59, see [She14a], [She11]. It seemed natural to assume that the first order theories of such pair is complicated if T is independent and "understandable" for dependent of T, but this is not so, see Kaplan-Shelah [KS14b].

But we shall leave open:

[6] the "$2^\lambda = \lambda^+$" is just for making the formulation more transparent

Conjecture 1.59. The Unique Limit Model Conjecture Assume if T is dependent, $|T| < \lambda = \lambda^{<\lambda}$ and $\lambda^+ = 2^\lambda$ and $\sigma = \mathrm{cf}(\sigma) < \lambda$. If $\langle M_\alpha : \alpha < \lambda^+\rangle$ is an increasing continuous of models of cardinality λ with λ^+-saturated union then for some club E of λ^+, all the models in $\{M_\alpha : \alpha$ is from E and has cofinality $\sigma\}$ are pairwise isomorphic.

* * *

Completions:

For linear order the notion of completion is very important, so it is natural to try to generalize it to dependent theories (if we accept thesis 0.4). Note that for stable theories as every type $p \in \mathbf{S}(A)$ is definable by formulas with parameters from A, this is not so necessary (and $\mathfrak{C}^{\mathrm{eq}}$ is a much less radical extension).

Definition 1.60. 1) $\mathrm{ai}(\mathfrak{C})$ is the set $\{\mathrm{Av}(\mathbf{I}, \mathfrak{C}) : \mathbf{I}$ an infinite indiscernible sequence of finite tuples in $\mathfrak{C}\}$.
2) $\mathrm{nsp}_\mu(\mathfrak{C})$ is the set of $p \in \mathbf{S}^m(\mathfrak{C})$ which does not split over some $A \subseteq \mathfrak{C}$ of cardinality $< \mu$.
3) $\mathrm{fs}_\mu(\mathfrak{C})$ is the set of $p \in \mathbf{S}^m(\mathfrak{C})$ which is $\mathrm{Av}(D, \mathfrak{C})$ where D is an ultrafilter on ${}^m A$ for some $m < \omega$ (or more) and $A \subseteq \mathfrak{C}$ a set of cardinality $< \mu$. If $\mu = \infty$, (i.e. $\|\mathfrak{C}\|$) we may omit it.

Thesis 1.61. So the types we considered as understandable, a base for analysis are $\mathrm{fs}_\mu(\mathfrak{C})$ or $\mathrm{nsp}_\mu(\mathfrak{C}), \mu$ for small enough (hopefully $|T|^+$) and $\mathrm{ai}(\mathfrak{C})$.

Question 1.62. Is it reasonable to add in the completion of $\mathfrak{C}, \mathrm{nsp}_\infty(\mathfrak{C})$ or just $\mathrm{nsp}_\mu(\mathfrak{C}) \cup \mathrm{ai}(\mathfrak{C})$.

Discussion 1.63. 0) So our main theorems say that any $p \in \mathbf{S}(M), M \in \mathrm{EC}_{\lambda,\lambda}(T)$ is definable over $\leq \beth_\omega + |T|$ elements from $\mathrm{ai}(\mathfrak{C}) \cup \mathrm{fs}_{<\beth_\omega}(\mathfrak{C})$.
1) We may prefer not to analyze complete types but ultrafilters, i.e. the $\bar{d}_{\mathbf{x}}$ and $\bar{c}_{\mathbf{x},i}$ are in the full completion! But there is no parallel to the "recounting of types" as there are dependent T with large directionality. However, given $D \in \mathrm{uf}({}^m M)$ we may choose an $\|N\|^+$-saturated elementary extension N of M and let $p = \mathrm{Av}(D, N)$, so analyzing p is very close.

It is still reasonable that in view of later developments we shall prefer to use the ultrafilter version.

Recall that if we succeed to use $\mu = |T|^+$ for countable T, then we can always use eventually indiscernible sequences, see below. This may be not just aesthetically nicer but helpful. Anyhow allowing constant though not so small μ, will give us the asymptotic behaviour.
2) To clarify our intension let us consider the class of linear orders. We like to deal with the class of complete linear orders; or at least $(< \kappa)$-complete. If I is the completion of a κ-saturated dense linear order, then it is natural to add predicate $P_{x,\theta}(x \in \{\mathrm{left}, \mathrm{right}\}, \theta = \mathrm{cf}(\theta) < \kappa)$ such that

- (a) (α) $I \models P_{\mathrm{left},\theta}(a)$ iff $\mathrm{cf}(I_{<a}) = \theta$
  - (β) $I \models P_{\mathrm{right},\theta}[a]$ iff the inverse of $I_{>a}$ has cofinality θ
- (b) $I_1 \leq_{\mathfrak{k}} I_2$ iff
  - (α) $I_1 \subseteq I_2$

$(\beta)_{\rm left}$ if $I_1 \models P_{{\rm left},\theta}(a)$ then $(I_1)_{<a}$ is cofinal in $(I_2)_{<a}$

$(\beta)_{\rm right}$ if $I_1 \models P_{{\rm right},\theta}[a]$ then $(I_1)_{>a}$ is cofinal in the inverse of $(I_2)_{>a}$.

Definition 1.64. 1) We say that $\mathbf{I} = \langle \bar a_t : t \in I \rangle$ is an eventually indiscernible sequence <u>when</u>: I is an endless linear order, $\ell g(\bar a_t)$ for $t \in I$ is constant and finite for transparency, and for every finite set $\Delta \subseteq \mathbb{L}(\tau_T)$ there is $t(\Delta) \in I$ such that $\langle \bar a_t : t \in I_{\geq t(\Delta)} \rangle$ is a Δ-indiscernible sequence (over A).

§ 2. Decompositions of types

We define $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$, which is a partial analysis of $\mathrm{tp}(\bar{d}_{\mathbf{x}}, M_{\mathbf{x}})$, it is related to $K_{\kappa,\theta}$ from [She15] but $B_{\mathbf{x}}$, which has cardinality $< \kappa$ there corresponds to $B^+_{\mathbf{x}} = B_{\mathbf{x}} \cup \{\mathbf{I}_{\mathbf{x},i} : i \in u_{\mathbf{x}}\}$ here. Moreover, the set $B_{\mathbf{x}}$ is of cardinality $< \mu_0$ rather than $< \kappa$ but in this section we do not really use this. We define "$\mathbf{x}$ is σ-active in $i < i_{\mathbf{x}}$", which cannot occur too many times. We define $\mathrm{qK}'_{\kappa,\bar{\mu},\theta}$, those for which we "exhaust the possible activities", this set is dense; and the related $\mathrm{qK}_{\kappa,\bar{\mu},\theta}$ is suppose to be the class of such $\mathbf{x}$'s in which we have fuller analysis. For the case $\kappa = \mu$ we have $\mathrm{qK}'_{\kappa,\mu,\theta} = \mathrm{qK}_{\kappa,\mu,\theta}$ so in this case $\mathrm{qK}_{\kappa,\mu,\theta}$ is dense and we define solvability, all are related to [She15]. But not so dealing with $(\bar{\mu},\theta)$-sets, over which the situation is similar to the one for stable T and any set $\subseteq \mathfrak{C}_T$; note that $B^+_{\mathbf{x}}$ is a so called $(\bar{\mu},\theta)$-set when $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ is smooth. Central here are the definitions of similarity of decompositions and their smoothness (points which are meaningless in [She15]) and we point out their basic properties. Those later ones indicate the possible advantages of Definition 2.2, i.e. the use of indiscernibles. Generally, we shall concentrate on the case $\kappa = \mu_2 \gg \mu_1 = \mu_0 > \theta \geq |T|$ so may not state claims in full generality concerning this point[7].

§ 2(A). **Decompositions - the basics.**

Convention 2.1. 1) In clause (i) of Definition 2.2 below we have three options, the choice is $\iota_{\mathbf{x}} \in \{0,1,2\}$, usually the choice does not matter and in those cases we suppress ι; so far we can use only $\iota_{\mathbf{x}} = 2$. Usually the set v can be a well ordering and even an ordinal but in disjoint amalgmation in $\mathrm{sK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ we shall need anti-well orderings whereas in proving density for qK' it is natural to use just well-ordering.
2) Also $\bar{c}_{\mathbf{x}}$ consists of finite sequences and sometimes we use $\bar{c} \subseteq \bar{d}$, normality, see Definition 2.6(7); we may demand that always $\bar{c} \subseteq \bar{d}$. We can work in $\mathfrak{C}^{\mathrm{eq}}$ hence use $\bar{c}$ a sequence of singletons but this is immaterial in Definition 2.4.
3) The notation is sometimes best understood as in the case when v is a set of ordinals, as the case "v an ordinal" is our prototype so abusing notation we let, e.g. $v \cap i := \{j \in v : j <_v i\}$.
4) Objects like $\mathbf{x}$ below will be called decompositions.

Definition 2.2. Assume $\bar{\mu} = (\mu_2, \mu_1, \mu_0)$ and $\lambda \geq \kappa \geq \mu_0, \lambda \geq \mu_2 \geq \mu_1 \geq \mu_0 > \theta$ but if not said otherwise in addition $\theta \geq |T|, \mathrm{cf}(\mu_2) > \theta$ and usually $\mu_1 = \mu_0$ and even $\kappa = \mu_2$.

We let $\mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ be the class of objects $\mathbf{x}$ consisting of:

- (a) $M \prec \mathfrak{C}$ which is κ-saturated of cardinality λ
- (b) $B = \cup\{B_i : i \in v \backslash u\}$ and each $B_i \subseteq M$ is of cardinality $< \mu_0$
- (c) $\bar{d} \in {}^{\theta^+ >}({}^{\omega >}\mathfrak{C})$ or even $\bar{d} \in {}^{w}({}^{\omega >}\mathfrak{C})$ where[8] w is a linear order (e.g. a set of ordinals) of cardinality $\leq \theta$; we may write w as $\ell g(\bar{d})$ or $\mathrm{Dom}(\bar{d})$ but we

[7]A debt: similarly replacing "M saturated of cardinality $\lambda^+, \lambda = \lambda^{<\lambda}$" by λ is strong limit singular, M_i, increasig $M = \cup\{M_i : i < \mathrm{cf}(\lambda)\}, M_{i+1}$ is $\|M_i\|^+$-saturated, $\|M_i\| < \lambda = \sum\limits_i \|M_i\|$.

[8]This is useful when we like to amalgamate such objects, but usually we may ignore this. We may work in $\mathfrak{C}^{\mathrm{eq}}$ and then use d_i instead of d_i. Similarly for the $\bar{c}_i$'s.

usually write $i < j$ instead of $i <_w j$ and $w \cap j$ or $w_{<j}$ for $\{i \in w : i <_w j\}$; similarly for v, u below

(d) $\bar{c} = \langle \bar{c}_i : i \in v\rangle \in {}^v({}^{\omega>}\mathfrak{C})$ which sometimes is treated as $(\ldots {}^\frown \bar{c}_i {}^\frown \ldots)_{i \in v}$ so[9] $\bar{c}_i \in {}^{\omega>}\mathfrak{C}$; where v is a linear order of cardinality $\le \theta$; we may write $v = \ell g(\bar{c})$ or $v = \mathrm{Dom}(\bar{c})$

(e) $u \subseteq v$

(f) $\bar{\kappa} = \langle \kappa_i : i \in u\rangle$ such that $\kappa_i = \mathrm{cf}(\kappa_i) \in [\mu_1, \mu_2)$

(g) $\bar{\mathbf{I}} = \langle \mathbf{I}_i : i \in u\rangle$

(h) $\mathbf{I}_i = \langle \bar{a}_{i,\alpha} : \alpha < \kappa_i\rangle$ is an indiscernible sequence in M for $i \in u$.

(i) $\iota_{\mathbf{x}} \le 2$ and[10] for $i \in v \backslash u$,

<u>Case 0</u>: $\iota_{\mathbf{x}} = 0, \mathrm{tp}(\bar{c}_i, M + \Sigma\{\bar{c}_j : j < i\})$ does not split over B_i

<u>Case 1</u>: $\iota_{\mathbf{x}} = 1 : \mathrm{tp}(\bar{c}_i, M + \Sigma\{\bar{c}_j : j < i\})$ does not locally split over

$B_i + \Sigma\{\bar{c}_j : j < i\}$, see Definition 2.3 below

<u>Case 2</u>: $\iota_{\mathbf{x}} = 2$, the type above is finitely satisfiable in B_i.

In short we may say $\mathrm{tp}(\bar{c}_i, M + \Sigma\{\bar{c}_j : j < i\})$ does not $\iota_{\mathbf{x}}$-split over B_i. Let $\mathrm{schm}_{\mathbf{x},i}$ be the scheme defining the type, (so the only parameters are $B_{\mathbf{x},i}$)

(j) for $i \in u$, the type $\mathrm{tp}(\bar{c}_i, M + \Sigma\{\bar{c}_j : j < i\})$ is $\mathrm{Av}(\mathbf{I}_i, M_{\mathbf{x}} + \Sigma\{\bar{c}_j : j < i\})$ hence $\ell g(\bar{a}_{i,\alpha}) = \ell g(\bar{c}_i)$ for $\alpha < \kappa_i$.

Definition 2.3. 1) We say that the type $p(\bar{x})$ locally splits over A <u>when</u> there is $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_t)$ such that for every finite $\Delta \subseteq \{\varphi : \varphi = \varphi(\bar{y}, \bar{z}) \in \mathbb{L}(\tau_t)\}$ there are formulas $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$ where $\bar{b}, \bar{c}$ realize the same Δ-type over A.
2) If $p \in \mathbf{S}^\varepsilon(B)$ does not split over $A \subseteq B$ let the scheme of p be the function H defined by: if $\varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \mathbb{L}(\tau_T)$ and $\bar{b} \in {}^{\ell g(\bar{y})}B$ then $H(\varphi(\bar{x}, \bar{y}),\ \mathrm{tp}(\bar{b}, A))$ is the truth value of $\varphi(\bar{x}, \bar{b}) \in p$; note that this defines the domain of H.

Definition 2.4. In Definition 2.2 we say $i \in v_{\mathbf{x}}$ is σ-active (in $\mathbf{x}$) <u>when</u> $1 \le \sigma \le \aleph_0$ and $\sigma = 1 \Rightarrow i \notin u_{\mathbf{x}}$ and (using notation of 2.6(1) below; the default value for σ is 1):

<u>Case 1</u>: $\sigma = 1$

We can find $\bar{b}_{i,0}, \bar{b}_{i,1}$ such that:

(a) $\bar{b}_{i,0}, \bar{b}_{i,1}$ realize the same type over $\bar{c}_{\mathbf{x},<i} + M_{\mathbf{x}}$, see 2.6(1)

(b) $\bar{b}_{i,0}, \bar{b}_{i,1}$ realize distinct types over $\bar{d}_{\mathbf{x}} + \bar{c}_{\mathbf{x},<i} + M_{\mathbf{x}}$

(c) $\bar{c}_{\mathbf{x},i} = \bar{b}_{i,0} {}^\frown \bar{b}_{i,1}$.

<u>Case 2</u>: $\sigma \ge 2$ and $i \notin u_{\mathbf{x}}$

We can find $\langle \bar{b}_{i,n} : n < \sigma\rangle$ such that:

[9] could demand $\bar{c}_i \in {}^\omega\mathfrak{C}$ or even $\bar{c}_i \in {}^{\theta^+>}\mathfrak{C}$, in this work usually it does not matter but not always; if we do this in 2.4 we can make $\bar{c}_{\mathbf{x},i} = (\ldots {}^\frown \bar{b}_{i,\ell} {}^\frown \ldots)_{\ell < \sigma}$ in Case A and a parallel demand in Case B

[10] we may use $\iota_{\mathbf{x},i}$ for $i \in v \backslash u$, that is possibly having different choice for each i; so far does not seem to matter.

(a) $\mathrm{tp}(\bar b_{i,n}, \cup\{\bar b_{i,m} : m \in (n,\sigma)\} + \bar c_{\mathbf{x},<i} + M_{\mathbf{x}}\})$ is $\subseteq$-decreasing with n and[11] does not $\iota_{\mathbf{x}}$-split over $B_{\mathbf{x},i}$; (i.e. does not split over $B_{\mathbf{x},i}$ if $\iota_{\mathbf{x}} = 0$, does not locally split over $B_{\mathbf{x},i}$ if $\iota_{\mathbf{x}} = 1$ and is finitely satisfiable in $B_{\mathbf{x},i}$ if $\iota_{\mathbf{x}} = 2$)

(b) $\mathrm{tp}(\bar b_{i,\ell}, \bar d_{\mathbf{x}} + \bar c_{\mathbf{x},<i} + M_{\mathbf{x}})$ for $\ell = 0, 1$ are distinct

(c) $\bar c_{\mathbf{x},i} = \bar b_{i,0}{}^\frown \bar b_{i,1}$.

Case 3: $\sigma \geq 2$ and $i \in u_{\mathbf{x}}$

We can find $\bar b_{i,\alpha,n}(\alpha \leq \kappa_{\mathbf{x},i}, n < \sigma)$ such that:

(a) $\mathrm{tp}(\bar b_{i,\kappa_{\mathbf{x},i},n}, \cup\{\bar a_{i,\kappa_{\mathbf{x},i},m} : m \in (n,\sigma)\} + \bar c_{\mathbf{x},<i} + M_{\mathbf{x}})$ does not $\iota_{\mathbf{x}}$-split over $\cup\{\bar a_{\mathbf{x},i,\alpha} : \alpha < \kappa_{\mathbf{x},i}\}$

(b) $\mathrm{tp}(\bar b_{i,\kappa_{\mathbf{x},i},\ell}, \bar d_{\mathbf{x}} + \bar c_{\mathbf{x},<i} + M_{\mathbf{x}})$ for $\ell = 0, 1$ are distinct

(c) $\bar c_i = \bar b_{i,\kappa_{\mathbf{x},i},0}{}^\frown \bar b_{i,\kappa_{\mathbf{x}},1}$ and $\bar a_{\mathbf{x},i,\alpha} = \bar b_{i,\alpha,0}{}^\frown \bar b_{i,\alpha,1}$ for $\alpha < \kappa_{\mathbf{x},i}$

(d) $\langle \bar b_{i,\alpha,n} : (\alpha,n) \in (\kappa_{\mathbf{x},i}+1)\times 2\rangle$ is an indiscernible sequence where $(\kappa_{\mathbf{x},i}+1)\times 2$ is ordered lexicographically.

Remark 2.5. 1) We shall return to this and to 2.14 and in 5.22.
2) We can use $\bar c_{\mathbf{x},\neq i} = \langle \bar c_{\mathbf{x},j} : j \in v_{\mathbf{x}} \backslash \{i\}\rangle$ instead $\bar c_{\mathbf{x},<i}$ in Definition 2.4, as in [She15]. Similarly we can allow $\bar c_{\mathbf{x},i} = \bar b_{i,0}{}^\frown \bar b_{0,1}{}^\frown \ldots$ in cases 2,3 in Definition 2.4; but so far this does not matter.
3) In Case (3) of Definition 2.4 note that it follows that for every $\bar e \in {}^{\omega >}(M_{\mathbf{x}})$ for some $\beta < \kappa$, $\langle \bar b_{i,\alpha,n} : (\alpha,n) \in [\beta, \kappa_{\mathbf{x},i}) \times 2\rangle$ is an indiscernible sequence over $\bar e$. Hence in clause (d) of case (3) of Definition 2.4, we can use "$(\alpha,n) \in \kappa_{\mathbf{x},i} \times 2$" or "$(\alpha,n) \in (\kappa_{\mathbf{x},i}+1) \times 2$", and get equivalence conditions.

Definition 2.6. 1) For $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar\mu,\theta}$ let $\mathbf{x} = (M_{\mathbf{x}}, \ldots)$ so $w_{\mathbf{x}} = w, v_{\mathbf{x}} = v, u_{\mathbf{x}} = u, \bar c_{\mathbf{x}} = \bar c[\mathbf{x}] = \bar c, \bar d_{\mathbf{x}} = \bar d[\mathbf{x}] = \bar d, \kappa_i = \kappa_{\mathbf{x},i} = \kappa(\mathbf{x},i), B_{\mathbf{x},i} = B_i, B_{\mathbf{x}} = B$, etc., and let $B^+_{\mathbf{x}} = \cup\{\mathbf{I}_{\mathbf{x},i} : i \in u_{\mathbf{x}}\} \cup B_{\mathbf{x}}$. Let $\bar c_{<i} = \bar c[<i] = \bar c_{\mathbf{x},<i} = (\ldots{}^\frown \bar c_{\mathbf{x},j}{}^\frown \ldots)_{j<i}, \bar c[u'] = \bar c_{\mathbf{x}}[u'] = (\ldots{}^\frown \bar c_{\mathbf{x},j}{}^\frown \ldots)_{j \in u'}$ and $\bar c_{\neq i} = \bar c_{\mathbf{x},\neq i} := (\ldots{}^\frown \bar c_{\mathbf{x},j}{}^\frown \ldots)_{j \in v_{\mathbf{x}} \backslash \{i\}}$ for $i \in v_{\mathbf{x}}$ and for $u' \subseteq v_{\mathbf{x}}$. We may write $\bar c, \bar d$ instead of $\bar c_{\mathbf{x}}, \bar d_{\mathbf{x}}$ when confusion is unlikely as there is only one $\mathbf{x}$ around, in particular avoiding using, e.g. $\bar x_{\bar d_{\mathbf{x}}}$; also we may write $\bar c[\mathbf{x}], \bar c[\mathbf{x}, <i]$, etc. Let $\mathfrak{a}_{\mathbf{x}} = \{\kappa_{\mathbf{x},i} : i \in u_{\mathbf{x}}\}$.
1A) For $i \in v_{\mathbf{x}}$ let $D_i = D_{\mathbf{x},i}$ be such that $\mathrm{tp}(\bar c_{\mathbf{x},i}, \bar c_{\mathbf{x},<i} + M_{\mathbf{x}}) = \mathrm{Av}(D_i, \bar c_{\mathbf{x},<i} + M_{\mathbf{x}})$ where D_i is an ultrafilter on ${}^{\ell g(\bar c(\mathbf{x},i))}(B_{\mathbf{x},i})$ if $\iota_{\mathbf{x}} = 2 \wedge i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$ and on $\mathbf{I}_{\mathbf{x},i}$ if $i \in u_{\mathbf{x}}$ such that $\alpha < \kappa_{\mathbf{x},i} \Rightarrow \{\bar a_{\mathbf{x},i,\beta} : \beta \in (\alpha, \kappa_{\mathbf{x},i})\} \in D_i$; but only $D'_i := D_i \cap \mathrm{Def}_{\ell g(\bar c(\mathbf{x},i))}(\mathrm{Dom}(D_i))$ matters so we normally use it.
2) Concerning $\mathrm{pK}_{\lambda,\kappa,\bar\mu,\theta}$, omitting λ means "for some λ"; omitting μ_0 means $\mu_0 = \mu_1$, omitting also μ_2 means $\mu_2 = \kappa$, then we may write μ instead of μ_1 and of $\bar\mu$; writing $*$ instead of μ_1 means $\mu_1 = (\theta^+ + |T|^+ + \beth_\omega)$; omitting $\lambda, \kappa, \mu_0, \mu_1, \mu_2, \theta$ means for some such cardinals. Similarly in parallel definitions later.
3) We say $i \in v_{\mathbf{x}}$ is active in $\mathbf{x}$ when it is σ-active for some σ, equivalently for $\sigma = 1$.
3A) We say $i \in v_{\mathbf{x}}$ is active in $\mathbf{x}$ over u when $i \in v_{\mathbf{x}}, u \subseteq v_{\mathbf{x}}(<i)$ and in Definition 2.4, in clause (b), in each of the cases we replace $\bar c_{\mathbf{x},<i}$ by $\bar c_{\mathbf{x}}[u]$. Similarly in the other versions.
4) We say $i \in v_{\mathbf{x}}$ is strongly active in $\mathbf{x}$ when it is $\aleph_0$-active.

[11] in the other cases the parallel statement follows

5) We say that $i \in v_{\mathbf{x}}$ is (σ, Δ)-active in $\mathbf{x}$ <u>when</u> $1 \le \sigma \le \aleph_0$ and $\Delta \subseteq \Gamma^1_{\mathbf{x},i} := \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}[<i]}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{y}, \bar{z}$ are finite$\}$ and in Definition 2.4 we replace clause (b), in all cases by

$(b)'$ for[12] some $\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}[<i]}, \bar{y}, \bar{z}) \in \Delta$ we have

$$\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x},<i}, \bar{b}_{i,\kappa_{\mathbf{x},i},0}, \bar{a}] \wedge \neg\varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x},<i}, \bar{b}_{i,\kappa_{\mathbf{x},i},1}, \bar{a}] \text{ for some } \bar{a} \in {}^{\ell g(\bar{z})}(M_{\mathbf{x}}).$$

5A) For $v \subseteq v_{\mathbf{x}}$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ let $\Gamma^1_{\mathbf{x},\Delta,i,v} = \{\varphi : \varphi \in \Gamma^1_{\mathbf{x},i}$ and $\varphi \in \Delta$ but $\bar{x}_{\bar{c}[\mathbf{x},<i]}$ is replaced by $\bar{x}_{\bar{c}\restriction(v \cap i)}\}$.
6) Let $M_{[\mathbf{x}]}$ be $(M_{\mathbf{x}})_{[B^+_{\mathbf{x}} + \bar{c}_{\mathbf{x}} + \bar{d}_{\mathbf{x}}]}$, see Definition 0.11.
7) We say $\mathbf{x}$ is normal when $\mathrm{Rang}(\bar{c}_{\mathbf{x}}) \subseteq \mathrm{Rang}(\bar{d}_{\mathbf{x}})$, pedantically $\cup\{\mathrm{Rang}(\bar{c}_{\mathbf{x},i}) : i \in v_{\mathbf{x}}\} \subseteq \cup\{\mathrm{Rang}(\bar{d}_{\mathbf{x},i}) : i \in w_{\mathbf{x}}\}$; note that usually there is no loss in assuming it.
8) Let $\Gamma^1_{\mathbf{x}}$ be $\{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \mathbb{L}(\tau_T)$ for some $\bar{y}$, finite if not said otherwise$\}$.
9) $\Gamma^0_{\mathbf{x}} = \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d},\eta}, \bar{y})$ and $\eta \in {}^n\ell g(\bar{d}_{\mathbf{x}})$ for some $n\}$, used in particular when $\mathbf{x}$ is normal, see 2.7(4) below.
10) For $v \subseteq v_{\mathbf{x}}, w \subseteq w_{\mathbf{x}}$ let $\mathbf{x}_{<v,w>} = (M_{\mathbf{x}}, \bar{B}_{\mathbf{x}}\restriction(v \backslash u_{\mathbf{x}}), \bar{c}_{\mathbf{x}}\restriction v, \bar{d}_{\mathbf{x}}\restriction w, \bar{\mathbf{I}}\restriction(v \cap u_{\mathbf{x}}))$, but if $w = \ell g(\bar{d}_{\mathbf{x}})$ we may omit it.
11) We say $\mathbf{x}$ is essentially well ordered <u>when</u> $\{i \in u_{\mathbf{x}} : \kappa_i = \kappa_j\}$ is well ordered (by $<_v$) for each $j \in u_{\mathbf{x}}$.

Notation 2.7. 0) We may write $\bar{d}, \bar{c}$ instead of $\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}$ when $\mathbf{x}$ is clear from the context, (usually in subscripts).
1) u, v, w are linear orders, members are i, j but, e.g. $w \cap j = w_{<j} := \{i \in w : i < j\}$.
2) If $v_1 \subseteq v_2$ let $[v_1, v_2) = \{i \in v_2 : j <_{v_2} i$ for every $j \in v_1\}$.
3) $\ell g(\bar{d}) = \mathrm{Dom}(\bar{d})$, etc.
4) $\bar{d}_\eta = \langle \bar{d}_{\eta(\ell)} : \ell < \ell g(\eta)\rangle$ if η is a function from $\ell g(\eta)$ to $\ell g(\bar{d})$.
5) $\bar{d}_{\mathbf{x},\eta} = \langle \bar{d}_{\mathbf{x},\eta(\ell)} : \ell < \ell g(\eta)\rangle$, see Definition 2.6(1).
6) $\bar{x}_{\bar{d},\eta} = \bar{x}_{\mathbf{x},\eta} = \langle \bar{x}_{d_{\eta(i)}} : i < \ell g(\eta)\rangle = \bar{x}_{\bar{d}_{\mathbf{x},\eta}}$ when $\bar{d} = \bar{d}_{\mathbf{x}}$.
7) $\bar{x}_{\bar{c},\eta}$ is defined similarly.
8) $\mathrm{tp}_\varphi(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus A) := \{\varphi(x_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}) : \bar{b} \in {}^{\ell g(\bar{y})}A$ and $\mathfrak{C} \models \varphi(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{b})\}$ when $\varphi = \varphi(\bar{x}_{\bar{d}[\mathbf{x}]}, \bar{x}_{\bar{c}[\mathbf{x}]}, \bar{y})$.
8A) We may above use $\pm\varphi, \Delta$ and/or $\varphi = \varphi(x_{\bar{d},\eta}, \bar{x}_{\bar{c},\eta}, \bar{y})$.
9) $v_2 = v_1 + 1$ is defined naturally.

Definition 2.8. Let $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$.
1) We say that $\bar{e}$ solves $(\mathbf{x}, \bar{\psi}, A)$ or $\bar{\psi}$-solves $(\mathbf{x}, A)$, or $\bar{\psi}$-solves $\mathbf{x}$ over A, (pedantically we should add θ) <u>when</u>:

- (a) $\bar{e} \in {}^\theta(M_{\mathbf{x}})$
- (b) $A \subseteq M_{\mathbf{x}}$
- (c) $\bar{\psi} = \langle \psi_\varphi = \psi_\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{[\theta]}) : \varphi \in \Gamma_{\bar{\psi}}\rangle$ where $\Gamma_{\bar{\psi}} \subseteq \Gamma^1_{\mathbf{x}}$, recalling 2.6(8)
- (d) $\psi_\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}) \vdash \{\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{a}) : \bar{a} \in {}^{\ell g(\bar{y}_\varphi)}A$ and $\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{a}]\}$ for $\varphi \in \Gamma_{\bar{\psi}}$
- (e) $\mathfrak{C} \models \psi_\varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}]$ for $\varphi \in \Gamma^1_{\mathbf{x}}$.

[12] No real loss if we replace $\bar{y}$ by $x_{\bar{c}_{\mathbf{x},i}}$. Also no real loss if we omit $\bar{z}$, absorbing $\bar{a}$ into $\bar{c}_i$ by cosmetic manipulations

1A) We say that $\psi(x_{\bar d}, \bar c_{\mathbf x}, \bar e)$ solves $(\mathbf x, A, \varphi)$ <u>when</u> $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y) \in \Gamma^1_{\mathbf x}$ and $\bar e \subseteq M_{\mathbf x}$ and $\psi(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e) \vdash \{\varphi(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar b) : b \in {}^{\ell g(\bar y_\varphi)}A$ and $\mathfrak C \models \varphi[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar b]\}$ and $\mathfrak C \models \psi_\varphi[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e]$.
1B) We say that $\psi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z)$ solves $(\mathbf x, A, \varphi)$ <u>when</u> $\psi(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e)$ solves $(\mathbf x, A, \varphi)$ for some $\bar e \in {}^{\ell g(\bar z)}M$.
1C) We let $\vartheta_{\mathbf x,\varphi}(\bar x_{\bar c}, \bar z, \bar y) = \vartheta_{\mathbf x,\varphi,\psi}(\bar x_{\bar c}, \bar z, \bar y)$ where $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y) \in \Gamma^1_{\mathbf x}$ and $\psi = \psi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z)$ be $(\forall \bar x_{\bar d})(\psi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z) \rightarrow \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y))$; we usually omit $\mathbf x$, being clear from the context and similarly ψ or ψ, φ.
1D) We say $\bar\psi = \langle \psi_\varphi(x_{\bar d}, \bar x_{\bar c}, \bar e) : \varphi \in \Gamma^1_{\bar\psi}\rangle$ solves $(\mathbf x, A)$ when $\Gamma^1_{\bar\psi} \subseteq \Gamma^1_{\mathbf x}$, so we usually write $\Gamma^1_{\bar\psi}$ instead of $\Gamma_{\bar\psi}$ to stress this (similarly in other cases), and for every $\varphi \in \Gamma^1_{\bar\psi}, \psi_\varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar e)$ solves $(\mathbf x, A, \varphi)$. We say $\bar\psi$ solves $(\mathbf x, A)$ when $\bar\psi = \langle \psi_\varphi : \varphi \in \Gamma^1_{\bar\psi}\rangle, \psi_\varphi = \psi_\varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[\theta]})$ and some $\bar e$ solves $(\mathbf x, \bar\psi, A)$.
2) We say $\bar\psi$ is full for $\mathbf x$ when $\Gamma^1_{\bar\psi} = \Gamma^1_{\mathbf x}$. Omitting $\bar\psi$ means for some $\bar\psi$ full for $\mathbf x$.
2A) Let "$\mathrm{tp}(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e) \vdash \mathrm{tp}(\bar d, \bar c_{\mathbf x} + M_{\mathbf x})$ according to $\bar\psi$" mean clause (d) of part (1) with $\Gamma^1_{\mathbf x}$ instead of $\Gamma_{\bar\psi}$.
3A) We say $\bar\psi$ illuminates $\mathbf x \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ <u>when</u> $\bar\psi$ is as in clause (c) of part (1) and for every $A \subseteq M_{\mathbf x}$ of cardinality $< \mu_2$ some $\bar e$ does solve $(\mathbf x, \bar\psi, A)$.
3B) We say $\psi(\bar x_{\bar d}, \bar x_{\bar c}, \bar x_{[\theta]})$ illuminates $(\mathbf x, \varphi)$ <u>when</u> $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_\varphi) \in \Gamma^1_{\mathbf x}$ and the above holds with (φ, ψ) standing for (φ, ψ_φ).
4) We say $\bar e$ solves $(\mathbf x, A)$ <u>when</u> for some $\bar\psi$ which is full for $\mathbf x, \bar e$ solves $(\mathbf x, \bar\psi, A)$.

Remark 2.9. 0) Note that we use "illuminate" rather than "solve" when we quantify on A.
1) For the case $\iota_{\mathbf x} = 2, \mu_1 = \aleph_1, \theta = \aleph_0$, i.e. for countable T, we can replace "$\mathrm{tp}(\bar c_{\mathbf x,i}, \bar c_{\mathbf x,<i} + M_{\mathbf x})$ is finitely satisfiable in some countable $B_{\mathbf x,i} \subseteq M_{\mathbf x}$" by: p is the average of an eventually indiscernible sequence $\mathbf I = \langle \bar a_n : n < \omega\rangle$ from $M_{\mathbf x}$ which means that for every finite Δ, some end-segment is Δ-indiscernible, see Definition 1.64. Also $\mathrm{Av}(\mathbf I, A)$ is well defined.
2) However, we cannot replace eventually indiscernible by indiscernible, e.g. for $= \mathrm{Th}(\mathbb R), \mathbb R$ the real field, there is an eventual indiscernible $\mathbf I = \langle a_n : n < \omega\rangle$ in $\mathbb R$ such that $a_n \geq n$; the cut it defines cannot be defined by a really indiscernible sequence, (well of length less than the saturation).
3) We can characterize when an eventually indiscernible sequence is equivalent, (see Definition 2.19(2)) to an indiscernible sequence, but this does not always occur, by the example above.
4) Being equivalent is well defined for eventually indiscernible sequences because their averages are well defined.
5) Usually no harm is done when below in 2.10(1)(b) we add "$w_{\mathbf x}$ is an initial segment of $w_{\mathbf y}$". Similarly in 2.10(1)(d).

Definition 2.10. 1) We define a two-place relation $\leq_1$ on pK : $\mathbf x \leq_1 \mathbf y$ <u>when</u>:

- (a) $M_{\mathbf x} = M_{\mathbf y}$
- (b) $\bar d_{\mathbf x} = \bar d_{\mathbf y} \restriction w_{\mathbf x}$ and $w_{\mathbf x} \subseteq w_{\mathbf y}$ as linear orders
- (c) $u_{\mathbf x} = u_{\mathbf y} \cap v_{\mathbf x}$ and $u_{\mathbf x} \subseteq u_{\mathbf y}$ as linear orders
- (d) $\bar c_{\mathbf x} = \bar c_{\mathbf y} \restriction v_{\mathbf x}$ and $v_{\mathbf x} \subseteq v_{\mathbf y}$ as linear orders.

2) We define $\leq_2$ similarly strengthening clause (b) to

$(b)^+$ $\bar{d}_{\mathbf{x}} = \bar{d}_{\mathbf{y}}$.

3) If $\mathbf{x} \le_1 \mathbf{y}$ and $\varphi = \varphi(\bar{x}_{\bar{d}[\mathbf{x}]}, \bar{x}_{\bar{c}[\mathbf{x}]}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$ then we may identify it with the $\varphi(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{y}) \in \Gamma^1_{\mathbf{y}}$ naturally.
4) For $\mathbf{x} \in \mathrm{pK}$ and $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$ let $\mathrm{supp}(\varphi)$ be the pair (w, v) such that $w \subseteq \ell g(\bar{d}_{\mathbf{x}}), v \subseteq \ell g(\bar{c}_{\mathbf{x}})$ are minimal (so finite) such that $\varphi \equiv \varphi(\bar{x}_{\bar{d}\restriction u}, \bar{x}_{\bar{c}\restriction v}, \bar{y})$, moreover the omitted variables are dummy (= does not appear in φ, not just "immaterial for satisfaction").
4A) Similarly for $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{\bar{d}}, \bar{x}'_{\bar{c}}, \bar{y})$, we define $\mathrm{supp}(\varphi) = (w_1, v_1, w_3, v_3)$; used in 3.3.
5) For $\Delta \subseteq \Gamma^1_{\mathbf{x}}$ let $\mathrm{supp}_{\mathbf{x}}(\Delta)$ be the pair $(\cup\{w : (w,v) = \mathrm{supp}(u)$ for some $\varphi \in \Delta\}, \cup\{v : (w,v) = \mathrm{supp}(\varphi)$ for some $\varphi \in \Delta\})$.

Definition 2.11. 0)

- (A) For $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ let $\Gamma^2_{\mathbf{x}} = \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, x'_{\bar{d}}, \bar{x}'_{\bar{c}}, \bar{y}) \in \mathbb{L}(\tau_T)$ so $\ell g(\bar{x}'_{\bar{c}}) = \ell g(\bar{x}_{\bar{c}}), \ell g(x'_{\bar{d}}) = \ell g(\bar{x}_{\bar{d}}), \bar{y}$ finite$\}$;used in 3.3(1).
- (B) For $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ let $\Gamma^3_{\mathbf{x}} := \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{[\theta]}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{z}$ finite$\}$, (used in (3A), close to $\Gamma^1_{\mathbf{x},i}$, see Definition 2.6(5),(5A)).

1) Let $\mathrm{qK}'_{\lambda,\kappa,\bar{\mu},\theta}[\Delta]$ be the class of $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ such that for no $\mathbf{y} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ do we have $\mathbf{x} \le_2 \mathbf{y}$ and $\mathbf{y}$ is Δ-active in some $i \in v_{\mathbf{x}} \backslash v_{\mathbf{y}}$ over $v_{\mathbf{x},<i}$, i.e. $v_{\mathbf{x}} \cap v_{\mathbf{y},<i}$, see[13] Definition 2.4, 2.6(3A),(5); if $\Delta = \Gamma^1_{\mathbf{x}}$ we may write $\mathrm{qK}'_{\lambda,\kappa,\bar{\mu},\theta}$; similarly below.
1A) We define $\mathrm{qK}''_{\lambda,\kappa,\bar{\mu},\theta}[\Delta]$ similarly but restricting ourselves to the case $v_{\mathbf{y}} = v_{\mathbf{x}}+1$.
2) Let $\mathrm{qK}_{\lambda,\kappa,\bar{\mu},\theta}$ be the class of $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ such that for every $A \in [M_{\mathbf{x}}]^{<\kappa}$ some $\bar{e}$ solves $\mathbf{x}$, see Definition 2.8(4).
3) Let $\mathrm{qK}^{\odot}_{\lambda,\kappa,\bar{\mu},\theta}$ be the class of triples $\mathbf{n} = (\mathbf{x}, \bar{\psi}, r)$ such that $\mathbf{x} \in \mathrm{qK}_{\lambda,\kappa,\bar{\mu},\theta}$ and $\bar{\psi} = \langle \psi_\varphi : \varphi \in \Gamma_{\bar{\psi}} = \Gamma^1_{\bar{\psi}} \subseteq \Gamma^1_{\mathbf{x}} \rangle$ illuminates $\mathbf{x}$ and $r = r(\bar{x}_{\bar{c}_{\mathbf{x}}}, \bar{x}_{\bar{d}_{\mathbf{x}}}, \bar{y}_{[\theta]})$ is a type over $M_{\mathbf{x}}$ such that: for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ there is a θ-tuple $\bar{e}$ from $M_{\mathbf{x}}$ such that:

$\bar{e}$ solves $(\mathbf{x}, \bar{\psi}, A)$ and the sequence $\bar{c}_{\mathbf{x}}{}^\frown \bar{d}_{\mathbf{x}}{}^\frown \bar{e}$ realizes r.

In this case we may say $\bar{e} \subseteq M_{\mathbf{x}}$ solves $(\mathbf{x}, A, \bar{\psi}, r)$ or solves $(\mathbf{n}, A)$. If not said otherwise, r is a type over $\emptyset$; in this case we may say $\mathbf{n}$ is pure.
3A) Let $\mathrm{qK}^{\oplus}$ be the class of $\mathbf{n} = (\mathbf{x}, \bar{\psi}, r) \in \mathrm{qK}^{\odot}$ such that $\Gamma^1_{\bar{\psi}} = \Gamma^1_{\mathbf{x}}$.
3B) Let $\mathrm{qK}^{\otimes}_{\kappa,\bar{\mu},\theta}$ be the class of triples $(\mathbf{x}, \bar{\psi}, r)$ such that[14]:

- (a) $\mathbf{x} \in \mathrm{qK}_{\lambda,\kappa,\bar{\mu},\theta}$
- (b) $r = r(\bar{x}_{\bar{c}}, \bar{x}_{\bar{d}}, \bar{y}_{[\theta]}), \bar{\psi} = \langle \psi_\varphi : \varphi \in \Gamma^3_{\mathbf{x}} \rangle$ satisfy
  - $\bullet_1$ $\psi_\varphi = \psi_\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{[\theta]})$
  - $\bullet_2$ for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ for some $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ realizing $r(\bar{c}, \bar{d}, \bar{y}_{[\theta]})$ we have: if $\varphi \in \Gamma^3_{\mathbf{x}}$ then $\mathfrak{C} \models \psi_\varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}]$ and $\psi_\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}) \vdash \{\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}, \bar{b}) : \mathfrak{C} \models$ "$\varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}, \bar{b}]$" and $\bar{b} \in {}^{\ell g(\bar{y})}A\}$.

4) We say $\bar{e}$ universally solves the triple $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{qK}^{\odot}_{\lambda,\kappa,\bar{\mu},\theta}$ when for every $A \in [M_{\mathbf{x}}]^{<\kappa}$ there is $\bar{e}'$ as in part (3) such that $\bar{e}, \bar{e}'$ realizes the same type over $\bar{d}_{\mathbf{x}} + \bar{c}_{\mathbf{x}} + A$, see 2.8(4) and Theorem 4.8.

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[13]In many places it suffices to use $[v_{\mathbf{y}}, v_{\mathbf{x}}]$. Note that qK' is like mxK in [She15]; as qK is related to a major corollary of being from mxK.

[14]What is the difference with part (3)? Here in the end, $\bar{e}$ appears in φ

4A) Similarly for $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{qK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$.
5) We define the partial orders $\le_1, \le_2$ on $\mathrm{qK}', \mathrm{qK}, \mathrm{qK}^{\odot}, \mathrm{qK}^{\oplus}$ naturally.

Remark 2.12. Concerning Definition 2.11(3),(4) note that $\mathrm{qK}^{\odot}$ is used in the end and in $(*)_5$ of the proof of 4.8 and in the proof of 4.12 only.

Observation 2.13. *Let $\ell = 0, 1, 2$.*
0) If the type $p(\bar{x})$ is finitely satisfiable in A then $p(\bar{x})$ does not locally split over A and this in turn implies that $p(\bar{x})$ does not split over A (hence the corresponding implications hold for the variants of Definition 2.2).
1) $\le_\ell$ is a partial order on pK.
2) If $\bar{\mathbf{x}} = \langle \mathbf{x}_\varepsilon : \varepsilon < \delta\rangle$ is $\le_\ell$-increasing sequence of members of $\mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ and $\delta < \theta^+$ is a limit ordinal then $\bar{\mathbf{x}}$ has a $\le_\ell$-lub, essentially the union, naturally defined and it belongs to $\mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$.
3) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and we define $\mathbf{y}$ like $\mathbf{x}$ replacing $\bar{d}_{\mathbf{x}}$ by $\bar{d}_{\mathbf{x}} {}^\frown \bar{c}_{\mathbf{x}}$, then $\mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ is normal and $\mathbf{x} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{y} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{x} \in \mathrm{qK}'_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{y} \in \mathrm{qK}'_{\kappa,\mu,\theta}$ and $\mathbf{x} \in \mathrm{qK}''_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{y} \in \mathrm{qK}''_{\kappa,\bar{\mu},\theta}$ and $\mathbf{x} \le_1 \mathbf{y}$ and no loss if systematically we use only normal $\mathbf{x}$ (except when we like e.g. $\ell g(\bar{d}_{\mathbf{x}})$ to be finite).
4) Parts (1),(2) apply[15] also to qK, qK$'$.
5) Assume $\kappa > \theta \ge |T|$ and $\bar{\mu}$ are as in 2.2. If M is κ-saturated, w a linear order of cardinality $< \theta^+$ and $\bar{d} \in {}^{w}({}^{\omega>}\mathfrak{C})$, then for one and only one $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ we have $M_{\mathbf{x}} = M, \bar{d}_{\mathbf{x}} = \bar{d}, v_{\mathbf{x}} = \emptyset$ hence $\bar{c}_{\mathbf{x}} = \langle\rangle, B_{\mathbf{x}} = \emptyset$.
6) Let $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$.

- (a) *If $\mathrm{cf}(\mu_0) > \theta$ then $B_{\mathbf{x}} = \cup\{B_{\mathbf{x},i} : i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}\} \subseteq M_{\mathbf{x}}$ has cardinality $< \mu_0$.*
- (b) *If $\mathrm{cf}(\mu_2) > \theta$ then $\cup\{\mathbf{I}_{\mathbf{x},i} : i \in u_{\mathbf{x}}\}$ has cardinality $< \mu_2$.*
- (c) *If $\mathrm{cf}(\mu_2) > \theta$, (hence $\mu_0 = \mu_2 \Rightarrow \mathrm{cf}(\mu_0) > \theta$) then also $|B^+_{\mathbf{x}}| < \mu_2$.*
- (d) *Always $|B_{\mathbf{x}}| \le \mu_0, |B^+_{\mathbf{x}}| \le \mu_2$.*

Proof. Easy, concerning part (2) for $\ell = 1, 2$ note that the union, it is not uniquely defined as if $i \in v_{\mathbf{x}_\varepsilon} \backslash u_{\mathbf{x}_\varepsilon}, \varepsilon < \delta$ then $\langle B_{\mathbf{x}_\zeta,i} : \zeta \in [\varepsilon, \delta)\rangle$ is not necessarily constant, but we can use any one of them. Similarly for $i \in u_{\mathbf{x}_\varepsilon}$. $\square_{2.13}$

Claim 2.14. *1) If $\theta \ge |T|$ then in $\mathrm{pK}_{\kappa,\bar{\mu},\theta}$ there is no $\le_2$-increasing sequence $\langle \mathbf{x}_\varepsilon : \varepsilon < \theta^+\rangle$ such that: if $\varepsilon < \theta^+$ then $\mathbf{x}_{\varepsilon+1}$ is active in some $i \in v(\mathbf{x}_{\varepsilon+1}) \backslash v(\mathbf{x}_\varepsilon)$.*
2) For finite[16] $\Delta \subseteq \Gamma^1_{\mathbf{x}}$, there is $n_\Delta = n_{\Delta,T} < \omega$ such that there is no $\le_2$-increasing chain $\langle \mathbf{x}_\ell : \ell \le n_\Delta\rangle$ of members of $\mathrm{pK}_{\kappa,\bar{\mu},\theta}$ such that $x_{\ell+1}$ is Δ-active in some $i \in [v(\mathbf{x}_\ell), v(\mathbf{x}_{\ell+1}))$.
3) In part (1), the sequence may be just $\le_1$-increasing if $\{\varepsilon < \theta^+ : \bar{d}_{\mathbf{x}_\varepsilon} = \bar{d}_{\mathbf{x}_{\varepsilon+1}} = \cup\{\bar{d}_{\mathbf{x}_\zeta} : \zeta < \varepsilon\}$ is a stationary subset of θ^+.

Proof. A similar proof appears in Case 1 of the proof of 8.4 or see [She15, 2.8=tp25.33] recalling Definition [She15, 2.6=tp25.32]. $\square_{2.14}$

Claim 2.15. *1) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ then there is $\mathbf{y} \in \mathrm{qK}'_{\kappa,\bar{\mu},\theta}$ such that $\mathbf{x} \le_2 \mathbf{y}$.*

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[15]However, while for $\mathrm{qK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{qK}^{\otimes}_{\kappa,\bar{\mu},\theta}, \mathrm{qK}^{\odot}_{\kappa,\bar{\mu},\theta}$ parts (1),(3) are O.K. but part (2) is a different, harder matter; for $\mathrm{qK}''_{\kappa,\bar{\mu},\theta}$ all are not clear.

[16]if we restrict ourselves to $\bar{d}_{\mathbf{x}} \restriction u$ for some finite $u \subseteq \ell g(\bar{d}_{\mathbf{x}})$ then any finite $\Delta \subseteq \mathbb{L}(\tau_T)$ is O.K.

2) If the finite Δ is as in 2.14(2) and $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ <u>then</u> there is $\mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ such that $\mathbf{x} \leq_2 \mathbf{y}$ and $v_{\mathbf{y}} \backslash v_{\mathbf{x}}$ is finite and there is no $\mathbf{z} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ such that $\mathbf{y} \leq_2 \mathbf{z}$ and some $i \in [v_{\mathbf{y}}, v_{\mathbf{z}})$ or just $i \in v_{\mathbf{z}} \backslash v_{\mathbf{x}}$ is Δ-active in $\mathbf{z}$.
2A) If above we restrict $\mathbf{z}$ to the case $v_{\mathbf{z}} = v_{\mathbf{y}}+1$, <u>then</u> we can demand $v_{\mathbf{y}} \subseteq v_{\mathbf{x}}+n_\Delta$ when n_Δ is[17] from 2.14(2).
2B) In part (2), if we restrict the assumption to the case $v_{\mathbf{y}} < v_{\mathbf{x}}+\omega$, i.e. $v_{\mathbf{y}} = v_{\mathbf{x}}+n$ for some n <u>then</u> this is O.K. provided that we restrict the conclusion to the case $v_{\mathbf{y}} \trianglelefteq v_{\mathbf{z}}$ (actually just $v_{\mathbf{y}} \subseteq v_{\mathbf{z}} \wedge v_{\mathbf{x}} \trianglelefteq v_{\mathbf{z}}$).
3) If $\mathbf{x} \in \mathrm{qK}'_{\kappa,\kappa,\theta}$ or just $\mathbf{x} \in \mathrm{qK}'_{\kappa,\bar{\mu},\theta}$ and $\mu_0 = \kappa$ <u>then</u> [18] $\mathbf{x} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$, that is, for every $A \in [M_{\mathbf{x}}]^{<\kappa}$ some $\bar{\psi}$ solves $(\mathbf{x}, A)$, see Definition 2.8(1D).
4) [Local version[19]]; if $\varphi \in \Gamma^1_{\mathbf{x}}$ and $\mathbf{x} \in \mathrm{qK}'_{\kappa,\bar{\mu},\theta}$ or just is as $\mathbf{y}$ in 2.15(2) or just 2.15(2A), <u>then</u> for every $A \in [M_{\mathbf{x}}]^{<\mu_0}$ there is $\psi = \psi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{z}$ finite and $\bar{e} \in {}^{\ell g(\bar{z})}M$ such that $\psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e})$ solves $(\mathbf{x}, A, \varphi)$.
5) Assume $\iota_{\mathbf{x}} = 2$ and $\mu_2 \leq \kappa$. The inverse of part (3) holds, i.e. if $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and $\bar{\mathbf{x}} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$, i.e. for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ there is a solution <u>then</u> $\mathbf{x} \in \mathrm{qK}'_{\kappa,\bar{\mu},\theta}$ (and see 3.7(2)).
6) Assume $\mu_0 = \kappa$. Assume $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ but $\mathbf{x} \notin \mathrm{qK}'_{\kappa,\bar{\mu},\theta}$ <u>then</u> there is a pair $(\mathbf{y}, \varphi)$ such that:

- (a) $\mathbf{x} \leq_1 \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$
- (b) $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}, \bar{z}) \in \Gamma^1_{\mathbf{x}}$
- (c) $\mathbf{y}$ *is* $\{\varphi\}$*-active in some* $i \in [v_{\mathbf{x}}, v_{\mathbf{y}})$, *so* $\bar{c}_i = \bar{c}_{i,0}{}^\frown\bar{c}_{i,1}, \bar{c} \subseteq \mathrm{Rang}(\bar{c}_{\mathbf{y},<i}), \ell g(\bar{c}_{i,0}) = \ell g(\bar{y}) = \ell g(\bar{c}_{i,1})$ *and* $\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{c}_{i,\ell}, \bar{c}]$ *iff* $\ell = 1$, *etc., see Definition 2.4.*

Remark 2.16. Note that in part (6), if $\bigwedge\limits_{\ell} \mu_\ell = \mu$ for transparency then we allow $\mu > \theta + |T| \geq \mathrm{cf}(\mu)$ and $|B_{\mathbf{x}}| = \mu$; also note that $B^+_{\mathbf{x}} = B_{\mathbf{x}}$.

Proof. By [She15, 2.14=tp25.36,2.15=tp25.38] this should be clear, still:
1) By 2.14(1).
5) Toward a contradiction assume that $\mathbf{y}, i \in v_{\mathbf{y}} \backslash v_{\mathbf{x}}, \varphi, \bar{b}_{i,0}, \bar{b}_{i,1}$ exemplify $\mathbf{x} \notin \mathrm{qK}'_{\kappa,\bar{\mu},\theta}$ so $\bar{b}_{i,0}, \bar{b}_{i,1}$ are as in Definition 2.4 in particular there are $\bar{b}^*$ from $M_{\mathbf{x}}$ and φ such that $\mathfrak{C} \models$ "$\varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x},<i}, \bar{b}_{i,0}, \bar{b}^*] \wedge \neg\varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x},<i}, \bar{b}_{i,1}, \bar{b}^*]$" and $\bar{c}_{\mathbf{y},i} = \bar{b}_{i,0}{}^\frown\bar{b}_{i,1}$ and $\bar{b}_{i,0}, \bar{b}_{i,1}$ realize the same type over $\bar{c}_{\mathbf{x},<i} + M_{\mathbf{x}}$ which is finitely satisfiable in $B_{\mathbf{y},i}$. Let A be $B_{\mathbf{y},i} + \bar{b}^*$ if $i \notin u_{\mathbf{y}}$ and be $\cup \mathbf{I}_{\mathbf{y},i} + \bar{b}^*$ if $i \in u_{\mathbf{x}}$; so $A \subseteq M_{\mathbf{x}}$ has cardinality $< \kappa$ hence for some $\psi = \psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{z}_\psi)$ and $\bar{e} \in {}^{\ell g(\bar{z}_\psi)}(M_{\mathbf{x}})$ we have

- $(*)$ $\psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}) \in \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ satisfies $\psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}) \vdash \mathrm{tp}_{\pm\varphi}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus A)$.

Hence

- $(*)'$ (a) $\mathfrak{C} \models \psi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}]$
  - (b) for every $\bar{b} \subseteq {}^{\ell g(\bar{b}_{i,\ell})}A$ for some truth value $\mathbf{t}$ we have $\mathfrak{C} \models (\forall \bar{x}_{\bar{d}})[\psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, e) \rightarrow \varphi(\bar{x}_{\bar{d}}, \bar{b})^{\mathbf{t}}]$.

Now for $\ell = 0, 1$ we know that $\mathrm{tp}(\bar{b}_{i,\ell}, M_{\mathbf{x}} + \bar{c}_{\mathbf{x},<i})$ is finitely satisfiable in A and does not depend on ℓ, easy contradiction. $\square_{2.15}$

[17] and see $\mathrm{ind}(\Delta)$ in §3

[18] the "$\mu_0 = \kappa$" is of course undesirable, but eliminating it is the reason of much of the work here.

[19] we may use 2.15(3),(4) replacing $A \in [M]^{<\mu_0}$ by $\in [M]^{<\kappa}$ as the definition of qK.

Observation 2.17. *1) Assume $\mu_0 = \kappa$ and $\mathrm{cf}(\kappa) > \theta + |T|$. If $\mathbf{x} \in \mathrm{qK}'_{\kappa,\bar\mu,\theta}$ or just $\mathbf{x} \in \mathrm{qK}_{\kappa,\bar\mu,\bar\theta}$ then for some full $\bar\psi$ we have $\mathbf{n} := (\mathbf{x}, \bar\psi, \emptyset) \in \mathrm{qK}^{\oplus}_{\kappa,\bar\mu,\theta}$, see 2.11(3A). Moreover there is $\mathbf{n} = (\mathbf{x}, \bar\psi', \emptyset) \in \mathrm{qK}^{\otimes}_{\kappa,\bar\mu,\theta}$.*
2) If $(\mathbf{x}, \bar\psi, r) \in \mathrm{qK}^{\oplus}_{\mu,\mu,\theta}$ and the model M is κ-saturated and $\kappa = \mu^+, \mu > \mathrm{cf}(\mu)$ then for some $\bar\psi'$ we have $(\mathbf{x}, \bar\psi', r) \in \mathrm{qK}^{\oplus}_{\kappa,\mu,\theta}$.

Proof. 1) First, for each $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y) \in \Gamma^1_{\mathbf{x}}$ there is $\psi_\varphi = \psi_\varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z_\varphi)$ illuminating $(\mathbf{x}, \varphi)$. Why? for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ by 2.15(4) there is $\psi(\bar x_{\bar d}, \bar c_{\mathbf{x}}, \bar e)$ solving $(\mathbf{x}, \varphi, A)$. The set Λ of candidates $\psi = \psi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z_\varphi)$ has cardinality $\le \theta + |T|$ and if $\psi \in \Lambda$ fails then we choose a set $A_{\varphi,\psi}$ exemplifying it. As $\mathrm{cf}(\kappa) > \theta + |T|$ the set $A_\varphi = \cup\{A_{\varphi,\psi} : \psi \in \Lambda_\varphi\}$ has cardinality $< \kappa$, so there are ψ and $\bar e$ such that $\psi(\bar x_{\bar d}, \bar c_{\mathbf{x}}, \bar e)$ solves $(\mathbf{x}, \varphi, A_\varphi)$, hence it contradicts the choice of $A_{\varphi,\psi}$. So ψ_φ exists.

Renaming the $\bar z_\varphi$'s we have $\langle \psi_\varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z_{[\theta]}) : \varphi \in \Gamma^1_{\mathbf{x}}\rangle$ as required for $\mathbf{n} := (\mathbf{x}, \bar\psi, \emptyset) \in \mathrm{qK}^{\oplus}_{\kappa,\bar\mu,\theta}$.

Second, to get the "moreover", let $\bar\varphi = \langle \varphi_i(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[\theta]}, \bar z_i) : i < \theta\rangle$ list the formulas of this form. For $i < \theta$ let $u_i \subseteq \theta$ be finite such that $\varphi_i = \varphi_i(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[u_i]}, \bar z_i)$ and without loss of generality we choose the sequence $\bar\varphi$ such that $u_i \subseteq i$. Let $\psi_i = \psi_i(\bar x_{\bar d}, \bar x_{\bar c}, \bar y^*_i)$ be as above for $\varphi = \varphi_i(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[u_i]}{}^\frown \bar z_i)$, so $\ell g(\bar y^*_i) < \omega$, let $\alpha_i = \Sigma\{\ell g(\bar y^*_j) : j < i\}$ and let $\bar y_i = \langle y_{\alpha_i + \ell} : \ell < \ell g(\bar y^*_i)\rangle$ and now let $\bar\psi^* = \langle \psi^*_{\varphi_i} : \psi_i(x_{\bar d}, \bar x_{\bar c}, \bar y^*_i) : i < \theta\rangle$.

Now given $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ we choose $\bar e_i = \langle e_{\alpha_i+\ell} : \ell < \ell g(\bar y^*_i)\rangle$ by induction on $i < \theta$ such that $\psi_i(\bar x_{\bar d}, \bar x_{\bar c}, \bar e_i)$ solves $(\mathbf{x}, A \cup \{e_\alpha : \alpha < \alpha_i\}, \varphi_i(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[u_i]}{}^\frown \bar z_i))$. So $\bar e \in {}^\theta(M_{\mathbf{x}})$ is well defined and satisfies the requirements.
2) Let $\bar\psi = \langle \psi_\varphi : \varphi \in \Gamma^1_{\mathbf{x}}\rangle$. For $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y) \in \Gamma^1_{\mathbf{x}}$ let $\vartheta_{0,\varphi} = \varphi, \vartheta_{1,\varphi} = \psi_{\vartheta_{0,\varphi}}$ and let $\vartheta_{2,\varphi} = \psi_{\vartheta_{1,\varphi}}$. Lastly, $\bar\psi := \langle \vartheta_{2,\varphi} : \varphi \in \Gamma^1_{\mathbf{x}}\rangle$ satisfies $(\mathbf{x}, \bar\psi', r) \in \mathrm{qK}^{\oplus}_{\kappa,\mu,\theta}$; compare with the proof of 2.27. $\square_{2.17}$

Note that we shall use 2.17 in 4.12.

§ 2(B). Smoothness, similarity and $(\bar\mu, \theta)$-sets.

We like to show that in some sense there are few decompositions, so toward this we define smooth ones, show that for a saturated model, the smooth decompositions are few up to being conjugate and every $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is equivalent to a smooth one modulo the relevant equivalence relation; this certainly helps.

Definition 2.18. 1) The decomposition $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is called smooth when: if $\kappa \in \mathfrak{a}_{\mathbf{x}}$, see end of 2.6(1), then $\mathbf{I}^+_{\mathbf{x},\kappa}$ is an indiscernible sequence over $\cup\{\mathbf{I}^+_{\mathbf{x},\kappa_1} : \kappa_1 \in \mathfrak{a}_{\mathbf{x}} \backslash \{\kappa\}\} \cup B_{\mathbf{x}} = \cup\{\mathbf{I}_{\mathbf{x},i,\alpha} : i \in u_{\mathbf{x}}$ and $\alpha < \kappa_{\mathbf{x},i}$ but $\kappa_{\mathbf{x},i} \ne \kappa\} \cup B_{\mathbf{x}}$ in the sense of Definitions 1.33(1), 1.39 and 2.17(2) where
1A) We define

$$\mathbf{I}^+_{\mathbf{x},\kappa} = \langle \bar a_{\mathbf{x},\kappa,\alpha} : \alpha \in I_{\kappa,u(\mathbf{x},\kappa)} = I_{\mathbf{x},\kappa}\rangle$$

for $\kappa \in \mathfrak{a}_{\mathbf{x}} := \{\kappa_{\mathbf{x},i} : i \in u_{\mathbf{x}}\}$ where:

(a) $u_{\mathbf{x},\kappa} = u(\mathbf{x}, \kappa) := \{i \in u_{\mathbf{x}} : \kappa_i = \kappa\}$
(b) $I_{\kappa,u(\mathbf{x},\kappa)} = (\kappa \times u_{\mathbf{x},\kappa}, <, P_\varepsilon)_{\varepsilon \in u(\mathbf{x},\kappa)}$, where

(α) $<$ ordered $\kappa \times u_{\mathbf{x},\kappa}$ lexicographically (so if $u_{\mathbf{x},\kappa}$ is well ordered we can use κ)

(β) $\langle P_\varepsilon : \varepsilon < \mathrm{otp}(u_{\mathbf{x},\kappa})\rangle$ is a partition to unbounded subsets, in fact, $P_\varepsilon = \kappa \times \{\varepsilon\}$

(c) $\bar{a}_{\mathbf{x},\kappa,\beta}$ is $\bar{a}_{\mathbf{x},i,\alpha}$ when $\beta = (\alpha, \varepsilon)$.

2) For $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}$ and $h \in \Pi\mathfrak{a}_{\mathbf{x}}$ let $\mathbf{x}_{[h]}$ be defined like $\mathbf{x}$ but $\bar{\mathbf{I}}_{\mathbf{x}}$ is replaced by $\bar{\mathbf{I}}_{\mathbf{x},h} = \langle \mathbf{I}_{\mathbf{x},i,h(\kappa(\mathbf{x},i))} : i \in u_{\mathbf{x}}\rangle$ where $\mathbf{I}_{\mathbf{x},i,\alpha} = \langle \bar{a}_{\mathbf{x},i,\beta} : \beta \in [\alpha, \kappa_{\mathbf{x},i})\rangle$.
3) We say $\bar{b}_1, \bar{b}_2$ are $\mathbf{x}$-similar when for every $n, (\forall i_0 \in u_{\mathbf{x}})(\forall^{\kappa_{i(0)}} \alpha_0 < \kappa_i)(\forall i_1 \in u_{\mathbf{x}}) \ldots (\forall i_{n-1} \in u_{\mathbf{x}})(\forall^{\kappa_{i(n-1)}} \alpha_{n-1} < \kappa_{i(n-1)})[\mathrm{tp}(\bar{b}_1, \cup\{\bar{a}_{\kappa_{i(\ell)},\alpha_\ell} : \ell < n\} \cup B_{\mathbf{x}}) = \mathrm{tp}(\bar{b}_2, \cup\{\bar{a}_{\kappa_{i(\ell)},\alpha_\ell} : \ell < n\} \cup B_{\mathbf{x}})]$, where we stipulate $i_\ell = i(\ell)$.

Definition 2.19. 1) We say the decompositions $\mathbf{x}, \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ are very similar when:

(a) $M_{\mathbf{x}} = M_{\mathbf{y}}, w_{\mathbf{x}} = w_{\mathbf{y}}, \bar{d}_{\mathbf{x}} = \bar{d}_{\mathbf{y}}, v_{\mathbf{x}} = v_{\mathbf{y}}, u_{\mathbf{x}} = u_{\mathbf{y}}, \bar{c}_{\mathbf{x}} = \bar{c}_{\mathbf{y}}$ (so $\bar{c}_{\mathbf{x},i} = \bar{c}_{\mathbf{y},i}$ for every i) and[20] $B_{\mathbf{x},i} = B_{\mathbf{y},i}$ for $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$

(b) for $i \in u_{\mathbf{x}}$, the indiscernible sequences $\mathbf{I}_{\mathbf{x},i}, \mathbf{I}_{\mathbf{y},i}$ are equivalent, (i.e. have the same average over $M_{\mathbf{x}}$, equivalently over $\mathfrak{C}$) and[21] $\kappa_{\mathbf{x},i} = \kappa_{\mathbf{y},i}$.

2) We say $\mathbf{x}, \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ are similar when $v_{\mathbf{x}} = v_{\mathbf{y}}, u_{\mathbf{x}} = u_{\mathbf{y}}$ and there is an elementary mapping g of $\mathfrak{C}$ witnessing it which means:

(a) $g(B_{\mathbf{x}}) = g(B_{\mathbf{y}}), g(\bar{c}_{\mathbf{x}}) = \bar{c}_{\mathbf{y}}, g(\bar{d}_{\mathbf{x}}) = \bar{d}_{\mathbf{y}}$ and $g(B_{\mathbf{x},i}) = B_{\mathbf{y},i}$ for $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$

(b) for $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}, g(\bar{c}_{\mathbf{x},i}) = \bar{c}_{\mathbf{y},i}$ and the scheme $\mathrm{schm}_{\mathbf{x},i}$ defining $\mathrm{tp}(\bar{c}_{\mathbf{x},i}, M_{\mathbf{x}})$ (equivalently $\mathrm{tp}(\bar{c}_{\mathbf{x},i}, \bar{c}_{\mathbf{x},<i} + M_{\mathbf{x}})$) is mapped to the scheme defining $\mathrm{tp}(\bar{c}_{\mathbf{y},i}, M_{\mathbf{y}})$; so if $\iota_{\mathbf{x}} = 2$ this means $g(D_{\mathbf{x},i}) = D_{\mathbf{y},i}$, i.e. $g(D'_{\mathbf{x},i}) = D'_{\mathbf{y},i}$, which means $g(D_{\mathbf{x}} \cap \mathrm{Def}_{\ell g(\bar{c}_{\mathbf{x},i})}(B_{\mathbf{x},i})) = D_{\mathbf{y},i} \cap \mathrm{Def}_{\mathbf{y},i} \cap \mathrm{Def}_{\ell g(\bar{c}_{\mathbf{x},i})}(B_{\mathbf{y},i})$

(c) $g(\mathbf{I}_{\mathbf{x},i}), \mathbf{I}_{\mathbf{y},i}$ are equivalent indiscernible sequences and $\kappa_{\mathbf{x},i} = \kappa_{\mathbf{y},i}$ for $i \in u_{\mathbf{x}}$.

3) Above we say weakly similar when (so possible $\mathfrak{a}_{\mathbf{x}} \neq \mathfrak{a}_{\mathbf{y}}$) as in part (2) but for each $i \in u_{\mathbf{x}}$ we replace the "are equivalent" in clause (c), by the indiscernible sequences $h(\mathbf{I}_{\mathbf{x},i}), \mathbf{I}_{\mathbf{y},i}$ being neighbors, (see here 1.36(6)) and $\kappa_{\mathbf{x},i} = \kappa_{\mathbf{x},j} \Leftrightarrow \kappa_{\mathbf{y},i} = \kappa_{\mathbf{y},j}$.
4) For $\mathbf{x}, \mathbf{y} \in \mathrm{pK}$ we say they are essentially similar when there are smooth $\bar{\mathbf{x}}', \mathbf{y}' \in \mathrm{pK}_{\kappa,\mu,\theta}$ which are very similar to $\mathbf{x}, \mathbf{y}$ respectively and are similar (by the definition in part (2); note that e.g. $B_{\mathbf{x},i}, B_{\mathbf{y},i}$ for $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$ may be different).

Remark 2.20. We may add: if $\mathbf{x}, \mathbf{y}$ are smooth we say they are smoothly immediately weakly similar when in part (2) we replace clause (c) by

(c)′ there is a one-to-one function h from $\mathfrak{a}_{\mathbf{x}}$ onto $\mathfrak{a}_{\mathbf{y}}$ such that for every $\kappa, \kappa_{\mathbf{x},i} = \kappa \Rightarrow \kappa_{\mathbf{y},i} = h(\kappa)$ and for some one-to-one order preserving function from some infinite $u \subseteq \kappa$ into $h(\kappa)$, we have $\alpha \in u \wedge \kappa_{\mathbf{x},i} = \kappa \Rightarrow \bar{a}_{\mathbf{x},\kappa_{\mathbf{x},i},\alpha} = \bar{a}_{\mathbf{y},\kappa_{\mathbf{y},i},\alpha}$.

Note that being smoothly immediately weakly similar implies being weakly similar.

[20] We may consider weakening it.
[21] usually this follows, but not for stable indiscernible sets

Claim 2.21. *Let $\kappa, \bar\mu$ and $\theta \ge |T|$ be as in 2.2 and we let μ'_0 be μ_0 if $\mathrm{cf}(\mu_0) > \theta$ and $\mu'_0 = \mu_0^+$ otherwise, similarly for μ'_2.*
1) Being similar, very similar, essentially similar and also weakly similar are equivalence relations.
1A) Being very similar implies being similar which implies being essentially similar which simplies being weakly similar.
2) For κ-saturated $M \prec \mathfrak{C}$, the number of $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ up to weak similarity is $\le 2^{<\mu'_0}$.
3) For κ-saturated $M \prec \mathfrak{C}$ if $\mu_2 = \mu_1^{+\alpha}$ and $\mu > |T| + \theta$, then the number of $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ up to similarity is $\le 2^{<\mu'_0} + |\alpha|^\theta$.
4) For $\mathbf{x}, \mathbf{y} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ we have: $\mathbf{x}, \mathbf{y}$ are very similar iff $\mathbf{x}, \mathbf{y}$ are $\le_1$-equivalent, i.e. $\mathbf{x} \le_1 \mathbf{y} \le_1 \mathbf{x}$.
5) If $\mathbf{x}, \mathbf{y}$ are very similar and $\bar b_1, \bar b_2 \in {}^\zeta\mathfrak{C}$ for some $\zeta < \mu_1$, then $\bar b_1, \bar b_2$ are $\mathbf{x}$-similar iff $\bar b_1, \bar b_2$ are $\mathbf{y}$-similar; see Definition 2.18(3).

Proof. Easy (for essentially similar use 2.22(1) below). $\square_{2.21}$

Claim 2.22. *1) For every $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ there is a smooth $\mathbf{y} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ very similar to $\mathbf{x}$.*
2) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}$ and $h \in \prod \mathfrak{a}_{\mathbf{x}}$ then $\mathbf{x}_{[h]} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is very similar to $\mathbf{x}$; see 2.18(2).
3) In part (2), if $\mathbf{x}$ is smooth then so is $\mathbf{x}_{[h]}$.
4) If $\mathbf{x} \in \mathrm{qK}_{\kappa,\bar\mu,\theta}$ and $\mathbf{y} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is very similar to $\mathbf{x}$ then $\mathbf{y} \in \mathrm{qK}_{\kappa,\bar\mu,\theta}$.
4A) Similarly for $\mathrm{qK}'_{\kappa,\bar\mu,\theta}$ and $\mathrm{qK}''_{\kappa,\bar\mu,\theta}$.
5) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is smooth then for every $\bar a \in {}^{\mu_0>}(M_{\mathbf{x}})$ for some $h \in \Pi\mathfrak{a}_{\mathbf{x}}$ also $((M_{\mathbf{x}})_{[\bar a]}, \bar B_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar d_{\mathbf{x}}, \bar{\mathbf{I}}_{\mathbf{x},h}) \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is smooth, pedantically replacing $\mathfrak{C}$ by $\mathfrak{C}_{[\bar a]}$; also if $\bar a = (\ldots {}^\frown \bar a_i {}^\frown \ldots)_{i \in v(\mathbf{x}) \backslash u(\mathbf{x})}$ where $\bar a_i \in {}^{\omega>}(M_{\mathbf{x}})$ or just $\bar a_i \in {}^{\mu_0>}(M_{\mathbf{x}})$ for every $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$ then for some $h \in \Pi\mathfrak{a}_{\mathbf{x}}$ the tuple $(M_{\mathbf{x}}, \langle B_{\mathbf{x},i} + \bar a_i : i \in u_{\mathbf{x}}\rangle, \bar c_{\mathbf{x}}, \bar d_{\mathbf{x}}, \bar{\mathbf{I}}_{\mathbf{x},h}) \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is smooth, see 2.18(2).
6) $\mathbf{S}^{<\theta^+}(B^+_{\mathbf{x}})$ has cardinality $\le |B^+_{\mathbf{x}}|^\theta$ when $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ is smooth.

Proof. E.g. for parts (5),(6) use 1.34(1) or see below, in particular 2.25(1). $\square_{2.22}$

We may formalize how "small" is $B^+_{\mathbf{x}}$ for smooth $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$.

Definition 2.23. We say that $\mathbf{f} = (\bar B, \bar{\mathbf{I}})$ is a $(\bar\mu, \theta)$-set or a $(\bar\mu, \theta)$-smooth set when $\bar\mu = (\mu_2, \mu_1, \mu_0)$ and for some u, v we have:

(a) v is a linear order of cardinality $< \theta^+$ and $u \subseteq v$
(b) $\bar B = \langle B_i : i \in v \backslash u\rangle$, we let $B = \cup\{B_i : i \in v\backslash u\}$ and each B_i is of cardinality $< \mu_0$; but $\mathbf{f} = (B, \bar{\mathbf{I}})$ means $i \in v\backslash u \Rightarrow B_i = B$ so in this case $|B| < \mu_0$
(c) $\bar{\mathbf{I}} = \langle \mathbf{I}_i : i \in u\rangle$
(d) $\mathbf{I}_i = \langle \bar a_{i,\alpha} : \alpha < \kappa_i\rangle$ is an indiscernible sequence of finite tuples, $\kappa_i \in \mathrm{Reg} \cap \mu_2 \backslash \mu_1$
(e) $(B, \bar{\mathbf{I}})$ satisfies the smoothness demand, clause (e) in Definition 2.18 and $\mathfrak{a}, I_\kappa, \mathbf{I}^+_\kappa, \bar a_{\kappa,\alpha}$ (for $\kappa \in \mathfrak{a}, \alpha \in I^+_\kappa$) are defined as there.

Definition 2.24. 1) For $\mathbf{f}$ as in 2.23 we let[22]: $\mu_{\mathbf{f},\ell} = \mu_\ell$ for $\ell = 0,1,2, v_{\mathbf{f}} = v, u_{\mathbf{f}} = u, B_{\mathbf{f},i} = B_i, B_{\mathbf{f}} = \cup\{B_{\mathbf{f},i} : i \in v_{\mathbf{f}} \backslash u_{\mathbf{f}}\}, \bar{\mathbf{I}}_{\mathbf{f}} = \bar{\mathbf{I}}$ so $\mathbf{I}_{\mathbf{f},i} = \mathbf{I}_i, \mathbf{I}^+_{\mathbf{f},\kappa} = \mathbf{I}^+_\kappa, \bar{a}_{\mathbf{f},i,\alpha} = \bar{a}_{i,\alpha}, \mathfrak{a}_{\mathbf{f}} = \mathfrak{a} = \{\kappa_i : i \in u\}$, etc.; for $u \subseteq u_{\mathbf{f}}$ let $\mathfrak{a}_{\mathbf{f},u} = \{\kappa_i : i \in u\}$.
1A) If $\mathbf{x} \in \mathrm{pK}$ is smooth <u>then</u> $\mathbf{f} = \mathbf{f}_{\mathbf{x}}$ is defined by $v_{\mathbf{f}} = v_{\mathbf{x}}, u_{\mathbf{f}} = u_{\mathbf{x}}, B_{\mathbf{f},i} = B_{\mathbf{x},i}, \mathbf{I}_{\mathbf{f},j} = \mathbf{I}_{\mathbf{x},j}$ hence $\bar{a}_{\mathbf{f},i,\alpha} = \bar{a}_{\mathbf{x},i,\alpha}, B_{\mathbf{f},j} = B_{\mathbf{x},j}$ for $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}, \alpha < \kappa_{\mathbf{x},j}$ and $j \in u_{\mathbf{x}}$.
2) For $u \subseteq u_{\mathbf{f}}$ let $B^+_{\mathbf{f},u} = \cup\{\bar{a}_{\mathbf{f},i,\alpha} : \alpha < \kappa_i$ and $i \in u\} \cup B_{\mathbf{f}}$, if we omit u we mean $u = u_{\mathbf{f}}$.
2A) For $v \subseteq v_{\mathbf{f}}$ let $B^{\pm}_{\mathbf{f},v} = \cup\{B_{\mathbf{f},i} : i \in v \backslash u_{\mathbf{f}}\} \cup \bigcup\{\bar{a}_{\mathbf{f},i,\alpha} : \alpha < \kappa_{\mathbf{f},i}$ and $i \in v \cap u_{\mathbf{f}}\}$.
3) We say $\mathbf{f}$ is an infinitary $(\bar{\mu},\theta)$-set <u>when</u> $\ell g(\bar{a}_{\mathbf{f},i,\alpha})$ is just $< \theta^+$ for every $i \in u_{\mathbf{f}}, \alpha < \kappa_{\mathbf{f},i}$ instead of being finite.
4) Let[23] $\bar{\mathbf{I}}_{\mathbf{f},h} = \langle \mathbf{I}_{\mathbf{f},i,h(\kappa_{\mathbf{f},i})} : i \in u_{\mathbf{f}}\rangle = \langle\langle \bar{a}_{\mathbf{f},i,\alpha} : \alpha < \kappa_i$ and $i \in \mathrm{Dom}(h) \Rightarrow h(\kappa_{\mathbf{f},i}) \le \alpha\rangle : i \in u_{\mathbf{f}}\rangle$ so $h \in \Pi\mathfrak{a}_{\mathbf{f}}$. Let $B^+_{\mathbf{f},u,h}$ be defined as in part (2) using $\bar{\mathbf{I}}_{\mathbf{f},h}$. Let $\mathbf{f}_{[h]} = (\bar{B}_{\mathbf{f}}, \bar{\mathbf{I}}_{\mathbf{f},h})$.
5) For $g \in \Pi\mathfrak{a}_{\mathbf{f}}$ let $\mathscr{F}_{\mathbf{f},u,g} = \{h : h \in \prod\limits_{i \in u}(\kappa_{\mathbf{f},i} \backslash g(\kappa_{\mathbf{f},i}))$; also for $h \in \mathscr{F}_{\mathbf{f},u,g}$ let $\bar{a}_{\mathbf{f},u,h} := \langle \bar{a}_{\mathbf{f},i,h(i)} : i \in u\rangle$.
6) We say that $\mathbf{f}$ is essentially well ordered <u>when</u> for each κ the set $\{i \in u_{\mathbf{f}} : \kappa_{\mathbf{f},i} = \kappa\}$ is well ordered by $\le_v$; compare with Definition 2.6(11).

Claim 2.25. *1) If $\mathbf{f}$ is a $(\bar{\mu},\theta)$-set, $|B^+_{\mathbf{f}}| \ge 2$ for simplicity and $\varepsilon < \theta^+$ <u>then</u> $\mathbf{S}^\varepsilon(B^+_{\mathbf{f}})$ has cardinality $\le |B^+_{\mathbf{f}}|^\theta$.*
1A) If $m < \omega$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite, <u>then</u> for some k we have $|\mathbf{S}^m_\Delta(B^+_{\mathbf{f}})| \le |B^+_{\mathbf{f}}|^k$ whenever $\mathbf{f}$ is a $(\bar{\mu},\theta)$-set, $u_{\mathbf{f}}$ is finite.
2) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ is smooth <u>then</u> $(\bar{B}_{\mathbf{x}}, \bar{\mathbf{I}}_{\mathbf{x}})$ is a $(\bar{\mu},\theta)$-set.
3) If $\mathbf{f}$ is an essentially well ordered $(\bar{\mu},\theta)$-set and $\bar{e} \in {}^{\mu_1 >}\mathfrak{C}$ <u>then</u> for some $h \in \Pi\mathfrak{a}_{\mathbf{f}}$ for some type q we have: $g \in \prod\limits_{i \in u_{\mathbf{f}}} \kappa_{\mathbf{f},i} \wedge \bigwedge\limits_{i \in u_{\mathbf{f}}} h(\kappa_i) \le g(i) \wedge \bigwedge\limits_{\kappa_i = \kappa_j, i <_v j} g(i) \le g(j) \Rightarrow$ $\mathrm{tp}((\ldots \hat{}\, \bar{a}_{\mathbf{f},i,g(\alpha)}\, \hat{}\, \ldots), B_{\mathbf{f}} + \bar{e}) = q$.
4) If $\mathbf{f} = (B_{\mathbf{f}}, \bar{\mathbf{I}}_{\mathbf{f}})$ is a $(\bar{\mu},\theta)$-set and $C \subseteq \mathfrak{C}$ has cardinality $< \mu_0$ <u>then</u> $(B_{\mathbf{f}} + C, \bar{\mathbf{I}}_{\mathbf{f},h})$ is a $(\bar{\mu},\theta)$-set for some $h \in \Pi\mathfrak{a}_{\mathbf{f}}$.
5) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ is smooth <u>then</u> $\mathbf{f}_{\mathbf{x}}$ is a $(\bar{\mu},\theta)$-set, see Definition 2.24(1A).
6) If $1 \le \varepsilon < \theta^+$ and $\mathbf{f}$ is a $(\bar{\mu},\theta)$-set <u>then</u>[24] $\mathbf{S}^\varepsilon(B^+_{\mathbf{f}})$ has cardinality $\le 2^{|B_{\mathbf{x}}|+|v|+|T|} + |B^+_{\mathbf{f}}|^{+\sigma} + |B^+_f|^{<\kappa_i|T|}$.
6A) If $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite, $m \ge 1$ then $\mathbf{S}^m_\Delta(B^+_{\mathbf{f}})$ has cardinality $\le 2^{|B_{\mathbf{f}}|+\aleph_0} + |B^+_f|$.

Proof. E.g. part (1) by 1.34(4) using $\{(\kappa,\alpha) : \kappa \in \mathfrak{a}_f$ and $\alpha < \kappa\}$ ordered lexicographically; part (4) by 1.34(1) as in 2.22(6), for part (3) recall the smoothness demand. As for parts (6),(6A) they are proved similarly to 1.34, noting that for $\mathbf{I}_i$ we use a well ordered index set. $\square_{2.25}$

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[22] This is an abuse of our notation as $\mathbf{f}$ does not determine μ_ℓ in Definition 2.23, pedantically we can expand $\mathbf{f}$ to have this information.

[23] Assume $\kappa \in \mathfrak{a}_{\mathbf{f}} \Rightarrow u \restriction \{i : \kappa_i = \kappa\}$ well ordered; which is reasonable. Then we can change a little the definition of small such that for $h \in \prod\limits_{i \in u} \kappa_i$ we can define $\mathbf{f}_{[h]}$ replacing $\mathbf{I}_{\mathbf{B},i}$ by $\langle \bar{a}_{\mathbf{B},i,\alpha} : \alpha \in [h(i),\kappa_i)\rangle$.

[24] We can use T being dependent (so, e.g. use $(\mathrm{Ded}(|B_{\mathbf{x}}|+|v|+|T|))^{|T|} + (|B^+_{\mathbf{f}}|+2)^{<(\kappa_r|T|+\sigma^+)}$. We can use Δ as in part (1A), so as we can decrease τ_T, really of interest.

§ 2(C). Measuring non-solvability and reducts.

The following is needed in §4, §5, it measures how far solutions are missing.

Definition 2.26. 1) For $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ let $\mathrm{ntr}(\mathbf{x})$, the non-transitivity of $\mathbf{x}$ be the minimal cardinal λ such that for some $A \subseteq M_{\mathbf{x}}$ of cardinality λ for no $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ do we have $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{e}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A)$.
2) For $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ let $\mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ be the minimal cardinal λ such that for every $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$, we have $\mathrm{ntr}_\varphi(\mathbf{x}) \leq \lambda$, see below.
3) For $\varphi \in \Gamma^1_\varphi$, let $\mathrm{ntr}_\varphi(\mathbf{x})$ be the minimal λ such that no ψ does λ^+-illuminates $(\mathbf{x}, \varphi)$, i.e. there is $A \subseteq M_{\mathbf{x}}$ of cardinality λ such that for no $\psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}) \in \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + M_{\mathbf{x}})$ do we have $\psi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{d}) \vdash \mathrm{tp}_\varphi(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A)$.
4) Let $\mathrm{ntr}_{\varphi,\psi}(\mathbf{x})$ be defined naturally.
5) We say that $\bar{\psi}$ does λ-illuminate $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ <u>when</u> $\Gamma_{\bar{\psi}} = \Gamma^1_{\mathbf{x}}, \bar{\psi} = \langle \psi_\varphi : \varphi \in \Gamma_{\bar{\psi}} \rangle$ and for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \lambda$, some $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ solves $(\mathbf{x}, \bar{\psi}, A)$, (see 2.8(1),(3A)).
6) We say $\bar{\psi}$ does λ-illuminate $(\mathbf{x}, \Gamma)$ or $\bar{\psi}$ illuminate $(\mathbf{x}, \lambda, \Gamma)$ <u>when</u> $\Gamma \subseteq \Gamma^1_{\mathbf{x}}, \bar{\psi} = \langle \psi_\varphi : \varphi \in \Gamma \rangle$ and for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \lambda$ for some $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ the sequence $\bar{e}$ solves $(\mathbf{x}, \bar{\psi}, A)$.
7) Similarly when $\Gamma = \Gamma_{\bar{\psi}} = \Gamma^3_{\bar{\psi}} \subseteq \Gamma^3_{\mathbf{x}}$; see 2.11(B),(3B).
As in 2.17.

Observation 2.27. *1) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and $\lambda = \mathrm{ntr}(\mathbf{x}) > \theta (\geq |T|)$ <u>then</u> $\mathrm{ntr}(\mathbf{x})$ is a regular cardinal.*
2) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and $\lambda = \mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ is singular <u>then</u> $\mathrm{cf}(\lambda) \leq \theta + |T|$.
3) If $\theta < \mathrm{cf}(\lambda) \leq \lambda \leq \mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ <u>then</u> some $\bar{\psi} = \langle \psi_\varphi : \varphi \in \Gamma^1_{\mathbf{x}} \rangle$ does λ-illuminate $\mathbf{x}$.
4) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ then $\mathrm{ntr}(\mathbf{x}) \leq \mathrm{ntr}_{\mathrm{lc}}(\mathbf{x}) \leq \mathrm{ntr}_\varphi(\mathbf{x}) \subseteq \mathrm{ntr}_{\varphi,\psi}(\mathbf{x})$ whenever φ, ψ are as in Definition 2.26.

Proof. 1) Why is λ regular? If $\lambda > \mathrm{cf}(\lambda)$, let $A \subseteq M_{\mathbf{x}}$ exemplify the choice of λ, let $\langle A_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ be $\subseteq$-increasing, each A_α being of cardinality $< \lambda$ such that $A = \cup \{A_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. For each $\alpha < \mathrm{cf}(\lambda)$ by the choice of λ there is $\bar{e}_\alpha \in {}^\theta(M_{\mathbf{x}})$ such that $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{e}_\alpha) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A_\alpha)$.

Let $A_* = \cup \{\bar{e}_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ so $|A_*| \leq \theta + \mathrm{cf}(\lambda) < \lambda$ hence for some $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ we have $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{e}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A_*)$. Clearly $\bar{e}$ contradicts the choice of A.
2) Similarly (as in 2.17(2)), changing the ψ_φ's. (In fact we can get $\mathrm{cf}(\lambda) \leq |T|$ and moving to a reduct, $\mathrm{cf}(\lambda) \leq \aleph_0$.)
3) Similarly, as in the proof of 2.17(1).
4) Easy. $\square_{2.26}$

Definition 2.28. 1) If $\tau \subseteq \tau_T$ and $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ let $\mathbf{x} {\restriction} \tau$ be defined like $\mathbf{x}$ but $\mathfrak{C}$ is replaced by $\mathfrak{C} {\restriction} \tau$ and $M_{\mathbf{x}}$ is replaced by $M_{\mathbf{x}} {\restriction} \tau$.
2) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and $v \subseteq v_{\mathbf{x}}, w \subseteq w_{\mathbf{x}}$ <u>then</u> $\mathbf{y} = \mathbf{x} {\restriction} (v, w)$ is defined by $M_{\mathbf{y}} = M_{\mathbf{x}}, w_{\mathbf{y}} = w, \bar{d}_{\mathbf{y}} = \bar{d}_{\mathbf{x}} {\restriction} w, v_{\mathbf{y}} = v, \bar{c}_{\mathbf{y}} = \bar{c}_{\mathbf{x}} {\restriction} v, u_{\mathbf{y}} = u_{\mathbf{x}} \cap v, B_{\mathbf{y},i} = B_{\mathbf{x},i}$ for $i \in v_{\mathbf{y}} \backslash u_{\mathbf{y}}$ and $\mathbf{I}_{\mathbf{y},i} = \mathbf{I}_{\mathbf{x},i}$ for $i \in u_{\mathbf{y}}$.

Observation 2.29. *Membership in $\mathrm{pK}_{\kappa,\bar{\mu},\theta}$ is preserved under reducts, i.e. if $\tau \subseteq \tau(T)$ then $\mathbf{x} {\restriction} \tau \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}[\mathfrak{C} {\restriction} \tau]$; also and $\mathbf{x} {\restriction} (v, w) \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ in the cases above. Also smoothness, "very similar", etc. are preserved.*

Proof. Straightforward. $\square_{2.29}$

Claim 2.30. *1) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ and $\iota_{\mathbf{x}} = 2$ then $\mathrm{tp}(\bar c_{\mathbf{x}}, M_{\mathbf{x}})$ is finitely satisfiable in $B^+_{\mathbf{x}}$ hence for some ultrafilter $D_{\mathbf{x}}$ on $\mathrm{Def}_{\ell g(\bar c_{\mathbf{x}})}(B^+_{\mathbf{x}})$, we have $\mathrm{tp}(\bar c_{\mathbf{x}}, M^+_{\mathbf{x}}) = \mathrm{Av}(D_{\mathbf{x}}, M_{\mathbf{x}})$ in fact $D_{\mathbf{x}}$ is unique.*
2) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ and $\iota_{\mathbf{x}} = 0$ then $\mathrm{tp}(\bar c_{\mathbf{x}}, M_{\mathbf{x}})$ does not split over $B^+_{\mathbf{x}}$.
3) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ and $\iota_{\mathbf{x}} = 1$ then $\mathrm{tp}(\bar c_{\mathbf{x}}, M_{\mathbf{x}})$ does not locally split over $B^+_{\mathbf{x}}$.

Proof. Straightforward. $\square_{2.30}$

We can elaborate 2.30(1)

Definition 2.31. 1) Let D_ℓ be an ultrafilter on the Boolean Algebra $\mathrm{Def}_\varepsilon(A_\ell)$ for $\ell = 1, 2$. We say D_1, D_2 are equivalent when $\mathrm{Av}(D_1, C) = \mathrm{Av}(D_2, C)$ for every set $C \subseteq \mathfrak{C}$.
2) We say an ultrafilter D on $\mathrm{Def}_\varepsilon(A)$ is $(\bar\mu, \theta)$-smooth when ${}^\varepsilon(B^+_{\mathbf{f}}) \in D$ for some $(\bar\mu, \theta)$-set $\mathbf{f}$.

Definition 2.32. For $\mathbf{x} \in \mathrm{pK}$ such that $\iota_{\mathbf{x}} = 2$ let $D_{\mathbf{x}}$ be the following ultrafilter:

- (a) $D_{\mathbf{x}}$ is an ultrafilter on $\mathscr{C}_{\mathbf{x}} = \{\langle \bar c'_i : i \in v_{\mathbf{x}}\rangle : \bar c'_i \in {}^{\ell g(\bar c_i)}(B_i)$ if $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$ and $\bar c'_i \in \{\bar a_{\mathbf{x},i,\alpha} : \alpha < \kappa_{\mathbf{x},i}\}$ if $i \in u_{\mathbf{x}}\}$
- (b) $\{\bar c' \in \mathscr{C}_{\mathbf{x}} : \mathfrak{C} \models \varphi[\bar c', \bar b]\} \in \mathrm{Av}(D, \mathfrak{C})$ iff letting φ depend just on $(\bar x_{\bar c_{i(0)}}, \dots, \bar x_{\bar c_{i(n-1)}}), i(0) > i(1) > \dots > i(n-1)$ and the formula $\varphi(\bar x_{\bar c_{i(0)}}, \dots, \bar x_{\bar c_{i(n-1)}}, \bar b)$ belongs to $\mathrm{Av}(D_{\mathbf{x},i(0)} \times D_{\mathbf{x},i(1)} \times \dots \times D_{\mathbf{x},i(n-1)}, \bar b)$

where above $D_{\mathbf{x},i}$ is the natural ultrafilter, see 2.6(1A).

Definition 2.33. For a $(\bar\mu, \theta)$-set $\mathbf{f}, u \subseteq v_{\mathbf{f}}$ (if $u = v_{\mathbf{f}}$ we may omit it), set v and $A \subseteq \mathfrak{C}$ of cardinality $< \mu_1$, we define an equivalence relation $\mathscr{E}^v_{\Delta,A}$ on $\mathbf{S}^v_\Delta(A + B^+_{\mathbf{f},u})$ as follows: (if $\Delta = \mathbb{L}(\tau_T)$ we may omit Δ)

$\mathrm{tp}_\Delta(\bar b_1, A + B^+_{\mathbf{f},u}) \mathscr{E}^v_\Delta \mathrm{tp}_\Delta(\bar b_2, A + B^+_{\mathbf{f},u})$ iff ($\ell g(\bar b_1) = v = \ell g(b_2)$ and) for some $h \in \Pi \mathfrak{a}_{\mathbf{f}}$, the types $\mathrm{tp}_\Delta(\bar b_2, A + B^+_{\mathbf{f},u,h}), \mathrm{tp}_\Delta(\bar b_2, A + B^+_{\mathbf{f},u,h})$ are equal.

Observation 2.34. *1) On $\mathbf{S}^v_{A,\mathbf{f},h} := \{\mathrm{tp}(\bar e, A + B^+_{\mathbf{f},u,h}) : \bar e \in {}^v\mathfrak{C}$ and $(B_{\mathbf{f},u_n} + \bar e, \bar{\mathbf{I}}_{\mathbf{f},h})$ is a $(\bar\mu, \theta)$-set$\}$ the equivalence relation $\mathscr{E}^v_A$ is the equality.*
2) Assume $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ and $(\forall \alpha < \kappa)(|\alpha|^\theta < \kappa)$ and $(\forall \mu < \mu_0)(2^\mu < \kappa)$ and $\mu_2 = \kappa$ and let $\mathbf{f} = \mathbf{f}_{\mathbf{x}}$ see Definition 2.21(1A). If $n < \omega$ and $A \subseteq M_{\mathbf{x}}$ is of cardinality $< \mu_0$ then the equivalence relation $\mathscr{E}^n_{\Delta,A}$ has $< \kappa$ equivalence classes.

Proof. Straightforward. $\square_{2.34}$

§ 3. Strong analysis

In §2 we have dealt with pK and qK, here we use tK and vK. Now tK is the "really analyzed" case, one essentially with "$\bar{d}_{\mathbf{x}}$ universally solve itself" so it is a central notion here. But we have problems in proving its density in enough cases (i.e. cardinals), so we use also a relative vK, weak enough for the density proof, strong enough for the main desired consequence. We do not forget tK as it is more transparent and says more. We give some consequences of $\mathbf{x} \in \mathrm{tK}_{\kappa,\bar{\mu},\theta}$ or $\mathbf{x} \in \mathrm{vK}_{\kappa,\bar{\mu},\theta}$. First, $M_{[\mathbf{x}]} = M_{[B^+_{\mathbf{x}}+\bar{c}_{\mathbf{x}}+\bar{d}_{\mathbf{x}}]}$ is $(\mathbf{D}_{\mathbf{x}},\kappa)$- sequence-homogenous (see 0.14(1); so pcf, see [She94] appears naturally when we try to analyze $\mathbf{D}_{\mathbf{x}}$ but this is not really used here). This implies uniqueness, so indirectly few types up to conjugacy; this will solve the recounting problems from §(1A) but only when we shall prove density of tK or vK in $(\mathrm{pK}_{\kappa,\bar{\mu},\theta},\le_1)$. We give sufficient condition for existence, using existence of universal solutions and prove it for κ weakly compact when $\|M_{\mathbf{x}}\| = \kappa$. We end with criterions for indiscernibility related to tK.

Note that tK is better than qK, but the relevant density result is for $\le_1$ rather than $\le_2$, i.e. you may say that we add more variables to the type analyzed.

§ 3(A). **Introducing** rK, tK, vK.

So a central definition is

Definition 3.1. Let $\mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$ be the class of $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ such that: for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ there is $(\bar{c}_*, \bar{d}_*)$ which strongly solves $(\mathbf{x}, A)$ which means: $\bar{c}_* {}^\frown \bar{d}_*$ is from $M_{\mathbf{x}}$ and it realizes $\mathrm{tp}(\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}}, A)$, of course $\ell g(\bar{c}_*) = \ell g(\bar{c}_{\mathbf{x}}), \ell g(\bar{d}_*) = \ell g(\bar{d}_{\mathbf{x}})$ and $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{d}_* + \bar{c}_*) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{d}_* + \bar{c}_* \dotplus A)$ by some $\bar{\psi}$.

Remark 3.2. 1) For $\le_1$-increasing chains in $\mathrm{tK}_{\kappa,\bar{\mu},\theta}$ the union is naturally defined (essentially see in 2.13(2)) but it is not a priori clear it belongs to $\mathrm{tK}_{\kappa,\bar{\mu},\theta}$, i.e. if $\langle \mathbf{x}_\alpha : \alpha < \delta\rangle$ is $\le_1$-increasing in $\mathrm{tK}_{\kappa,\bar{\mu},\theta}$ and $\delta < \theta^+$ <u>then</u> does the union belongs to $\mathrm{tK}_{\kappa,\bar{\mu},\theta}$?
2) To have enough cases when this holds we define a relative of pK which carries more information.
3) Note that below:

- (a) (α) $\mathrm{rK}, \mathrm{tK}, \mathrm{uK}, \mathrm{vK}$ as well as $\mathrm{qK}, \mathrm{qK}', qK''$ are subsets of pK
  - (β) $\mathrm{qK}^{\odot}, \mathrm{qK}^{\oplus}$ from 2.11(3),(3A) have members of the form $(\mathbf{x}, \bar{\psi}, r)$ with $\Gamma_{\bar{\psi}} \subseteq \Gamma^1_{\mathbf{x}}$
- (b) $\mathrm{rK}^{\oplus}, \mathrm{tK}^{\oplus}, \mathrm{vK}^{\oplus}, \mathrm{vK}^{\odot}$ has members of the form $(\mathbf{x}, \bar{\psi}, r)$ such that $\Gamma_{\bar{\psi}} = \Gamma^2_{\bar{\psi}} \subseteq \Gamma^2_{\mathbf{x}}$ where $\bar{\psi} = \langle \psi_\varphi : \varphi \in \Gamma^2_{\bar{\psi}}\rangle$
- (c) $\mathrm{sK}^{\oplus}, \mathrm{uK}^{\otimes}, \mathrm{uK}^{\otimes}$ as well as $\mathrm{qK}^{\otimes}$ are similar but with $\Gamma_{\bar{\psi}} = \Gamma^3_{\psi} \subseteq \Gamma^3_{\mathbf{x}}$
- (d) the vK's and uK's use so-called duplicates (defined below)
- (e) $\mathrm{uK}_{\kappa,\bar{\mu},\theta}, \mathrm{vK}_{\kappa,\bar{\mu},\theta}$ is a parallel of $\mathrm{qK}_{\kappa,\bar{\mu},\theta}, \mathrm{tK}_{\kappa,\bar{\mu},\theta}$ respectively when we allow duplication, see 2.11(1), 3.6(3c).

Definition 3.3. 1) Let $\mathrm{rK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$ be the class of triples $\mathbf{n} = (\mathbf{x}, \bar{\psi}, r)$ such that

- (a) $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$

(b) r is a type in the variables $\bar{x}_{\bar{d}} {}^\frown \bar{x}_{\bar{c}} {}^\frown \bar{x}'_{\bar{d}} {}^\frown \bar{x}'_{\bar{c}}$, over $\emptyset$ if not said otherwise and always over some $A \subseteq M_{\mathbf{x}}$

(c) $\bar{\psi} = \langle \psi_\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{\bar{d}}, \bar{x}'_{\bar{c}}) : \varphi \in \Gamma^2_{\bar{\psi}} \rangle$ recalling[25] 2.11(0)(A)

(d) $\Gamma_{\bar{\psi}} = \Gamma^2_{\bar{\psi}} \subseteq \Gamma^2_{\mathbf{x}} := \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{\bar{d}}, \bar{x}'_{\bar{c}}, \bar{y}) \in \mathbb{L}(\tau_T)\}$ recalling 2.11(0)(A)

(e) $\psi_\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{\bar{d}}, \bar{x}'_{\bar{c}}) \in r$ for every $\varphi \in \Gamma^2_{\bar{\psi}}$

(f) if $A \subseteq M_{\mathbf{x}}$ has cardinality $< \kappa$ <u>then</u> some $(\bar{c}', \bar{d}')$ solves $(\mathbf{x}, \bar{\psi}, r, A)$ or solves $(\mathbf{n}, A)$; we may write $\bar{c}' {}^\frown \bar{d}'$ instead $(\bar{c}', \bar{d}')$; which means:
  - (α) $\bar{c}' {}^\frown \bar{d}'$ is from $M_{\mathbf{x}}$ and realizes $\mathrm{tp}(\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}}, A)$
  - (β) $\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}} {}^\frown \bar{c}' {}^\frown \bar{d}'$ realizes r, of course, $\ell g(\bar{c}') = \ell g(\bar{c}_{\mathbf{x}}), \ell g(\bar{d}') = \ell g(\bar{d}_{\mathbf{x}})$
  - (γ) if $\varphi \in \Gamma^2_{\mathbf{x}}$ then $\psi_\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{d}', \bar{c}') \vdash \mathrm{tp}_\varphi(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} {}^\frown \bar{d}' {}^\frown \bar{c}' \dotplus A)$ recalling the latter means $\{\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{d}', \bar{c}', \bar{b}) : \bar{b} \in {}^{\ell g(\bar{y}_\varphi)}A$ and $\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{d}', \bar{c}', \bar{b}]\}$.

2) Let $\mathrm{sK}^\oplus_{\lambda,\kappa,\bar{\mu},\theta}$ be the class of tuples $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r)$ such that:

(a) $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$

(b) r is a type in the variables $\bar{x}_{\bar{c}} {}^\frown \bar{x}_{\bar{d}} {}^\frown \bar{y}_{[\theta]}$, over $\emptyset$ if not said otherwise and always over some $A \subseteq M_{\mathbf{x}}$

(c) $\bar{\psi} = \langle \psi_\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{[\theta]}) : \varphi \in \Gamma^3_{\bar{\psi}} \rangle$
  where recalling 2.11(0)(B)

(d) $\Gamma_{\bar{\psi}} = \Gamma^3_{\bar{\psi}} \subseteq \Gamma^3_{\mathbf{x}} = \{\varphi : \varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{[\theta]}, \bar{z}) \in \mathbb{L}(\tau_T)\}$

(e) $\psi_\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{[\theta]}) \in r$ for every $\varphi \in \Gamma^3_{\bar{\psi}}$

(f) if $A \subseteq M_{\mathbf{x}}$ has cardinality $< \kappa$ <u>then</u> some $\bar{e}$ solves $(\mathbf{x}, \bar{\psi}, r, A)$ or solves $(\mathbf{m}, A)$ which means:
  - (α) $\bar{c} {}^\frown \bar{d} {}^\frown \bar{e}$ realizes r
  - (β) $\psi_\varphi(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}) \vdash \mathrm{tp}_\varphi(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} {}^\frown \bar{e} \dotplus A)$ for every $\varphi \in \Gamma^3_{\bar{\varphi}}$.

3) We define "very similar/similar/weakly similar" on $\mathrm{rK}^\oplus$ and $\mathrm{sK}^\oplus$ naturally as in Definition 2.19, (and they are equivalence relations).

Remark 3.4. 1) So arbitrary $\bar{b} \subseteq \mathrm{Rang}(\bar{c}_{\mathbf{x}})$ is not allowed in clauses $(f)(\gamma)$ of 3.3(1) and $(f)(\beta)$ of 3.3(2). The reason is in the proof of 2.15(3),(4), i.e. [She15, 2.14=tp25.36,2.15=tp25.38]. We can partially allow it, see 2.15(4), the "moreover", but not needed now.
2) Note that for singular μ_2 we get a better result for free (as in the case $\kappa = \mu^+, \mu$ strong limit singular of cofinality $> \theta$ is easier, see 2.17(2) and the proof of 4.12.
3) In Definition 3.6 below note that vK is a weak form of tK and uK a weak form of qK.

Discussion 3.5. Concerning Definitions 3.1, 3.3 and 3.6 below:
0) For the vK's, uK's instead of dealing with some $\varphi \in \Gamma^2_{\mathbf{x}}/\Gamma^3_{\mathbf{x}}$ we allow ourselves to deal with a so called duplicate.
1) Note that $\mathrm{tK}^\oplus, \mathrm{rK}^\oplus, \mathrm{vK}^\oplus, \mathrm{vK}^\odot$ deals with $\Gamma^2_{\mathbf{x}}$ while $\mathrm{sK}^\oplus, \mathrm{uK}^\oplus, \mathrm{uK}^\otimes$ as well as $\mathrm{qK}^\otimes$ deals with $\Gamma^3_{\mathbf{x}}$.
2) Note that $\mathrm{uK}^\otimes, \mathrm{vK}^\odot$ have the witness $\bar{\mathbf{w}}$ as part of the $\mathbf{m}$ while $\mathrm{uK}^\oplus, \mathrm{vK}^\oplus$ do not.

[25] but we may allow one φ to appear more than once

3) $\mathrm{tK}, \mathrm{uK}^\oplus, \mathrm{uK}^\otimes, \mathrm{vK}^\oplus, \mathrm{vK}^\odot$ deal with all formulas unlike $\mathrm{rK}^\oplus, \mathrm{sK}^\oplus$.
4) vK, uK is the projection of $\mathrm{vK}^\oplus, \mathrm{uK}^\oplus$ respectively to pK.
5) What is the point of $\mathrm{vK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$? We do not deal with every $\varphi = \varphi_0 \in \Gamma^2_{\mathbf{x}}$, we "translate" the problem of φ_0 to a "duplicate" φ_2 similar enough which is in $\Gamma^2_{\bar\psi}$.
6) We can formulate 3.6(3A)(d), more like 3.6(3C)(α).
7) rK is intended as a step toward tK (or vK).
8) Note $\mathrm{uK}/\mathrm{uK}^\odot/\mathrm{uK}^\oplus/\mathrm{uK}^\otimes$ relate like $\mathrm{qK}/\mathrm{qK}^\odot/\mathrm{qK}^\oplus/\mathrm{qK}^\otimes$.
9) $\mathrm{tK}^\oplus$ is parallel to $\mathrm{vK}^\odot$ just as it is parallel to $\mathrm{vK}^\oplus$.
10) Note that vK is defined as the projection of $\mathrm{vK}^\oplus$ whereas tK is only provably the projection of $\mathrm{tK}^\oplus$ when $\mathrm{cf}(\kappa) > 2^\theta$.
11) So $\mathrm{vK}^\oplus/\mathrm{vK}^\odot$ is not parallel to $\mathrm{uK}^\oplus/\mathrm{uK}^\otimes$ but the latter is parallel to $\mathrm{qK}^\oplus/\mathrm{qK}^\otimes$.

Definition 3.6. 1) In 3.1, 3.3 we adopt the conventions of 2.6(2) concerning the cardinals.
1A) If $\mathbf{m}$ belongs to $\mathrm{rK}^\oplus$, let $\mathbf{m} = (\mathbf{x}_{\mathbf{m}}, \bar\psi_{\mathbf{m}}, r_{\mathbf{m}}) = (\mathbf{x}[\mathbf{m}], \bar\psi[\mathbf{m}], r[\mathbf{m}])$ and $M_{\mathbf{m}} = M_{\mathbf{x}[\mathbf{m}]}$, etc. and $\Gamma^2_{\mathbf{m}} = \Gamma^2_{\mathbf{x}[\mathbf{m}]}$, this may well be $\neq \Gamma^2_{\bar\psi[\mathbf{m}]}$, see 3.3(1)(d).
1B) Similarly for $\mathbf{n} \in \mathrm{sK}^\oplus$ let $\mathbf{n} = (\mathbf{x}_{\mathbf{n}}, \bar\psi_{\mathbf{n}}, r_{\mathbf{n}}) = (\mathbf{x}[\mathbf{n}], \bar\psi[\mathbf{n}], r[\mathbf{n}])$.
2) We define a two-place relation $\le_1$ on $\mathrm{rK}^\oplus_{\kappa,\bar\mu,\theta} : (\mathbf{x}_1, \bar\psi_1, r_1) \le_1 (\mathbf{x}_2, \bar\psi_2, r_2)$ <u>when</u> $\mathbf{x}_1 \le_1 \mathbf{x}_2$ (in $\mathrm{pK}_{\kappa,\bar\mu,\theta}$), $\bar\psi_1 = \bar\psi_2 \restriction \Gamma^2_{\bar\psi_1}$ (but dummy variables may be added) and $r_1 \subseteq r_2$.
2A) Similarly for $\mathrm{sK}^\oplus_{\kappa,\bar\mu,\theta}$.
3) Let $\mathrm{tK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ be the class of $(\mathbf{x}, \bar\psi, r) \in \mathrm{rK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ such that $\Gamma^2_{\bar\psi} = \Gamma^2_{\mathbf{x}}$ and r is a complete type, over $\emptyset$ if not said otherwise.
3A) Let $\mathrm{vK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ be the class of $\mathbf{m} = (\mathbf{x}, \bar\psi, r) \in \mathrm{rK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ such that: for every $\varphi \in \Gamma^2_{\mathbf{x}}$ there is an $(\mathbf{m}, \varphi)$-duplicate $\mathbf{w} = (\eta_0, \nu_0, \eta_1, \nu_1, \eta_2, \nu_2, \eta_3, \nu_3, \varphi_0, \varphi_1, \varphi_2)$ which means[26] (treating $\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}$ as sequences of singletons, similarly[27] later)

- (a) $\varphi = \varphi_0 = \varphi_0(\bar x_{\bar d}, \bar x_{\bar c}, \bar x'_{\bar d}, \bar x'_{\bar c}, \bar y) \in \Gamma^2_{\mathbf{x}}$ (as in Definition 2.11(0)(A), 3.3(1)(d))
- (b) $\eta_0, \eta_1, \eta_2, \eta_3 \in {}^{\omega>}\ell g(\bar d_{\mathbf{x}})$ and $\ell g(\eta_0) = \ell g(\eta_1), \ell g(\eta_2) = \ell g(\eta_3)$
- (c) $\nu_0, \nu_1, \nu_2, \nu_3 \in {}^{\omega>}\ell g(\bar c_{\mathbf{x}})$ and $\ell g(\nu_0) = \ell g(\nu_1), \ell g(\nu_2) = \ell g(\nu_3)$
- (d) $\varphi_1 = \varphi_1(\bar x_{\bar d,\eta_1}, \bar x_{\bar c,\nu_1}, \bar x'_{\bar d,\eta_3}, \bar x'_{\bar c,\nu_3}, \bar y) \equiv \varphi_0$
- (e) $\varphi_1(\bar x_{\bar d,\eta_0}, \bar x_{\bar c,\nu_0}, \bar x'_{\bar d,\eta_2}, \bar x'_{\bar c,\nu_2}, \bar y) \equiv \varphi_2 = \varphi_2(\bar x_{\bar d}, \bar x_{\bar c}, \bar x'_{\bar d}, \bar x'_{\bar c}, \bar y) \in \Gamma^2_{\mathbf{x}}$
- (f) $\mathfrak{C} \models$ "$\varphi_1[\bar d_{\mathbf{x},\eta_0}, \bar c_{\mathbf{x},\nu_0}, \bar d'_{\eta_2}, \bar c'_{\nu_2}, \bar b] \equiv \varphi_1[\bar d_{\mathbf{x},\eta_1}, \bar c_{\mathbf{x},\nu_1}, \bar d'_{\eta_3}, \bar c'_{\nu_3}, \bar b]$" for every $\bar d'_{\eta_3} \in {}^{\ell g(\eta_3)}(M_{\mathbf{x}}), \bar c'_{\nu_3} \in {}^{\ell g(\nu_3)}(M_{\mathbf{x}}), \bar b \in {}^{\ell g(\bar y)}(M_{\mathbf{x}})$
- (g) $\varphi_2 \in \Gamma^2_{\bar\psi}$.

3B) For $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ we say Γ is $\mathbf{x}-\mathrm{vK}$-large where $\Gamma \subseteq \Gamma^2_{\mathbf{x}}$ <u>when</u> for every $\varphi \in \Gamma^2_{\mathbf{x}}$ there is $\mathbf{w}$ satisfying clause (a)-(g) of part (3A) and $\varphi_2 \in \Gamma$.
3C) Let $\mathrm{uK}_{\lambda,\kappa,\bar\mu,\theta}$ be the class of $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar\mu,\theta}$ such that: for every $\varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar y) \in \Gamma^1_{\mathbf{x}}$ there is a weak $(\mathbf{x}, \varphi)$-duplicate $\mathbf{w} = (\eta_0, \nu_0, \eta_1, \nu_1, \varphi_0, \varphi_1, \varphi_2)$ meaning $\varphi_0 = \varphi$ and[28]:

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[26] we may use a $\bar d_\eta \restriction u, \bar x_{\bar d_\eta \restriction u} = \bar x_u$ instead of $\bar d_\eta, \bar x_{\bar d,\eta}$; as we may omit $\bar x'_{\bar c}$, no real change, in particular for normal $\mathbf{x}$ it is the same; used in proof of 3.10(1A), 3.24, a degenerate case without $\eta_2, \nu_2, \eta_3, \nu_3$.

[27] Below, you may wonder what is the difference between φ_0 and φ_2. <u>Example</u>: $d = (d_0, d_1), \bar c = \langle c_0, c_1 \rangle, \varphi_0(\bar x_{\bar d}, \bar x_{\bar c}, \ldots) = (x_{d,0} R x_{c,0}), \varphi_2 = (x_{d,1} R x_{c,1})$.

[28] We may demand $\nu_1 = \nu_0$; it seems there is no serious diference.

(a) $(\eta_1,\nu_1) = \mathrm{supp}(\varphi)$, (see 2.10(4)), i.e. η_1,ν_1 list w,v respectively, in increasing order for some $(w,v) \in \mathrm{supp}(\varphi)$

(b) $\eta_0,\eta_1 \in {}^{\omega>}\ell g(\bar d_{\mathbf{x}})$ and $\ell g(\eta_0) = \ell g(\eta_1)$, pedantically also $\ell g(\bar d_{\mathbf{x},\eta_0(\ell)}) = \ell g(\bar d_{\mathbf{x},\eta_1(\ell)})$ for every $\ell < \ell g(\eta_0)$ but usually we ignore this

(c) $\nu_0,\nu_1 \in {}^{\omega>}\ell g(\bar c_{\mathbf{x}})$ and $\ell g(\nu_0) = \ell g(\nu_1)$

(d) $\varphi_1 = \varphi_1(\bar x_{\bar d,\eta_1},\bar x_{\bar c,\nu_1},\bar y) \equiv \varphi_0 = \varphi_0(x_{\bar d},\bar x_{\bar c},\bar y) \in \Gamma^1_{\mathbf{x}}$

(e) $\varphi_1(\bar x_{\bar d,\eta_0},\bar x_{\bar c,\nu_0},\bar y) \equiv \varphi_2 = \varphi_2(\bar x_{\bar d},\bar x_{\bar c},\bar y) \in \Gamma^1_{\mathbf{x}}$

(f) $\mathfrak{C} \models$ "$\varphi_1[\bar d_{\mathbf{x},\eta_0},\bar c_{\mathbf{x},\nu_0},\bar b] \equiv \varphi_2[\bar d_{\mathbf{x},\eta_1},\bar c_{\mathbf{x},\nu_1},\bar b]$" for every $\bar b \in {}^{\ell g(\bar y)}(M_{\mathbf{x}})$

(g) some ψ illuminates $(\mathbf{x},\varphi_2)$.

3D) For $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$, we say $\Gamma \subseteq \Gamma^1_{\mathbf{x}}$ is $\mathbf{x}-$uK-large <u>when</u> for every $\varphi \in \Gamma^1_{\mathbf{x}}$ there is a weak $(\mathbf{x},\varphi)$-duplicate $\mathbf{w}$, see part (3C).

3E) Let $\mathrm{vK}_{\lambda,\kappa,\bar\mu,\theta}$ be the class of $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar\mu,\theta}$ such that for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ we can find $\bar\psi = \langle \psi_\varphi : \varphi \in \Gamma^2_{\bar\psi}\rangle$ and $(\bar c',\bar d')$ such that:

(α) as in 3.3(1)(f)

(β) if $\varphi \in \Gamma^2_{\bar\psi}$ then $\psi_\varphi(\bar x_{\bar d},\bar c_{\mathbf{x}},\bar d',\bar c') \vdash \mathrm{tp}_\varphi(\bar d_{\mathbf{x}},\bar c_{\mathbf{x}}{}^\frown\bar d'{}^\frown\bar c' + A)$

(γ) $\Gamma^2_{\bar\psi}$ is $\mathbf{x}-$vK large, see part (3B).

3F) We define $\mathrm{uK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ as the class of triples $\mathbf{n} = (\mathbf{x},\bar\psi,r) \in \mathrm{sK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ such that $\Gamma^3_{\bar\psi} \subseteq \Gamma^3_{\mathbf{x}}$ is $\mathbf{x}-$uK-large. We define $\mathrm{vK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ as the class of $\mathbf{n} = (\mathbf{x},\bar\psi,r)$ which belongs to $\mathrm{rK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ and $\Gamma^2_{\bar\psi} \subseteq \Gamma^2_{\mathbf{x}}$ is $\mathbf{x}-$vK-large.

3G) We define $\mathrm{uK}^\otimes_{\lambda,\kappa,\bar\mu,\theta}$ as the class of $\mathbf{n} = (\mathbf{x},\bar\psi,r,\bar{\mathbf{w}})$ such that $(\mathbf{x},\bar\psi,r) \in \mathrm{sK}^\oplus_{\lambda,\kappa,\bar\mu,\theta}$ so $\Gamma^3_{\bar\psi} \subseteq \Gamma^3_{\mathbf{x}}, \bar\psi = \langle \psi_\varphi(\bar x_{\bar d},\bar x_{\bar c},\bar y_{[\theta]}) : \varphi = \varphi(\bar x_{\bar d},\bar x_{\bar c},\bar y_{[\theta]},\bar z) \in \Gamma^3_{\bar\psi}\rangle, \bar{\mathbf{w}} = \langle \mathbf{w}_\varphi : \varphi \in \Gamma^3_{\mathbf{x}}\rangle$ recalling 2.11(0)(B) and: for every $\varphi = \varphi(\bar x_{\bar d},\bar x_{\bar c},\bar y_{[\theta]},\bar z) \in \Gamma^3_{\mathbf{x}}$, $\mathbf{w}_\varphi$ is a witness of the form $(\eta_0,\nu_0,\eta_1,\nu_1,\varphi_0,\varphi_1,\varphi_2)$ (see below) and for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ there is a solution $\bar e$, i.e. $\bar e$ solves $(\mathbf{n},A)$ which means: $\bar e \in {}^\theta(M_{\mathbf{x}})$ solves $(\mathbf{x},\bar\psi,r)$ recalling 3.3(2)(f) and for every $\varphi \in \Gamma^3_{\mathbf{x}}$ the witness $\mathbf{w}_\varphi$ satisfies:

(a) $(\eta_1,\nu_1) \in \mathrm{supp}(\varphi(\bar x_{\bar d},\bar x_{\bar c};\bar y_{[\theta]},\bar z))$

(b) $\eta_0,\eta_1 \in {}^{\omega>}\ell g(\bar d_{\mathbf{x}})$ and $\ell g(\eta_0) = \ell g(\eta_1)$

(c) $\nu_0,\nu_1 \in {}^{\omega>}\ell g(\bar c_{\mathbf{x}})$ and $\ell g(\nu_0) = \ell g(\nu_1)$

(d) $\varphi_2 = \varphi_2(\bar x_{\bar d,\eta_1},\bar x_{\bar c,\nu_0},\bar y_{[\theta]},\bar z) \equiv \varphi_0 = \varphi(\bar x_{\bar d},\bar x_{\bar c},\bar y_{[\theta]},\bar z) \in \Gamma^3_{\mathbf{x}}$

(e) $\varphi_2(\bar x_{\bar d,\eta_0},\bar x_{\bar c,\nu_0},\bar y_{[\theta]},\bar z) \equiv \varphi_2 = \varphi_2(\bar x_{\bar d},\bar x_{\bar c},\bar y_{[\theta]},\bar z) \in \Gamma^3_{\mathbf{x}}$

(f) $\mathfrak{C} \models$ "$\varphi_1[\bar d_{\mathbf{x},\eta_0},\bar c_{\mathbf{x},\nu_0},\bar e,\bar b] \equiv \varphi_2[\bar d_{\mathbf{x},\eta_1},\bar c_{\mathbf{x},\nu_2},\bar e,\bar b]$" for every $\bar b \in {}^{\ell g(\bar z)}A$

(g) $\varphi_2 \in \Gamma^3_{\bar\psi}$.

4) Let $\le^+_1$ be the following two-place relation on $\mathrm{rK}^\oplus$:
$(\mathbf{x}_1,\bar\psi_1,r_1) \le^+_1 (\mathbf{x}_2,\bar\psi_2,r_2)$ <u>iff</u> $(\mathbf{x}_1,\bar\psi_1,r_1) \le_1 (\mathbf{x}_1,\bar\psi_2,r_2)$ and $\Gamma^2_{\mathbf{x}_1} \subseteq \Gamma^2_{\bar\psi_2}$, not just $\Gamma^2_{\bar\psi_1} \subseteq \Gamma^2_{\bar\psi_2}$!

4A) Let $\le^\odot_1$ be the following two-place relation on $\mathrm{rK}^\oplus$:

$\mathbf{m} \le^\odot_1 \mathbf{n}$ <u>iff</u> $\mathbf{m} \le_1 \mathbf{n}$ and if $\varphi = \varphi(x_{\bar d[\mathbf{m}]},\bar x_{\bar c[\mathbf{m}]},\bar x'_{\bar d[\mathbf{m}]},\bar x'_{\bar c[\mathbf{m}]},\bar y) \in \Gamma^2_{\mathbf{x}[\mathbf{m}]}$ <u>then</u> some $\mathbf{w}$ is an $(\mathbf{n},\varphi)$-duplicate, see part (3A).

4B) We define $\le^+_{1,\Delta}, <^\odot_{1,\Delta}$ similarly where $\Delta \subseteq \Gamma^2_{\mathbf{x}_1}$ and we deal only with $\varphi \in \Delta$.

4C) We define $\mathrm{vK}^\otimes_{\lambda,\kappa,\bar\mu,\theta}$ as the class of $\mathbf{n} = (\mathbf{x},\bar\psi,r,\bar{\mathbf{w}})$ such that:

(a) $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{vK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$
(b) $\bar{\mathbf{w}} = \langle \mathbf{w}_\varphi : \varphi \in \Gamma^2_{\mathbf{x}} \rangle$
(c) for $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{\bar{d}}, \bar{x}'_{\bar{c}}, \bar{y}) \in \Gamma^2_{\mathbf{x}}$ we have[29] $\mathbf{w}_\varphi$ is a[30] $((\mathbf{x}, \bar{\psi}, r), \varphi)$-duplicate, see part (3A).

5) If $\langle \mathbf{m}_\varepsilon : \varepsilon < \delta \rangle$ is $\leq_1$-increasing in $\mathrm{rK}^{\oplus}$ (see Definition 3.6(c)) <u>then</u> we let $\mathbf{m}_\delta := \cup\{\mathbf{m}_\varepsilon : \varepsilon < \delta\}$ be naturally defined (uniquely up to "very similar") but it is not clear that $\mathbf{m}_\delta \in \mathrm{rK}^{\oplus}$, the problem is with 3.3(1)(f). Similarly in the other cases.
6) We define reducts, $\mathbf{m} \restriction \tau$ for $\tau \subseteq \tau(T)$ naturally.

Note

Observation 3.7. *Let $\kappa > \theta$ and $\bar{\mu}$ be as in Definition 2.2.*
1) If $\mathbf{x} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$ and $\mathbf{y}$ is defined like $\mathbf{x}$ replacing $\bar{d}_{\mathbf{x}}$ by $\bar{d}_{\mathbf{x}} {}^\frown \bar{c}_{\mathbf{x}}$ <u>then</u> $\mathbf{y} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$ and is normal. Similarly for $\mathrm{vK}_{\lambda,\kappa,\bar{\mu},\theta}, \mathrm{uK}_{\lambda,\kappa,\bar{\mu},\theta}$.
2) $\mathbf{x} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta} \Rightarrow \mathbf{x} \in \mathrm{qK}_{\lambda,\kappa,\bar{\mu},\theta}$ and $\mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta} \subseteq \mathrm{uK}_{\lambda,\kappa,\bar{\mu},\theta}$ and $\mathrm{tK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta} \subseteq \mathrm{vK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta} \subseteq \mathrm{vK}^{\otimes}_{\lambda,\kappa,\bar{\mu},\theta}$.
3) For every κ-saturated M there is $\mathbf{x} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$ with $M_{\mathbf{x}} = M, \bar{d}_{\mathbf{x}} = \langle\rangle = \bar{c}_{\mathbf{x}}$ hence $w_{\mathbf{x}} = \emptyset = v_{\mathbf{x}}, B_{\mathbf{x}} = \emptyset$.
4) Assume $\mathrm{cf}(\kappa) > 2^\theta + |T|$. <u>Then</u> $\mathbf{x} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$ iff for some $\bar{\psi}, r$ we have $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{tK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$ with r a complete type over $\emptyset$.
4A) Similarly for $\mathrm{vK}_{\lambda,\kappa,\bar{\mu},\theta}, \mathrm{vK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$.
4B) If $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r) \in \mathrm{vK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$ <u>then</u> for some $\mathbf{w}$ we have $(\mathbf{x}, \bar{\psi}, r, \mathbf{w}) \in \mathrm{vK}^{\odot}_{\lambda,\kappa,\bar{\mu},\theta}$.
5) If $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\lambda,\kappa,\bar{\mu},\theta}$ <u>then</u> $\mathbf{x}_{\mathbf{m}} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$.
5A) $\mathbf{m} \in \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ <u>then</u> $\mathbf{x}_{\mathbf{m}} \in \mathrm{vK}_{\kappa,\mu,\theta}$.

Proof. 1) Straight (as in 2.13(3)).
2) For the first statement recall (from Definition 3.1) that a consequence of $\mathbf{x} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta}$ is the existence of solutions, but this consequence for $\mathbf{x} \in \mathrm{pK}_{\lambda,\kappa,\bar{\mu},\theta}$ implies $\mathbf{x} \in \mathrm{qK}_{\lambda,\kappa,\bar{\mu},\theta}$ by Definition 2.11(2) so indeed $\mathbf{x} \in \mathrm{tK}_{\lambda,\kappa,\bar{\mu},\theta} \Rightarrow \mathbf{x} \in \mathrm{qK}_{\lambda,\kappa,\bar{\mu},\theta}$. Also for the other statements see the definitions.
3) Obvious (and see 2.13(5)).
4),4A),4B),5), 5A) Easy. Read the definitions for $\Leftarrow$ and immitate 2.17 for $\Rightarrow$.
$\square_{3.7}$

Observation 3.8. *0) If $\mathbf{m} \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ <u>then</u> $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{m} \leq^+_1 \mathbf{m}$ and $\mathbf{m} \in \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{m} \leq^{\odot}_1 \mathbf{m}$.*
1) $\leq^+_1$ partially ordered $\mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ except that possibly $\neg(\mathbf{m} \leq^+_1 \mathbf{m})$.
1A) Similarly $\leq^{\odot}_1$.
2) Also on $\mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ we have $\leq^+_1 \subseteq \leq^{\odot}_1 \subseteq \leq_1$ and $\mathbf{m}_1 \leq_1 \mathbf{m}_2 \leq^+_1 \mathbf{m}_3 \leq_1 \mathbf{m}_4$ implies $\mathbf{m}_1 \leq^+_1 \mathbf{m}_4$ and $\mathbf{m}_1 \leq_1 \mathbf{m}_2 \leq^{\odot}_1 \mathbf{m}_3 \leq_1 \mathbf{m}_4$ implies $\mathbf{m}_1 \leq^{\odot}_1 \mathbf{m}_4$.
3) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ <u>then</u> $(\mathbf{x}, \langle\rangle, \emptyset) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$.
4) If $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{x}, \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ are very similar <u>then</u> $\mathbf{n} := (\mathbf{y}, \bar{\psi}, r) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{n} \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{m} \in \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta} \Leftrightarrow \mathbf{n} \in \mathrm{vK}^{\oplus}_{\kappa,\mu,\theta}$.

[29] may use $\mathbf{u}(\varphi, u_0)$ but this can be absorbed as we consider $u_1 = \{0\}$ for $\varphi = (x_{d_0} = x_{d_0})$
[30] we may restrict ourselves to normal $\mathbf{x}$ (and $\mathbf{m}$) and then demand $v_1 = w_1 \cap \ell g(\bar{c}_{\mathbf{x}})$

Observation 3.9. *1) If* $\mathbf{x}, \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ *are very similar* <u>*then*</u> $\mathbf{x} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$ *iff* $\mathbf{y} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$.
2) Similarly for $\mathrm{qK}'_{\kappa,\bar{\mu},\theta}, \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{uK}_{\kappa,\bar{\mu},\theta}$ *and* $\mathrm{uK}^{\oplus}_{\kappa,\bar{\mu},\theta}$. *E.g., for such* $\mathbf{x}, \mathbf{y}$*: for any* $\bar{\psi}, r$ *we have* $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ *iff* $(\mathbf{y}, \bar{\psi}, r) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$.

§ 3(B). **Sequence homogeneity and indiscernibles.**

We now try to prove that decompositions from tK and vK are "good" and "helpful". We prove for $\mathbf{x} \in \mathrm{tK}_{\kappa,\bar{\mu},\theta}$ that $M_{[\mathbf{x}]} = M_{[B^+_{\mathbf{x}} + \bar{c}_{\mathbf{x}} + \bar{d}_{\mathbf{x}}]}$ defined in 2.6(6), is κ-sequence-homogeneous, see 0.11, this is nice, and help to prove that there are few types up to conjugacy because if M, N are $(\mathbf{D}, \kappa)$-sequence homogeneous models of cardinality κ then they are isomorphic.

Theorem 3.10. <u>*The sequence homogeneous Theorem*</u>
1) If $\mathbf{x} \in \mathrm{tK}_{\kappa,\bar{\mu},\theta}$ <u>*then*</u> $M_{[\mathbf{x}]}$ *is a* κ*-sequence-homogeneous model for the finite diagram which we call* $\mathbf{D}_{\mathbf{x}}$*; see Definition 0.14(1), 2.6(6).*
1A) Similarly for $\mathbf{x} \in \mathrm{vK}_{\kappa,\bar{\mu},\theta}$.
2) Moreover, if $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ *or just* $(\mathbf{x}, \bar{\psi}, r, \bar{\mathbf{w}}) \in \mathrm{vK}^{\otimes}_{\kappa,\bar{\mu},\theta}$ *and* r *is a complete type* <u>*then*</u> $\mathbf{D}_{\mathbf{x}}$ *depends just on* $T, \bar{\psi}, r, \bar{\mathbf{w}}, B^+_{\mathbf{x}}, \langle \mathrm{schm}_{\mathbf{x},i} : i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}} \rangle$ *and* $\mathrm{tp}(\bar{d}_{\mathbf{x}} \hat{\ } \bar{c}_{\mathbf{x}}, B^+_{\mathbf{x}})$*. That is, if* $\mathbf{m}_\ell = (\mathbf{x}_\ell, \bar{\psi}, r, \bar{\mathbf{w}}) \in \mathrm{vK}^{\otimes}_{\kappa,\mu,\theta}$ *for* $\ell = 1, 2$ *and* $\mathbf{x}_1, \mathbf{x}_2$ *are smooth and similar as witnessed by* g*, see Definition 2.19(2)* <u>*then*</u> g *maps* $\mathbf{D}_{\mathbf{x}_1}$ *onto* $\mathbf{D}_{\mathbf{x}_2}$.

Remark 3.11. 1) We use a little less than the requirements in the definitions of $\mathrm{tK}_{\kappa,\bar{\mu},\theta}, \mathrm{vK}_{\kappa,\bar{\mu},\theta}$; see the proof, i.e. in $(*)_1$ below there is $\psi(\bar{x}_{\bar{d}}, \bar{c}, \bar{d}_*, \bar{c}_*) \in \mathrm{tp}(\bar{d}, \bar{c} \hat{\ } \bar{d}_* \hat{\ } \bar{c}_*)$ such that $\psi(\bar{x}_{\bar{d}}, \bar{c} \hat{\ } \bar{d}_* \hat{\ } \bar{c}_*) \vdash \varphi(\bar{x}_{\bar{d}}, \bar{c}, \bar{b}, a_1)$ <u>but</u> ψ may depend on b_1, a_1 and in $\varphi, \bar{c}_*, \bar{d}_*$ does not appear.
2) In 3.10(A),(1A) can we replace $\mathrm{tK}_{\kappa,\bar{\mu},\theta}, \mathrm{vK}_{\kappa,\bar{\mu},\theta}, \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ by $\mathrm{uK}_{\kappa,\mu,\theta}, \mathrm{qK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{uK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ respectively? No! We need universal solution, i.e. inside the proof that the type of $\bar{d}_* \hat{\ } \bar{c}_*$ over A is the same as that of $\bar{d}_{\mathbf{x}} \hat{\ } \bar{c}_{\mathbf{x}}$, so still this is less than stated.

Proof. 1) Let $B = B^+_{\mathbf{x}}$ and as usual let $\bar{c} = \bar{c}_{\mathbf{x}}, \bar{d} = \bar{d}_{\mathbf{x}}$. So it suffices to prove that $M^+ := M_{[\mathbf{x}]} = M_{[B + \bar{c} + \bar{d}]}$ is a κ-sequence-homogeneous model.

Let f be an elementary mapping from $A_1 \subseteq M^+$ onto $A_2 \subseteq M^+$ in the sense of M^+ and $|A_1| < \kappa$ and $b_1 \in M$ and we should find such $g \supseteq f$ for which $b_1 \in \mathrm{Dom}(g)$, this suffices. Let $A = B + A_1 + A_2 + b_1$. Let $f_0 = f, f_1 = f \cup \mathrm{id}_B$ and $f_2 = f_1 \cup \mathrm{id}_{\bar{c} + \bar{d}}$. By the definition of M^+ the mappings f_1, f_2 are elementary (in the sense of $\mathfrak{C}$, the default value). As $A \subsetneq M$ has cardinality $< \kappa$, recalling $\mathbf{x} \in \mathrm{tK}_{\kappa,\bar{\mu},\theta}$ there is $\bar{c}_* \hat{\ } \bar{d}_*$ in $M_{\mathbf{x}}$ realizing $\mathrm{tp}(\bar{c} \hat{\ } \bar{d}, A)$ such that:

$\odot_0$ $\mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_*) \vdash \mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_* + A)$.

But actually we need just

$\odot'_0$ $\mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_*) \vdash \mathrm{tp}(\bar{d}, \bar{c} + A)$.

By the choice of $(\bar{c}_*, \bar{d}_*)$, clearly the following function h is elementary for $\mathfrak{C}$:

$\odot''_0$ $\mathrm{Dom}(h) = A + \bar{c} + \bar{d}$ and $h \restriction A$ is the identity, $h(\bar{c} \hat{\ } \bar{d}) = \bar{c}_* \hat{\ } \bar{d}_*$.

Let $f'_2 = f_1 \cup \mathrm{id}_{\bar{c}_* + \bar{d}_*}$, so $f'_2 = h \circ f_2 \circ h^{-1}$ but h and f_2 are elementary so f'_2 is elementary too. Clearly $\mathrm{Dom}(f'_2) = \mathrm{Dom}(f_1) + \bar{c}_* + \bar{d}_* = B + A_1 + \bar{c}_* + \bar{d}_* \subseteq M_{\mathbf{x}}$ and $\mathrm{Rang}(f'_2) = \mathrm{Rang}(f_1) + \bar{c}_* + \bar{d}_* = B + A_2 + \bar{c}_* + \bar{d}_* \subseteq M_{\mathbf{x}}$. Hence there is an elementary mapping g_1 such that $g_1 \supseteq f'_2$ and $\mathrm{Dom}(g_1) = B + A_1 + \bar{c}_* + \bar{d}_* + b_1$ and without loss of generality $b_2 := g_1(b_1)$ belongs to $M_{\mathbf{x}}$ recalling $\mathrm{Rang}(f'_2) \subseteq M_{\mathbf{x}}$ and $M_{\mathbf{x}}$ is κ-saturated.

Let $g_2 = g_1 \cup \mathrm{id}_{\bar{c}}$, next:

$\odot_1$ (a) g_2 is an elementary mapping
(b) g_2 is with domain $B + A_1 + \bar{c}_* + \bar{d}_* + b_1 + \bar{c}$.

[Why? Clause (a) as $\mathrm{tp}(\bar{c}, M_{\mathbf{x}})$ does not split over B and $g_2 \supseteq g_1 \supseteq f'_2 \supseteq f_1 \supseteq \mathrm{id}_B$. Clause (b) holds as $\mathrm{Dom}(g_2) = \mathrm{Dom}(g_1) \cup \bar{c}$ by the choice of g_2 and $\mathrm{Dom}(g_1) = B + A + \bar{c}_* + \bar{d}_* + b_1$ as said above.]

Now assume for awhile:

$\odot_2$ $\bar{a}_1 \in {}^{\omega>}(B + A_1)$ and[31] $\mathfrak{C} \models \varphi[\bar{d}, \bar{c}, b_1, \bar{a}_1]$; let $\bar{a}_2 = f_1(\bar{a}_1)$.

Now

$(*)_0$ (a) $f_2 \supseteq f_1$ and $g_2 \supseteq g_1 \supseteq f'_2 \supseteq f_1$
(b) $\bar{a}_1 \subseteq (B + A_1) = \mathrm{Dom}(f_1)$, hence
(c) $g_2(\bar{a}_1) = \bar{a}_2$; also
(d) $g_2(b_1) = g_1(b_1) = b_2$
(e) g_2 is the identity on $B + \bar{c}_* + \bar{d}_* + \bar{c}$.

[E.g. why clause (e)? By their choices, f_1 is the identity on B, f'_2 is the identity on $\bar{c}_* + \bar{d}_*$ and g_2 is the identity on $\bar{c}$ hence by clause (a) we are done.]

We know that $\mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_*) \vdash \mathrm{tp}(\bar{d}, \bar{c} + A)$ by $\odot_0$ or $\odot'_0$ (why? $\varphi(\bar{x}_{\bar{d}}, \bar{c}, b_1, \bar{a}_1) \in \mathrm{tp}(\bar{d}, \bar{c} + b_2 + \bar{a}_2)$ by $\odot_2$; and $\mathrm{tp}(\bar{d}, \bar{c} + b_1 + \bar{a}_1) \subseteq \mathrm{tp}(\bar{d}, \bar{c} + A)$ because $\bar{c} + b_1 + a_1 \subseteq \bar{c}$ (why? $\bar{a}_1 \subseteq B + A_1 \subseteq A$, see $(*)_0(b)$ and $b_1 \in A$ by the choice of A)):

$(*)_1$ $\mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_*) \vdash \varphi(\bar{x}_{\bar{d}}, \bar{c}, b_1, \bar{a}_1)$.

So applying g_2 recalling $(*)_0(e)$

$(*)_2$ g_2 maps $\mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_*)$ to itself

As $(*)_1 + (*)_2$ hold and $g_2(b_1) = b_2, g_2(\bar{a}_1) = \bar{a}_2$ and $g_2(\bar{c}) = \bar{c}$ (by $(*)_0(d), (c), (e)$ respectively) recalling g_2 is an elementary mapping by $\odot_1(a)$ we get

$(*)_3$ $\mathrm{tp}(\bar{d}, \bar{c} + \bar{d}_* + \bar{c}_*) \vdash \varphi(\bar{x}_{\bar{d}}, \bar{c}, b_2, \bar{a}_2)$.

So it follows that:

$\odot_3$ $\mathfrak{C} \models \varphi[\bar{d}, \bar{c}, b_2, \bar{a}_2]$.

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[31] we can strengthen the demand on $\bar{a}_1$ to $\bar{a}_1 \in {}^{\omega>}(A_1 + B + \bar{d}_* + \bar{c}_*)$ and change according in later cases

We have proved $\odot_2 \Rightarrow \odot_3$ when $\bar a_1$ was any finite sequence from $B + A_1$. Recalling $(*)_0(e)$ and $g_2(b_1) = b_2, g_2(\bar a_1) = \bar a_2$ and $g_2(\bar c) = \bar c$, this means that $g_3 := (g_2{\restriction}(B + A_1 + b_1 + \bar c)) \cup \mathrm{id}_{\bar d}$ is an elementary mapping, so the function $g_3 = g_3{\restriction}(B + A_1 + b_1 + \bar c + \bar d)$ is an elementary mapping of $\mathfrak{C}$, so as $g_3{\restriction}(B + \bar c + \bar d)$ is the identity clearly $g := g_3{\restriction}(A_1 + b_1)$ is an elementary mapping in the sense of M^+, so g is as required.
1A) The proof above works now, too, except that not necessarily $\odot_0$ holds (and so $\odot'_0$, too) which was used only in proving $(*)_1$ so in proving $\odot_2 \Rightarrow \odot_3$, hence it suffices to prove $\odot_3$ assuming $\odot_2$. Let $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}; z, \bar y)$ so there is $\varphi_0 = \varphi_0(\bar x_{\bar d,\eta_1}, \bar x_{\bar c,\nu_1}; z, \bar y)$ equivalent to φ for some $\eta_1 \in {}^{\omega>}\ell g(\bar d_{\mathbf{x}}), \nu_1 \in {}^{\omega>}\ell g(\bar c_{\mathbf{x}})$, hence by the Definition 3.6(3A), a degenerated case[32], there are η_0, ν_0 such that:

$\oplus$ (a) $\eta_0, \eta_1 \in {}^{\omega>}\ell g(\bar d_{\mathbf{x}})$ and $\nu_0, \nu_1 \in {}^{\omega>}\ell g(\bar c_{\mathbf{x}})$
(b) $\ell g(\eta_0) = \ell g(\eta_1)$ and $\ell g(\nu_0) = \ell g(\nu_1)$, all finite
(c) $\mathfrak{C} \models$ "$\varphi_0[\bar d_{\mathbf{x},\eta_1}, \bar c_{\mathbf{x},\nu_1}, b', \bar a] \equiv \varphi_0[\bar d_{\mathbf{x},\eta_0}, \bar c_{\mathbf{x},\nu_0}, b', \bar a]$" for every $b' \in M_{\mathbf{x}}, \bar a' \in {}^{\ell g(\bar y)}(M_{\mathbf{x}})$

$\oplus'$ there is $\bar d_* {}^\frown \bar c_*$ from $M_{\mathbf{x}}$ realizing $\mathrm{tp}(\bar d_{\mathbf{x}} {}^\frown \bar c_{\mathbf{x}}, A)$ such that
(d) $\mathrm{tp}(\bar d_{\mathbf{x},\eta_0}, \bar c_{\mathbf{x},\nu_0} + \bar d_* + \bar c_*) \vdash \{\varphi_0(\bar x_{\bar d,\eta_0}, \bar c_{\mathbf{x},\nu_0}, b', \bar a') : b' \in A, \bar a' \in {}^{\ell g(\bar a_1)}A$ and $\mathfrak{C} \models \varphi_0[\bar d_{\mathbf{x},\eta_0}, \bar c_{\mathbf{x},\nu_0}, b', \bar a']\}$.

Now as in the proof of part (1) above we are assuming

$\odot_2$ $\bar a_1 \in {}^{\omega>}(B + A_1)$ and $\mathfrak{C} \models \varphi[\bar d, \bar c, b_1, \bar a_1]$.

By the choice of φ_0 this means that $\mathfrak{C} \models \varphi_0[\bar d_{\mathbf{x},\eta_1}, \bar c_{\mathbf{x},\nu_1}, b_1, \bar a_1]$ hence by $\oplus(c)$ we have:

$\odot'_2$ $\mathfrak{C} \models \varphi_0[\bar d_{\mathbf{x},\eta_0}, \bar c_{\mathbf{x},\nu_0}, b_1, \bar a_2]$.

Recalling $\oplus'(d)$, for this formula the proof of $\odot_2 \Rightarrow \odot_3$ in part (1) works so $\mathfrak{C} \models \varphi_0[\bar d_{\mathbf{x},\eta_0}, \bar c_{\mathbf{x},\nu_0}, b_2, \bar a_2]$. Using $\oplus(c)$ again this implies $\mathfrak{C} \models \varphi_0[\bar d_{\mathbf{x},\eta_1}, \bar c_{\mathbf{x},\nu_1}, b_2, \bar a_2]$. As this holds for any $\bar a_1 \in {}^{\omega>}(B + A_1)$ we finish as in part (1).
2) Assume

$\boxplus_{\mathrm{tK}}$ $\mathbf{m}_\ell = (\mathbf{x}_\ell, \bar\psi, r) \in \mathrm{tK}^{\oplus}_{\kappa,\bar\mu,\theta}$ for $\ell = 1, 2$
<u>or</u>
$\boxplus_{\mathrm{vK}}$ $\mathbf{m}_\ell = (\bar{\mathbf{x}}_\ell, \bar\psi, r, \bar{\mathbf{w}}) \in \mathrm{vK}^{\otimes}_{\kappa,\bar\mu,\theta}$ for $\ell = 1, 2$.

Assume further that g witnesses $\mathbf{x}_1, \mathbf{x}_2$ are similar; the proof is like the proof of parts (1),(1A) but we give some details so $g_0 = g{\restriction}B^+_{\mathbf{x}_1}$ is an elementary mapping from $B^+_{\mathbf{x}_1}$ onto $B^+_{\mathbf{x}_2}$.

Let $\bar c_\ell = \bar c_{\mathbf{x}_\ell}, \bar d_\ell = \bar d_{\mathbf{x}_\ell}$. Assume $\bar a_\ell \in {}^{\omega>}(M_{\mathbf{x}_\ell})$ for $\ell = 1, 2$. Let $\ell \in \{1, 2\}$ choose $\bar c^\ell_* {}^\frown \bar d^\ell_*$ as in Definition for $A_\ell := B^+_{\mathbf{x}_\ell} \cup \bar a_\ell$, so

$(*)_{1,\ell}$ (a) $\bar d^\ell_*, \bar c^\ell_*$ are from $M_{\mathbf{x}_\ell}$
(b) $\bar d^\ell_*, \bar c^\ell_*$ realize $\mathrm{tp}(\bar d_\ell {}^\frown \bar c_\ell, A_\ell)$
(c) $\bar d_\ell {}^\frown \bar c_\ell {}^\frown \bar d^\ell_* {}^\frown \bar c^\ell_*$ realizes $r_\ell := r[\mathbf{m}_\ell]$
(d)(α) if $\boxplus_{\mathrm{tK}}$ then $\mathrm{tp}(\bar d_\ell, \bar c + \bar d^\ell_* + \bar c^\ell_*) \vdash \mathrm{tp}(\bar d_\ell, \bar c + A_\ell)$
(β) if $\boxplus_{\mathrm{vK}}$ then $(\bar c^\ell_*, \bar d^\ell_*)$ solves $(\mathbf{x}_\ell, \bar\psi, r, \bar{\mathbf{w}})$, see 3.3(1)(f).

[32]this is a weak version of 3.6(3A) as $\eta_2, \nu_2, \eta_3, \nu_3$ does not appear

Let h_ℓ be the elementary mapping with domain $A_\ell + \bar c_\ell + \bar d_\ell, h_\ell \restriction A_\ell = \mathrm{id}_{A_\ell}, h_\ell(\bar c_\ell {}^\frown \bar d_\ell) = \bar c^\ell_* {}^\frown \bar d^\ell_*$, it is well defined and elementary by the choice of $\bar c^\ell_*, \bar d^\ell_*$.

Now $g_1 := h_2 \circ g \circ h_1^{-1}$ is an elementary mapping by $(*)_{1,\ell}(b)$, for $\ell = 1,2$. As g_1's domain is $\subseteq M_{\mathbf{x}_1}$ and its range is $\subseteq M_{\mathbf{x}_2}$ there is an extension g_2 of g_1 to an elementary mapping with domain $\supseteq A_1$ but $\subseteq M_{\mathbf{x}_1}$ and range $\supseteq A_2$ but $\subseteq M_{\mathbf{x}_2}$. Next extend g_2 to the mapping g_3 by letting $g_3(\bar c) = \bar c$, by the definition of "g witnesses the similarity of ..." easily also g_3 is an elementary mapping. Let g_4 be the mapping with domain $\mathrm{Rang}(\bar d_1 {}^\frown \bar c_1 {}^\frown \bar d^1_* {}^\frown \bar c^1_*)$ mapping $\bar d_1, \bar c_1, \bar d^1_*, \bar c^1_*$ to $\bar d_2, \bar c_2, \bar d^2_*, \bar c^2_*$ respectively, now g_4 is an elementary mapping by the assumption on r. Easily $g_3(\bar a_1)$ witness g maps $\mathrm{tp}(\bar a_1, \emptyset, M_{\mathbf{x}_1}) \in \mathbf{D}_{\mathbf{x}_1}$ to the member $\mathrm{tp}(g_3(\bar a_1), \emptyset, M_{[\mathbf{x}_1]})$ of $\mathbf{D}_{\mathbf{x}_2}$. Similarly for $g_3^{-1}, \bar a_2, M_{[\mathbf{x}_2]}, M_{[\mathbf{x}_1]}$.

So we are done. $\square_{3.10}$

Discussion 3.12. 1) Now we can start to see the relevance of tK, vK to the recounting of types; of course, the following conclusion will be helpful only if we prove the density of tK (or of vK).
2) Note that if we below like to use 3.10(1),(1A) rather than 3.10(2), we lose little using $\Sigma\{2^{2^\partial} : \partial < \mu_0\}$ instead $2^{<\mu_0}$.

Conclusion 3.13. *1) If $\kappa, \bar\mu, \theta$ are as in Definition 2.2, $\kappa = \kappa^{<\kappa} = \mu^{+\alpha}$ and*[33] *$2^\theta < \kappa$ and $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ <u>then</u> the number of $\{\mathrm{tp}(\bar d, M)$: for some $\mathbf{x} \in \mathrm{tK}_{\kappa,\bar\mu,\theta}$ we have $\bar d \trianglelefteq \bar d_{\mathbf{x}}, M_{\mathbf{x}} = M\}$ up to conjugacy is $\le 2^{<\mu_0} + |\alpha|^\theta$.*
2) Similarly for $\mathbf{x} \in \mathrm{vK}_{\kappa,\bar\mu,\theta}$.

Proof. 1) By part (2) recalling 3.7(2).
2) Let $\lambda = \kappa$; by 2.22(1) + 3.8(4) we can restrict ourselves to smooth $\mathbf{x} \in \mathrm{vK}_{\lambda,\kappa,\bar\mu,\theta}$. By 3.7(4A),(4B) we can deal with $\{\mathrm{tp}(\bar d, M_{\mathbf{m}}) : \mathbf{m} \in \mathrm{vK}^\otimes_{\kappa,\bar\mu,\theta}$ satisfies $\bar d \trianglelefteq \bar d_{\mathbf{m}}, M_{\mathbf{m}} = M$ and, as said above, $\mathbf{x}_{\mathbf{m}}$ is smooth$\}$.

Now if $(\mathbf{x}_\ell, \bar\psi, r, \bar{\mathbf{w}}) \in \mathrm{vK}^\otimes_{\kappa,\bar\mu,\theta}$, see 3.6(4C) are smooth for $\ell = 1,2$ and $\mathbf{x}_1, \mathbf{x}_2$ are similar as witnessed by g then g maps $\mathbf{D}_{\mathbf{x}_1}$ onto $\mathbf{D}_{\mathbf{x}_2}$, see 3.10(2) hence by the uniqueness of the $(\mathbf{D}, \kappa)$-sequence-homogeneous model of cardinality κ there is an automorphism f of M such that $g \cup f$ is an elementary mapping. Hence $\mathrm{tp}(\bar d_{\mathbf{x}_1}, M), \mathrm{tp}(\bar d_{\mathbf{x}_2}, M)$ are conjugate. We are done as: the number of relevant triples $(\bar\psi, r, \bar{\mathbf{w}})$ is $\le 2^\theta$ and the number of $\mathbf{m} \in \mathrm{vK}^\otimes_{\kappa,\bar\mu,\theta}$ with $M_{\mathbf{m}} = M, (\bar\psi_{\mathbf{m}}, r_{\mathbf{m}}, \bar{\mathbf{w}}_{\mathbf{m}}) = (\bar\psi, r, \mathbf{w})$ up to similarly is $\le 2^{<\mu_0} + |\alpha|^\theta$ if $\mathrm{cf}(\mu_0) > \theta$ and $2^{\mu_0} + |\alpha|^\theta$ if $\mathrm{cf}(\mu_0) < \theta$. The $2^{\mu_0}/2^{<\mu_0}$ comes from the type of $\bar b_{\mathbf{x}}$ where $\bar b_{\mathbf{x}}$ consists of: $\bar b_{\mathbf{x},i}$ listing B_i for $i \in v_{\mathbf{m}} \backslash u_{\mathbf{m}}$ and $(\bar a_{\mathbf{x},i,0} {}^\frown \ldots {}^\frown \bar a_{\mathbf{x},i,n} {}^\frown \ldots)_{n<\omega}$ if $i \in u_{\mathbf{m}}$ (and of course the respective lengths, etc.); the $|\alpha|^\theta$ is for the choice of the $\langle \kappa_{\mathbf{x},i} : i \in u_{\mathbf{x}} \rangle$.

Now for each $\mathbf{x} \in \mathrm{vK}_{\kappa,\bar\mu,\theta}$ the set $\{\bar d : \bar d \trianglelefteq \bar d_{\mathbf{x}}\}$ is $\le \theta$ (even allowing $\{\bar d : \bar d$ a sub-sequence of $\bar d_{\mathbf{x}}\}$ gives $2^\theta \le 2^{<\mu_0}$). $\square_{3.13}$

* * *

Now we turn to proving sufficient conditions for (some versions of) indiscernibility, they are naturally related to tK and vK.

Claim 3.14. *$\langle \bar c_s {}^\frown \bar d_s : s \in I \rangle$ is an indiscernible sequence over B <u>when</u>:*

[33] without assuming it we have just to replace $\mathrm{tK}_{\kappa,\bar\mu,\theta}/\mathrm{vK}_{\kappa,\bar\mu,\theta}$ by $\mathrm{tK}^\oplus_{\kappa,\bar\mu,\theta}/\mathrm{vK}^\otimes_{\kappa,\bar\mu,\theta}$.

(a) *$I \in K_{p,\sigma}$, see Definition 1.39*

(b) *if $s <_I t$ are E_I-equivalent then* $\mathrm{tp}(\bar c_s \hat{}\, \bar d_s, B_s) \subseteq \mathrm{tp}(\bar c_t \hat{}\, \bar d_t, B_t)$ *where*

  (α) $E_I = \{(s,t) : (\exists i < \sigma)(s,t \in P^I_i)\}$

  (β) $B_t = \cup\{\bar c_s \hat{}\, \bar b_s : s <_I t\} \cup B$

  (γ) $\ell g(\bar d_s), \ell g(\bar c_s)$ *for $s \in I$ depend just on s/E_I.*

(c) $\mathrm{tp}(\bar c_s, B_s)$ *does not split over B*

(d) *if $s \in P^I_i$ and $t \in P^I_j$ and $s <_I t$ then $r_{i,j} = \mathrm{tp}(\bar c_s \hat{}\, \bar d_s \hat{}\, \bar c_t \hat{}\, \bar d_t, \emptyset)$, i.e. depend only on (i,j)*

(e) $\mathrm{tp}(\bar d_t, \bar c_t + \bar d_s + \bar c_s) \vdash \mathrm{tp}(\bar d_t, \bar c_t + \bar d_s + \bar c_s + B_s)$ *when $s <_I t$*
  or just

(e)′ *if $s <_I t, \eta_1 \in {}^{\omega >}\ell g(\bar d_t), \nu_1 \in {}^{\omega >}\ell g(\bar c_t), \eta_3 \in {}^{\omega >}\ell g(\bar d_s), \nu_3 \in {}^{\omega >}\ell g(\bar c_s)$ and $\varphi = \varphi(\bar x_{\bar d_t,\eta_1}, \bar x_{\bar c_t,\nu_1}, \bar x_{\bar d_s,\eta_3}, \bar x_{\bar d_s,\nu_3}, \bar y)$ then for some $\eta_0,\nu_0,\eta_2,\nu_2$ (depending on $(s/E_I, t/E_I, \eta_1,\nu_1,\eta_3,\nu_3)$ but not on (s,t)) we have*

  (α) *$\eta_0 \in {}^{\ell g(\eta_1)}(\ell g(\bar d_t))$ and $\nu_0 \in {}^{\ell g(\nu_1)}(\ell g(\bar c_t))$ and $\eta_2 \in {}^{\ell g(\eta_3)}(\ell g(\bar d_s))$ and $\nu_2 \in {}^{\ell g(\nu_3)}(\ell g(\bar c_s))$*

  (β) *if $\bar b \in {}^{\ell g(\bar y)}(B_s)$ then*
  $\mathfrak{C} \models$ "$\varphi[\bar d_{t,\eta_0}, \bar c_{t,\nu_0}, \bar d_{s,\eta_2}, \bar c_{s,\nu_2}, \bar b] \equiv \varphi[\bar d_{t,\eta_1}, \bar c_{t,\nu_1}, \bar d_{s,\eta_3}, \bar c_{s,\nu_3}, \bar b]$"

  (γ) $\mathrm{tp}(\bar d_{t,\eta_0}, \bar c_t + \bar d_s + \bar c_s) \vdash \{\varphi(\bar x_{\bar d_{t,\eta_0}}, \bar c_{t,\nu_0}, \bar d_{s,\eta_2}, \bar c_{s,\nu_2}, \bar b) : \bar b \in {}^{\ell g(\bar y)}(B_s)$ *and* $\mathfrak{C} \models \varphi[\bar d_{t,\eta_0}, \bar c_{t,\nu_0}, \bar d_{s,\eta_2}, \bar c_{s,\nu_2}, \bar b]\}$.

Proof. Recall $E = \{(s,t) : s,t \in P^I_i$ for some $i < \sigma\}$. We prove by induction on n that

$(*)_n$ if $s_0 < \ldots < s_{n-1}$ and $t_0 < \ldots < t_{n-1}$ and $\ell < n \Rightarrow s_\ell E t_\ell$ then the sequence $\bar c_{s_0} \hat{}\, \bar d_{s_0} \hat{}\, \ldots \hat{}\, \bar c_{s_{n-1}} \hat{}\, \bar d_{s_{n-1}}$ and the sequence $\bar c_{t_0} \hat{}\, \bar d_{t_0} \hat{}\, \ldots \hat{}\, \bar c_{t_{n-1}} \hat{}\, \bar d_{t_{n-1}}$, realize the same type over $B_{\min\{s_0,t_0\}}$.

The case $n = 0$: The desired conclusion is trivial.

The case $n = 1$: By clause (b) of the assumption, i.e. for any $t_* \in I$ and $i < \sigma$, the type $p_t = \mathrm{tp}(\bar c_t \hat{}\, \bar d_t, B_{t_*})$ is constant for $t \in \{s : s \in P^I_i$ and $t_* \leq_I s\}$.

The case $n = m+1, m \neq 0$:

By clause (b) of the claim assumption, without loss of generality $s_m = t_m$ call it $t(*)$ and let $s(*) = \min\{s_0, t_0\}$.

Let $f_0 = \mathrm{id}_{B_{s(*)}}$, let f_1 be the function with domain $B_{s(*)} + \bar c_{s_0} \hat{}\, \bar d_{s_0} + \ldots + \bar c_{s_{m-1}} \hat{}\, \bar d_{s_{m-1}}$ such that $f_1 \supseteq f_0$ and $f_1(\bar c_{s_\ell} \hat{}\, \bar d_{s_\ell}) = \bar c_{t_\ell} \hat{}\, \bar d_{t_\ell}$ for $\ell < m$, it is an elementary mapping by the induction hypothesis. Let $f_2 = f_1 \cup \mathrm{id}_{\bar c_{t(*)}}$, it is an elementary mapping as $\mathrm{tp}(\bar c_{t(*)}, B_{t(*)})$ does not split over B by clause (c) of the claim assumption.

Let f_3 be an elementary mapping (in $\mathfrak{C}$) extending f_2 with domain $\mathrm{Dom}(f_2) + \bar d_{t(*)} = \mathrm{Dom}(f_1) + \bar c_{s_m} \hat{}\, \bar d_{s_m} = B_{s(*)} + \bar c_{s_0} \hat{}\, \bar d_{s_0} \hat{}\, \ldots \hat{}\, \bar c_{s_m} \hat{}\, \bar d_{s_m}$ and let $\bar d'_{t_m} = f_3(\bar d_{s_m})$. Let $i,j < \sigma$ be such that $s_{m-1} \in P^I_i, s_m \in P^I_j$.

So

$\boxplus_1$ $\mathrm{tp}(\bar d'_{t_m}, \bar c_{t_m} + \bar d_{t_{m-1}} + \bar c_{t_{m-1}}) = \mathrm{tp}(\bar d_{t_m}, \bar c_{t_m} + \bar d_{t_{m-1}} + \bar c_{t_{m-1}})$.

[Why? By clause (d) of the claim assumption as $s_\ell E t_\ell$ for $\ell = m-1, m$ and $s_{m-1} <_I s_m, t_{m-1} <_I t_m$ we have $\mathrm{tp}(\bar d_{t_m} \hat{}\, \bar c_{t_m} \hat{}\, \bar d_{t_{m-1}} \hat{}\, \bar c_{t_{m-1}}, \emptyset, \mathfrak{C}) = \mathrm{tp}(\bar d_{s_m} \hat{}\, \bar c_{s_m} \hat{}\, \bar d_{s_{m-1}} \hat{}\, \bar c_{s_{m-1}}, \emptyset, \mathfrak{C})$. By recalling the choice of f_2, f_3 and $\bar d'_{t_m}$ we have $\mathrm{tp}(\bar d'_{t_m} \hat{}\, \bar c_{t_m} \hat{}\, \bar d_{t_{m-1}} \hat{}\, \bar c_{t_{m-1}}, \emptyset, \mathfrak{C}) = \mathrm{tp}(\bar d_{s_m} \hat{}\, \bar c_{s_m} \hat{}\, \bar d_{s_{m-1}} \hat{}\, \bar c_{s_{m-1}}, \emptyset, \mathfrak{C})$.

Together we are done.]

The proof now splits into two cases.

Case 1: Clause (e) of the claim assumption holds.

$\boxplus_2$ $\bar d_{t_m}, \bar d'_{t_m}$ realize the same type over $\bar c_{t_m} + C$ where $C = B_{s(*)} + \bar c_{t_0} \hat{}\, \bar d_{t_0} + \ldots + \bar c_{t_{m-1}} \hat{}\, \bar d_{t_{m-1}}$ that is $\mathrm{Rang}(f_1)$.

Why $\boxplus_2$ holds?

Clearly $\mathrm{tp}(\bar d_{t_m}, \bar c_{t_m} + \bar d_{t_{m-1}} + \bar c_{t_{m-1}}) \vdash \mathrm{tp}(\bar d_{t_m}, \bar c_{t_m} + C)$ and together with $\boxplus_1$ we are done.

Case 2: Clause (e)$'$ of the claim assumption holds.

So assume

$\odot_1$ $\mathfrak{C} \models \varphi[\bar d_{t_m,\eta_1}, \bar c_{t_m,\nu_1}, \bar d_{t_{m-1},\eta_3}, \bar c_{t_{m-1},\nu_3}, \bar b]$ where $\bar b \in {}^{\ell g(\bar y)}C$ and $\eta_1 \in {}^{\omega >}\ell g(d_{t_m}), \nu_1 \in {}^{\omega >}\ell g(\bar c_{t_m}), \eta_3 \in {}^{\omega >}\ell g(\bar d_{t_{m-1}}), \nu_3 \in {}^{\omega >}\ell g(\bar c_{t_{m-1}})$ all finite.

By clause (e)$'$ we can find $\eta_0, \nu_0, \eta_2, \nu_2$ as there.

By $\odot_1$ and subclause (β) of $(e)'$ we have

$\odot_2$ $\mathfrak{C} \models \varphi[\bar d_{t_m,\eta_0}, \bar c_{t_m,\nu_0}, \bar d_{t_{m-1},\eta_2}, \bar c_{t_{m-1},\nu_2}, \bar b]$.

By $\odot_2$ and subclause (γ) of (e)$'$ we have:

$\odot_3$ $\mathrm{tp}(\bar d_{t_m,\eta_0}, \bar c_{t_m} + \bar d_{t_{m-1}} + \bar c_{t_{m-1}}) \vdash \varphi(\bar x_{\bar d_m,\eta_0}, \bar c_{t_m,\nu_0}, \bar d_{t_{m-1},\eta_2}, \bar c_{t_{m-1},\nu_2}, \bar b)$.

But $\mathrm{tp}(\bar d'_{t_m,\eta_0}, \bar c_{t_m} + \bar d_{t_{m-1}} + \bar c_{t_{m-1}}) = \mathrm{tp}(\bar d_{t_m,\eta_0}, \bar c_{t_{m-1}} + \bar d_{t_m} + \bar c_{t_{m-1}})$ by $\boxplus_1$ so by $\odot_3$ we have

$\odot_4$ $\mathrm{tp}(\bar d'_{t_m,\eta_0}, \bar c_{t_m} + \bar d_{t_{m-1}} + \bar c_{t_{m-1}}) \vdash \varphi(\bar x_{\bar d_{t(n)},\eta_0}, \bar c_{t_m,\nu_0}, \bar d_{t_{m-1},\eta_2}, \bar c_{t_{m-1},\nu_2}, \bar b)$

hence

$\odot_5$ $\mathfrak{C} \models \varphi[\bar d'_{t_m,\eta_0}, \bar c_{t_m,\nu_0}, \bar d_{t_{m-1},\eta_2}, \bar c_{t_{m-1},\nu_2}, \bar b]$.

We can apply the elementary mapping f_3^{-1} whose range include all of the elements of $\mathfrak{C}$ appearing in $\odot_5$ hence we get

$\odot_6$ $\mathfrak{C} \models \varphi[\bar d_{s_m,\eta_0}, \bar c_{s_m,\nu_0}, \bar d_{s_{m-1},\eta_2}, \bar c_{s_{m-1},\nu_2}, f_s^{-1}(\bar b)]$.

By subclause (β) of $(e)'$ and $\odot_6$ we have (recalling from $(e)'$ depending on s/E_J, but not on (s,t))

$\odot_7$ $\mathfrak{C} \models \varphi[\bar d_{s_m,\eta_1}, \bar c_{s_m,\nu_1}, \bar d_{t_{m-1},\eta_2}, \bar c_{s_{m-1},\nu_2}, \bar b]$.

As this holds for any such φ we have finished proving $(*)_n$ also in Case 2, so we are done. $\square_{3.14}$

Discussion 3.15. 1) Naturally we can prove finitary versions of 3.14 in some senses. Below we deal with **k**-indiscernibility; another variant deals with Δ-indiscernible.
2) See 3.18.

Claim 3.16. *The sequence $\langle \bar c_{s,0}\hat{}\bar d_{s,0} : s \in I\rangle$ is $\mathbf{k}$-indiscernible over B_0 <u>when</u> the sequences $\langle(\bar d_{s,\ell},\bar c_{s,\ell}) : \ell \le \mathbf{k}, s \in I\rangle, \langle B_\ell : \ell \le \mathbf{k}\rangle$ satisfy:*

(a) $I \in K_{p,\sigma}$ *and* $\ell g(\bar c_{s,\ell})$ *and* $\ell g(\bar d_{s,\ell})$ *depend just on* ℓ *and* $\mathbf{i}(s) :=$ *the unique* i *such that* $s \in P^I_i$; *also* $\bar c_{s,\ell} \trianglelefteq \bar c_{s,\ell+1}, \bar d_{s,\ell} \trianglelefteq \bar d_{s,\ell+1}$ *for* $\ell < \mathbf{k}, s \in I$

(b) $\mathrm{tp}(\bar d_{s,\mathbf{k}}\hat{}\bar c_{s,\mathbf{k}}, B_s)$ *is* $\subseteq \mathrm{tp}(\bar c_{t,\mathbf{k}}\hat{}\bar d_{t,\mathbf{k}}, B_t)$ *when* $s <_I t \wedge \mathbf{i}(s) = \mathbf{i}(t)$ *(hence this holds for* $\ell < \mathbf{k}$, *too) where* $B_t = B_{t,\mathbf{k}}$ *and*
$B_{t,k} = \cup\{\bar d_{s,k}\hat{}\bar c_{s,k} : s <_I t\} \cup B_k$ *for* $k \le \mathbf{k}$

(c) $\mathrm{tp}(\bar c_{s,k}, B_{s,k})$ *does not split over* B_k *and* $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_{\mathbf{k}}$

(d) *if* $s \in P^I_i, t \in P^I_j$ *and* $s <_I t$ *then* $r_{i,j} = \mathrm{tp}(\bar c_{s,\mathbf{k}}\hat{}\bar d_{s,\mathbf{k}}\hat{}\bar c_{t,\mathbf{k}}\hat{}\bar d_{t,\mathbf{k}},\emptyset)$

(e) $\mathrm{tp}(\bar d_{t,\ell},\bar c_{t,\ell+1}+\bar d_{s,\ell+1}+\bar c_{s,\ell+1}) \vdash \mathrm{tp}(\bar d_{t,\ell},\bar c_{t,\ell}+\bar d_{s,\ell}+\bar c_{s,\ell}+B_{s,\ell})$ *when* $s <_I t$ <u>*or just*</u>

(e)′ *if* $s <_I t, \ell < \mathbf{k}, \eta_1 \in {}^{\omega>}\ell g(\bar d_{t,\ell}), \nu_1 \in {}^{\omega>}\ell g(\bar c_{t,\ell}), \eta_3 \in {}^{\omega>}\ell g(\bar d_{s,\ell}), \nu_3 \in {}^{\omega>}\ell g(\bar c_{s,\ell})$ *are all finite and* $\varphi'(\bar x_{\bar d_{t,\ell}},\bar x_{\bar c_{t,\ell}},\bar x'_{\bar d_{t,\ell}},x'_{\bar c_{t,\ell}},\bar y) = \varphi = \varphi(\bar x_{\bar d_{t,\ell,\eta_1}},\bar x_{\bar c_{t,\ell,\nu_1}},\bar x'_{\bar d_{s,\ell,\eta_3}},\bar x'_{\bar c_{s,\ell,\nu_3}},\bar y) \in \mathbb{L}(\tau_T)$, <u>*then*</u> *we can find* $\eta_0 \in {}^{\omega>}\ell g(\bar d_{t,\ell}), \nu_0 \in {}^{\omega>}\ell g(\bar c_{t,\ell+1}), \eta_2 \in {}^{\omega>}\ell g(\bar d_{s,\ell+1}), \nu_2 \in {}^{\omega>}\ell g(\bar c_{s,\ell+1})$ *(depending on* $s/E_I, t/E_I, \ell, \eta_1, \nu_1, \eta_3, \nu_3$ *but not on* (s,t)*) such that:*

   (α) $\ell g(\eta_0) = \ell g(\eta_1)$ *and* $\ell g(\nu_0) = \ell g(\eta_1)$ *and* $\ell g(\eta_2) = \ell g(\eta_3)$ *and* $\ell g(\nu_2) = \ell g(\nu_3)$

   (β) *if* $\bar b \in {}^{\ell g(\bar y)}(B_s)$ *then*
   $\mathfrak{C} \models \varphi[\bar d_{t,\ell,\eta_0},\bar c_{t,\ell+1,\nu_0},\bar d_{s,\ell+1,\eta_2},\bar c_{s,\ell+1,\nu_2},\bar b] \equiv \varphi[\bar d_{t,\ell,\eta_1},\bar c_{t,\ell,\nu_1},\bar d_{s,\ell,\eta_3},\bar c_{s,\ell,\nu_3},\bar b]$

   (γ) $\mathrm{tp}(\bar d_{t,\ell,\eta_0},\bar c_{t,\ell+1}+\bar d_{s,\ell+1}+\bar c_{s,\ell+1}) \vdash \mathrm{tp}_\varphi(\bar d_{t,\ell,\eta_0},(\bar c_{t,\ell+1,\nu_0}+\bar d_{s,\ell+1,\eta_2}+\bar c_{s,\ell+1,\nu_2}) \dot{+} B_{s,\ell})$.

Remark 3.17. 1) Note that $(e) \Rightarrow (e)'$.
2) In clause $(e)'$ we may use "$\eta_0 \in {}^{\omega>}\ell g(\bar d^*_{t,\ell})$" rather than "$\eta_0 \in {}^{\omega>}\ell g(\bar d_{t,\ell+1})$" when we add $\bar d^*_{s,\ell}$ such that $\bar d_{s,\ell} \trianglelefteq \bar d^*_{s,\ell} \trianglelefteq \bar d_{s,\ell+1}$ and use $\bar d^*_{s,\ell,\eta_0}$ instead $\bar d_{s,\ell,\eta_0}$ in clause $(e)'(\gamma)$.

Proof. We prove by induction on $k < \mathbf{k}$ that:

(∗) if $\varrho_1,\varrho_2 \in {}^{k+1}I$ are $<_I$-decreasing, $(\forall \ell \le k)(\exists i < \sigma)[\varrho_1(\ell),\varrho_2(\ell) \in P^I_i]$ and $s \le_I \varrho_1(k),\varrho_2(k)$ <u>then</u> the sequences $\bar d_{\varrho_\ell(0),\mathbf{k}-k}\hat{}\bar c_{\varrho_\ell(0),\mathbf{k}-k}\hat{}\ldots\hat{}\bar d_{\varrho_\ell(k),\mathbf{k}-k}\hat{}\bar c_{\varrho_\ell(k),\mathbf{k}-k}$ for $\ell = 1,2$ realize the same type over $B_{s,\mathbf{k}-k}$.

<u>Case $k = 0$</u>: This holds by clause (b) of the claim as $\ell g(\varrho_1) = 1 = \ell g(\varrho_2)$.

<u>Case $k > 0$</u>: For $\iota = 1,2$, let $\rho_\iota = \langle \varrho_\iota(1+m) : m < k\rangle$ and for $i \le \mathbf{k}$ let $\bar c_{\rho_\iota,i} = \bar c_{\rho_\iota(0),i}\hat{}\ldots\hat{}\bar c_{\rho_\iota(k-1),i}$ and $\bar d_{\rho_\iota,i} = \bar d_{\rho_\iota(0),i}\hat{}\ldots\hat{}\bar d_{\rho_\iota(k-1),i}$ so the induction hypothesis applies and if $k = 1$ then $\bar c_{\rho_\iota,i} = \bar c_{\eta_\iota(1),i}, \bar d_{\rho_\iota,i} = \bar d_{\eta_\iota(1)}$ for $i = 0$.

Note that $\bar c_{\rho_\iota,i}$ is a subsequence of $\bar c_{\rho_\iota,i+1}$ and $\bar d_{\rho_\iota,i}$ is a subsequence of $\bar d_{\rho_\iota,i+1}$.

By the case $k = 0$, that is by clause (b) without loss of generality $\varrho_1(0) = \varrho_2(0)$ call it t. So assume

$(*)_1$ $\varphi = \varphi(\bar x_{\bar d_{t,\mathbf{n}-k}},\bar x_{\bar c_{t,\mathbf{n}-k}},\bar x'_{\bar d_{\rho_1,\mathbf{n}-k}},\bar x'_{\bar c_{\rho_1,\mathbf{n}-k}},\bar z)$ and
$\bar b \in {}^{\ell g(\bar z)}(B_{s,\mathbf{k}-k})$ and $\mathfrak{C} \models \varphi[\bar d_{t,\mathbf{n}-k},\bar c_{t,\mathbf{n}-k},\bar d_{\rho_1,\mathbf{n}-k},\bar c_{\rho_1,\mathbf{n}-k},\bar b]$.

We should prove the parallel statement for ϱ_2, i.e. for t and ρ_2.

Subcase 1: Clause (e) of the assumption.

Hence by clause (e) there is a formula $\psi_\varphi = \psi_\varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{x}_{\bar{c}_{t,\mathbf{k}-k+1}}, \bar{x}'_{\bar{d}_{\rho_1(0),\mathbf{k}-k+1}}, \bar{x}'_{\bar{c}_{\rho_1(0),\mathbf{k}-k+1}})$ such that

$(*)_2$ (a) $\mathfrak{C} \models \psi_\varphi[\bar{d}_{t,\mathbf{k}-k}, \bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_1(0),\mathbf{k}-k+1}, \bar{c}_{\rho_1(0),\mathbf{k}-k+1}]$

(b) $\psi_\varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_1(0),\mathbf{k}-k+1}, \bar{c}_{\rho_1(0),\mathbf{k}-k+1}) \vdash \varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{c}_{t,\mathbf{k}-k}, \bar{d}_{\rho_1,\mathbf{k}-k}, \bar{c}_{\rho_1,\mathbf{k}-k}, \bar{b})$.

Hence

$(*)_3$ (a) $\mathfrak{C} \models \vartheta_\varphi[\bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_1,\mathbf{k}-k+1}, \bar{c}_{\rho_1,\mathbf{k}-k+1}, \bar{b}]$ where

(b) $\vartheta_\varphi(\bar{x}_{\bar{c}_{t,\mathbf{k}-k+1}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{k}-k+1}}, \bar{x}'_{\bar{c}_{\rho_1,\mathbf{k}-k+1}}, \bar{z}) :=$
$(\forall \bar{x}_{\bar{d}_{s,\mathbf{k}-k}})[\psi_\varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{x}_{\bar{c}_{t,\mathbf{k}-k+1}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{k}-k+1}}, \bar{x}'_{\bar{c}_{\rho_1,,\mathbf{k}-k+1}})$
$\rightarrow \varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{x}_{\bar{c}_{t,\mathbf{k}-k}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{k}-k}}, \bar{x}'_{\bar{c}_{\rho_1,\mathbf{k}-k}}, \bar{z})]$.

Now

$(*)_4$ $\bar{d}_{\rho_\iota,\mathbf{k}-k+1} \hat{\ } \bar{c}_{\rho_\iota,\mathbf{k}-k+1} \hat{\ } \bar{b}$ realize the same type over $B_{s,\mathbf{k}-k+1}$ for $\iota = 1,2$.

[Why? By the induction hypothesis as $\bar{b}$ is from $B_{s,\mathbf{k}-k} \subseteq B_{s,\mathbf{k}-k+1}$.]

$(*)_5$ $\bar{c}_{t,\mathbf{n}-k+1} \hat{\ } \bar{d}_{\rho_\iota,\mathbf{k}-k+1} \hat{\ } \bar{c}_{\rho_\iota,\mathbf{k}-k+1} \hat{\ } \bar{b}$ realize the same type over $B_{s,\mathbf{k}-k+1}$ for $\iota = 1,2$.

[Why? As first, $\mathrm{tp}(\bar{c}_{t,\mathbf{k}-k+1}, B_{t,k+1})$ does not split over $B_{s,k+1}$ by clause (c) of the assumption, second $\bar{d}_{\rho_\ell,\mathbf{k}-k+1}, \bar{c}_{\rho_\ell,\mathbf{k}-k+1}, \bar{b}$ are included in $B_{t,k+1}$ and third $(*)_4$.]

$(*)_6$ in $(*)_3(a)$ we can replace ρ_1 by ρ_2, i.e. $\mathfrak{C} \models \vartheta_\varphi[\bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_2,\mathbf{k}-k+1}, \bar{c}_{\rho_2,\mathbf{k}-k+1}, \bar{b}]$.

[Why? By $(*)_5$ and $(*)_3(a)$.]

$(*)_7$ $\mathfrak{C} \models \psi_\varphi[\bar{d}_{t,\mathbf{k}-k}, \bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_2(0),\mathbf{k}-k+1}, \bar{c}_{\rho_2(0),\mathbf{k}-k+1}]$.

[Why? By clause (d) of the hypothesis of the claim and $(*)_2(a)$.]

$(*)_8$ $\mathfrak{C} \models \varphi[\bar{d}_{t,\mathbf{k}-k}, \bar{c}_{t,\mathbf{k}-k}, \bar{d}_{\rho_2,\mathbf{k}-k}, \bar{c}_{\rho_2,\mathbf{k}-k}, \bar{b}]$.

[Why? By $(*)_6 + (*)_7$ and the definition of ϑ in $(*)_3(b)$.]

So we are done.

Subcase 2: Clause (e)$'$ of the assumption holds.

Similarly as in the proof of 3.14 and see the proof of 3.21. $\square_{3.16}$

Claim 3.18. *The conclusions of 3.14, 3.16 and 3.21 below still hold (and even $(*)$ from its proof holds) even under the following weaker assumptions*

(a) (α) $I \in K_{p,\sigma}$

(β) *we add* $I_\iota \subseteq I$ *for* $\iota = 1,2$ *and* $t \in I \Rightarrow t \in I_1 \vee t \in I_2$

$(b),(c),(d)$ *the same*

$(e),(e)'$ *the same but only for* I_1 *and for* I_2 *(but not for* $I_1 \cup I_2$*)*

(f) *if* $t_1 <_I t_2, \iota \in \{1,2\}$ *and* $t_1 \in I_\iota \backslash I_{3-\iota}$ *and* $t_2 \in I_{3-\iota} \backslash I_\iota$ *and* $\eta \in {}^{\mathbf{k}}\sigma$ *(for 3.16, 3.21* $\mathbf{k}$ *is given, otherwise any* $\mathbf{k} < \omega$*)* *then* *we can find* $s_0 < \ldots < s_{\mathbf{k}-1}$ *from* $(t_1, t_2)_{I_1 \cap I_2}$ *such that* $s_\iota \in P^I_{\eta(\iota)}$ *for* $\ell < \mathbf{k}$.

Remark 3.19. 1) The case which suffice in 3.23 below is $I_1 = [0, \omega + \omega), I_2 = [0, \omega + \omega + 1) \backslash \{\omega\}$ which is somewhat easier.
2) In the natural case, for decreasing $\varrho \in {}^{n+1}I$ we have $\mathrm{tp}(\bar d_{\varrho(0)} \hat{} \ldots \hat{} \bar d_{\varrho(n-1)}, \bar c_{\varrho(0)} \hat{} \ldots \hat{} \bar c_{\varrho(n)} + \bar d_{\varrho(n)}) \vdash \mathrm{tp}(\bar d_{\varrho(0)} \hat{} \ldots \hat{} \bar d_{\varrho(n-1)-1}, \bar c_{\varrho(0)} \hat{} \ldots \hat{} \bar c_{\varrho(n)} + d_{\varrho(n)} + B_{\varrho(n)})$ and it is quite natural to use this.
3) A variant is: e.g. $\langle \bar c_s \hat{} \bar d_s : s \in I \rangle$ is an indiscernible sequence over B when we assume $(a) + (b)$ of 3.14 and (a),(f) of 3.18 and

(g) if $s \in I$ and $\iota \in \{1,2\}$ then $\langle \bar c_t \hat{} \bar d_t : t \in I_\iota$ and $t \geq s \rangle$ is an indiscernible sequence over B_s.

Proof. It is enough to prove this when $I_1 \backslash I_2, I_2 \backslash I_1$ is finite by induction on $|I_1 \backslash I_2| + |I_2 \backslash I_1|$ (probably losing appropriately in ℓ for 3.16). So without loss of generality this number is 2. By symmetry without loss of generality $I_1 \backslash I_2 = \{t_1\}, I_2 \backslash I_1 = \{t_2\}$ and $t_1 <_I t_2$. The rest should be clear by the transitivity of the equality of types. I.e. for notational simplicity concerning 3.14, by it we know

$\boxplus$ if $\iota \in \{1,2\}, t \in I$, then $\langle \bar c_s \hat{} \bar d_s : s \in (I_\iota)_{\geq t} \rangle$ is an indiscernible sequence over $\cup \{\bar c_s \hat{} \bar d_s : s \in I_{<t}\} \cup B$.

It suffices to prove

$\oplus$ if $s_0 <_I \ldots <_I s_{n-1}$ then for some $r_0 <_{I_1} \ldots <_{I_1} r_{n-1}$, (so all from I_1) the sequence $\bar c_{s_0} \hat{} \bar d_{s_0} \hat{} \ldots \hat{} \bar c_{s_{n-1}} \hat{} \bar d_{s_{n-1}}$ realizes the same type over B as $\bar c_{r_0} \hat{} \bar d_{r_0} \hat{} \ldots \hat{} \bar c_{r_{n-1}} \hat{} \bar d_{r_{n-1}}$.

Why $\oplus$ holds? Now if $t_2 \notin \{s_0, \ldots, s_{n-1}\}$ this is obvious, so assume $t_2 = s_{k(2)}$, and let $k(1)$ be minimal such that $t_1 <_I s_{k(1)}$, so $k(1) \leq k(2)$; (we can even demand $t_1 = s_{k(1)-1}$, but not used). By clause (f) there are $r_{k(1)} <_I \ldots <_I r_{k(2)}$ from $(t_1, t_2)_{I_1 \cap I_2}$ such that $k \in [k(1), k(1)] \wedge i < \sigma \Rightarrow r_k \in P^I_i \Leftrightarrow s_k \in P^I_i$ and let $r_k = s_k$ for $k < n$ such that $k \notin [k(1), k(2)]$.

So applying $\boxplus$ for $\iota = 2, t = t_{k(1)}$ we know that $\bar c_{r_{k(1)}} \hat{} \bar d_{r_{k(1)}} \hat{} \ldots \hat{} \bar c_{r_{n-1}} \hat{} \bar d_{r_{n-1}}$ realizes over $B_{\min\{s_{k(1)}, r_{k(1)}\}}$ the same type as $\bar c_{s_{k(1)}} \hat{} \bar d_{s_{k(1)}} \hat{} \ldots \hat{} \bar c_{s_{n-1}} \hat{} \bar d_{r_{n-1}}$. As $\bar c_{s_k} \hat{} \bar d_{s_k} = \bar c_{r_k} \hat{} \bar d_{r_k}$ is from $B_{\min\{s_{k(1)}, r_{k(1)}\}}$ for $k < k(1)$ and $\{r_k : k < n\} \subseteq I_1$ we are done proving $\oplus$ hence the claim. $\square_{3.18}$

The following may be used in 3.16, 3.21.

Definition 3.20. Assume $\mathbf{b} = \langle (\bar c_s, \bar d_s) : s \in I) \rangle$ where $I \in K_{p,\sigma}$ but below we omit $\mathbf{b}$ if clear from the context, and if we have $\langle (\bar c_{s,\ell}, \bar d_{s,\ell}) : s \in I \rangle$ for $\ell \leq \mathbf{k}$ we may write $\mathbf{b}_\ell$ instead of $\mathbf{b}$ but below may write $\bar d_{\varrho,\ell}, \bar c_{\varrho,\ell}$.
1) For $k < \omega$ and $\varrho \in {}^k I$ let

$$\bar d_{\varrho,\mathbf{b}} = \bar d_{\varrho(0),\mathbf{b}} \hat{} \ldots \hat{} \bar d_{\varrho(k-1),\mathbf{b}}$$

$$\bar c_{\varrho,\mathbf{b}} = \bar c_{\varrho(0),\mathbf{b}} \hat{} \ldots \hat{} \bar c_{\varrho(k-1),\mathbf{b}}.$$

2) The sequences $\eta_1, \eta_2 \in {}^k I$ are called similar when they realize the same quantifier-free types in I.

The following generalizes 3.16: using only formulas for some Δ's following the quantifier-free types in I and using a parallel of vK rather than of tK.

Claim 3.21. *The sequence $\langle \bar d_{s,0} \hat{}\, \bar c_{s,0} : s \in I\rangle$ is $(\Delta_{\mathbf k}, \mathbf k)$-indiscernible over B_0 <u>when</u> the sequence $\langle (\bar d_{s,\ell}, \bar c_{s,\ell}) : \ell \le \mathbf k, s \in I\rangle$ satisfies*

(a) (α) $I \in K_{p,\sigma}$ *and*[34] $\ell g(\bar d_{s,\ell}), \ell g(\bar c^*_{s,\ell}), \ell g(\bar d^*_{s,\ell})$ *depend just on ℓ and $\mathbf i(s) :=$ the unique $i < \sigma$ such that $s \in P^I_i$; also $\bar c_{s,\ell} \trianglelefteq \bar c_{s,\ell+1}, \bar d_{s,\ell} \trianglelefteq \bar d^*_{s,\ell} \trianglelefteq \bar d_{s,\ell+1}$ for $\ell < \mathbf k, s \in I$ and $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n$; and $\bar c_{s,\ell}, \bar d_{s,\ell}$ are finite*[35]

(β) *for $i < \sigma$ and $k \le \mathbf k$ let $w_{i,k} = \ell g(\bar d_{s,k}), w^*_{i,k} = \ell g(\bar d^*_{s,k})$, $v_{i,k} = \ell g(\bar c_{s,k})$ for any $s \in P^I_i$*

(γ) *for $k \le \mathbf k$ let $R_k = \{\mathrm{tp}_{\mathrm{qf}}(\varrho, \emptyset, I) : \varrho \in {}^{\mathbf k - k}I$ is $<_I$-decreasing$\}$ and let $\mathbf i(r,\ell) = \mathbf i(\eta(\ell)) \Leftrightarrow \varrho(\ell) \in P^I_{\mathbf i(t,\ell)}, \ell g(r) = \ell g(\varrho)$ so $= k$ when $r = \mathrm{tp}_{\mathrm{qf}}(\varrho,\emptyset, I) \in R_k$*

(δ) *for $\ell \le \mathbf k, \Delta_\ell$ and also Δ^*_ℓ is the closure of a finite set of formulas (each with finite set of variables) under permuting the variables, negation and adding dummy variables*

(ε) $\Delta_\ell \subseteq \Delta^*_\ell \subseteq \Delta_{\ell+1} \subseteq \Delta_{\ell+1}$

(b) $\mathrm{tp}_{\Delta_{\mathbf k}}(\bar d_{s,\mathbf k} \hat{}\, \bar c_{s,\mathbf k}, B_s)$ *is* $\subseteq \mathrm{tp}_{\Delta_k}(\bar c_{t,\mathbf k} \hat{}\, \bar d_{t,\mathbf k}, B_t)$ *when $s <_I t \wedge \mathbf i(s) = \mathbf i(t)$ (hence this holds for $k \le \mathbf k$, too) where $B_t = B_{t,\mathbf k}$ and $B_{t,k} = \cup\{\bar d_{s,k} \hat{}\, \bar c_{s,k} : s <_I t\} \cup B_k$ for $k \le \mathbf k$*

(c) $\mathrm{tp}_{\Delta^*_\ell}(\bar c_{s,\ell}, B_{s,\ell+1})$ *does not $\Delta_{\ell+1}$-split over B_s*

(d) *if $s \in P^I_i, t \in P^I_j$ and $s <_I t$ then $r_{i,j} = \mathrm{tp}_{\Delta_k}(\bar c_{s,\mathbf k} \hat{}\, \bar d_{s,\mathbf k} \hat{}\, \bar c_{t,\mathbf k} \hat{}\, \bar d_{t,\mathbf k}, \emptyset)$*

(e) $\mathrm{tp}(\bar d_{t,\ell}, \bar c_{t,\ell+1} + \bar d_{s,\ell} + \bar c_{s,\ell+1}) \vdash \mathrm{tp}(\bar d_{t,\ell}, \bar c_{t,\ell} + \bar d_{s,\ell} + \bar c_{s,\ell}, B_\ell)$ *when $s <_I t$* <u>*or just*</u>

(e)$'$ *if $s <_I t, \ell < \mathbf k, \eta_1 \in {}^{\omega >}\ell g(\bar d_{t,\ell}), \nu_1 \in {}^{\omega >}\ell g(\bar c_{t,\ell}), \eta_3 \in {}^{\omega >}\ell g(\bar d_{s,\ell}), \nu_3 \in {}^{\omega >}\ell g(\bar c_{s,\ell})$ are all finite and $\varphi_0 = \varphi_0(\bar x_{\bar d_{t,\ell},\eta_1}, \bar x_{\bar c_{t,\ell},\nu_1}, \bar x'_{\bar d_{s,\ell},\eta_3}, \bar x'_{\bar c_{s,\ell},\nu_3}, \bar y) \in \mathbb L(\tau_T)$, <u>then</u> we can find $\eta_0 \in {}^{\omega >}\ell g(\bar d^*_{t,\ell}), \nu_0 \in {}^{\omega >}\ell g(\bar c^*_{t,\ell}), \eta_2 \in {}^{\omega >}\ell g(\bar d_{s,\ell+1}), \nu_2 \in {}^{\omega >}\ell g(\bar c_{s,\ell+1})$, (depending on $\ell, s/E_1, t/E_I, \eta_1, \nu_1, \eta_3, \nu_3$ and φ_0 but not on (s,t)) such that:*

(α) $\ell g(\eta_0) = \ell g(\eta_1)$ *and* $\ell g(\nu_0) = \ell g(\eta_1)$ *and* $\ell g(\eta_2) = \ell g(\eta_3)$ *and* $\ell g(\nu_2) = \ell g(\nu_3)$

(β) *if $\bar b \in {}^{\ell g(\bar y)}(B_t)$ then*
$\mathfrak C \models \varphi_0[\bar d_{t,\ell+1,\eta_0}, \bar c_{t,\ell+1,\nu_0}, \bar d_{s,\ell,\eta_2}, \bar c_{s,\ell,\nu_2}, \bar b] \equiv \varphi_1[\bar d^*_{t,\ell,\eta_1}, \bar c^*_{t,\ell,\nu_1}, \bar d_{s,\ell,\eta_3}, \bar c_{s,\ell,\nu_3}, \bar b]$

(γ) $\mathrm{tp}(\bar d^*_{t,\ell,\eta_0}, \bar c_{t,\ell+1} + \bar d_{s,\ell+1}, \bar c_{s,\ell+1}) \vdash \mathrm{tp}_\varphi(\bar d^*_{t,\ell,\eta_0}, (\bar c_{t,\ell+1,\nu_0} + \bar d_{s,\ell+1,\eta_2} + \bar c_{s,\ell+1,\nu_2}) + B_s)$

<u>*or*</u> *(we rephrase some clauses recalling the assumptions on the Δ_ℓ's):*

(e)$''$ *if $s <_I t, \ell < \mathbf k$ and $\varphi = \varphi_0 = \varphi_0(\bar x_{\bar d_{t,\ell}}, \bar x_{c_{t,\ell}}, \bar x_{\bar d_{s,\ell}}, \bar x_{\bar c_{s,\ell}}, \bar z) \in \Delta_\ell$ <u>then</u> we can find $\varphi_1 = \varphi_1(\bar x_{\bar d^*_{t,\ell}}, \bar x_{\bar c^*_{t,\ell}}, \bar x'_{\bar d_{s,\ell+1}}, \bar x'_{\bar c_{s,\ell+1}}, \bar z)$ and $\psi = \psi_\varphi = \psi_\varphi(\bar x_{\bar d^*_{t,\ell}}, \bar x_{\bar c_{t,\ell+1}}, \bar x'_{\bar d_{s,\ell+1}}, \bar x'_{\bar c_{s,\ell+1}})$ depending only if $\mathbf i(s), \mathbf i(t), \ell$ and φ such that:*

---

[34] if we assume (e) then without loss of generality $\bar d^*_{s,\ell} = \bar d_{s,\ell}$

[35] this helps in phrasing the demands on the Δ_ℓ's

(a) $\mathfrak{C} \models \psi[\bar{d}^*_{t,\ell}, \bar{c}_{t,\ell+1}, \bar{d}_{s,\ell+1}, \bar{c}_{s,\ell+1}]$

(b) *for every* $\bar{b} \in {}^{\ell g(\bar{z})}(B_{s,\ell})$ *we have* $\mathfrak{C} \models \varphi_0[\bar{d}_{t,\ell}, \bar{c}_{t,\ell}, \bar{d}_{s,\ell}, \bar{c}_{s,\ell}, \bar{b}]$ *iff* $\mathfrak{C} \models \varphi_1[\bar{d}^*_{t,\ell}, \bar{c}^*_{t,\ell}, \bar{d}_{s,\ell+1}, \bar{c}_{s,\ell+1}]$

(c) $\mathfrak{C} \models \psi_\varphi[\bar{d}^*_{t,\ell}, \bar{c}_{t,\ell+1}, \bar{d}_{s,\ell+1}, \bar{c}_{s,\ell+1}]$

(d) $\psi(\bar{x}_{\bar{d}^*_{t,\ell}}, \bar{c}_{t,\ell+1}, \bar{d}_{s,\ell+1}, \bar{c}_{s,\ell+1}) \vdash \{\varphi_1(\bar{x}_{\bar{d}^*_{t,\ell}}, \bar{c}^*_{t,\ell}, \bar{d}_{s,\ell+1}, \bar{c}_{s,\ell+1}, \bar{b}) : \bar{b} \in {}^{\ell g(\bar{z})}(B_{s,\ell})$ *and* $\mathfrak{C} \models \varphi_1[\bar{d}^*_{t,\ell}, \bar{c}^*_{t,\ell}, \bar{d}_{s,\ell+1}, \bar{e}_{s,\ell+1}; b]\}$

(e) $\vartheta_\varphi = \vartheta_\varphi(\bar{x}_{\bar{c}_{t,\ell+1}}, \bar{x}'_{\bar{d}_{s,\ell+1}}, \bar{x}'_{\bar{c}_{s,\ell+1}}, \bar{z}) \in \Delta^*_\ell$ *where* ϑ_φ *is* $(\forall \bar{x}_{\bar{d}^*_{t,\ell}})[\psi_\varphi(\bar{x}_{\bar{d}^*_{t,\ell}}, \bar{x}_{\bar{c}_{t,\ell+1}}, \bar{x}_{\bar{d}_{s,\ell+1}}, \bar{x}_{\bar{c}_{s,\ell+1}}) \to \varphi_1(\bar{x}_{\bar{d}^*_{t,\ell}}, \bar{x}_{\bar{c}^*_{t,\ell}}, \bar{x}_{\bar{d}_{s,\ell+1}}, \bar{x}_{\bar{c}_{s,\ell+1}}, \bar{z})]$.

Remark 3.22. Used in 3.23, 3.24 below (for the case we use $(c)'$, (c); respectively).

Proof. We prove by induction on $k < \mathbf{k}$ that:

$(*)^1_k$ if $\varrho_1, \varrho_2 \in {}^{k+1}I$ are $<_I$-decreasing, $(\forall \ell \leq k)(\exists i < \sigma)[\varrho_1(\ell), \varrho_2(\ell) \in P^I_i]$ and $s \leq_I \varrho_1(k), \varrho_2(k)$ <u>then</u> the sequences $\bar{d}_{\varrho_\ell(0),\mathbf{k}-k} {}^\frown \bar{c}_{\varrho_\ell(0),\mathbf{k}-k} {}^\frown \ldots {}^\frown \bar{d}_{\varrho_\ell(k),\mathbf{k}-k} {}^\frown \bar{c}_{\varrho_\ell(k),\mathbf{k}-k}$ for $\ell = 1,2$ realize the same $\Delta_{\mathbf{k}-k}$-type over $B_{s,\mathbf{k}-k}$.

<u>Case $k = 0$</u>: This holds by clause (b) of the claim as $\ell g(\varrho_1) = 1 = \ell g(\varrho_2)$.

<u>Case $k > 0$</u>: For $\iota = 1,2$, let $\rho_\iota = \langle \varrho_\iota(1+m) : m < k \rangle$ and for $i \leq \mathbf{k}$ recall $\bar{c}_{\rho_\iota,i} = \bar{c}_{\rho_\iota(0),i} {}^\frown \ldots {}^\frown \bar{c}_{\rho_\iota(k-1),i}$ and $\bar{d}_{\rho_\iota,i} = \bar{d}_{\rho_\iota(0),i} {}^\frown \ldots {}^\frown \bar{d}_{\rho_\iota(k-1),i}$ so the induction hypothesis applies and if $k = 1$ then $\bar{c}_{\rho_\iota,i} = \bar{c}_{\eta_\iota(1),i}, \bar{d}_{\rho_\iota,i} = \bar{d}_{\eta_\iota(1)}$ for $i = 0$.

Note that $\bar{c}_{\rho_\iota,i}$ is a subsequence of $\bar{c}_{\rho_\iota,i+1}$ and $\bar{d}_{\rho_\iota,i}$ is a subsequence of $\bar{d}_{\rho_\iota,i+1}$.

By the case $k = 0$, i.e. by clause (b) without loss of generality $\varrho_1(0) = \varrho_2(0)$ call it t. So assume

$(*)_1$ $\varphi = \varphi_0 = \varphi_0(\bar{x}_{\bar{d}_{t,\mathbf{n}-k}}, \bar{x}_{\bar{c}_{t,\mathbf{n}-k}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{n}-k}}, \bar{x}'_{\bar{c}_{\rho_2,\mathbf{n}-k}}, \bar{z}) \in \Delta_{\mathbf{k}-k}$ and $\bar{b} \in {}^{\ell g(\bar{z})}(B_{s,\mathbf{k}-k})$ and $\mathfrak{C} \models \varphi[\bar{d}_{t,\mathbf{n}-k}, \bar{c}_{t,\mathbf{n}-k}, \bar{d}_{\rho_1,\mathbf{n}-k}, \bar{c}_{\rho_1,\mathbf{n}-k}, \bar{b}]$.

We should prove the parallel statement for ϱ_2, i.e. for t and ρ_2; this will suffice.

<u>Subcase 1</u>: Clause (e) of the assumption.

Follows by the second subcase, (and has easier proof).

<u>Subcase 2</u>: Clause $(e)'$ of the assumption but similar to Subcase 3.

<u>Subcase 3</u>: Clause $(e)''$ of the assumption.

Hence

$(*)_2$ choose φ_1, ψ_φ and ϑ_φ as in clause $(e)''$ for $\rho(0), t$ (chosen above) and φ from $(*)_1$, hence in particular

(a) $\mathfrak{C} \models \psi_\varphi[\bar{d}^*_{t,\mathbf{k}-k}, \bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_1(0),\mathbf{k}-k+1}, \bar{c}_{\rho_1(0),\mathbf{k}-k+1}]$

(b) $\psi_\varphi(\bar{x}_{\bar{d}^*_{t,\mathbf{k}-k}}, \bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_1(0),\mathbf{k}-k+1}, \bar{c}_{\rho_1(0),\mathbf{k}-k+1}) \vdash \varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{c}_{t,\mathbf{k}-k}, \bar{d}_{\nu_1,\mathbf{k}-k}, \bar{c}_{\nu_1,\mathbf{k}-k}, \bar{b})$.

Hence

$(*)_3$ (a) $\mathfrak{C} \models \vartheta_\varphi[\bar{c}_{t,\mathbf{k}-k+1}, \bar{d}_{\rho_1,\mathbf{k}-k+1}, \bar{c}_{\rho_1,\mathbf{k}-k+1}, \bar{b}]$ where

(b) $\vartheta_\varphi(\bar{x}_{\bar{c}_{t,\mathbf{k}-k+1}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{k}-k+1}}, \bar{x}'_{\bar{c}_{\rho_1,\mathbf{k}-k+1}}, \bar{z}) :=$
$(\forall \bar{x}_{\bar{d}^*_{s,\mathbf{k}-k}})[\psi_\varphi(\bar{x}_{\bar{d}^*_{t,\mathbf{k}-k}}, \bar{x}_{\bar{c}_{t,\mathbf{k}-k+1}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{k}-k+1}}, \bar{x}'_{\bar{c}_{\rho_1,,\mathbf{k}-k+1}})$
$\to \varphi(\bar{x}_{\bar{d}_{t,\mathbf{k}-k}}, \bar{x}_{\bar{c}_{t,\mathbf{k}-k}}, \bar{x}'_{\bar{d}_{\rho_1,\mathbf{k}-k}}, \bar{x}'_{\bar{c}_{\rho_1,\mathbf{k}-k}}, \bar{z})]$.

Now

$(*)_4$ $\bar d_{\rho_\iota,\mathbf{k}-k+1}{}^\frown \bar c_{\rho_\iota,\mathbf{k}-k+1}{}^\frown \bar b$ realize the same $\Delta_{\ell+1}$- type over $B_{s,\mathbf{k}-k+1}$ for $\iota = 1,2$.

[Why? By the induction hypothesis as $\bar b$ is from $B_{s,\mathbf{k}-k} \subseteq B_{s,\mathbf{k}-k+1}$.]

$(*)_5$ $\bar c_{t,\mathbf{n}-k+1}{}^\frown \bar d_{\nu_\iota,\mathbf{k}-k+1}{}^\frown \bar c_{\nu_\iota,\mathbf{k}-k+1}{}^\frown \bar b$ realizes the same Δ_ℓ-type over $B_{s,\mathbf{k}-k+1}$ for $\iota = 1,2$.

[Why? As first, $\mathrm{tp}_{\Delta^*_\ell}(\bar c_{t,\mathbf{k}-k+1}, B_{t,k+1})$ does not $\Delta_{\ell+1}$-split over $B_{s,k+1}$ by clause (c) of the assumption, second $\bar d_{\rho_\ell,\mathbf{k}-k+1}, \bar c_{\rho_\ell,\mathbf{k}-k+1}, \bar b$ are included in $B_{t,k+1}$ and third $(*)_4$.]

$(*)_6$ in $(*)_3(a)$ we can replace ρ_1 by ρ_2, i.e. $\mathfrak{C} \models \vartheta_\varphi[\bar c_{t,\mathbf{k}-k+1}, \bar d_{\rho_2,\mathbf{k}-k+1}, \bar c_{\rho_2,\mathbf{k}-k+1}, \bar b]$.

[Why? By $(*)_5$ and $(*)_3(a)$.]

$(*)_7$ $\mathfrak{C} \models \psi_\varphi[\bar d_{t,\mathbf{k}-k}, \bar c^*_{t,\mathbf{k}-k+1}, \bar d_{\rho_2(0),\mathbf{k}-k+1}, \bar c_{\rho_2(0),\mathbf{k}-k+1}]$.

[Why? By clause (d) of the hypothesis of the claim and $(*)_2(a)$.]

$(*)_8$ $\mathfrak{C} \models \varphi[\bar d_{t,\mathbf{k}-k}, \bar c_{t,\mathbf{k}-k}, \bar d_{\rho_2,\mathbf{k}-k}, \bar c_{\rho_2,\mathbf{k}-k}, \bar b]$.

[Why? By $(*)_6 + (*)_7$ and the definition of ϑ in $(*)_3(b)$.]

So we are done. $\square_{3.22}$

§ 3(C). **Toward Density of** tK.

We first show that the existence of $\le^+_1$-extension for every $\mathbf{x} \in \mathrm{rK}^\oplus_{\kappa,\mu,\theta}$ suffice for existence (i.e. for density) for tK. The main case in 3.23, 3.24 is $\sigma = \omega$. Then in 3.27 we prove this sufficient condition for weakly compact κ. Note that for $\mathrm{rK}^\oplus$ closure under union is not obviously true.

Claim 3.23. *We have $\mathbf{x}_\delta \in \mathrm{tK}_{\kappa,\bar\mu,\theta}$, moreover $\mathbf{m}_\delta = (\mathbf{x}_\delta, \bar\psi_\delta, r_\delta) \in \mathrm{tK}^\oplus_{\kappa,\bar\mu,\theta}$ and $\varepsilon < \delta \Rightarrow \mathbf{x}_\varepsilon \le_1 \mathbf{x}_\delta$ <u>when</u> ($\delta < \theta^+$ is a limit ordinal and):*

$\boxplus$ (a) $\mathbf{m}_\varepsilon = (\mathbf{x}_\varepsilon, \bar\psi_\varepsilon, r_\varepsilon) \in \mathrm{rK}^\oplus_{\kappa,\bar\mu,\theta}$ *for $\varepsilon < \delta$ is $\le_1$-increasing*

(b) *r_ε is a complete type, (over the empty set)*

(c) *$\mathbf{m}_\varepsilon \le^+_1 \mathbf{m}_{\varepsilon+1}$, see Definition 3.6(4) or just*

(c)′ *if $\varepsilon < \delta$ and $\varphi \in \Gamma^2_{\mathbf{m}_\varepsilon}$ <u>then</u> for some $\zeta \in [\varepsilon,\delta)$ we have $\varphi \in \Gamma^2_{\bar\psi[\mathbf{m}_\zeta]}$*

(d) *$\mathbf{m}_\delta = \cup\{\mathbf{m}_\varepsilon : \varepsilon < \delta\}$, see 3.6(6), i.e.*

  (α) *$\mathbf{x}_\delta = \cup\{\mathbf{x}_\varepsilon : \varepsilon < \delta\}$, see 2.13(2)*

  (β) *$\bar\psi_\delta$ is the limit*[36] *of $\langle \bar\psi_\varepsilon : \varepsilon < \delta\rangle$*

  (γ) *$r_\delta = \cup\{r_\varepsilon : \varepsilon < \delta\}$.*

We shall prove 3.23 together with

Claim 3.24. *We have $\mathbf{x}_\delta \in \mathrm{vK}_{\kappa,\bar\mu,\theta}$ moreover $\mathbf{m}_\delta = (\mathbf{x}_\delta, \bar\psi_\delta, r_\delta) \in \mathrm{vK}^\oplus_{\kappa,\bar\mu,\theta}$ and $\varepsilon < \delta \Rightarrow \mathbf{x}_\varepsilon \le_1 \mathbf{x}_\delta$ <u>when</u> as in 3.23 except that we replace (c), (c)′ by*

[36] But note that for $\varepsilon < \zeta$ the formulas in $\bar\psi_\zeta$ has more dummy variables than those in $\bar\psi_\varepsilon$

$(c)_1$ $\mathbf{m}_\varepsilon \le^{\odot}_1 \mathbf{m}_{\varepsilon+1}$, *see 3.6(4A)*
or at least

$(c)'_1$ *for every* $\varepsilon < \delta$ *and* $\varphi \in \Gamma^2_{\mathbf{m}_\varepsilon}$ *for some* $\zeta \in (\varepsilon, \delta)$ *we have* $\mathbf{m}_\varepsilon \le^{\odot}_{1,\varphi} \mathbf{m}_\zeta$, *see Definition 3.6(4B).*

Remark 3.25. 1) We may weaken clause (b), i.e. r_ε, r_δ are not necessarily complete, still need sufficient condition for indiscernibility in proving $\boxplus_3$ below.
2) In 3.24 we can use $\mathrm{vk}^{\otimes}_{\kappa,\bar{\mu},\theta}$.

Proof. Proof of 3.23, 3.24 For simplicity we assume (c),(c)$_1$ in 3.23, 3.24, respectively, otherwise we have to use 3.21 (or use compactness). Let $\bar{d}_\varepsilon = \bar{d}_{\mathbf{x}_\varepsilon}, \bar{d}_\delta = \bar{d}_{\mathbf{x}_\delta}, \bar{c}_\varepsilon = \bar{c}_{\mathbf{x}_\varepsilon}$ and $\bar{c}_\delta = \bar{c}_{\mathbf{x}_\delta}$ for $\varepsilon < \delta$. The main point is proving clause (f) from Definition 3.3(1).

Let $A \subseteq M_{\mathbf{x}}$ be of cardinality $< \kappa$ and without loss of generality $\varepsilon < \delta \Rightarrow B^+_{\mathbf{x}_\varepsilon} \subseteq A$. We now choose $A_\alpha, \bar{d}_{\alpha,\delta}, \bar{c}_{\alpha,\delta}, \bar{d}_{\alpha,\varepsilon}, \bar{c}_{\alpha,\varepsilon}$ (for $\varepsilon < \delta$) by induction on $\alpha < \theta^+$, really $\alpha < \delta + \delta$ suffice, such that:

$\oplus$ (a) $\bar{d}_{\alpha,\delta}, \bar{d}_{\alpha,\varepsilon}, \bar{c}_{\alpha,\varepsilon}$ are sequences from $M_{\mathbf{x}}$

(b) $\ell g(\bar{d}_{\alpha,\varepsilon}) = \ell g(\bar{d}_\varepsilon)$ and $\varepsilon < \zeta < \delta \Rightarrow \bar{d}_{\alpha,\varepsilon} = \bar{d}_{\alpha,\zeta} \restriction \ell g(\bar{d}_\varepsilon)$

(c) $\ell g(\bar{c}_{\alpha,\varepsilon}) = \ell g(\bar{c}_\varepsilon)$ and $\varepsilon < \zeta < \delta \Rightarrow \bar{c}_{\alpha,\varepsilon} = \bar{c}_{\alpha,\zeta} \restriction \ell g(\bar{c}_\varepsilon)$

(d) $\bar{d}_{\alpha,\delta} = \cup\{\bar{d}_{\alpha,\varepsilon} : \varepsilon < \delta\}$ and $\bar{c}_{\alpha,\delta} = \cup\{\bar{c}_{\alpha,\varepsilon} : \varepsilon < \delta\}$;

(e) if $\varepsilon \le \delta$ then $\bar{c}_{\alpha,\varepsilon}{}^\frown \bar{d}_{\alpha,\varepsilon}$ and $\bar{c}_\varepsilon{}^\frown \bar{d}_\varepsilon$ realize the same type over $A_\beta := A + \Sigma\{\bar{c}_{\beta,\delta}{}^\frown \bar{d}_{\beta,\delta} : \beta < \alpha\}$

(f) if $\varepsilon < \delta$ and $\alpha = \varepsilon \mod \delta$ then the sequence $\bar{c}_\varepsilon{}^\frown \bar{d}_\varepsilon{}^\frown \bar{c}_{\alpha,\varepsilon}{}^\frown \bar{d}_{\alpha,\varepsilon}$ realizes r_ε

(g) if $\varepsilon < \delta$ and $\alpha = \varepsilon \mod \delta$ and $\varphi = \varphi(\bar{x}_{\bar{d}_\varepsilon}, \bar{x}_{\bar{c}_\varepsilon}, \bar{x}'_{\bar{d}_{\varepsilon+1}}, \bar{x}'_{\bar{c}_{\varepsilon+1}}, \bar{y}) \in \Gamma^2_{\psi[\mathbf{m}_\varepsilon]}$ then $\psi_\varphi(\bar{x}_{\bar{d}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_{\alpha,\varepsilon}, \bar{c}_{\alpha,\varepsilon}) \vdash \mathrm{tp}_\varphi(\bar{d}_\varepsilon, (\bar{c}_\varepsilon{}^\frown \bar{d}_{\alpha,\varepsilon}{}^\frown \bar{d}_{\alpha,\varepsilon}) \dotplus A_\alpha)$.

This is possible by the assumptions recalling the definitions, that is, if $\varepsilon < \delta, \alpha < \delta + \delta$ and $\alpha = \varepsilon \mod \delta$ then first we choose $\bar{d}_{\alpha,\varepsilon}, \bar{c}_{\alpha,\varepsilon}$ as required in clauses (e),(f),(g), this is possibly by the assumption on $\mathbf{m}_\varepsilon$; second we choose $(\bar{d}_{\alpha,\delta}, \bar{c}_{\alpha,\delta})$ from $M_{\mathbf{x}}$ realizing $\mathrm{tp}(\bar{d}_\delta{}^\frown \bar{c}_\delta, A_\alpha)$ and $\bar{d}_{\alpha,\varepsilon} = \bar{d}_{\alpha,\delta} \restriction \ell g(\bar{d}_\varepsilon), \bar{c}_{\alpha,\varepsilon} = \bar{c}_{\alpha,\delta} \restriction \ell g(\bar{c}_\varepsilon)$, possible as $M_{\mathbf{x}}$ is κ-saturated and as clause (e) is satisfied; and third define $\bar{d}_{\alpha,\zeta}, \bar{c}_{\alpha,\zeta}$ as $\bar{d}_{\alpha,\delta} \restriction \ell g(\bar{d}_\zeta), \bar{c}_{\alpha,\delta} \restriction \ell g(\bar{c}_\zeta)$ for $\zeta < \delta$ so clauses (a),(b),(c) hold.

So let $u_\varepsilon = (\varepsilon, \delta) \cup (\delta + \varepsilon, \delta + \delta)$; now

$\boxplus_1$ if $\zeta < \delta$ and $\alpha \in u_{\zeta+1}$, then

(a) $\bar{c}_\zeta{}^\frown \bar{d}_\zeta{}^\frown \bar{c}_{\alpha,\zeta}{}^\frown \bar{d}_{\alpha,\zeta}$ realizes r_ζ

(b) $\bar{c}_\zeta{}^\frown \bar{d}_\zeta$ and $\bar{c}_{\alpha,\zeta}{}^\frown \bar{d}_{\alpha,\zeta}$ realize the same type over A_α

(c) for 3.23 $\mathrm{tp}(\bar{d}_{\alpha+1,\zeta}, \bar{c}_{\alpha+1,\zeta+1} + \bar{d}_{\alpha,\zeta+1} + \bar{c}_{\alpha,\zeta+1}) \vdash \mathrm{tp}(\bar{d}_{\alpha+1,\zeta}, \bar{c}_{\alpha+1,\zeta+1} + \bar{d}_{\alpha,\zeta+1} + \bar{c}_{\alpha,\zeta+1} + A_\alpha)$.
for 3.24: if $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{\bar{d}}, \bar{x}'_{\bar{c}}, \bar{y}) \in \Gamma^2_{\mathbf{x}_\zeta}$ then $\mathrm{tp}(\bar{d}_{\alpha+1,\zeta}, \bar{c}_{\alpha+1,\zeta} + \bar{d}_{\alpha,\zeta} + \bar{c}_{\alpha,\zeta}) \vdash \mathrm{tp}_\varphi(\bar{d}_{\alpha+1,\zeta}, (\bar{c}_{\alpha+1,\zeta}{}^\frown \bar{d}_{\alpha,\zeta}{}^\frown \bar{c}_{\alpha,\zeta}) \dotplus A_\alpha)$.

[Why? Let $\alpha = \varepsilon \mod \delta$ so $\zeta < \varepsilon$; clause (b) holds by clause (e) of $\oplus$, for clause (a) uses clause (f) of $\oplus$ noting that $\mathbf{m}_\zeta \le_1 \mathbf{m}_\varepsilon$ hence $r_\zeta \subseteq r_\varepsilon$ (so $\zeta \le \varepsilon$ suffices for (a),(b)). For clause (c), first assume clause (c) of 3.23. Note that $\mathbf{m}_\zeta \le^+_1 \mathbf{m}_{\zeta+1}$ hence (by Definition 3.6(4)) we have $\Gamma^2_{\mathbf{x}[\mathbf{m}_\varepsilon]} \subseteq \Gamma^2_{\psi[\mathbf{m}_\zeta]}$. Second, assume clause (c) of 3.24: similarly using $\mathbf{m}_\varepsilon \le^{\odot}_1 \mathbf{m}_\zeta$.]

Let D be an ultrafilter on δ to which every co-bounded subset of δ belongs and let

$\boxplus_2$ $q = q(\bar{x}_{\bar{d}_\delta}, \bar{x}_{\bar{c}_\delta}) := \{\vartheta(\bar{x}_{\bar{d}_\delta}, \bar{x}_{\bar{c}_\delta}, \bar{b}) : \bar{b} \in {}^{\delta>}(A_{\delta+\delta})$ and for some $\mathscr{U} \in D$ we have $\alpha \in \mathscr{U} \Rightarrow M_{\mathbf{x}} \models \vartheta[\bar{d}_{\alpha,\delta}, \bar{c}_{\alpha,\delta}, \bar{b}]\}$ so $q(\bar{x}_{\bar{d}_\delta}, \bar{x}_{\bar{c}_\delta})$ is a complete type over $A_{\delta+\delta} \subseteq M_{\mathbf{x}}$.

Let $\bar{d}'_\delta \char`\^ \bar{c}'_\delta$ be a sequence from $M_{\mathbf{x}}$ realizing $q(\bar{x}_{\bar{d}_\delta}, \bar{x}_{\bar{c}_\delta})$.

Let $\bar{d}'_\varepsilon, \bar{c}'_\varepsilon$ be such that $\bar{d}'_\varepsilon \triangleleft \bar{d}'_\delta, \ell g(\bar{d}'_\varepsilon) = \ell g(\bar{d}_\varepsilon)$ and $\bar{c}'_\varepsilon \triangleleft \bar{c}'_\delta, \ell g(\bar{c}'_\varepsilon) = \ell g(\bar{c}_\varepsilon)$.

$\boxplus_3$ if $\varepsilon < \delta$ and $\varepsilon = n \mod \omega$ and $\gamma \in u_\varepsilon$ then $\langle \bar{d}_{\alpha,\varepsilon} \char`\^ \bar{c}_{\alpha,\varepsilon} : \alpha \in u_{\varepsilon+2n} \backslash \gamma \rangle \char`\^ \langle \bar{d}_\varepsilon \char`\^ \bar{c}_\varepsilon \rangle$ is an n-indiscernible sequence over A_γ.

[Why? For claim 3.23, by claim 3.16 the version with clause (e), for claim 3.24 by claim 3.16 the version with clause (e)$'$.]

$\boxplus_4$ if $\varepsilon < \delta$ and $\varepsilon = n \bmod \omega$ then $\langle \bar{d}_{\alpha,\varepsilon} \char`\^ \bar{c}_{\alpha,\varepsilon} : \alpha \in u_{2\varepsilon+2n} \cap \delta \rangle \char`\^ \langle \bar{d}'_\varepsilon \char`\^ \bar{c}'_\varepsilon \rangle \char`\^ \langle \bar{d}_{\alpha,\varepsilon} \char`\^ \bar{c}_{\alpha,\varepsilon} : \alpha \in u_{\varepsilon+2n} \backslash \delta \rangle$ is an n-indiscernible sequence over $A_{\varepsilon+2n}$.

[Why? By $\boxplus_3$, the choice of $q(\bar{x}_{\bar{d}_\delta}, \bar{x}_{\bar{c}_\delta})$ and the choice of $(\bar{d}'_\delta, \bar{c}'_\delta), (\bar{d}'_\varepsilon, \bar{c}'_\varepsilon)$. Note that $\delta \notin u_\varepsilon$ by the definition of u_ε.]

$\boxplus_5$ if $\varepsilon < \delta$ and $\varepsilon = n$ mod ω and $\beta \in u_{\varepsilon+2n} \backslash \delta$ and $v \subseteq u_{\varepsilon+2n} \cap \beta$ and $|v| < n$ <u>then</u> $\bar{d}_\varepsilon \char`\^ \bar{c}_\varepsilon$ and $\bar{d}_{\beta+1,\varepsilon} \char`\^ \bar{c}_{\beta+1,\varepsilon}$ realize the same type over $A_{v,\varepsilon} + \bar{d}'_\varepsilon + \bar{c}'_\varepsilon$ where $A_{v,\varepsilon} = A_0 + \Sigma\{\bar{d}_{\alpha,\varepsilon} \char`\^ \bar{c}_{\alpha,\varepsilon} : \alpha \in v\}$.

We elaborate the more complicated case.

<u>Proof of $\boxtimes_5$ for 3.24</u>:

Let $v_1 = v, v_2 = v \cup \gamma$ where $\varepsilon + 2n + 3 < \gamma < \delta$.

So assume

$\odot_1$ (a) $\varphi = \varphi(\bar{x}_{\bar{d}_\varepsilon}, \bar{x}_{\bar{c}_\varepsilon}, \bar{y})$

(b) $\bar{b}_1 \in {}^{\ell g(\bar{y})}(A_{v,\varepsilon} + \bar{d}'_\varepsilon + \bar{c}'_\varepsilon)$

(c) $\mathfrak{C} \models \varphi[\bar{d}_{\beta+1,\varepsilon}, \bar{c}_{\beta+1,\varepsilon}, \bar{b}_1]$.

We choose $\bar{b}_2$ such that

$\odot_2$ $\bar{b}_2 \in {}^{\ell g(\bar{y})}(A_{v_2,\varepsilon})$ and $\bar{b}_1, \bar{b}_2$ realize the same type over $A_0 + \bar{c}_{\beta+1,\varepsilon+1} + \bar{d}_{\beta+1,\varepsilon+1} + \bar{c}_{\beta,\varepsilon+1} + \bar{d}_{\beta,\varepsilon+1}$.

[Why possible? By $\boxplus_4$ and the choice of γ.]

So

$\odot_3$ (a) $\mathfrak{C} \models \varphi[\bar{d}_{\beta+1,\varepsilon}, \bar{c}_{\beta+1,\varepsilon}, \bar{b}_2]$

(b) $\mathfrak{C} \models \varphi[\bar{d}_\varepsilon, \bar{c}_\varepsilon, \bar{b}_2]$.

[Why? Clause (a) follows by clause $\odot_1(c)$ and the choice of $\bar{b}_2$, i.e. $\odot_2$. Clause (b) follows from clause (a) by $\oplus(e)$.]

Let $(\eta_1, \nu_1) = \text{supp}_{\mathbf{x}}(\varphi)$ and let $(\eta_0, \nu_0), \psi$ be as guaranteed in (a degenerated case of) Definition 3.6(4A), 3.3(1) for φ and $\varphi \equiv \varphi'(\bar{x}_{\bar{d}_\varepsilon,\eta_1}, \bar{x}_{\bar{c}_\varepsilon,\nu_1}, \bar{y})$.

But $\bar{d}_{\varepsilon+1,\eta_0} = \bar{d}_{\varepsilon,\eta_1}, \bar{c}_{\varepsilon+1,\nu_1} = \bar{c}_{\varepsilon+1,\nu_1}$ hence

$\odot_4$ $\mathfrak{C} \models \varphi'[\bar{d}_{\varepsilon+1,\eta_1}, \bar{c}_{\varepsilon+1,\nu_1}, \bar{b}_2]$.

So by the choice of (η_0, ν_0) and $\psi = \psi_{\varphi'}$

$\odot_5$ (a) $\mathfrak{C} \models \varphi'[\bar{d}_{\varepsilon+1,\eta_0}, \bar{c}_{\varepsilon+1,\nu_0}, \bar{b}_2]$

(b) $\psi = \psi(\bar{x}_{\bar{d}_{\varepsilon+1,\eta_0}}, x_{\bar{c}_{\varepsilon+1,\nu_0}}, \bar{x}'_{\bar{d}_{\varepsilon+1}}, \bar{x}'_{\bar{c}_{\varepsilon+1}})$

(c) $\mathfrak{C} \models$ "$\psi(\bar{x}_{\bar{d}_{\varepsilon+1}}, \bar{x}_{\bar{c}_{\varepsilon+1}}, \bar{x}'_{\bar{d}_{\varepsilon+1}}, \bar{x}'_{\bar{c}_{\varepsilon+1}})$" so for some η, ν we have $\eta_0 \trianglelefteq \eta \in {}^{\omega>}(\ell g(\bar{d}_{\varepsilon+1})), \nu_0 \trianglelefteq \nu \in {}^{\omega>}\ell g((\bar{c}_{\varepsilon+1}))$ we have $\psi = \psi[\bar{d}_{\varepsilon+1,\eta}, \bar{c}_{\varepsilon+1,\nu}, \bar{d}_{\beta,\varepsilon+1}, \bar{c}_{\beta,\varepsilon+1}]$

(d) $\mathfrak{C} \models \vartheta[\bar{c}_{\varepsilon+1,\nu_0}, \bar{d}_{\beta,\varepsilon+1}, \bar{c}_{\beta,\varepsilon+1}, \bar{b}_2]$ where
$\vartheta(\bar{x}_{\bar{c}_{\varepsilon+1,\nu}}, \bar{x}'_{\bar{d}_{\varepsilon+1}}, \bar{x}'_{\bar{c}_{\varepsilon+1}}, \bar{y}) :=$
$(\forall \bar{x}_{\bar{d}_{\varepsilon+1,\eta}})(\psi(\bar{x}_{\bar{d}_{\varepsilon+1,\eta}}, \bar{x}_{\bar{c}_{\varepsilon+1,\nu}}, \bar{x}'_{\bar{d}_{\varepsilon+1}}, \bar{x}'_{\bar{c}_{\varepsilon+1}}) \rightarrow \varphi'(\bar{x}_{\bar{d}_{\varepsilon+1,\eta_0}}, \bar{x}_{\bar{c}_{\varepsilon+1,\nu_0}}, \bar{y}))$.

Next

$\odot_6$ $\mathfrak{C} \models \vartheta[\bar{c}_{\varepsilon+1,\nu}, \bar{d}_{\beta,\varepsilon+1}, \bar{c}_{\beta,\varepsilon+1}, \bar{b}_1]$.

[Why? By $\odot_2$ the sequences $\bar{d}_{\beta,\varepsilon+1} \hat{\ } \bar{c}_{\beta,\varepsilon+1} \hat{\ } \bar{b}_1, \bar{d}_{\beta,\varepsilon+1} \hat{\ } \bar{c}_{\beta,\varepsilon+1} \hat{\ } \bar{b}_2$ realize the same type over $A_0 \supseteq B^+_{\mathbf{x}}$ hence also over $B^+_{\mathbf{x}} + \bar{c}_{\varepsilon+1}$, so by $\odot_5(d)$ we get the statement in $\odot_6$.]

$\odot_7$ $\mathfrak{C} \models \varphi'[\bar{d}_{\varepsilon+1,\eta_0}, \bar{c}_{\varepsilon+1,\nu_0}, \bar{b}_1]$.

[Why? Recall by $\odot_5(c)$ we have $\mathfrak{C} \models \psi[\bar{d}_{\varepsilon+1,\eta}, \bar{c}_{\varepsilon+1,\nu}, \bar{d}_{\beta,\varepsilon+1}, \bar{c}_{\beta,\varepsilon+1}]$ so by $\odot_6$ and the definition of ϑ, see $\odot_5(d)$, we get $\odot_7$.]

By the choice of (η_0, ν_0)

$\odot_8$ $\mathfrak{C} \models \varphi'[\bar{d}_{\varepsilon,\eta_1}, \bar{c}_{\varepsilon,\nu_1}, \bar{b}_1]$.

This means

$\odot_9$ $\mathfrak{C} \models \varphi[\bar{d}_\varepsilon, \bar{c}_\varepsilon, \bar{b}_1]$.

As for any $\varphi = \varphi(\bar{x}_{\bar{d}_\varepsilon}, \bar{x}_{\bar{c}_\varepsilon}, \bar{y})$ and $\bar{b}_1 \in {}^{\ell g(\bar{y})}(A_{v,\varepsilon} + \bar{d}' + \bar{c}')$ for some truth value $\mathbf{t}$ the statement $\odot_1(c)$ holds for $\varphi^{\mathbf{t}}$, i.e. $\mathfrak{C} \models \varphi^{\mathbf{t}}[\bar{d}_{\beta+1,\varepsilon}, \bar{c}_{\beta+1,\varepsilon}, \bar{b}_1]$ hence by the above, see $\odot_9$, we get $\mathfrak{C} \models \varphi^{\mathbf{t}}[\bar{d}_\varepsilon, \bar{c}_\varepsilon, \bar{b}_1]$. Hence we get the equality of types stated in $\boxplus_5$, so indeed $\boxplus_5$ holds.

$\boxplus_6$ if $\varepsilon < \delta$ and $\varepsilon = n \bmod \omega$ then $\langle \bar{d}_\alpha \hat{\ } \bar{c}_\alpha : \alpha \in u_{2\varepsilon+n+2} \cap \delta \rangle \hat{\ } \langle \bar{d}' \hat{\ } \bar{c}' \rangle \hat{\ } \langle \bar{d}_\alpha \hat{\ } \bar{c}_\alpha : \alpha \in u_{\varepsilon+2n+2} \backslash \delta \rangle \hat{\ } \langle \bar{d}_\varepsilon \hat{\ } \bar{c}_\varepsilon \rangle$ is an n-indiscernible sequence over A_0.

[Why? As $u_{\varepsilon+2,n+2} \backslash \delta$ is with no last member hence is infinite it suffices to for each $\beta \in u_{\varepsilon+2n+2} \backslash \delta$ to prove this statement replacing $u_{\varepsilon+2n+2} \backslash \delta$ by $u_{\varepsilon+2n+2} \cap \beta \backslash \delta$. But by $\boxplus_5$ this is equivalent to proving the statement omitting $\bar{d}_\varepsilon \hat{\ } \bar{c}_\varepsilon$ and replacing $u_{\varepsilon+2n+2} \cap \beta \backslash \delta$ by $(u_{\varepsilon+2n+2} \cap \beta \backslash \delta) \cup \{\beta\}$, which holds by $\boxplus_4$.

Alternatively, by $\boxplus_5$ recalling 3.18; the main point is that clause (b) there holds, except that for $\bar{d}_\varepsilon \hat{\ } \bar{c}_\varepsilon$ we can use $\boxplus_4$ and for this case we use $\boxplus_5$.]

This shows that for each finite $v \subseteq \delta$ and $\varepsilon < \delta$, the pair $(\bar{d}'_\varepsilon, \bar{c}'_\varepsilon)$ solves $(\mathbf{m}_\varepsilon, A_v)$, but this means that $(\bar{d}', \bar{c}')$ solves $(\mathbf{m}_\delta, A)$ which is what we need. $\square_{3.23}$

Conclusion 3.26. *1) If $\delta < \theta^+$ is a limit ordinal and $\langle (\mathbf{x}_\varepsilon, \bar{\psi}_\varepsilon, \bar{r}_\varepsilon) : \varepsilon < \delta \rangle$ is a $\leq_1$-increasing sequence of members of $\mathrm{tK}^\oplus_{\kappa,\mu,\theta}$, see Definition 3.1, <u>then</u> the limit $(\mathbf{x}_\delta, \bar{\psi}_\delta, r_\delta)$ belongs to $\mathrm{tK}^\oplus_{\kappa,\bar{\mu},\theta}$ and is a $\leq_1$-lub of the sequence.*
2) Similarly for $\mathrm{vK}^\oplus_{\kappa,\bar{\mu},\theta}$.

Proof. 1) By 3.23 as $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}, \mathbf{m} \le_1 \mathbf{n} \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta} \Rightarrow \mathbf{m} \le^+_1 \mathbf{n}$ so we use (c) rather than (c)$'$ there.
2) Similarly by 3.24(2), so we use (c)$_1$ rather than (c)$'_1$ there. $\square_{3.26}$

Claim 3.27. *Assume κ is weakly compact $> \theta \ge |T|$.*
1) If $(\mathbf{x}_1, \bar\psi_1, \bar r_1) \in \mathrm{rK}_{\kappa,\kappa,\theta}$ and $M_{\mathbf{x}}$ has cardinality κ, then there is $(\mathbf{x}_2, \bar\psi_2, r_2) \in \mathrm{rK}^{\oplus}_{\kappa,\kappa,\theta}$ which is $\le^+_1$-above $(\mathbf{x}_1, \bar\psi_1, r_1)$.
2) If $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ and $\bar d \in {}^{\theta^+>}\mathfrak{C}$ then for some $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\kappa,\theta}$ we have $M_{\mathbf{x}[\mathbf{m}]} = M$ and $\bar d \trianglelefteq d_{\mathbf{x}[\mathbf{m}]}$.
3) If $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ then $\mathfrak{S}^{\theta}_{\mathrm{aut}}(M)$ has cardinality $\le \kappa$.

Remark 3.28. 1) Compare with [She15, §4], we here replace "measurable" by "weakly compact".

So for κ weakly compact we can prove the density of $\mathrm{tK}^{\oplus}_{\kappa,\kappa,\theta}$ (by 3.23 + 3.27 above), hence using the $(\mathbf{D}_{\mathbf{x}}, \kappa)$-sequence homogeneity (see Theorem 3.10, Conclusion 3.13) we can prove that there are few types (i.e. $\le \kappa$) up to conjugacy on saturated model (the proof in the end of §4 use only this). To get it for some smaller cardinals we shall need a replacement of weak compactness which is the major point of §4 and to get it for all large enough κ we use $\mathrm{vK}_{\kappa,\bar\mu,\theta}$.
2) Note that in 3.27, for parts (2),(3) it is enough in part (1) to have $\Gamma_{\bar\psi_1} = \emptyset = r_1$.

Proof. 1) By 2.15(1) there is $\mathbf{y} \in \mathrm{qK}'_{\kappa,\kappa,\theta}$ such that $\mathbf{x}_1 \le_2 \mathbf{y}$ so $\bar d_{\mathbf{y}} = \bar d_{\mathbf{x}_1}$ and as we are using $\mathrm{rK}_{\kappa,\kappa,\theta}$, that is $\mu_0 = \mu_1 = \mu_2 = \kappa$ without loss of generality $u_{\mathbf{y}} = \emptyset$. Let $\langle M_\alpha : \alpha < \kappa\rangle$ be $\prec$-increasing continuous with union $M_{\mathbf{x}}$ such that $\|M_\alpha\| < \kappa$ for $\alpha < \kappa$.

As $\mathbf{y} \in \mathrm{qK}'_{\kappa,\kappa,\theta}$ by 2.15(3) we have $\mathbf{y} \in \mathrm{qK}_{\kappa,\kappa,\theta}$ so, see Definition 2.11(2) for each $\alpha < \kappa$ there are $\bar e_\alpha \in {}^{\theta}(M_{\mathbf{x}_1})$ and $\bar\psi_\alpha$ such that $\mathrm{tp}(\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}} + \bar e_\alpha) \vdash \mathrm{tp}(\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}} \dotplus M_\alpha))$ according to $\bar\psi_\alpha$. As $(\mathbf{x}_1, \bar\psi_1, r_1) \in \mathrm{rK}_{\kappa,\kappa,\theta}$, for every $\alpha < \kappa$ we can choose $(\bar c_\alpha, \bar d_\alpha)$ from $M_{\mathbf{x}}$ solving $(\mathbf{x}_1, \bar\psi_1, r_1, \bar e_\alpha + M_\alpha)$, see 3.3(1)(f) by an assumption of the claim. As κ is weakly compact we can find $(\bar\psi_*, f)$ such that

- $(*)$ (a) f is an increasing function from κ to κ so $\alpha \le f(\alpha)$,
  - (b) $\bar\psi_{f(\alpha)} = \bar\psi_*$ for $\alpha < \kappa$
  - (c) $\langle \mathrm{tp}(\bar d_{f(\alpha)}{}^\frown \bar c_{f(\alpha)}{}^\frown \bar e_{f(\alpha)}, \bar d_{\mathbf{y}} + \bar c_{\mathbf{y}} + M_\alpha) : \alpha < \kappa\rangle$ is $\subseteq$-increasing.

Let $(\bar c, \bar d, \bar e)$ from $\mathfrak{C}$ be such that $\bar c{}^\frown \bar d{}^\frown \bar e$ realize $\cup\{\mathrm{tp}(\bar c_{f(\alpha)}{}^\frown \bar d_{f(\alpha)}{}^\frown \bar e_{f(\alpha)}, \bar c_{\mathbf{x}} + \bar d_{\mathbf{x}} + M_\alpha) : \alpha < \kappa\}$, but the pairs $(\bar c, \bar d), (\bar c_{\mathbf{y}} \restriction v_{\mathbf{x}}, \bar d_{\mathbf{y}})$ realize the same type over $M_{\mathbf{x}}$ so without loss of generality $(\bar c, \bar d) = (\bar c_{\mathbf{y}} \restriction v_{\mathbf{x}}, \bar d_{\mathbf{y}})$.

Hence

- $(*)$ for $\alpha < \kappa$ the sequences $(\bar c_{\mathbf{y}} \restriction v_{\mathbf{x}}){}^\frown \bar d_{\mathbf{y}}{}^\frown \bar e$ and $\bar c_{f(\alpha)}{}^\frown \bar d_{f(\alpha)}{}^\frown \bar e_{f(\alpha)}$ realize the same type over M_α.

Now we can define $(\mathbf{x}_2, \bar\psi_2, r_2)$ as follows:

- $\circledast$ (a) $M_{\mathbf{x}_2} = M_{\mathbf{x}_1}$
  - (b) $\bar d_{\mathbf{x}_2} = \bar d_{\mathbf{x}_1}{}^\frown \bar e = \bar d_{\mathbf{y}}{}^\frown \bar e$ and[37]
  - (c) $\bar c_{\mathbf{x}_2} = \bar c_{\mathbf{y}}$

[37] no harm in demanding $u_{\mathbf{x}_1} = \emptyset$

(d) $\bar{B}_{\mathbf{x}_2} = \bar{B}_{\mathbf{y}}$ and $v_{\mathbf{x}_2} = v_{\mathbf{y}}$ and $u_{\mathbf{x}_2} = u_{\mathbf{x}_1}$

(e) $\bar{\psi}_2$ is just putting together $\bar{\psi}_1$ and $\bar{\psi}_*$

(f) r_2 is such that $r_2 = \mathrm{tp}(\bar{c}_{\mathbf{x}_2} \hat{\ } \bar{d}_{\mathbf{x}_2} \hat{\ } \bar{e} \hat{\ } \bar{c}_{f(\alpha)} \hat{\ } \bar{d}_{f(\alpha)} \hat{\ } \bar{e}_{f(\alpha)}, \emptyset)$ for unboundedly many $\alpha < \kappa$.

Clearly $(\mathbf{x}_2, \bar{\psi}_2, r_2)$ is as required.
2) By part (1) and 3.26(1).
3) By part (2) and 3.10. $\square_{3.27}$

§ 4. Density

Our immediate goal is, concentrating on countable T, is to prove density for $\mathrm{tK}_{\kappa,\mu,\theta}$ in some ZFC cases: $\kappa = \mu^{+n}, \mu = \beth_\delta, \mathrm{cf}(\delta) > \theta$. We do it in 4.12 when $n = 1$ and shall do it in §(5A) when $n \geq 1$. The point is proving some $\bar{e}$ universally solves a given $\mathbf{x} \in \mathrm{qK}_{\kappa,\kappa,\theta}$ done in §(4B) and for this we use the partition Theorem 4.1. Theorem 4.6 is a partition theorem which is nicer per se, and is more transparent (and stronger in some respects, see also 8.1), but it is not enough for helping in the proofs about decompositions.

§ 4(A). **Partition theorems for Dependent T.**

The following partition theorem will be crucial (in the proof of 4.8 and also will be used in 5.2). We prove a nicer one later, but not useful here. We can below use "v finite, $\mathbf{k} = 1$" in 4.1, see 4.4(3). For a case when the conclusion of 4.1 can be nicely phrased, see 4.5. In 4.1 we do not explicitly demand T to be dependent but clause (i) holds if T is dependent.

Theorem 4.1. *The partition Theorem*

There are $\mathscr{D}_1$-positive sets $\mathscr{S}_{1,n}$ for $n < \mathbf{k}$ and a $\mathscr{D}_2$-positive set $\mathscr{S}_2$ and $h \in \Pi\mathfrak{a}_{\mathbf{f}}$ and $q_n \in \mathbf{S}^{v_n+w_n}_{\Delta_n}(B^+_{\mathbf{f},h} + \Sigma_n C_n)$ such that for every $n < \mathbf{k}, \mathscr{S}_{1,n+1} \subseteq \mathscr{S}_{1,n}$ and[38] *for each $s \in \mathscr{S}_{1,n}$ for $\mathscr{D}_2$-almost every $t \in \mathscr{S}_2$ we have*[39] $q_n = \mathrm{tp}_{\Delta_n}((\bar{e}^1_s{\restriction}v_n)\hat{\ }(\bar{e}^2_t{\restriction}w_n), B^+_{\mathbf{f},u_n,h} + C_n)$*, see 2.24(2) when:*

$\oplus$ (a) $\mathbf{k} \leq \omega$ *and* $k_n < \omega, k_n \geq 1$ *for* $n < \mathbf{k}$

(b) *for* $n < \mathbf{k}, \Delta_n \subseteq \mathbb{L}(\tau_T)$ *is finite, each* $\varphi \in \Delta_n$ *has the form* $\varphi(\bar{x}_{[v_n]}, \bar{y}_{[w_n]}, \bar{z})$

(c)(α) $\mathbf{f}$ *is a* $(\bar{\mu}, \theta)$*-set, see Definitions 2.23, 2.24, we use their notations*

(β) $v_n \subseteq v$ *is finite*[40] *and* $\subseteq$*-increasing with* $n < \mathbf{k}$

(γ) $u_n \subseteq v_{\mathbf{f}}$ *for* $n < \mathbf{k}$, *note that* $v, v_{\mathbf{f}}$ *are unrelated objects and let* $u_{n,2} = u_n \cap u_{\mathbf{f}}$ *and* $u_{n,1} = u_n \backslash u_{\mathbf{f}}$

(δ) $\mathrm{cf}(\Pi\{\kappa_{\mathbf{f},i} : i \in u_{n,2}\}) = \mathrm{cf}(\Pi\mathfrak{a}_{\mathbf{f},u_{n,2}}) < \kappa$

(ε) $w_n \subseteq w_{n+1} \subseteq w = \cup\{w_m : m < \omega\}$

(ζ) *for notational simplicity*[41] $\bar{a}_{\mathbf{f},i,\alpha}$ *is a singleton for* $i \in u_{\mathbf{f}}, \alpha < \kappa_{\mathbf{f},i}$

(d) $\Delta^1_n \subseteq \mathbb{L}(\tau_T)$ *and* $C_n \subseteq \mathfrak{C}$ *for* $n < \mathbf{k}$

(e)(α) $\kappa = \mathrm{cf}(\kappa)$ *and* $\min\{\kappa_{\mathbf{f},i} : i \in u_{n,2} \cap u_{\mathbf{f}}\}$ *are* $> \beth_{k_n}(|B_{\mathbf{f},u_n\backslash u_{\mathbf{f}}}| + |C_n| + \theta), \theta \geq \aleph_0 + |v_n| + |u_n| + |\Delta^1_n|$ *and* $C_n \subseteq \mathfrak{C}$

or

(β) $\kappa = \mathrm{cf}(\kappa)$ *and* $B^+_{\mathbf{f},u_n\backslash u_{\mathbf{f}}}, v_n, w_n, C_n, \Delta^1_n$ *are finite*

(g) $\bar{e}^1_s \in {}^v\mathfrak{C}$ *for* $s \in I_1$

(h) $\bar{e}^2_t \in {}^w\mathfrak{C}$ *for* $t \in I_2$

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[38] can add $s \in \mathscr{S}_{1,n} \Rightarrow \mathrm{tp}(\bar{e}_s, B^+_{\mathbf{f},u_n,h}) = q_n$

[39] could ask just $q{\restriction}\Delta_n = \mathrm{tp}_{\Delta_n}(\bar{e}_s\hat{\ }\bar{e}_i\hat{\ }\bar{a}_{\mathbf{f},u_n,h}, B_{\mathbf{f},u_n} + C)$ for every $h \in \Pi\{\kappa_{\mathbf{f},i}\backslash h_n(i) : i \in u_n\}$, does not really matter.

[40] instead "v_n finite" we can use $v_n = \varepsilon$ but $\Delta^1_n \subseteq \Gamma_{[\varepsilon],n} \subseteq \mathbb{L}(\tau_T)$, see 0.13(4)

[41] As we can work in $\mathfrak{C}^{\mathrm{eq}}$ this is not a loss.

(i) *if* $n < \omega, B \subseteq \mathfrak{C}, \bar{e} \in {}^{w_n}\mathfrak{C}$ *and* $\langle \bar{a}_\ell : \ell < \omega \rangle$ *is a* (Δ^1_n, k_n)*-indiscernible sequence over* B *where* $\ell < \omega \Rightarrow \bar{a}_\ell \in {}^{v_n}\mathfrak{C}$ *<u>then</u> the set* $\{\ell < \omega : \mathrm{tp}_{\Delta_n}(\bar{a}_\ell {}^\frown \bar{e}, B) \neq \mathrm{tp}_{\Delta_n}(\bar{a}_{\ell+1} {}^\frown \bar{e}, B)\}$ *is finite*

(j) $\mathscr{D}_2$ *is a* κ*-complete filter on* I_2

$(k)(\alpha)$ $\mathscr{D}_1$ *is a* κ*-complete filter on* I_1

(β) *if* $\mathbf{k} = \omega$ *then* $\alpha < \kappa \Rightarrow |\alpha|^{\aleph_0} < \kappa$.

Remark 4.2. 1) Similarly if T is strongly dependent (hence by [She14b, 4.1] we already get some existence of indiscernibles) we can get more.
2) Assume c_n as well as Δ_n, Δ_{n+1} so without loss of generality Δ^1_n are finite. If $\mathbf{k}$ is finite, then it is enough that D_1 is a filter, if $\mathbf{k} = \omega$ then it is enough that D_1 is $\aleph_1$-complete filter.

Claim 4.3. *1) Assume* T *is strongly dependent. In 4.1 we can use:* $\Delta_n = \Delta^1_n = \{\varphi : \varphi = \{\varphi(\bar{x}_{[v_n]}, \bar{y}_{[w]}, \bar{z}) \in \mathbb{L}(\tau_T)\}$ *but demand* $\kappa = \mathrm{cf}(\kappa) > \beth_{|T|^+}(|B_{\mathbf{f},u_n}| + |C_n| + \theta)$.
2) Similarly in 4.6.

Proof. 1) Like the proof 4.1 but first we just use [She14b, 4.1] instead Erdös-Rado theorem in the proof of $(*)_2$ and second, we use the definition of strongly dependent (see [She14b, 2.1]), in the proof of $(*)_6$, justifying why we are stuck for some k.
2) Similarly below. $\square_{4.3}$

Remark 4.4. 1) A nice case is $\cup\{\Delta_n : n < \mathbf{k} = \omega\} = \{\varphi(\bar{x}_{[v_n]}, \bar{x}_{[w]}) : \varphi \in \mathbb{L}(\tau_T)$ and $n < \omega\}$ and $\cup\{v_n : n < \omega\} = v$ and $\cup\{\Delta^1_n : n < \omega\} = \mathbb{L}(\tau_T)$ and $\Delta_n \subseteq \Delta_{n+1}, \Delta^1_n \subseteq \Delta^1_{n+1}$ and each Δ_n, Δ^1_n is finite so T is countable.
2) If we first replace $(\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}})$ by $(\langle\rangle, \bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}})$ the way back is problematic!
3) We could use v finite. Also we may use $I_\ell = I = [M]^{<\kappa}$ and $\mathscr{D}_\ell$ is a normal filter on I, it is natural in the application here (similarly for the definition of a $(\bar{\mu}, \theta)$-sets!)
4) In clause (c) of 4.1, by 2.25(1A), it suffices to demand

$(c)'$ $\mathbf{f}$ is a $(\bar{\mu}, \theta)$-set, $\Delta_n \subseteq \mathbb{L}(\tau_T)$ for $n < \mathbf{n}$ and $i \in v_{\mathbf{f}} \backslash u_{\mathbf{f}} \Rightarrow B_{\mathbf{f},i} = B_{\mathbf{f}}$.

5) We have considerable leeway in the proof.
6) In order to use infinite Δ_n at present we need a stronger assumption on T, see 4.3.
7) For transparency assuming $\mathbf{k} = 1$, we can get also that for some q' for each $t \in \mathscr{S}_2$ for $\mathscr{D}$-almost $s \in \mathscr{S}_{1,0}$ we have $q' = \mathrm{tp}_{\Delta_0}(\bar{e}^1_s {}^\frown \bar{e}^2_t, B^+_{\mathbf{f},u_n,h} + C)$. How? Applying 4.1 twice.
8) In the proof we can demand that $\bar{s}_{n,k}$ has length $k+1$ and so can demand that the game is of a fix finite number of moves, e.g. $\Sigma\{2 \times \mathrm{ind}(\varphi) : \varphi \in \Delta_n\} + 1$, on $\mathrm{ind}(\varphi)$, see 5.22.
9) We can assume $\mathscr{S}^*_\ell \in \mathscr{D}^+_\ell$ for $\ell = 1, 2$ and demand $\mathscr{S}_{1,n} \subseteq \mathscr{S}^*_1, \mathscr{S}_2 \subseteq \mathscr{S}^*_2$ but this does not add anything because we may just use $\mathscr{D}'_\ell = \mathscr{D}^*_\ell \restriction \mathscr{S}^*_\ell := \{\mathscr{S} \cap \mathscr{S}^*_\ell : \mathscr{S} \in \mathscr{D}_\ell\}$.
10) There is no real harm if in 4.1 we assume $v = w$, i.e. $\mathrm{Dom}(\bar{e}^1_s) = \mathrm{Dom}(\bar{e}^2_t)$ for $s \in I_1, t \in I_2$.
11) Assume $\mathbf{f}_1$ satisfies $\mathrm{cf}(\Pi \mathfrak{a}_{\mathbf{f}_1}) < \kappa$ (or $\kappa \notin \mathrm{pcf}(\mathfrak{a}_{\mathbf{f}_1})$); so we have $h_{\bar{e}_1,\bar{e}_2} \in \Pi \mathfrak{a}_{\mathbf{f}_1}$. Can we find h satisfying $(\forall s_0 \in \mathscr{S}^1_{\alpha,0})(\forall^{\mathscr{D}_2} s_1 \in \mathscr{S}_1)(h_{\bar{e}_{s_0},\bar{e}_{s_1}} \le h)$? see 5.1.
12) We could have asked $u_n \subseteq v_{\mathbf{f}}$ instead $u_n \subseteq u_{\mathbf{f}}$ and use $B^{\pm}_{\mathbf{f},u_n}$ instead of $B_{\mathbf{f},u_n}$.

Remark 4.5. If you are interested in weakening the generality of the theorem for having a somewhat more transparent proof, note that the statement of 4.1 is simplified when we use a model M of cardinality κ to which all relevant elements belong (as in the proof). Let $\langle M_\alpha : \alpha < \kappa\rangle$ be $\prec$-increasing continuous with union M such that $\alpha < \kappa \Rightarrow \|M_\alpha\| < \kappa$. So we can decide $I_\ell = \kappa, \mathscr{D}_\ell$ a normal filter on κ, e.g. the club filter hence instead of "for every $s \in \mathscr{S}_{1,m}$ for $\mathscr{D}_2$-almost every $t \in \mathscr{S}_2$" we have: if $s \in \mathscr{S}_{1,n}, t \in \mathscr{S}_2$ and $s < t$ as ordinals <u>then</u> $q_n = \text{tp}_{\Delta_n}((\bar{e}^1_s \restriction v_n)\hat{}(\bar{e}^2_t \restriction w_n), B^+_{\mathbf{f},u_n,h} + C_n)$. This is by the normality of the filter.

Proof. <u>Proof of 4.1</u> Let $M \prec \mathfrak{C}$ include $\cup\{\bar{e}^\ell_s : s \in I_\ell$ and $\ell = 1,2\} \cup \{C_n : n < \omega\} \cup B^+_{\mathbf{f}}$. Let $\mu = \mu_0$ hence $\mu \le \min(\mathfrak{a}_{\mathbf{f}})$. We can choose $\bar{a}_{*,k} = \langle \bar{a}_{*,i,k} : i \in u_{\mathbf{f}}\rangle$ for $k < \omega$ such that (recalling Definition 2.24(5)):

$\boxplus_1$ for every $A \subseteq M$ of cardinality $< \mu$ and $n < \mathbf{k}, k < \omega$, for some $g \in \Pi\mathfrak{a}_{\mathbf{f},u_{\mathbf{f}}}$ we have: $g \le h \in \Pi\mathfrak{a}_{\mathbf{f},u_{\mathbf{f}}} \Rightarrow \text{tp}(\bar{a}_{\mathbf{f},u_{n,2},h}, A + B_{\mathbf{f}} + C_n + \sum\limits_{m>k} \bar{a}_{*,m}) = \text{tp}(\bar{a}_{*,k,u_{n,2}}, A + B_{\mathbf{f}} + C_n + \sum\limits_{m>k} \bar{a}_{*,m})$.

Note that

$\boxplus'_1$ if for some $j \in u_{n,2}$ we have $\kappa_{\mathbf{f},j} = \min\{\kappa_{\mathbf{f},i} : i \in u_{\mathbf{f}}\}$ <u>then</u> in $\boxplus_1$ we may replace $u_{n,2}$ by $u_{\mathbf{f}}$ and C_n by $\sum\limits_m C_m$.

Next

$\boxplus_2$ for $n < \mathbf{k}$ let $C^+_n = \cup\{\bar{a}_{*,m} : m < \omega\} \cup B_{\mathbf{f}} \cup C_n$ but if the assumptions of sub-clause (β) of clause (e) of $\oplus$ fails, <u>then</u> $C^+_n = \cup\{a_{*,i,k} : i \in u_n$ and $k < k_n\} \cup B_{\mathbf{f}} \cup C_n$

hence

$\boxplus_3$ if $n < \mathbf{k}, \zeta < \mu$ and $\bar{e}_1, \bar{e}_2 \in {}^\zeta M$ realize the same type over C^+_n <u>then</u> for some $g \in \Pi\mathfrak{a}_{\mathbf{f},u_n}$, we have $(C^+_n + \bar{e}_1 + \bar{e}_2, \bar{\mathbf{I}}_{\mathbf{f},u_{n,2},g})$ is a $(\bar{\mu},\theta)$-set and $\bar{e}_1, \bar{e}_2$ realize the same type over $C^+_n + \bar{\mathbf{I}}_{\mathbf{f},u_{n,2},g}$.

Choose $\mathscr{F}_n$ such that

$\boxplus_4$ $\mathscr{F}_n$ is a cofinal subset of $\Pi\mathfrak{a}_{\mathbf{f},u_{n,2}}$ of cardinality $\text{cf}(\Pi\mathfrak{a}_{\mathbf{f},u_{n,2}})$,

hence by clause $(c)(\delta)$ of the assumption

$\boxplus_5$ $|\mathscr{F}_n| < \kappa$.

Without loss of generality (recalling $\mu_0 > \theta \ge |T| \ge \aleph_0$):

$\boxplus_6$ $(B_{\mathbf{f},u_{n,1}} + C^+_n, \bar{\mathbf{I}}_{\mathbf{f},u_{n,2}})$ is a $(\bar{\mu},\theta)$-set for each $n < \mathbf{k}$

$\boxplus_7$ $|\tau_T| = \Sigma\{|\Delta_n| + |\Delta^1_n| : n < \mathbf{k}\}$ or both are $\le \aleph_0$.

Next

$(*)_1$ if $\ell = 1,2$ and $n < \mathbf{k}$ and $\mathscr{S} \in \mathscr{D}^+_\ell$ <u>then</u> for some $q = q_{\mathscr{S},n}$ and $h = h_{\mathscr{S},n} \in \mathscr{F}_n \subseteq \Pi\mathfrak{a}_{\mathbf{f},u_n}$ we have $\mathscr{S}_{h,q} \in \mathscr{D}^+_\ell$ where $\mathscr{S}_{h,q} := \{s \in \mathscr{S} : (C^+_n + \bar{e}^\ell_s, \bar{\mathbf{I}}_{\mathbf{f},u_n,h})$ is a $(\bar{\mu},\theta)$-set and $q = \text{tp}_{\mathbb{L}(\tau_n)}(\bar{e}^\ell_s \restriction v_n, B^+_{\mathbf{f},u_n,h} + C^+_n)\}$.

[Why? For each $s \in \mathscr{S}$ by 2.25(4) there is a function $h_s \in \Pi \mathfrak{a}_{\mathbf{f},u_n}$ such that $(C_n^+ + \bar{e}_s^\ell, \bar{\mathbf{I}}_{\mathbf{f},h})$ is a $(\bar{\mu},\theta)$-set; without loss of generality $h_s \in \mathscr{F}_n$.

But by clause (j) of the assumption, i.e. the κ-completeness of $\mathscr{D}_\ell$ and $\boxplus_5$ there is a function $h \in \mathscr{F}_n \subseteq \Pi \mathfrak{a}_{\mathbf{f},u_n}$ such that $\mathscr{S}' := \{s \in \mathscr{S} : h_s \le h$ and we can add $h_s = h\} \in \mathscr{D}_\ell^+$. Now $2^{|C_n^+|+|\Delta_n^1|} < \kappa$ by clauses (a),(d),(e) of $\oplus$, hence for some $(\mathscr{S}'', q)$ we have: $\mathscr{S}'' \subseteq \mathscr{S}', \mathscr{S}'' \in \mathscr{D}_\ell^+$ and $s \in \mathscr{S}'' \Rightarrow q = \mathrm{tp}_{\mathbb{L}(\tau_n)}(\bar{e}_s^\ell, C_n^+)$. By the choice of $h_s = h$ and C_n^+ it follows that $\mathscr{S}'' \subseteq \mathscr{S}_{h,q}$ so $\mathscr{S}_{h,q} \in \mathscr{D}_\ell^+$ is as required in $(*)_1$.]

We now define some games; for any $n < \mathbf{k}, g \in \mathscr{F}_n, q \in \mathbf{S}^{v_n}_{\mathbb{L}(\tau_n)}(B^+_{\mathbf{f},u_{n,2},g})$ and $\mathscr{S} \in \mathscr{D}_1^+$ we define a game $\eth_{\mathscr{S},n,g,q}$: a play last ω moves, in the ℓ-th move the antagonist chooses $\mathscr{X}_\ell \in \mathscr{D}_1$ and the protagonist chooses $s_\ell \in \mathscr{X}_\ell \cap \mathscr{S}$.

In the end of a play the protagonist wins the play <u>iff</u>:

$\otimes$ (a) $(C_n^+ + \Sigma_{\ell<\omega} \bar{e}^1_{s_\ell}, \bar{\mathbf{I}}_{\mathbf{f},u_n,g})$ is a $(\bar{\mu},\theta)$-set

(b) $\langle \bar{e}^1_{s_\ell} \restriction v_n : \ell < \omega \rangle$ is an $(\mathbb{L}(\tau_T), k_n)$-indiscernible sequence over $B^+_{\mathbf{f},u_n,g} + C_n^+$ and[42]

(c) $q = \mathrm{tp}_{\mathbb{L}(\tau_n)}(\bar{e}^1_{s_\ell} \restriction v_n, B^+_{\mathbf{f},u_n,g} + C_n^+)$ so is the same for every $\ell < \omega$.

Alternatively[43] (by $(e)(\beta)$ of $\oplus$) we can define $\eth'_{\mathscr{S},n,g,q}$ similarly but in the end of the play the protagonist wins the play <u>iff</u>:

$\otimes'$ $(a)'$ as in $\otimes$

$(b)'$ $\langle \bar{e}^1_{s_\ell} \restriction v_n : \ell < \omega \rangle$ is a (Δ_n^1, k_n)-indiscernible sequence over $B^+_{\mathbf{f},u_n,g} + C_n^+$

$(c)'$ $\mathrm{tp}_{\Delta_n}(\bar{e}^1_{s_\ell} \restriction v_n, C_n^+) = (q \restriction \Delta_n) \restriction C_n^+$ hence is the same for all $\ell < \omega$.

So only $q_n := (q \restriction \Delta_n) \restriction C_n^+$ matters and we may write $\eth'_{\mathscr{S},n,g,q_n}$.

$(*)_2$ if $\mathscr{S} \in \mathscr{D}_1^+$ and $n < \mathbf{k}$ <u>then</u> for some $h \in \mathscr{F}_n$ and q, the protagonist wins in the game $\eth'_{\mathscr{S},n,h,q}$.

[Why? Let $\lambda = 2^{|C_n^+|+|\Delta_n^1|} + \aleph_1$, so by Erdös-Rado theorem and $\oplus(e)$ clearly $\kappa \to (\omega)^{k_n}_\lambda$.

For each $h \in \mathscr{F}_n$ and $q \in \mathbf{S}^{v_n}(C_n^+)$ the game $\eth'_{\mathscr{S},n,h,q}$ is determined being closed for the protagonist, so toward contradiction let $\mathbf{st}_{\mathscr{S},n,h,q}$ be a winning strategy for the antagonist. We choose $s_\alpha \in \mathscr{S} \subseteq I_1$ by induction on $\alpha < \kappa$ such that: for any relevant h and q in any finite initial segment of a play of $\eth'_{\mathscr{S},n,h,q}$ in which the antagonist uses the strategy $\mathbf{st}_{\mathscr{S},n,h,q}$ and the protagonist chooses members of I_1 from $\{s_\beta : \beta < \alpha\}$, the last move of the antagonist is a member $\mathscr{X}$ of $\mathscr{D}_1$ to which s_α belongs. So s_α just have to belong to $\le |[\alpha]^{<\aleph_0}| + |\mathscr{F}_n| + |\mathbf{S}^{v_n}_{\Delta_n}(C_n^+)| < \kappa$ sets[44] $\mathscr{X} \in \mathscr{D}_1$, but $\mathscr{D}_1$ is a κ-complete filter so this is possible. As $\kappa = \mathrm{cf}(\kappa)$ is large enough without loss of generality $\langle \mathrm{tp}_{\Delta_n}(\bar{e}^1_{s_\alpha} \restriction v_n, C_n^+) : \alpha < \kappa \rangle$ is constant. Now letting $\lambda_n = |\mathbf{S}^{k_n \times |v_n|}(C_n^+)|$, by clause (e) of the assumption we have $\kappa \to (\omega)^{k_n}_{\lambda_n}$, so for some increasing sequence $\langle \alpha(i) : i < \omega \rangle$ of ordinals $< \kappa$ the sequence $\langle \bar{e}^1_{s_{\alpha(i)}} : i < \omega \rangle$ is (Δ_n^1, k_n)-indiscernible over C_n^+. We can find $h \in \mathscr{F}_n$ such that $(\cup\{\bar{e}^1_{s_{\alpha(i)}} : i < \omega\} + C_n^+, \bar{\mathbf{I}}_{\mathbf{f},u_n,h})$ is a $(\bar{\mu},\theta)$-set. By the choice of C_n^+ (in particular

[42] we may omit clause (c)

[43] we use the second; presently it does not make a difference what we use

[44] Why is it $< \kappa$? Recall $|\mathscr{F}_n| < \kappa$ by $\boxplus_4$ and $|\mathbf{S}^{v_n}_{\Delta_n}(C_n^+)| < \kappa$ by clause (e) of the assumption.

enough $a_{*,i,k}$'s belong to it) and of $\langle \alpha(i) : i < \omega \rangle$ it follows that $\langle \bar e^1_{s_{\alpha(i)}} : i < \omega \rangle$ is (Δ^1_n, k_n)-indiscernible sequence also over $B^+_{\mathbf{f},u_n,h} + C^+_n$. So for some q the sequence $\langle \bar e^1_{s_{\alpha(i)}} : i < \omega \rangle$ is a result of a play of the game $\Game'_{\mathscr{S},n,h,q}$ in which the protagonist wins, easily a contradiction.]

$(*)_3$ for any $\mathscr{S} \in \mathscr{D}^+_1$ and $n < \omega$ we choose $(\mathscr{W}, q, h, \mathbf{st}, \mathbf{T}) = (\mathscr{W}_{\mathscr{S},n}, q_{\mathscr{S},n}, h_{\mathscr{S},n}, \mathbf{st}_{\mathscr{S},n}, \mathbf{T}_{\mathscr{S},n})$ satisfying:
- (a) $h \in \mathscr{F}_n$
- (b) $\mathscr{W} \in \mathscr{D}^+_1$ and $\mathscr{W} \subseteq \mathscr{S}$
- (c) if $s \in \mathscr{W}$ then $q = \mathrm{tp}_{\Delta_n}(\bar e^1_s, B^+_{\mathbf{f},u_n,h} + C^+_n)$
- (d) $\mathbf{st}$ is a winning strategy in $\Game'_{\mathscr{W},n,h,q}$ for the protagonist
- (e) $\mathbf{T} = \{\bar s$: for some $m < \omega$ the sequence $\bar s = \langle \mathscr{X}_\ell, s_\ell : \ell < m \rangle$ is a finite initial segment of the play in which the protagonist uses $\mathbf{st}\}$.

[Why? Easy by $(*)_1 + (*)_2$.]

Let χ be large enough and

$(*)_4$ let $\mathscr{N}$ be the set of $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ such that:
- (a) $\mathfrak{C}, \mathbf{f}, M, \langle C_n : n < \mathbf{k} \rangle, \langle \bar e^\ell_t : t \in I_\ell \rangle$ for $\ell = 1,2$ and $\langle \mathscr{F}_n : n < \mathbf{k} \rangle$ belongs to N
- (b) the following function belong to N: $(\mathscr{S}, n) \mapsto (\mathscr{W}_{\mathscr{S},n}, q_{\mathscr{S},n}, h_{\mathscr{S},n}, \mathbf{st}_{\mathscr{S},n}, \mathbf{T}_{\mathscr{S},n})$
- (c) N has cardinality $< \kappa$ and $N \cap \kappa \in \kappa$ (hence, e.g. $n < \mathbf{k} \Rightarrow \mathscr{F}_n \subseteq N$)
- (d) if $\mathbf{k} = \omega$ <u>then</u> $[N]^{\aleph_0} \subseteq N$.

Now

$(*)_5$ for every $N \in \mathscr{N}$ we can choose $t_N \in \cap\{\mathscr{S} \in \mathscr{D}_2 : \mathscr{S} \in N\}$.

[Why? Because $\mathscr{D}_2$ is a κ-complete filter and N is of cardinality $< \kappa$.]

$(*)_6$ for every $N \in \mathscr{N}$ choose $(\mathscr{S}_{1,n}, q_n, h_n, \mathbf{st}_n, \mathbf{T}_n) = (\mathscr{S}_{N,n}, q_{N,n}, h_{N,n}, \mathbf{st}_{N,n}, \mathbf{T}_{N,n})$ by induction on $n < \mathbf{k}$ such that:
- (a) $(\mathscr{S}_{1,n}, q_n, h_n, \mathbf{st}_n, \mathbf{T}_n) \in N$ is as in $(*)_3$
- (b) $\mathscr{S}_{1,n} \supseteq \mathscr{S}_{1,n+1}$
- (c) $\mathscr{S}_{1,n} \in \mathscr{D}^+_1 \cap N$
- (d) if $s \in \mathscr{S}_{1,n+1} \cap N$ then $q_n = \mathrm{tp}_{\Delta_n}((\bar e^1_s \restriction v_n)\hat{\ }(\bar e^2_{t_N} \restriction w_n), B^+_{\mathbf{f},u_n,h_n} + C^+_n)$.

We can carry the inductive construction.
[Why? For $n = 0$ choose $\mathscr{S}_{1,n}, q_n, h_n, \mathbf{st}_n, \mathbf{T}_n$ as in $(*)_3$ with (I_1, n) here standing for $(\mathscr{S}, n)$ there and as we are assuming $N \in \mathscr{N}$ without loss of generality they belong to N. Assume that the tuple $(\mathscr{S}_{1,n}, q_n, h_n, \mathbf{st}_n, \mathbf{T}_n)$ was chosen and $n+1 < \mathbf{k}$. We try to choose $\bar s_{n,k} = \langle \mathscr{X}_\ell, s_\ell : \ell \le \ell_k \rangle$ by induction on $k < \omega$ (so $k = m+1 \Rightarrow \bar s_{n,m} \trianglelefteq \bar s_{n,k}$) such that: $\bar s_{n,k}$ is a finite initial segment of a play of the game $\Game'_{\mathscr{S}_{1,n},n,h_n,q_n}$ in which the antagonist uses the strategy $\mathbf{st}_n$ and $\bar s_{n,k} \in N$ and if $k = m+1$, then for some $\ell \in [\ell_m, \ell_k - 1)$ we have $\mathrm{tp}_{\Delta_n}((e^1_{s_\ell} \restriction v_n)\hat{\ }(\bar e^2_{t_N} \restriction w_n), B^+_{\mathbf{f},u_n,h_n} + C^+_n) \ne \mathrm{tp}_{\Delta_n}((\bar e^1_{s_{\ell+1}} \restriction v_n)\hat{\ }(\bar e^2_{t_N} \restriction w_n), B^+_{\mathbf{f},u_n,h_n} + C^+_n)$.

If we can choose[45] all $\bar{s}_{n,k}, k < \omega$ we get a contradiction to clause (i) of the assumption of the theorem. Obviously, we can choose $\bar{s}_{n,0}$. So for some $k, \bar{s}_{n,k}$ is well defined but we cannot choose $\bar{s}_{n,k+1}$.

Let

$$\mathscr{S}'_{1,n} := \{s_{\ell_k+1} : \text{ for some } \bar{s} = \langle \mathscr{X}_\ell, t_\ell : \ell \leq \ell_k + 1\rangle \in \mathbf{T}_n \text{ we have } \bar{s}_{n,k} \triangleleft \bar{s}\}.$$

Let $q_n = \mathrm{tp}_{\Delta_n}((\bar{e}^1_{s_{n,\ell_k}} \restriction v_n)\hat{}(\bar{e}^2_{t_N} \restriction w_n), B^+_{\mathbf{f},u_n,h_n} + C^+_n)$.

Lastly, choose $(\mathscr{S}_{N,n+1}, \mathbf{q}_{n+1}, h_{n+1}, \mathbf{st}_{n+1}, \mathbf{T}_{n+1})$ as in $(*)_3$ with $(\mathscr{S}_{1,n+1}, n+1)$ here standing for $(\mathscr{S}, n)$ there. Clearly we are done proving $(*)_6$, in particular Clause (f) of $(*)_6$ holds because we cannot choose $\bar{s}_{n,k+1}$ as required above. So we can carry the induction.]

So we have chosen $\langle (h_{N,n}, q_{N,n}, \mathscr{S}_{1,n}) : n < \mathbf{k}\rangle$ for each $N \in \mathscr{N}$ and it belongs to N: if $\mathbf{k} < \omega$ trivially by $(*)_6$ and if $\mathbf{k} = \omega$ by clause (d) of $(*)_4$ and let $q_N = \cup\{q_{N,n} : n < \mathbf{k}\}$. Also $\mathscr{N}$ is a stationary subset of $[\mathscr{H}(\chi)]^{<\kappa}$ by $(*)_4$ and clause (k) of the assumption. Hence using Fodor's lemma on the club filter on $\mathscr{N}$:

- $(*)_7$ for some $\mathscr{S}_{1,n}, h_n, r_n, q^*_n$ for $n < \mathbf{k}$ the set $\mathscr{N}_2$ is a stationary subset of $[\mathscr{H}(\chi)]^{<\kappa}$ where $\mathscr{N}_2 := \{N \in \mathscr{N} : q_{N,n} = q^*_n$ and $\mathscr{S}_{N,n} = \mathscr{S}_{1,n}$ for every $n < \mathbf{k}\}$
- $(*)_8$ $\mathscr{S}_2 \in \mathscr{D}^+_2$ where $\mathscr{S}_2 := \{t_N : N \in N_2\}$.

[Why? Clearly $\mathscr{S}_2 \in \mathscr{P}(I_2) \in \mathscr{H}(\chi)$ hence $\mathscr{N}' = \{N \in \mathscr{N} : \mathscr{S}_2 \in N\}$ belongs to the club filter on $\mathscr{N}$, hence $\mathscr{N}_2 \cap \mathscr{N}' \neq \emptyset$, choose N in this intersection so $t_N \in \mathscr{S}_2 \in N$ hence by the choice of t_N we have $I_2 \backslash \mathscr{S}_2 \notin \mathscr{D}_2$, so $\mathscr{S}_2 \in D^+_2$ that is $(*)_8$ holds.]

- $(*)_9$ if $n < \mathbf{k}$ and $s \in \mathscr{S}_{1,n}$ then $\mathscr{S}_2 \backslash \mathscr{S}_{2,n,s} = \emptyset \mod \mathscr{D}_2$ where $\mathscr{S}_{2,n,s} = \{t \in \mathscr{S}_2 : q_n = \mathrm{tp}_{\Delta_n}((\bar{e}^1_s \restriction v_n)\hat{}(e^2_t \restriction w_n), B^+_{\mathbf{f},u_n,h_n} + C_n)\}$.

[Why? Similar to the proof of $(*)_8$.]

Let $h \in \Pi\mathfrak{a}_\mathbf{f}$ be $\sup\{h_n : n < \omega\}$. So clearly $\langle q^*_n : n < \mathbf{k}\rangle, h, \langle \mathscr{S}_{1,n} : n < \omega\rangle$ and $\mathscr{S}_2$ are as required. $\square_{4.1}$

The following is a transparent "$n(*)$-dimensional" relative of 4.1

Theorem 4.6. *1) Assume κ is regular uncountable, $\mathscr{D}_0$ is a filter on $I_0, \mathscr{D}_\ell$ is a κ-complete filter on I_ℓ for non-zero $\ell < n, \bar{e}^\ell_s \in {}^{m(\ell)}\mathfrak{C}$ for $\ell < n, s \in I_\ell$ and $\Delta \subseteq \mathbb{L}(\tau_T), C \subseteq \mathfrak{C}$ are finite. Then there are a type q and $\mathscr{S}_\ell \in \mathscr{D}^+_\ell$ for $\ell < n$ such that $(\forall^{\mathscr{D}_0} s_0 \in \mathscr{S}_0)(\forall^{\mathscr{D}_1} s_1 \in \mathscr{S}_1) \ldots (\forall^{\mathscr{D}_{n-1}} s_{n-1} \in \mathscr{S}_{n-1})[q = \mathrm{tp}_\Delta(\bar{e}^0_{s(0)}\hat{}\bar{e}^1_{s(1)}\hat{} \ldots \hat{}\bar{e}^{n-1}_{s_{n-1}}, C)]$.*
2) If above $\mathscr{D}_\ell$ is a normal filter on κ for $\ell < n$ then for some q and $\mathscr{S}_\ell \in \mathscr{D}^+$ we have: if $s_0 < \ldots < s_{n-1}$ and s_ℓ belongs to $\mathscr{S}$ then $q \equiv tp_\Delta(\bar{e}^0_{s_0}\hat{} \ldots \hat{}\bar{e}^{n-1}_{s_{n-1}}, c)$.
3) Moreover, if $s_\ell \in \mathscr{S}_\ell$ for $\ell < n$ then $\mathrm{tp}(\bar{e}^0_{s_0}\hat{} \ldots \hat{}\bar{e}^{n-1}_{s_{n-1}}, C)$ depends just on the permutation π of n such that $s_{\pi(0)} < s_{\pi(1)} < \ldots$.

Proof. 1) Let $m(< i) = \Sigma\{m(j) : j < i\}$.

Stage A: We prove it by induction on n; for $n = 0$ it says nothing, for $n = 1$ it holds by the pigeon-hull principle, i.e., because $\mathscr{D}_0$ is a filter and the set $\{\mathrm{tp}_\Delta(\bar{e}^0_{s_0}, c) : s_0 \in I_0\}$ is finite. So assume we know it for $n \geq 1$ and we shall prove it for $n + 1$.

[45] of course, as Δ^1_n is finite we can use a finite long enough game; part of our leeway

Let $I = \prod\limits_{\ell<n} I_\ell$ and $\bar e_{\bar s} = \bar e^0_{s_0} \hat{} \ldots \hat{} \bar e^{n-1}_{s_{n-1}} \in {}^{m(<n)}\mathfrak{C}$ for $\bar s \in I$ and let $\mathscr{P} := \{\{\bar s \in I : \mathfrak{C} \models \varphi[\bar e_{\bar s}, \bar c]\} : \varphi = \varphi(\bar x_{[m(<n)]}, \bar y) \in \mathbb{L}(\tau_T)$ and $\bar c \in {}^{\ell g(\bar y)}\mathfrak{C}$ and for some finite $C_1 \subseteq \mathfrak{C}$ and finite $\Delta_1 \subseteq \mathbb{L}(\tau_T)$ there are no $(\Delta_1, m(<n))$-type q on C_1 and sequence $\langle \mathscr{S}_\ell : \ell < n\rangle \in \prod\limits_{\ell<n} \mathscr{D}^+_\ell$ such that $(\forall^{\mathscr{D}_0} s_0 \in \mathscr{S}_0) \ldots (\forall^{\mathscr{D}_{n-1}} s_{n-1} \in \mathscr{S}_{n-1})[q = \mathrm{tp}_{\Delta_1}(\bar e_{\langle s_\ell:\ell<n\rangle}, C_1)]$ and $\neg\varphi(\bar x_{[m(<n)]}, \bar c) \in q\}$.

By the induction hypothesis Γ generates a filter on I hence there is an ultrafilter $\mathscr{D}_*$ on I extending it.

Stage B:

Choose finite $\Delta^1 \subseteq \mathbb{L}(\tau_T)$ large enough, i.e. such that

$(*)_1$ if $\bar e_\ell \in {}^{m(<n)}\mathfrak{C}$ for $\ell < \omega$ and $\langle \bar e_\ell : \ell < \omega\rangle$ is a Δ^1-indiscernible sequence over some set $C_1 \subseteq \mathfrak{C}$ then for no formula $\varphi(\bar x, \bar y, \bar z) \in \Delta, \ell g(\bar x) = m(<n)$ and $\bar b \in {}^{\ell g(\bar y)}\mathfrak{C}$ is the set $\{\ell < \omega$: for some $\bar c \in {}^{\ell g(\bar z)}C_1$ we have $\mathfrak{C} \models \varphi(\bar e_\ell, \bar b, \bar c) \equiv \neg\varphi(\bar e_{\ell+1}, \bar b, \bar c) : \ell < \omega\}$ infinite.

Choose χ and define $\mathscr{N}$ as in $(*)_4$ from the proof of 4.1. For $C \subseteq \mathfrak{C}$ define a game $\eth_C$. A play last ω moves (really $n_* < \omega$ large enough suffice). In the ℓ-th move the protagonist player chooses $\mathscr{X}_\ell \in \mathscr{D}_* \cap \mathrm{Def}_{m(<n)}(M)$ and the antagonist chooses $\bar s_\ell \in \prod\limits_{m<n} I_m$ such that $\bar e_{\bar s_\ell} \in \mathscr{X}_\ell$. In the end of the play the protagonist player wins the play when $\langle \bar e_\ell : \ell < \omega\rangle$ is a Δ^1-indiscernible sequence over C.

As in the proof of 4.1, see $(*)_2$ there, the protagonist player has a winning strategy **st**, and let $\mathbf{N} \in \mathscr{N}$ be such that $\mathbf{st} \in \mathbf{N}$ and choose $t_* \in I_n$ such that $\mathscr{X} \in \mathscr{D}_n \cap \mathbf{N} \Rightarrow t_* \in \mathscr{X}$, possible as $\mathscr{D}_n$ is κ-complete because $n \geq 1$. We now simulate a play of $\eth_C$ called $\langle(\mathscr{X}_\ell, \bar s_\ell) : \ell < \omega\rangle$ such that:

$(*)_2$ (a) the protagonist player uses **st** to choose $\mathscr{X}_\ell$

(b) the antagonist chooses $\bar s_\ell \in I \cap \mathbf{N}$ and in $\mathscr{X}_\ell$ such that if $\ell > 0$ and it is possible then $\mathrm{tp}_{\Delta^1}(\bar e^1_{\bar s_\ell} \hat{} \bar e^2_{t_*}, C^+) \neq \mathrm{tp}_{\Delta^1}(\bar e^1_{\bar s_{\ell-1}} \hat{} \bar e^2_{t_*}, C)$.

It follows that $(\mathscr{X}_\ell, \bar s_\ell) \in \mathbf{N}$ for $\ell < \omega$ and that for some $\ell(*) > 0$ the demand in clause (b) of $(*)_2$ is not possible. So for some q

$(*)_3$ (a) $\mathscr{X}_{\ell(*)} \in D$

(b) $\mathrm{tp}_{\Delta^1}(\bar e^1_{\bar s} \hat{} \bar e^2_{t_*}, C) = q$ for every $\bar s \in \mathscr{X}_{\ell(*)} \cap \mathbf{N}$.

By the definition of $\mathscr{D}_*$ and of the game there is $\langle \mathscr{S}_{\mathbf{N},\ell} : \ell < n\rangle \in \prod\limits_{\ell<n} \mathscr{D}^+_\ell$ as there, such that $\prod\limits_{\ell<n} \mathscr{S}_{\mathbf{N},\ell} \subseteq \mathscr{X}$ and without loss of generality $\langle \mathscr{S}_{\mathbf{N},\ell} : \ell < n\rangle \in \mathbf{N}$.

We continue as in the proof of 4.1 after proving $(*)_6$.

2),3) By $\mathscr{D}_\ell$ being a normal filter on κ for $\ell < n$. $\square_{4.6}$

In 4.6, if $\theta \leq \kappa$ is a compact cardinal then we allow C to be of cardinality $< \theta$.

Theorem 4.7. *Assume θ is a compact cardinal (or $\aleph_0$), $\mathscr{D}_0$ is a θ-complete filter on $I_0, \mathscr{D}_\ell$ is a κ-complete filter on I_ℓ for $\ell = 1, \ldots, n_* - 1$.*

Assume further $\varepsilon(\ell) < \theta$ for $\ell \leq n$ and $\bar e^\ell_s \in {}^{\varepsilon(\ell)}\mathfrak{C}$ for $\ell < n, s \in I_\ell$.

Then there are a type q and $\mathscr{S}_\ell \in \mathscr{D}^+_\ell$ for $\ell < n$ such that $(\forall^{\mathscr{D}_0} s_0 \in \mathscr{S}_0)(\forall^{\mathscr{D}_1} s_1 \in \mathscr{S}_1) \ldots (\forall^{\mathscr{D}_{n-1}} s_{n-1} \in \mathscr{S}_{n-1})[q = \mathrm{tp}(\bar e^0_{s(0)} \hat{} \bar e^1_{s(1)} \hat{} \ldots \hat{} \bar e^{n-1}_{s(n-1)}, C)]$.

Proof. The difference from the proof of 4.6 is that:

(a) $\mathscr{D}_*$ is now a θ-complete ultrafilter
(b) the game $\Game_C$ last θ moves
(c) we prove the protagonist player has a winning strategy directly (not via the game is determined).

[Why clause (c)? For every m and $\mathscr{Y}_0 \subseteq {}^m I$ letting $\mathscr{Y}_1 = ({}^m I)\backslash\mathscr{Y}_0$ for some $\mathbf{t} \in \{0,1\}$ we have $(\forall^{\mathscr{D}_*} \bar{s}_0 \in I)\ldots(\forall^{\mathscr{D}_*} \bar{s}_{m-1} \in I)[(\bar{s}_0,\ldots,\bar{s}_{m-1}) \in \mathscr{Y}_{\mathbf{t}}]$. $\square_{4.7}$

§ 4(B). Density of tK in ZFC occurs.

Theorem 4.8. *The universal solution theorem* *Assume T is countable, $(\kappa,\bar{\mu},\theta)$ as usual, $\mu_2 \le \kappa, \mu_0 \ge \beth_\omega$ and $\theta = \aleph_0$ and $\mathrm{cf}(\kappa) > 2^\theta$.*
1) If $\mathbf{m}_1 = (\mathbf{x}_1, \bar{\psi}_1, r_1) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{x}_1 \le_1 \mathbf{y} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$ then we can find $\mathbf{m}_2$ such that $\mathbf{m}_1 \le^+_1 \mathbf{m}_2 \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{y} \le_1 \mathbf{x}_{\mathbf{m}_2}$.
2) Similarly but in the assumption $\mathbf{y} \in \mathrm{uK}_{\kappa,\bar{\mu},\theta}$ and in the conclusion $\mathbf{m}_1 \le^{\odot}_1 \mathbf{m}_2$.

Remark 4.9. 0) Note that this theorem restricts the cardinals lightly, but for density of tK we shall have quite heavy restrictions, still ZFC ones.
1) Part (2) is not needed for this subsection.
2) If $M_{\mathbf{x}} \in \mathrm{EC}_{\kappa,\kappa}(T)$ the proof is somewhat easier, similarly in 4.1.
3) There is no real difference between the two parts. We just deal with the set of pairs (φ,ψ) where $\varphi \in \Gamma_{\mathbf{x}[\mathbf{m}_1]}$ and ψ illuminate $(\mathbf{m},\varphi)$.
4) In 4.8 we use $\iota(\mathbf{x}_1) = 2$ but with minor changes $\iota(\mathbf{x}_1) = 1$ is O.K., too; the changes are in $\oplus_5 - \oplus_7$ in the proof.
5) Concerning 4.8(2) see 3.2(3)(e).

Before proving note

Observation 4.10. *1) If $(\mathbf{x}_1, \bar{\psi}_1, r_1) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{x}_1 \le_1 \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$, then $(\mathbf{x}_1, \bar{\psi}_1, r_1) \le_1 (\mathbf{y}, \bar{\psi}_1, r_1) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$.*
1A) If $\mathbf{m}_1 \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathrm{cf}(\kappa) > 2^\theta, \theta \ge |T|$ then for some $r_2 = r_2(\bar{x}_{\bar{c}_{\mathbf{y}}}, \bar{x}_{\bar{d}_{\mathbf{y}}}, \bar{x}'_{\bar{c}_{\mathbf{y}}}, \bar{x}'_{\bar{d}_{\mathbf{y}}})$ which extends $r_{\mathbf{m}}$ and is complete (over $\emptyset$) we have $\mathbf{m} \le_1 (\mathbf{x}_{\mathbf{m}}, \bar{\psi}_{\mathbf{m}}, r_2) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$.
1B) If in (1A) in addition $\mathrm{cf}(\kappa) > 2^{|B_{\mathbf{x}}|}$ we can demand r_2 is a complete type over $B_{\mathbf{y}}$; (similarly for $B^+_{\mathbf{x}}$ when $|\mathbf{S}^\theta(B^+_{\mathbf{x}})| < \kappa$).
2) If $\mathbf{m}_1 \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{x}_{\mathbf{m}_1} \le_1 \mathbf{y} \in \mathrm{qK}_{\kappa,\bar{\mu},\theta}$ and $\mathrm{cf}(\kappa) > \theta \ge |T|$ then

(a) for some pair $(\bar{\psi}, r)$ we have $\bar{\psi}_{\mathbf{m}_1} \trianglelefteq \bar{\psi}$ and $\Gamma_{\bar{\psi}} = \Gamma_{\mathbf{x}[\mathbf{m}_1]} \cup \Gamma^1_{\mathbf{x}[\mathbf{m}]}$ and $(\mathbf{y}, \bar{\psi}, r_{\mathbf{m}_1}) \in \mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$
(b) similarly replacing $\Gamma^1_{\mathbf{x}[\mathbf{m}_1]}$ by $\Gamma^3_{\mathbf{x}[\mathbf{m}_1]}$.

2A) Like part (2) replacing $\mathrm{qK}_{\kappa,\bar{\mu},\theta}, \mathrm{qK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ by $\mathrm{uK}^{\oplus}_{\kappa,\bar{\mu},\theta}, \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ respectively.
3) Assume $\mathscr{D}$ is a κ-complete filter on a set $I, \bar{e}_t \in {}^\zeta\mathfrak{C}$ for $t \in I, \kappa = \mathrm{cf}(\kappa) > 2^{|B|+|\zeta|}$ and $\mathbf{f}$ is a $(\bar{\mu},\theta)$-set and $\kappa > \mathrm{cf}(\Pi\{\kappa_{\mathbf{f},i} : i \in u_{\mathbf{x}}\})$. Then for some $q \in \mathbf{S}^\zeta(B_{\mathbf{f},h}), \mathscr{S} \in \mathscr{D}^+$ and $h \in \Pi\mathfrak{a}_{\mathbf{f}}$ we have: $q = \mathrm{tp}(\bar{e}_t, B^+_{\mathbf{f},h})$ for every $t \in \mathscr{S}$.
4) Similarly for $\mathrm{sK}^{\oplus}_{\kappa,\bar{\mu},\theta}$.

Remark 4.11. 1) Variants: for $\bar{\psi}$ enough if $\mathrm{cf}(\kappa) > \theta$, see 2.27.
2) Compare with 2.17 and 3.7.
3) The reader may wonder: in the proof of 4.8 we deal with Γ^1 but the conclusion is on Γ^2? But see the proof of $\boxtimes$.

Proof. Straightforward, e.g. part (2) as in the proof of 3.27(1). $\square_{4.10}$

Proof. Proof of 4.8

1) Without loss of generality $\mathbf{y}$ is smooth and for notational simplicity ω is disjoint to $v_{\mathbf{y}}, w_{\mathbf{y}}$ and let $\mathbf{x} = \mathbf{x}_1$. Now

$(*)_0$ let $\bar{v}, \bar{w}, \bar{u}$ be such that
- (a) $\bar{v} = \langle v_n : n < \omega \rangle$ and $\bar{w} = \langle w_n : n < \omega \rangle$
- (b) $v_n \subseteq v_{n+1} \subseteq v_{\mathbf{y}}$ and $w_n \subseteq w_{n+1} \subseteq w_{\mathbf{y}}$
- (c) v_n is finite and w_n is finite
- (d) $v_{\mathbf{y}} = \cup\{v_n : n < \omega\}$ and $w_{\mathbf{y}} = \cup\{w_n : n < \omega\}$
- (e) $u = v_{\mathbf{y}} \cup w_{\mathbf{y}} \cup \omega$
- (f) let $\bar{u} = \langle u_n : n < \omega \rangle$ where $u_n = v_n \cup w_n \cup \{0, \dots, n-1\}$
- (g) let $u(n) = u_n, v(n) = v_n, w(n) = w_n$.

We choose $\langle (\Delta_n, \Delta^1_n, k_n, m_n) : n < \omega \rangle$ such that:

$(*)_1$
- (a) $m_n < \omega$ is increasing with n
- (b) $\Delta_n \subseteq \{\varphi(\bar{x}_{[u_n]}, \bar{z}_{[n]}) : \varphi \in \mathbb{L}(\tau_T)\}$ is finite
- (c) $\Delta_n \subseteq \Delta_{n+1}$ in the natural sense, i.e. up to equivalence
- (d) $\Delta_\omega = \cup\{\Delta_n : n < \omega\} = \{\varphi(\bar{x}_{[u]}, z_{[\omega]}) : \varphi \in \mathbb{L}(\tau_T)\}$ up to equivalence

$(*)_2$
- (a) $\Delta^1_n \subseteq \{\varphi(\bar{x}_{[u_n]}, \bar{y}_{[u_n]}, \bar{z}_{[m_n]}) : \varphi \in \mathbb{L}(\tau_T)\}$ is finite
- (b) $\Delta^1_n \subseteq \Delta^1_{n+1}$
- (c) $\Delta^1_\omega = \cup\{\Delta^1_n : n < \omega\} = \{\varphi(\bar{x}_{[u]}, \bar{y}_{[u]}, \bar{z}) : \bar{z} = \bar{z}_{[n]}$ for some $n, \varphi \in \mathbb{L}(\tau_T)\}$ up to equivalence, i.e. adding dummy variables
- (d) if $\varphi(\bar{x}_{[u_n]}, \bar{z}_{[n]}) \in \Delta_n$ then some $\varphi'(\bar{x}_{[u_n]}, \bar{y}_{[u_n]}, \bar{z}_{[m_n]}) \in \Delta^1_n$ is equivalent to $\varphi(\bar{x}_{[u_n]}, \bar{z}_{[n]})$ and some $\varphi''(\bar{x}_{[u_n]}, \bar{y}_{[u_n]}, \bar{z}_{[m_n]})$ is equivalent to $\varphi(\bar{y}_{[u_n]}, \bar{z}_{[n]})$
- (e) the finite Δ^1_n and $k_n < \omega$ are such that clause (i) of $\oplus$ of 4.1 holds for Δ_n, see the way we use this proving $(*)_6$ below.

This is possible as T is countable, and for clause (e) of $(*)_2$ as T is dependent.

$(*)_3$ $I := ([M_{\mathbf{x}}]^{<\kappa}, \subseteq)$ is $\mathrm{cf}(\kappa)$-directed and let $\mathscr{D}_I$ be the club filter on I.

Clearly

$(*)_4$
- (a) if $\alpha < \kappa \Rightarrow |\alpha|^{\aleph_0} < \kappa$ then $\mathbf{S}^\omega_{\mathbb{L}(\tau_T)}(B^+_{\mathbf{y}} + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}})$ has cardinality $< \kappa$
- (b) $\mathbf{S}^{u(*)}_{\Delta_n}(B^+_{\mathbf{y}, v_n} + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}})$ has cardinality $< \kappa$ for any finite $u(*)$.

[Why? Recall that $\mu_2 \le \kappa$ by an assumption of 4.8 and we are assuming that $\mathbf{y}$ is smooth hence $(B^+_{\mathbf{y}} + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}}, \bar{\mathbf{I}}_{\mathbf{y}})$ is a $(\bar{\mu}, \theta)$-set by 2.25(2); now for clause (a) apply 2.25(1) and for clause (b) apply 2.25(1A).]

So together by observation 4.10(2).

$(*)_5$ there are $\bar{\psi}_2$ and r_2 such that, (recall 2.11(3),(3A)):

(a) $(\mathbf{y}, \bar{\psi}_2, r_2)$ belongs[46] to $\mathrm{qK}^{\odot}_{\kappa,\bar{\mu},\theta}$ for part (1) and belongs to $\mathrm{uK}^{\odot}_{\kappa,\bar{\mu},\theta}$ for part (2)

(b) $\bar{\psi}_2 = \langle \psi_{2,\varphi} : \varphi \in \Gamma_1 \rangle$ where $\Gamma_1 = \Gamma^1_{\mathbf{y}}$ for part (1) and $\Gamma_1 \subseteq \Gamma^1_{\mathbf{y}}$ is $\mathbf{y} - \mathrm{uK}$-large for part (2), see Definition 3.6(3B)

(c) $r_2 = r_2(\bar{x}_{\bar{d}_{\mathbf{y}}}, \bar{x}_{\bar{c}_{\mathbf{y}}}, \bar{x}_{[\omega]})$ is a type over $\emptyset$ in $\mathfrak{C}$; may add r is complete but should at least contain $\psi_{2,\varphi}(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{x}_{[\omega]})$ for $\varphi \in \Gamma_1$

(d) for every $A \in [M_{\mathbf{x}}]^{<\kappa}$ some $\bar{e} \in {}^{\omega}(M_{\mathbf{x}})$ solve $(\mathbf{y}, \bar{\psi}_2, r_2, A)$, see Definition 2.11(3) or 3.3(2)(f).

For $A \in I$ first choose $(\bar{d}_A, \bar{c}_A)$ solving $(\mathbf{y}, \bar{\psi}, r, A)$, possible by 4.10(1) and second choose $\bar{e}_A$ as in $(*)_5(d)$ for $A + \bar{d}_1 + \bar{c}_A$ and let $\bar{e}^+_A = \bar{e}_A{}^\frown \bar{d}_A{}^\frown \bar{c}_A$.

Next

$(*)_6$ there are q^0_n, q^1_n, h_* and $\mathscr{S}_{\ell,n}$ (for $\ell < 3, n < \omega$) such that:

(a) $\mathscr{S}_{\ell,n} \in \mathscr{D}^+_I$

(b) $\mathscr{S}_{\ell,n} \subseteq \mathscr{S}_{\ell,m}$ when $n = m+1$

(c) $q^\ell_n \in \mathscr{S}^{u(n)+u(n)}_{\Delta^1_n}(B^+_{\mathbf{y},u_n,h_{*,n}} + \bar{d}_{\mathbf{y}} + \bar{c}_{\mathbf{y}})$

(d) if $s_0 \in \mathscr{S}_{0,n}$ then for the $(\mathscr{D}_I + \mathscr{S}_{1,n})$-majority of $s_1 \in \mathscr{S}_{1,n}$ (say for every $s_1 \in \mathscr{S}_{1,n,s_0}$) we have $q^0_n = \mathrm{tp}_{\Delta^1_n}((\bar{e}^+_{s_0} \restriction m_n)^\frown (\bar{e}^+_{s_1} \restriction m_n), B^+_{\mathbf{y},v_n,h_*} + \bar{d}_{\mathbf{y}} + \bar{c}_{\mathbf{y}})$ so $\mathscr{S}_{1,n,s_0} \subseteq \mathscr{S}_{1,n}$ belongs to $\mathscr{D}_I + \mathscr{S}_{1,n}$

(e) if $s_1 \in \mathscr{S}_{1,n}$ then for the $(\mathscr{D}_I + \mathscr{S}_{2,n})$-majority of $s_2 \in \mathscr{S}_{2,n}$ (say for every $s_2 \in \mathscr{S}_{2,n,s_1}$) we have $q^1_n = \mathrm{tp}_{\Delta^1_n}((\bar{e}^+_{s_1} \restriction m_n)^\frown (\bar{e}^+_{s_2} \restriction m_n), B^+_{\mathbf{y},v_n,h_*} + \bar{d}_{\mathbf{y}} + \bar{c}_{\mathbf{y}})$.

[Why? We do it by induction on n replacing h_* by h_n. For $n = 0$, without loss of generality $m_n = 0, h_0$ constantly zero and we can let $\mathscr{S}_{\ell,n} = I$ for $\ell = 0,1,2$. For $n = m+1$ we do it in two steps. First, letting $\mathbf{f}_n = (\langle B_{\mathbf{y},i} : i \in v_n \backslash u_{\mathbf{y}} \rangle, \langle \mathbf{I}_{\mathbf{y},i} : i \in v_n \cap u_{\mathbf{y}} \rangle)$ and applying Theorem 4.1 for $\mathbf{k} = 1$ with $(\mathscr{D}_I + \mathscr{S}_{\ell,m}, \mathscr{S}_{\ell,m}, \langle \bar{e}_A \restriction m_n : A \in \mathscr{S}_{\ell,m} \rangle, M_{\mathbf{x}}, \mathbf{f}_n, \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}}, \Delta_n, \Delta^1_n, k_n))_{\ell<2}$ here standing for $(\mathscr{D}_\ell, \mathscr{S}_\ell, \langle \bar{e}^\ell_A : A \in I_\ell \rangle, M, \mathbf{f}, C, \Delta_0, \Delta^1_0, k_0)_{\ell<2}$ there.

We get $h^0_n, \mathscr{S}_{0,n}, \mathscr{S}'_{1,m}, q^0_n \in \mathbf{S}^{2m_n}_{\Delta_n}(B^+_{\mathbf{y},v_n,h^0_n} + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{x}})$.

Second, let $\mathscr{S}'_{2,m} = \mathscr{S}_{2,m}$ and we apply Theorem 4.1 for $\mathbf{k} = 1$ with $(\mathscr{D}_I + \mathscr{S}'_{\ell,m}, \langle \bar{e}_A : A \in \mathscr{S}'_{\ell,m} \rangle, M, \mathbf{f}_n, \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}}, \Delta_n, \Delta^1_n, \mathbf{k})_{\ell=1,2}$ here standing for $(\mathscr{D}_\ell, \mathscr{S}_\ell, \langle \bar{e}^\ell_A : A \in I_\ell \rangle, M, \mathbf{f}, C, \Delta_0, \Delta^1_0, k_0)_{\ell=0,1}$ there. We get $h^1_n, \mathscr{S}_{1,n}, \mathscr{S}_{2,m}, q^1_n \in \mathscr{S}^{2m_n}_{\Delta_n}(B^+_{\mathbf{y},v_n,h^1_n} + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}})$.

Let $h_* = \sup\{h^\ell_n : n < \omega \text{ and } \ell = 0,1\}$, i.e. $\kappa \in \mathfrak{a}_{\mathbf{y}} \Rightarrow h_*(\kappa) = \sup\{h^\ell_n(\kappa) : n < \omega$ and $\ell = 0,1\}$. Now for $\ell = 0,1$ and $A_\ell \in \mathscr{S}_{\ell,n}$ let $\mathscr{S}_{\ell,n,A_\ell} = \{A_{\ell+1} \in \mathscr{S}_{\ell+1,n}$: we have $q^\ell_n = \mathrm{tp}_{\Delta_n}((\bar{e}_{A_\ell} \restriction m_n)^\frown (\bar{e}_{A_{\ell+1}} \restriction m_n), B^+_{\mathbf{y},v_n,h_*} + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}})\}$. So $(*)_6$ holds indeed.]

We choose an ultrafilter $\mathscr{D}$ on I extending $\mathscr{D}_I + \mathscr{S}_{1,n}$ for every $n < \omega$ so clearly $A \in I \Rightarrow \{B : A \subseteq B \in I\} \in \mathscr{D}$. Let $\bar{e}_* \in {}^{\omega}\mathfrak{C}$ realizes $p_*(\bar{y}_{[\omega]}) :=$ $\mathrm{Av}(\mathscr{D}, \langle \bar{e}_A : A \in I \rangle, \bar{d}_{\mathbf{y}} + \bar{c}_{\mathbf{y}} + M_{\mathbf{x}})$, i.e. (clause (a) by the definition of Av and clause (b) follows)

[46] moreover we can demand it belongs to $\mathrm{qK}^{\oplus}_{\kappa,\bar{\mu},\theta}$.

$(*)_7$ (a) for $\bar{b} \in {}^{\omega>}(M_{\mathbf{x}})$ we have $\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, \bar{e}_*, \bar{b}]$ iff
$\varphi(\bar{d}_{\mathbf{y}}, \mathbf{c}_{\mathbf{y}}, \bar{y}_{[\omega]}, \bar{b}) \in p_*(\bar{y}_{[\omega]})$ iff
$\{A \in I : \mathfrak{C} \models \varphi[\bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, \bar{e}_A, \bar{b}]\} \in \mathscr{D}$

(b) $\bar{e}_*$ exemplify $\bar{\psi}$, i.e. is as in $(*)_5(d)$ except that it may be $\notin M_{\mathbf{x}}$

$(*)_8$ (a) let Γ_1 be $\Gamma^1_{\mathbf{y}}$

(b) we define $\bar{\psi}^*_2 = \langle \psi^*_{2,\varphi} : \varphi \in \Gamma_1\rangle$ as follows: letting $\varphi = \varphi(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{z}) \in \Gamma_3$, for some $n_0(\varphi), \varphi \equiv \varphi_0(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{y}_{[n_0(\varphi)]}, \bar{z})$ and let $\varphi_1 = \psi_{2,\varphi_0}(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{y})$ where $\bar{\psi}_2$ is from $(*)_5(a)$
so really $\varphi_1 \equiv \varphi_2(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{y}_{[n_1(\varphi)]})$ for some $n_1(\varphi) \geq n_0(\varphi)$
and lastly let $\psi^*_{2,\varphi} = \psi_{\varphi_2}(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{y}_{[\omega]})$.

$(*)_9$ without loss of generality $\mathrm{tp}(\bar{e}_* {}^\frown \bar{d}_{\mathbf{y}} {}^\frown \bar{c}_{\mathbf{y}}, M_{\mathbf{x}})$ is recalling u is from $(*)_0(e)$
$p_{**}(\bar{x}_{[u]}) = \mathrm{Av}(\mathscr{D}, \langle \bar{e}_A {}^\frown \bar{d}_A {}^\frown c_A : A \in I\rangle, M_{\mathbf{x}})$.

[Why? By the choice of $\bar{e}_*$ it is enough to have $\mathrm{tp}(\bar{d}_{\mathbf{y}} {}^\frown \bar{c}_{\mathbf{y}}, M_{\mathbf{x}}) = \mathrm{Av}(\mathscr{D}, \langle \bar{d}_A {}^\frown \bar{c}_A : A \in I\rangle, M_{\mathbf{x}})$ which is easy by $\mathscr{D} \supseteq \mathscr{D}_I$ and the choice of $(\bar{d}_A, \bar{c}_A)$ for $A \in I$ after $(*)_5$.]

$(*)_{10}$ let $e^+_* = \bar{e}_* {}^\frown \bar{d}_{\mathbf{y}} {}^\frown \bar{c}_{\mathbf{y}}$.

We now consider the statement

$\boxtimes$ for every $A_* \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$, i.e. $A_* \in I$ there are $\bar{e} \in {}^{\omega}(M_{\mathbf{x}}), \bar{d} \in {}^{\ell g(\bar{d}[\mathbf{y}])}(M_{\mathbf{x}})$ and $\bar{c} \in {}^{\ell g(\bar{c}[\mathbf{x}])}(M_{\mathbf{x}})$ such that

(a) $\bar{e}$ solves $(\mathbf{y}, \bar{\psi}^*, r_2, A_*)$

(b) $(\bar{d}, \bar{c})$ solves $(\mathbf{m}_1, A_*)$ so $\ell g(\bar{c}) = \ell g(\bar{c}_{\mathbf{x}}), \ell g(\bar{d}) = \ell g(\bar{d}_{\mathbf{x}})$

(c) $\bar{e} {}^\frown \bar{d} {}^\frown \bar{c}$ realizes $\mathrm{tp}(\bar{e}_* {}^\frown \bar{d}_{\mathbf{y}} {}^\frown \bar{c}_{\mathbf{x}}, A_*)$, but we do not say "$\mathrm{tp}(\bar{e}_* {}^\frown \bar{d}_{\mathbf{y}} {}^\frown \bar{c}_{\mathbf{y}}, A_*)$".

Why proving $\boxtimes$ is enough?

We define $\mathbf{x}_2$ as $(M_{\mathbf{x}_1}, \bar{B}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, \bar{d}_{\mathbf{x}_1} {}^\frown \bar{e}_*, \bar{\mathbf{I}}_{\mathbf{y}})$; so clearly $\mathbf{y} \leq_1 \bar{\mathbf{x}}_2 \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$. Note that $W_{\mathbf{x}_2} = W_{\mathbf{x}_1} \cup W$ and $\omega = \mathrm{Dom}(\bar{e}_*)$.

Next

$(*)_{11}$ we define r_3 by: $r_3 = r_1 \cup \{\varphi(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{x}'_{\bar{e}_*}) : \varphi(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\bar{\mathbf{y}}]}, \bar{x}_{[\omega]}) \in r_2\}$ recalling $r_1 = r[\mathbf{m}_1]$ and r is from $(*)_5$.

Lastly, let

$(*)_{12}$ $\bar{\psi}_3 = \langle \psi_{3,\varphi} : \varphi \in \Gamma_{\bar{\psi}_3}\rangle$, where[47] $\Gamma_{\bar{\psi}_3} = \Gamma_{\bar{\psi}_1} \cup \Gamma_1$, see $(*)_5(b)$ (here it is convenient to allow repetitions of φ's in $\bar{\psi}$; for part (2) we have to change more) where:

(a) $\psi_{3,\varphi} = \psi_{1,\varphi}$ if $\varphi \in \Gamma^2_{\bar{\psi}_1}$, adding dummy variables recalling $\bar{\psi}_1 = \bar{\psi}_{\mathbf{m}_1}$
so $\bar{\psi}_1 = \langle \psi_{1,\varphi} : \varphi \in \Gamma^2_{\bar{\psi}_1}\rangle$

(b) $\psi_{3,\varphi}$ is $\psi^*_{2,\varphi}$ if $\varphi \in \Gamma^1_{\mathbf{y}}$ using $w_{\mathbf{y}} + \omega$ instead $w_{\mathbf{y}}$, i.e.
$\psi_{3,\varphi}(\bar{x}_{\bar{d}[\mathbf{y}]} {}^\frown \bar{x}_{[\omega]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{x}'_{\bar{d}[\mathbf{y}]} {}^\frown \bar{x}'_{[\omega]}, \bar{x}'_{\bar{c}[\mathbf{y}]})$
$= \psi^*_{2,\varphi}(\bar{x}_{\bar{d}[\mathbf{y}]}, \bar{x}_{\bar{c}[\mathbf{y}]}, \bar{x}'_{[\omega]})$.

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[47] if $I = \kappa$, $\langle M_\alpha : \alpha < \kappa\rangle$ our problem will be to choose the $\bar{e}'_\alpha$ such that $\langle \mathrm{tp}(\bar{e}'_\alpha, M_\alpha + \bar{c}_{\mathbf{y}} + \bar{d}_{\mathbf{y}}) : \alpha < \kappa\rangle$ is increasing

We shall show that the triple $\mathbf{m}_2 = (\mathbf{x}_2, \bar\psi_3, r_3)$ is as required. First, $\mathbf{m}_2 \in \mathrm{rK}^\oplus_{\kappa,\bar\mu,\theta}$: all the requirements are obviously satisfied, e.g. for clause (f) of Definition 3.3(1), given $A \in I$ we can choose $(\bar e, \bar d, \bar c)$ as in $\boxtimes$ so $\ell g(\bar c) = \ell g(\bar c_{\mathbf{x}_1}), \ell g(\bar d) = \ell g(\bar d_{\mathbf{x}_1}) = \ell g(\bar d_{\mathbf{y}})$ and let $\bar c' \in {}^{\ell g(\bar c[\mathbf{y}])}(M_{\mathbf{x}})$ be such that $\bar c = \bar c' \restriction \ell g(\bar c)$ and $\bar c' {}^\frown \bar d {}^\frown \bar e$ realizes $\mathrm{tp}(\bar c_{\mathbf{y}} {}^\frown \bar d_{\mathbf{y}} {}^\frown \bar e_*, A)$; this is possible as $\bar d {}^\frown \bar c$ realizes $\mathrm{tp}(\bar d_{\mathbf{x}} {}^\frown \bar c_{\mathbf{x}}, A)$ because by $\boxtimes(b)$ the pair $(\bar d, \bar c)$ solves $\mathbf{m}_1$ and $\bar d, \bar c$ are from $M_{\mathbf{x}}$. We shall now show that $(\bar d {}^\frown \bar e, \bar c)$ solves $\mathbf{m}_2$.

We have to check clauses $(\alpha), (\beta), (\gamma)$ of Definition 3.3(1)(f).

By the choice of $\bar c', \bar c' {}^\frown (\bar d {}^\frown \bar e)$ realizes $\mathrm{tp}(\bar c_{\mathbf{x}_2} {}^\frown \bar d_{\mathbf{x}_2}, A) = \mathrm{tp}(\bar c_{\mathbf{y}} {}^\frown (\bar d_{\mathbf{y}} {}^\frown \bar e_*), A)$ so clause (α) there holds.

Second, $\bar d_{\mathbf{x}_2} {}^\frown \bar c_{\mathbf{x}} {}^\frown \bar d {}^\frown \bar c = \bar d_{\mathbf{y}} {}^\frown \bar c_{\mathbf{x}} {}^\frown \bar d {}^\frown \bar c$, realizes $r_1 = r_{\mathbf{m}_1}$ by clause (b) of $\boxtimes$ and the definition; in addition $\bar d_{\mathbf{y}} {}^\frown \bar c_{\mathbf{y}} {}^\frown \bar e$ realizes r_2 by clause (a) of $\boxtimes$. Together $(\bar d_{\mathbf{y}} {}^\frown \bar e^+_*) {}^\frown \bar c_{\mathbf{y}} {}^\frown (\bar d {}^\frown \bar e) {}^\frown \bar c'$ realizes r_3 by the choice of r_2 above and the previous sentence; so clause (β) there holds.

For clause (γ) there, recalling that $\Gamma^2_{\bar\psi_3} = \Gamma^2_{\bar\psi_1} \cup \Gamma^1_{\mathbf{y}}$ and we have to check a condition for each $\varphi \in \Gamma^2_{\bar\psi_3}$. Now if $\varphi \in \Gamma^2_{\bar\psi_1}$ the desired conclusion holds by clause (b) of $\boxtimes$ and the definition of "solve".

If $\varphi \in \Gamma^1_{\mathbf{y}}$, use clause (a) of $\boxtimes$.

So we have proved indeed that $\mathbf{m}_2 = (\mathbf{x}_2, \bar\psi_3, r_3) \in \mathrm{rK}^\oplus_{\kappa,\bar\mu,\theta}$. In addition obviously $\mathbf{m}_1 \le_1 \mathbf{m}_2$. Lastly, $\mathbf{m}_1 \le^\odot_1 \mathbf{m}_2$ by the choice of $\bar\psi_3 \restriction \Gamma^1_{\mathbf{y}}$, so we are done.

So we are left with
<u>proving $\boxtimes$ holds:</u>

Let $A_* \subseteq M_{\mathbf{x}}$ be of cardinality $< \kappa$ and we shall show that there are sequences $\bar e, \bar d, \bar c$ as required for A_* in $\boxtimes$, this suffices. We can choose $\langle A_{*,n} : n < \omega \rangle$ and A_{**} such that, without loss of generality

$\boxplus_0$ $A_* \cup B^+_{\mathbf{y}} \subseteq A_{*,n} \in \mathscr{S}_{0,n}$ and let $A_{**} = \cup\{\bar e^+_{A_{*,n}} + A_{*,n} : n < \omega\} \in [M]^{<\kappa}$.

Recalling $(*)_6$ let $\mathscr{S}'_{1,n} = \cap\{\mathscr{S}_{1,m,A_{*,m}} : m \le n\}$ but $\mathscr{S}'_{1,n} \subseteq \mathscr{S}_{1,n,A_{*,m}} \in \mathscr{D}_I + \mathscr{S}_{1,m} \subseteq \mathscr{D}_I + \mathscr{S}_{1,n}$ for $m \le n < \omega$ hence $\mathscr{S}'_{1,n} \in \mathscr{D}_I + \mathscr{S}_{1,m}$.

Recalling $(*)_8$ let[48] $\Lambda = \{p : p$ a finite subset of $p_{**}(\bar y_{[u]})$ with parameters from $A_{**}\}$; so clearly $|\Lambda| < \kappa$ and let $\Lambda_{\ge n} = \{p \in \Lambda : |p| \ge n\}$. By the choice of $\bar e_*, \mathscr{D}$ we can find $\langle A(p) : p \in \Lambda \rangle$ such that:

$\boxplus_1$ $A(p) \in \mathscr{S}'_{1,n} \subseteq I$ and $\bar e_{A(p)}$ realizes p and $A_{**} \subseteq A(p)$ <u>when</u> $p \in \Lambda$ and $|p| = n$.

For $n < \omega$ let C_n be a member of $\mathscr{S}_{2,n}$ which includes $\cup\{\bar e^+_{A(p)} + A(p) : p \in \Lambda\} \cup A_{**}$ such that $p \in \Lambda \wedge |p| = n \Rightarrow C_n \in \mathscr{S}_{2,n,A(p)}$, possible by 4.1 the "$\mathscr{D}_2$-almost", i.e. recalling $\mathscr{D}_I$ is from $(*)_3$ as $\mathscr{S}_{2,n,A(p)} \in \mathscr{D}_I + \mathscr{S}_{2,n}$ by $(*)_6(e)$ and the choice of $\mathscr{D}$. Let $\mathscr{D}_*$ be an ultrafilter on Λ such that $p_1 \in \Lambda \Rightarrow \{p \in \Lambda : p_1 \subseteq p\} \in \mathscr{D}_*$. Let $q_*(\bar x_{[\omega]}) = \mathrm{Av}(\mathscr{D}_*, \langle \bar e_{A(p)} : p \in \Lambda \rangle, \cup\{C_n + \bar e_{C_n} : n < \omega\})$, it is a type in $M_{\mathbf{x}}$ of cardinality $< \kappa$. Recalling $(*)_8$ let $q_{**}(\bar x_{[u]}) = q_{**}(\bar x_{[\omega]}, \bar x_{\bar d[\mathbf{y}]}, \bar x_{\bar c[\mathbf{y}]}) = \mathrm{Av}(\mathscr{D}_*, \langle \bar e_{A(p)} {}^\frown \bar d_{A(p)} {}^\frown \bar c_{A(p)} = e^+_{A(p)} : p \in \Lambda \rangle, \cup\{C_n + \bar e_{C_n} + \bar d_{C_n} + \bar c_{C_n} : n < \omega\})$, it is a type in $M_{\mathbf{x}}$ of cardinality $< \kappa$ and it extends $q_*(\bar x_{[\omega]})$. Lastly, let $\bar c \in {}^{\ell g(\bar c_{\mathbf{y}})}M_{\mathbf{x}}, \bar d \in {}^{\ell g(\bar d_{\mathbf{y}})}(M_{\mathbf{x}})$ and $\bar e \in {}^\omega(M_{\mathbf{x}})$ be such that the sequence $e^+ = \bar e {}^\frown \bar d {}^\frown \bar c$ realizes $q_{**}(\bar x_{[\omega]}, \bar x_{\bar d[\mathbf{y}]}, \bar x_{\bar c[\mathbf{y}]})$, we shall prove that $\bar e, \bar d, \bar c$ are as required in $\boxtimes$; and let $\bar e^+ = \bar e {}^\frown \bar d {}^\frown \bar c$. That is, we have three

[48] we could use parameters just from $\bar d_{\mathbf{y}} + \bar c_{\mathbf{y}} + \Sigma_n \bar e_{A_{*,n}}$

demands on $\bar e^+$, i.e. on $\bar e, \bar d, \bar c$ in $\boxtimes$, noting $\bar e^+ \in {}^u(M_{\mathbf x})$ recalling u is from $(*)_0(e)$, in other words, $\bar e, \bar d, \bar c$ are sequences of elements of $M_{\mathbf x}$ of the right lengths; let $u = \ell g(\bar e^+) = \ell g(\bar e^+_*)$.

First, clause (c) there says"realizing $\mathrm{tp}(\bar e^+_*, A_*) = \mathrm{tp}(\bar e_* {}^\frown \bar d_{\mathbf y} {}^\frown \mathbf c_y, A_*)$" recalling $(*)_9$; we shall show that moreover $\bar e^+$ realizes $\mathrm{tp}(\bar e^+_*, A_{**})$. This sufices as $A_* \subseteq A_{**}$. Why "$\bar e^+$ realizes $\mathrm{tp}(\bar e^+_*, A_{**})$" holds? Just recall $q_{**}(\bar y_{[u]}) = \mathrm{tp}(\bar e^+_*, M_{\mathbf x})$ by the choice of $\bar e^+_*$ (see $(*)_7(a) + (*)_8 + (*)_9$), by the definition of Λ and the choice of $\mathscr D_*$ and $\bar e^+$ above. We shall deal with clause (b) in $\boxplus_5$ below and with clause (a) in $\boxplus_7$ below.

Now

$\boxplus_2$ $\bar e^+_* \restriction u_n, \bar e^+ \restriction u_n$ and all $\bar e^+_A \restriction u_n$ for $A \in \mathscr S'_{1,n}$ realize the same complete Δ_n-type (can add same complete type) over $B^+_{\mathbf y, \upsilon_n, h_*}$.

[Why? First, recaling h_* is from $(*)_6$ there is $p_n \in \mathbf S^{u(n)}_{\Delta_n}(B^+_{\mathbf y, \upsilon_n, h_*})$ such that $\bar e^+_A$ (equivalently $\bar e^+_A \restriction u_n$), realizes p_n when $A \in \mathscr S'_{1,n}$ by $(*)_6(d)$ recalling $(*)_2(d)$ and $\mathscr S'_{1,n} \subseteq \mathscr S_{1,n}$, see after $\boxplus_0$. For $\bar e^+_*$ by its choice in $(*)_7$, the choice of $\mathscr D$ and the previous sentence; lastly, for $\bar e^+$ it realizes $q_{**}(\bar x_{[u]})$ and recall that $B^+_{\mathbf y} \cup A_* \subseteq C_n \subseteq \mathrm{Dom}(q_{**})$, the definition of q_{**} and the previous sentence.]

$\boxplus_3$ $\bar e^+_* \restriction u_n, \bar e^+ \restriction u_n$ and $\bar e^+_A \restriction u_n$ (for $A \in \mathscr S'_{1,n+1}$) all realize the same Δ_n-type over $\bar c_{\mathbf y} + B^+_{\mathbf y, \upsilon_n, h_*}$.

[Why? Compared to $\boxplus_2$ we add $\bar c_{\mathbf y}$. First, all the $\bar e^+_A$ for $A \in \mathscr S'_{1,n}$ realizes the same Δ_n-type over $\bar c_{\mathbf y} + B^+_{y, \upsilon_n, h_*}$ as this holds for $B^+_{\mathbf y, \upsilon_n, h_*}$ and recalling 3.11(4) the type $\mathrm{tp}(\bar c_{\mathbf y}, M_{\mathbf x})$ is finitely satisfiable in $B^+_{\mathbf y, \upsilon_n, h_*}$ and all those sequences are from $M_{\mathbf x}$. Second, the equality for $\bar e^+_*$ and $\bar e^+_A$'s as in the proof of $\boxplus_2$. Third, for $\bar e^+$ and the $\bar e^+_A$'s as $\mathrm{tp}(\bar c_{\mathbf y, \upsilon_n}, M_{\mathbf x})$ is finitely satisfiable in $B^+_{\mathbf y, \upsilon_n}$ when $\iota(\mathbf x) = 2$ (also it is locally does not split over $B^+_{\mathbf y, \upsilon_n}$ but have to do more earlier if $\iota_{\mathbf x} = 1$) and those sequences are from $M_{\mathbf x}$, using $\boxplus_2$ of course.]

$\boxplus_4$ if $\varphi = \varphi(\bar x_{\bar d[\mathbf x]}, \bar x_{\bar c[\mathbf x]}, \bar y'_{[u]})$ and $\Lambda_\varphi = \{p \in \Lambda : \mathfrak C \models \varphi[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e^+_{A(p)}]\}$ belongs to $\mathscr D_*$ then $\mathfrak C \models \varphi[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e^+]$.

This will take awhile.

Recalling $(*)_{11}$ note that

$\oplus_{4.1}$ (a) $\varphi = \varphi(\bar x_{\bar d[\mathbf x]}, \bar x_{\bar c[\mathbf x]}, \bar y'_{[u]}) \in \Gamma^1_{\mathbf x}$

(b) $\varphi_1 = \varphi_1(\bar x_{\bar d[\mathbf x]}, \bar x_{\bar c[\mathbf x]}, \bar y_{[\omega]}) := \psi_{2,\varphi}$

(c) $\varphi_2 = \varphi_2(\bar x_{\bar d[\mathbf x]}, \bar x_{\bar c[\mathbf x]}, \bar y_{[\omega]}) = \psi_{2,\varphi_1}$.

[Why? For clause (a) just reflect, for clause (b) and (c) recall $(*)_5$.]

Now:

$\oplus_{4.2}$ (a) $\mathfrak C \models$ "$\varphi_1[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e_{C_n}]$" for $n < \omega$

(b) $\varphi_1(\bar x_{\bar d[\mathbf x]}, \bar c_{\mathbf x}, \bar e_{C_n}) \vdash \varphi(\bar x_{\bar d[\mathbf x]}, \bar c_{\mathbf x}, \bar e^+_{A(p)})$ for $p \in \Lambda_\varphi, n < \omega$.

[Why? Clause (a) holds by the choice of φ_1 (in $\oplus_{4.1}$), the choice of $\bar\psi_2$ (in $(*)_5$) and the choice of the $\bar e_A$'s, in particular, $\bar e_{C_n}$.

Clause (b) holds by the choice of φ_1 the choice of $\bar\psi_2$ (in $(*)_5$) and the choice of $\bar e_{C_n}$ recalling $\bar e^+_{A(p)} \subseteq C_n$ for $p \in \Lambda$ by the choice of C_n after $\boxplus_1$.]

Hence letting $\vartheta_2 = \vartheta_2(\bar x_{\bar c[\mathbf{x}]}, \bar y''_{[\omega]}, \bar y'_{[u]}) := (\forall \bar x_{\bar d[\mathbf{x}]})(\varphi_1(\bar x_{\bar d[\mathbf{x}]}, \bar x_{\bar c[\mathbf{x}]}, \bar y''_{[\omega]}) \rightarrow \varphi(\bar x_{\bar d[\mathbf{x}]}, \bar x_{\bar c[\mathbf{x}]}, \bar y'_{[u]}))$, we have

$\oplus_{4.3}$ (a) $\mathfrak{C} \models$ "$\varphi_1[\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar e_{C_n}]$" for $n < \omega$

(b) $\mathfrak{C} \models$ "$\vartheta_2[\bar c_{\mathbf{x}}, \bar e_{C_n}, \bar e^+_{A(p)}]$" for $p \in \Lambda_\varphi, n < \omega$.

[Why? As $\bar e^+_{A(p)} \subseteq C_n$ for $p \in \Lambda, n < \omega$.]

By 4.1, that is by $(*)_6(d)$, recalling $A(p) \in \mathscr{S}'_{1,|p|} \subseteq \mathscr{S}_{1,|p|,A}$ from $\oplus_{4.3}(b)$ as $\mathrm{tp}(\bar c_{\mathbf{x}}, M_{\mathbf{x}})$ is finitely satisfiable in $B^+_{\mathbf{x}}$ it follows that for some $n_1 < \omega$ (recalling $\Delta^1_\omega = \cup\{\Delta^1_n : n < \omega\}, \Delta^1_n \subseteq \Delta^1_{n+1}$ by $(*)_1$) we have

$\oplus_{4.4}$ if $\bar c \in {}^{\ell g(\bar c[\mathbf{x}])}(B^+_{\mathbf{x}})$ and $p, q \in \Lambda_{\geq n_1}$ and $n < \omega$ <u>then</u> $\mathfrak{C} \models$ "$\vartheta_2[\bar c', \bar e_{C_n}, \bar e^+_{A(q)}] \equiv \vartheta_2[\bar c', \bar e^+_{C_n}, \bar e^+_{A(p)}]$".

Hence by the choice of $\bar e^+$ (after $\boxplus_1$)

$\oplus_{4.5}$ if $\bar c' \in {}^{\ell g(\bar c[\mathbf{x}])}(B^+_{\mathbf{x}})$ and $p \in \Lambda_{\geq n_1}$ and $n < \omega$ <u>then</u> $\mathfrak{C} \models$ "$\vartheta_2[\bar c', \bar e_{C_n}, \bar e^+] \equiv \vartheta_2[\bar c', \bar e_{C_n}, \bar e^+_{A(p)}]$".

As $\mathrm{tp}(\bar c_{\mathbf{x}}, M_{\mathbf{x}})$ is finitely satisfiable in $B^+_{\mathbf{x}} \subseteq A_{**}$, clearly

$\oplus_{4.6}$ if $p \in \Lambda_{\geq n_1}$ and $n < \omega$ <u>then</u> $\mathfrak{C} \models$ "$\vartheta_2[\bar c_{\mathbf{x}}, \bar e_{C_n}, \bar e^+] \equiv \vartheta_2[\bar c_{\mathbf{x}}, \bar e_{C_n}, \bar e^+_{A(p)}]$".

By $\oplus_{4.6}$ and $\oplus_{4.3}(b)$ we get

$\oplus_{4.7}$ $\mathfrak{C} \models$ "$\vartheta_2[\bar c_{\mathbf{x}}, \bar e_{C_n}, \bar e^+]$"

which means

$\oplus_{4.8}$ $\varphi_1(\bar x_{\bar d[\mathbf{x}]}, \bar c_{\mathbf{x}}, \bar e_{C_n}) \vdash \varphi(\bar x_{\bar d[\mathbf{x}]}, \bar c_{\mathbf{x}}, \bar e^+)$.

By $\oplus_{4.3}(a)$ we have

$\oplus_{4.9}$ $\mathfrak{C} \models \varphi_1[\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar e_{C_n}]$ for $n < \omega$.

By $\oplus_{4.8} + \oplus_{4.9}$ we have

$\oplus_{4.10}$ $\mathfrak{C} \models$ "$\varphi[\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar e^+]$"

So $\boxplus_4$ has been proved indeed.

$\boxplus_5$ clause (b) of $\boxtimes$ holds (for our choice of $\bar e^+$).

[Why? We have to check clauses $(\alpha), (\beta), (\gamma)$ Definition 3.3(1)(f). For every $A \in I$ the pair $(\bar d_A, \bar c_A)$ solve $(\mathbf{m}_1, A_*)$ hence $\bar d_{\mathbf{x}} \hat{\ } \bar c_{\mathbf{x}} \hat{\ } \bar d_A \hat{\ } \bar c_A$ realizes r_1 so recalling $e^+_A = \bar e_A \hat{\ } \bar d_A \hat{\ } \bar c_A$ by $\boxplus_4$ also $(\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar d, \bar c)$ realizes r_1 so clause (β) there holds. Also as in the proof of $\boxplus_2$ just easier, $\bar d \hat{\ } \bar c$ realizes $\mathrm{tp}(\bar d_{\mathbf{x}} \hat{\ } \bar c_{\mathbf{x}}, A_{**})$ and is from $M_{\mathbf{x}}$ by the choices of $\bar e, \bar d, \bar c$ after $\boxplus_1$, so clause (α) there holds. As in the proof of $\boxplus_4$ easily $\bar d \hat{\ } \bar c$ and $\bar d_{A(p)} \hat{\ } \bar c_{A(p)}$ realize the same type over $\bar c_{\mathbf{x}} + A_{**}$ for $p \in \Lambda$.

Let $\varphi = \varphi(\bar x_{\bar d[\mathbf{x}]}, \bar x_{\bar c[\mathbf{x}]}, \bar x'_{\bar d[\mathbf{x}]}, \bar x'_{\bar c[\mathbf{x}]}, \bar y) \in \Gamma^2_{\bar\psi_1}$ and $\bar b \in {}^{\ell g(\bar y)}(A_*)$ be such that $\models \varphi[\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar d, \bar e, \bar b]$ so $\psi_{1,\varphi} = \psi_{1,\varphi}(\bar x_{\bar d[\mathbf{x}]}, \bar x_{\bar c[\mathbf{x}]}, \bar x'_{\bar d[\mathbf{x}]}, \bar x'_{\bar c[\mathbf{x}]}) \in r_1$ and $\psi_{1,\varphi}(\bar x_{\bar d[\mathbf{x}]}, \bar c_{\mathbf{x}}, \bar d_{A(p)}, \bar c_{A(p)}) \vdash \varphi(\bar x_{\bar d[\mathbf{x}]}, \bar c_{\mathbf{x}}, \bar d_{A(p)}, \bar c_{A(p)}, \bar b)$.

As $\psi_{1,\varphi} \in r_1$ necessarily $\models$ "$\psi_{1,\varphi}[\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar d_{A(p)}, \bar c_{A(p)}] \wedge \psi_{1,\varphi}[\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}}, \bar d, \bar c]$" and as $\bar d_{A(p)}{}^\frown \bar c_{A(p)}, \bar d^\frown \bar c$ realize the same type over A_* and as $\bar b \subseteq A_*$ and the end of the previous sentence, $\psi_{1,\varphi}(\bar x_{\bar d[\mathbf{x}]}, \bar c_{\mathbf{x}}, \bar d, \bar c) \vdash \varphi(\bar x_{\bar d[\mathbf{x}]}, \bar c_{\mathbf{x}}, \bar d, \bar c, \bar b)$.

The last sentence says that clause (γ) from 3.3(1)(f) holds. Together indeed we have proved that $\bar d, \bar c$ satisfies clause (b) of $\boxtimes$, i.e. $(\bar d, \bar c)$ solves $(\bar{\mathbf{m}}_1, A_{**})$ as promised.]

We are left with clause (a) of $\boxtimes$. For the rest of the proof let $\bar x_{\bar d} = \bar x_{\bar d[\mathbf{y}]}, \bar x_{\bar c} = \bar x_{\bar c[\mathbf{y}]}$.

$\boxplus_6$ for $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}, \bar z)$ let $\varphi_0 = \varphi, \varphi_1 = \varphi_1(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[\omega]}) = \psi_{2,\varphi}$ where $\bar\psi_2$ is from $(*)_5$ and $\varphi_2 = \varphi_2(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[\omega]}) := \psi_{2,\varphi_1}$ so $\varphi_2 = \psi^*_{2,\varphi}$, see $(*)_{11}$.

Now to finish the proof of $\boxtimes(a)$ hence of the theorem, it suffices to show:

$\boxplus_7$ $\mathfrak{C} \models \varphi_2[\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar e]$ and $\varphi_2(x_{\bar d}, \bar c_{\mathbf{y}}, \bar e) \vdash \varphi(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar b)$ <u>when</u> (if we add $\bar e$ in φ we have a problem in $\oplus_{7.10}$ as the $\bar e$ is changed):
- (a) $\varphi = \varphi(\bar x_{\bar d}, \bar x_{\bar c}; \bar z) \in \Gamma_1$
- (b) $\mathfrak{C} \models \varphi[\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar b]$ and $\bar b \in {}^{\ell g(\bar z)}(A_*)$.

Why? So assume clauses (a),(b) of $\boxplus_7$ and eventually we shall prove the desired conclusions of $\boxplus_7$. The first part, $\mathfrak{C} \models \varphi_2[\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar e]$ holds by $\boxplus_4$ by $\oplus_{7.2}(a)$ and the second part by $\oplus_{7.10}$ below.

Recalling the choice of $\bar\psi_2$ in $(*)_5$ recalling $\varphi_0 = \varphi$ and the formula $\varphi_1(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[\omega]})$ is equal to ψ_φ, clearly we have: letting $I_{\bar b} := \{A \in I : \bar b \in {}^{\ell g(\bar z)}A\}$ and $J = \{\bar b \in {}^{\ell g(\bar b)}(M_{\mathbf{x}}) : \mathfrak{C} \models \varphi_0[\bar d, \bar c, \bar b]\}$

$\oplus_{7.1}$ (a) $\mathfrak{C} \models \varphi_1[\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar e_A]$ when $A \in I$
- (b) $\varphi_1(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar e_A) \vdash \varphi_0(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar b)$ when $A \in I_{\bar b} \cap J$
- (c) $\varphi_1(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar e_*) \vdash \varphi_0(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar b)$ when $A \in J$.

Hence by $(*)_7(b)$, recalling $\varphi_2 = \psi_{2,\varphi_1}$

$\oplus_{7.2}$ (a) $\mathfrak{C} \models \varphi_2[\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar e_A]$ for $A \in I$ hence $\mathfrak{C} \models \varphi_2[\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar e_*]$ so $\{\varphi_2(\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar y_{[\omega]})\} \in \Lambda$ pedentically $\{\varphi'_2(\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar y_{[u]})\} \in \Lambda$ where $\varphi'_2(\bar x_{\bar d}, x_{\bar c}, y_{[u]}) = \varphi_2(\bar x_{\bar d}, x_{\bar c}, \bar y_{[u]})$
- (b) $\varphi_2(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar e_*) \vdash \varphi_1(\bar x_{\bar d}, \bar c_{\mathbf{y}}, \bar e_{A_{*,n}})$ for $n < \omega$

moreover

$\oplus_{7.3}$ (a) if $p \in \Lambda$ satisfies $\varphi_2(\bar d, \bar c, \bar y_{[\omega]}) \in p$ then $\mathfrak{C} \models \varphi_2[\bar d, \bar c, \bar e_{A(p)}]$
- (b) if $A \in I$ then $\varphi_2(\bar x_{\bar d}, \bar c, \bar e_A) \vdash \{\varphi_1(\bar x_{\bar d}, \bar c, \bar e') : \mathfrak{C} \models \varphi_1[\bar d, \bar c, \bar e']$ and $\bar e' \in {}^\omega A\}$
- (c) like (a) replacing $\bar e_{A(p)}$ by $\bar e_*$.

So letting $\Lambda_* = \{p \in \Lambda : \varphi_2(\bar d_{\mathbf{y}}, \bar c_{\mathbf{y}}, \bar y_{[\omega]}) \in p\}$ we have $\Lambda_* \in \mathscr{D}_*$ and let

$$\vartheta_1 = \vartheta_1(\bar x_{\bar c}, \bar y'_{[\omega]}, \bar y_{[\omega]}) := (\forall \bar x_{\bar d})(\varphi_2(\bar x_{\bar d}, \bar x_{\bar c}, \bar y'_{[\omega]}) \rightarrow \varphi_1(\bar x_{\bar d}, \bar x_{\bar c}, \bar y_{[\omega]})).$$

So we have, by $\oplus_{7.1}(b)$ and $\oplus_{7.3}(c)$ as $\bar e_{A_*,n} \subseteq A_{**} \subseteq A(p)$ by $\boxplus_1$

$\oplus_{7.4}$ $\mathfrak{C} \models \vartheta_1[\bar{c}_{\mathbf{y}}, \bar{e}_{A(p)}, \bar{e}_{A_*,n}]$ for $p \in \Lambda_*, n < \omega$.

But by 4.1, that is, by $(*)_6(d)$ recalling $\bar{e}_{A(p)} \in \mathscr{S}'_{1,|p|} \subseteq \mathscr{S}_{1,|p|,A_*}$ it follows that for some $n_1 < \omega$ (as $\cup\{\Delta^1_n : n < \omega\} = \Delta^1_\omega$) we have:

$\oplus_{7.5}$ if $\bar{c}' \in {}^{\ell g(\bar{c}[\mathbf{y}])}(B^+_{\mathbf{x}})$ and $p, q \in \Lambda_{\geq n_1}$ <u>then</u>
$\mathfrak{C} \models$ "$\vartheta_1[\bar{c}', \bar{e}_{A(p)}, \bar{e}_{A_*,n}] \equiv \vartheta_1[\bar{c}', \bar{e}_{A(q)}, \bar{e}_{A_*,n}]$" for $n < \omega$.

Hence by the choice of $\bar{e}$

$\oplus_{7.6}$ if $\bar{c}' \in {}^{\ell g(\bar{c}[\mathbf{y}])}(B^+_{\mathbf{x}})$ and $p \in \Lambda_{\geq n_1}$ and $n < \omega$, <u>then</u>
$\mathfrak{C} \models \vartheta_1[\bar{c}', \bar{e}, \bar{e}_{A_*,n}] \equiv \vartheta_1[\bar{c}', \bar{e}_{A(p)}, \bar{e}_{A_*,n}]$.

As $\mathrm{tp}(\bar{c}_{\mathbf{y}}, M_{\mathbf{x}})$ is finitely satisfiable in $B^+_{\mathbf{x},h_*}$, clearly

$\oplus_{7.7}$ if $p \in \Lambda_{\geq n_1}$ and $n < \omega$ then $\mathfrak{C} \models$ "$\vartheta_1[\bar{c}_{\mathbf{y}}, \bar{e}, \bar{e}_{A_*,n}] \equiv \vartheta_1[\bar{c}_{\mathbf{y}}, \bar{e}_{A(p)}, \bar{e}_{A_*,n}]$".

Next

$\oplus_{7.8}$ if $n < \omega$ then $\mathfrak{C} \models \vartheta_1[\bar{c}_{\mathbf{y}}, \bar{e}, \bar{e}_{A_*,n}]$.

[Why? By $\oplus_{7.4} + \oplus_{7.7}$ because there is $p \in \Lambda_{\geq n_1} \cap \Lambda_*$ which holds as $\Lambda_{\geq n_1} \in \mathscr{D}_*$ and $\Lambda_* \in \mathscr{D}_*$ and $\mathscr{D}_*$ is an ultrafilter on Λ.]

So by the choice of ϑ_1

$\oplus_{7.9}$ $\varphi_2(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{y}}, \bar{e}) \vdash \varphi_1(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{y}}, \bar{e}_{A_*,n})$.

By $\oplus_{7.9} + \oplus_{7.1}(b)$ applied to $A = A_{*,n}$ we have (recall $\bar{b}$ is from $\boxplus_7$ hence $\bar{b} \subseteq A_* \subseteq A_{**}$)

$\oplus_{7.10}$ $\varphi_2(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{y}}, \bar{e}) \vdash \varphi_0(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{y}}, \bar{b})$.

This proves the second clause in the desired conclusion of $\boxplus_7$.

So we are done proving $\boxplus_7$.

As said above (before $\boxplus_7$) proving $\boxplus_7$ finish the proof.

2) We repeat the proof above with some changes. In $(*)_5(a)$ we replace $\mathrm{qK}^{\odot}_{\kappa,\bar{\mu},\theta}$ by $\mathrm{uK}^{\odot}_{\kappa,\bar{\mu},\theta}$ respectively. We change $(*)_8 + (*)_9$ naturally and also the rest should be clear. $\square_{4.8}$

Now we get a "density of $\mathrm{tK}_{\kappa,\mu,\theta}$ in ZFC" for $\theta = \aleph_0$ and some pairs κ, μ.

Conclusion 4.12. *If T is countable, $\theta = \aleph_0, \mu$ is strong limit and $(\mu > \mathrm{cf}(\mu) \geq \aleph_1 \wedge \kappa = \mu^+)$ or $(\mu = \mathrm{cf}(\mu) = \kappa)$, <u>then</u> for every $\mathbf{m} \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ there is $\mathbf{n} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ such that $\mathbf{m} \leq_1 \mathbf{n}$.*

Remark 4.13. 1) Do we need $\mathrm{cf}(\mu) > 2^\theta$? No, see 2.15(1) and 2.19 but "μ is strong limit" is assumed above.
2) Recall that $\mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ means $\mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ with $\bar{\mu} = (\mu_2, \mu_1, \mu_0) = (\kappa, \mu, \mu)$.
3) This is enough for the recounting of types for κ strongly inaccessible. Also for $\kappa = \mu^+, \mu$ strong limit singular of uncountable cofinality, but only if $\mu = \aleph_\mu$ we can deduce the correct upper bound on the number of types up to conjugacy in $\mathbf{S}(M), M \in \mathrm{EC}_{\lambda,\lambda}(T)$, still if $\mu < \aleph_\mu$ the upper bound is μ, smaller than the value for independent T.

Proof. We choose $\mathbf{m}_n$ by induction on $n < \omega$ such that

$\boxplus_1$ (a) $\mathbf{m}_n \in \mathrm{rK}_{\kappa,\mu,\theta}$

(b) $\mathbf{m}_0 = \mathbf{m}$

(c) $r[\mathbf{m}_{n+1}]$ is complete

(d) $\mathbf{m}_m \leq_1^+ \mathbf{m}_n$ when $n = m + 1$.

Why can we carry the induction?

For $n = 0$, by clause (b) this is trivial.

For $n = m + 1$ by 2.15(1) there is $\mathbf{y}_m$ such that $\mathbf{x}_{\mathbf{m}_m} \leq_2 \mathbf{y}_m \in \mathrm{qK}'_{\kappa,\mu,\theta}$ hence to $\mathrm{qK}'_{\mu,\mu,\theta}$ hence by 2.15(3) using our use of (μ, μ, θ) rather than (κ, μ, θ) we have $\mathbf{y}_\mathbf{m} \in \mathrm{qK}_{\mu,\mu,\theta}$; as $\mathrm{cf}(\mu) > \aleph_0$ by 2.17(1) for some $\bar{\psi}_m$, we have $(\mathbf{y}_m, \bar{\psi}_m, \emptyset) \in \mathrm{qK}^{\odot}_{\mu,\mu,\theta}$. Hence $\mathbf{y}_m \in \mathrm{qK}_{\kappa,\mu,\theta}$, why? If $\kappa = \mu$ trivially and if $\kappa = \mu^+$ by 2.17(2), which is O.K. by the assumptions $\theta < \mathrm{cf}(\mu) < \mu$. As $\mathrm{cf}(\mu) > \theta$ we have $|B^+_\mathbf{y}| < \mu$ and as μ is strong limit we have $\beth_k(|B_\mathbf{f}| + \theta) < \mu$ for $k < \omega$, so we can apply 4.8.

By 4.8 there is $\mathbf{n}_m \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ such that $\mathbf{m}_m \leq_1^+ \mathbf{n}_m$ such that $\mathbf{y}_m \leq_1 \mathbf{x}_{\mathbf{n}_m}$.

Lastly, as κ is regular $> 2^\theta$, by Observation 4.10(1A) there is a complete $r_n \supseteq r[\mathbf{n}_n]$ such that $\mathbf{m}_n := (\mathbf{x}_{\mathbf{n}_m}, \bar{\psi}_{\mathbf{n}_m}, r_n) \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ is $\leq_1$-above $\mathbf{n}_m$. So $\mathbf{m}_n$ is as required.

Now $\mathbf{n} = \lim\langle \mathbf{m}_n : n < \omega\rangle$ is as required by 3.23. $\square_{4.12}$

Remark 4.14. Also 4.12 is enough for the "generic pair conjecture" for the relevant cardinals.

§ 5. Stronger Density

§ 5(A). **More density of** tK.

The following will help us to prove density of $\mathrm{tK}_{\kappa,\mu,\theta}$ replacing κ by κ^{+n} in 4.12. Unfortunately, we are stuck in $\kappa^{+\omega}$, still this gives more cases for the recounting of types.

Claim 5.1. Crucial Claim *There is an indiscernible sequence $\mathbf{I} = \langle \bar{a}_\alpha : \alpha < \lambda \rangle$ in $M_{\mathbf{x}}$ such that letting $\bar{a}_\lambda$ realizes $\mathrm{Av}(\mathbf{I}, M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$, the types of $\bar{a}_\lambda$ and $\bar{d}_{\mathbf{x}}$ over $M_{\mathbf{x}} + \bar{c}_{\mathbf{x}}$ are not weakly orthogonal when:*

- (a) κ *is regular*
- (b) $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ *and* $\iota_{\mathbf{x}} = 2$
- (c) $\lambda = \mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ *is regular, see Definition 2.26(2)*
- (d) (α) $u_{\mathbf{x}}$ *is finite*
  or just
  (β) $\alpha < \lambda \Rightarrow |\alpha|^{|T|} < \lambda$
- (e) $\kappa > \lambda \geq \mu_1 > |B_{\mathbf{x}}|$
- (f) $\lambda > 2^{|B_{\mathbf{x}}|+\theta}$ *and* $\kappa \geq \beth_\omega(|B_{\mathbf{x}}| + \theta)$.

Discussion 5.2. 1) Recall $\mathrm{ntr}(\mathbf{x})$ is regular or is $\leq \theta$, see Definition 2.26, Observation 2.27(1) but this is not necessarily so for $\mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$, on it we know only that it is regular or its cofinality is $\leq \theta$.
2) Why above "$u_{\mathbf{x}}$ is finite"? Otherwise in 5.1 there is a problem. The reason is a pcf one: maybe $\lambda \in \mathrm{pcf}(\mathfrak{a}_{\mathbf{x},<\lambda})$ where we let $\mathfrak{a}_{\mathbf{x},<\lambda} = \{\kappa_{\mathbf{x},i} : i \in u_{\mathbf{x}}$ and $\kappa_{\mathbf{x},i} < \lambda\}$, even the case $\lambda \in \{\kappa_{\mathbf{x},i} : i \in u_{\mathbf{x}}\}$ need care.

Even under G.C.H., if $\lambda = \chi^+, \mathrm{cf}(\chi) \leq \theta$ we have a problem. The problem is in fixing the "essential" type of $\bar{e}_\alpha$ for $\alpha \in [\alpha_\varepsilon, \alpha_{\varepsilon+1})$ over $\mathbf{x}$; which has more information than its type over $B_{\mathbf{x}}$ but less than its type over $B^+_{\mathbf{x}}$ and is preserved if we replace $\mathbf{x}$ by a very similar $\mathbf{x}'$, we can use just $\mathbf{x}_{[h]}$ which is smooth see Definition 2.18(1),(2) and 2.19 and 2.22.

The first idea for saving the day was to get $\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle$ tree indiscernible for some $\bar{g} = \langle g_\alpha \in \Pi \mathfrak{a}_{\mathbf{x},\lambda} : \alpha < \lambda \rangle <_{J_{<\lambda}[\mathfrak{a}_{\mathbf{x},\lambda}]}$-increasing and cofinal which is "nice". Did not seem to work.

Second is a weaker version: demand something on $\langle \bar{e}^*_{\varepsilon_0}, \ldots, \bar{e}^*_{\varepsilon_{n-1}} \rangle$ only when $g_{\varepsilon_0} < g_{\varepsilon_1} < \ldots$.

The second is not good enough to classify $f^{\mathrm{aut}}_{T,\theta}(-)$. Still, when $\kappa = \mu^{+n}, \mu$ regular, $\mu = \mu^\theta$ this may help but we prefer not too, when we can.

The solution is to do it locally, i.e. to deal with local density for qK (in pK), deal with one φ, then pretend you have no φ and deal with the case $u_{\mathbf{x}}$ is finite, i.e. 4.1 whose original aim was to help 4.8.
3) The proof serves also for a related more local result, 5.3, there we just replace stage A; it also serves §(5B).
4) We may use normal $\mathbf{x}$ so $\bar{x}_{\bar{c}}$ disappears.

Proof. Stage A: By Claim 2.22, without loss of generality

$\otimes_0$ $\mathbf{x}$ is smooth (see Definition 2.18) so $\langle \mathbf{I}_{\mathbf{x},\kappa} : \kappa \in \mathfrak{a}_{\mathbf{x}} \rangle$ are well defined.

As we are assuming $\mathrm{ntr}_{\mathrm{lc}}(\mathbf{x}) = \lambda$ is regular, so, in particular, of cofinality $> \theta$, see Definition 2.26(2) and by 2.27(3), there are $\bar{\psi}, \varphi_*$ such that (see Definition 2.26(5) on λ-illuminate, 2.6(8) for $\Gamma^1_{\mathbf{x}}$, 2.8(3A),(3B) on illuminate):

$\otimes_1$ (a) $\bar{\psi}$ does λ-illuminate $\mathbf{x}$ so $\Gamma^1_{\bar{\psi}} = \Gamma^1_{\mathbf{x}}$,

(b) $\varphi_* = \varphi_*(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$

(c) ψ_{φ_*} does not λ^+-illuminate[49] $(\mathbf{x}, \varphi_*)$

(d) without loss of generality $\psi_{\neg\varphi_*} = \psi_{\varphi_*}$

(e) if $A \subseteq M_{\mathbf{x}}$ has cardinality $< \lambda$ then there is $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ such that $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \hat{\ } \bar{e}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \hat{\ } + A)$ according to $\bar{\psi}$; follows by (a).

Hence for some A

$\otimes_2$ (a) $A \subseteq M_{\mathbf{x}}$ has cardinality λ

(b) for no $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ does $\psi_{\varphi_*}(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{e})$ solves $(\mathbf{x}, A, \varphi_*)$, see 2.8(1A)

(c) let $\langle a_\alpha : \alpha < \lambda \rangle$ list A

$\otimes_3$ (a) let $\varphi_0 = \varphi_*(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_0)$

(b) $\varphi_0^{\mathbf{t}} = \varphi_0^{\mathrm{if}(\mathbf{t})}$ is φ_0 if $\mathbf{t} = 1$ and $\neg\varphi_0$ if $\mathbf{t} = 0$, so $\vartheta_{\varphi_0^{\mathbf{t}}}$ are well defined

(c) let $\varphi_1 = \varphi_1(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}_{[\theta]}) \in \Gamma^1_{\bar{\psi}}$ be $\psi_{\varphi_0} = \psi_{\varphi_*}$

(d) let $\varphi_2 = \varphi_2(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}'_{[\theta]}) \in \Gamma^1_{\bar{\psi}}$ be ψ_{φ_1}

(e) let $\varphi_3 = \varphi_3(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}''_{[\theta]}) \in \Gamma^1_{\bar{\psi}}$ be ψ_{φ_2}

$\otimes_4$ (a) let $\Delta_0 = \{\vartheta'_{\varphi_1}(\bar{x}'_{[\theta]}, x''_{[\theta]}, \bar{x}_{\bar{c}})\}$ where $\vartheta'_{\varphi_1} = \vartheta_{\varphi_1}(\bar{x}_{\bar{c}}, \bar{x}'_{[\theta]}, \bar{x}_{[\theta]})$, see 2.8(1C)

(b) let $\Delta_1 = \{\vartheta'_{\varphi_2}(\bar{x}'_{[\theta]}, \bar{x}''_{[\theta]}, \bar{x}_{\bar{c}})\}$ where $\vartheta'_{\varphi_2} = \vartheta_{\varphi_2}(\bar{x}_{\bar{c}}, \bar{x}''_{[\theta]}, \bar{x}'_{[\theta]})$

(c) let $\Delta_2 \subseteq \mathbb{L}(\tau_T)$ be finite large enough such that clause (i) of 4.1 holds with (Δ_n, Δ^1_n) there standing for (Δ_0, Δ_2) here and for (Δ_1, Δ_2) here.

Note that

$\otimes_5$ $\Pi\{\lambda_i : i \in u_{\mathbf{x}}$ and $\lambda_i < \lambda\} < \lambda$.

[Why it holds? By clause (d) of the assumption; important for 5.3.]

Stage B: Let $I = ([M_{\mathbf{x}}]^{<\kappa}, \subseteq)$. Recall that for every $\bar{e} \in {}^\theta(M_{\mathbf{x}})$ for some $h \in \Pi\{\kappa_{\mathbf{x},i} : i \in u_*\}$ the pair $(B_{\mathbf{x}} + \bar{e}, \bar{\mathbf{I}}_{\mathbf{x},h})$ is a $(\bar{\mu}, \theta)$-set.

Now let $\bar{e}_A$ be as guaranteed by $\otimes_1(e)$ above for $A \in I$. Let $\mathscr{D}_I$ be the club filter on I.

We apply[50] Theorem 4.1 with $1, M_{\mathbf{x}}, (B_{\mathbf{x}}, \bar{\mathbf{I}}_{\mathbf{x}}), \langle \bar{e}_A : A \in I \rangle, \langle \bar{e}_A : A \in I \rangle, \langle \Delta_0 \rangle, \langle \Delta_2 \rangle, \mathscr{D}_I, \mathscr{D}_I$ here standing for $\mathbf{k}, M, \mathbf{f}, \langle \bar{e}^1_A : A \in I \rangle, \langle \bar{e}^2_t : t \in I \rangle, \langle \Delta_n : n < \mathbf{k} \rangle, \langle \Delta^1_n : n < \mathbf{k} \rangle, \mathscr{D}_1, \mathscr{D}_2$ there. We get $h^*_0, q_0, \mathscr{S}'_1, \mathscr{S}_2$ here standing for $h_*, q, \mathscr{S}_{1,0}, \mathscr{S}_2$ there.

Next we apply 4.1 again with $1, M_{\mathbf{x}}, (B_{\mathbf{x}}, \bar{\mathbf{I}}_{\mathbf{x}}), \langle \bar{e}_A : A \in I \rangle, \langle \bar{e}_A : A \in I \rangle, \langle \Delta_1 \rangle, \langle \Delta_2 \rangle, \mathscr{D}_I \restriction \mathscr{S}'_1, \mathscr{D}_I \restriction \mathscr{S}'_1$ here standing for $\mathbf{k}, M, \mathbf{f}, \langle \bar{e}^1_A : A \in I \rangle, \langle \bar{e}^2_s : s \in I_\ell \rangle, \langle \Delta_n : n < \mathbf{k} \rangle, \langle \Delta^1_n : n <$

[49] In the present proof, we can demand that no ψ does λ^+-illuminate $(\mathbf{x}, \varphi)$

[50] We could use Δ_n with union $\mathbb{L}(\tau_T)$ if $\theta = \aleph_0 = |T|$, if so we do not have to care in choosing Δ_0.

$\mathbf{k}\rangle, \mathscr{D}_1, \mathscr{D}_2$ there. We get $h^*_1, q_1, \mathscr{S}_0, \mathscr{S}_1$ here standing for $h_*, q, \mathscr{S}_{1,0}, \mathscr{S}_2$ there and without loss of generality $\mathscr{S}_0, \mathscr{S}_1 \subseteq \mathscr{S}'_1$. Note that $q_\ell \in \mathbf{S}^{\theta+\theta}_{\Delta_\ell}(B^+_{\mathbf{x}})$ for $\ell = 0, 1$.

Let $h = \max\{h^*_0, h^*_1\}$ and $B_* = B_{\mathbf{x},u_*,h}$ and let $S_\ell = \{3\alpha + \ell : \alpha < \lambda\}$ for $\ell = 0, 1, 2$.

We shall show that there is a quadruple $(\bar{N}, \mathbf{I}, \bar{A}, B_*)$ such that:

$\boxplus_1$ (a) $\bar{N} = \langle N_\alpha : \alpha < \lambda\rangle$ and $\mathbf{I} = \langle \bar{e}_\alpha : \alpha < \lambda\rangle$ and $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$
- (b) $N_\alpha \prec M_{\mathbf{x}}$ is $\prec$-increasing
- (c) $\|N_\alpha\| < \lambda$
- (d) $B_{\mathbf{x}} \subseteq N_0$ and $a_\alpha \in N_{\alpha+1}$; hence $A + B_{\mathbf{x}} \subseteq N_\lambda := \cup\{N_\alpha : \alpha < \lambda\}$
- (e) $i \in u_{\mathbf{x}} \wedge |\mathbf{I}_{\mathbf{x},i}| < \lambda \Rightarrow \mathbf{I}_{\mathbf{x},i} \subseteq N_0$
- (f) if $i \in u_{\mathbf{x}} \wedge \kappa_{\mathbf{x},i} = \lambda$ then $\bar{a}_{\mathbf{x},i,\alpha} \subseteq N_{\alpha+1}$ for $\alpha < \kappa_{\mathbf{x},i}$, hence $\mathbf{I}_{\mathbf{x},i} \subseteq N_\lambda$
- (g) $\bar{e}_\alpha \in {}^\theta(N_{\alpha+1})$
- (h) $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{e}_\alpha) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus (N_\alpha + B^+_{\mathbf{x}}))$ according to $\bar{\psi}$
- (i) $\bar{c}_\alpha{}^\frown\bar{d}_\alpha \subseteq {}^{\theta^+>}(N_{\alpha+1})$ realize $\mathrm{tp}(\bar{c}_{\mathbf{x}}{}^\frown\bar{d}_{\mathbf{x}}, N_\alpha + B^+_{\mathbf{x}} + \bar{e}_\alpha)$ where $\ell g(\bar{c}_\alpha) = \ell g(\bar{c}_{\mathbf{x}}), \ell g(\bar{d}_\alpha) = \ell g(\bar{d}_{\mathbf{x}})$
- (j) $N_\alpha \subseteq A_\alpha \in N_{\alpha+1}$
- (k) $A_\alpha \in \mathscr{S}_\ell \Leftrightarrow \alpha \in S_\ell$ and $\bar{e}_\alpha = \bar{e}_{A_\alpha}$
- (l) if $\beta < \alpha$ and $\beta \in S_0, \alpha \in S_1$ then $q_0 = \mathrm{tp}_{\Delta_0}(\bar{e}_\beta{}^\frown\bar{e}_\alpha, B_*)$
- (m) if $\beta < \alpha$ and $\beta \in S_1, \alpha \in S_2$ then $q_1 = \mathrm{tp}_{\Delta_1}(\bar{e}_\beta{}^\frown\bar{e}_\alpha, B_*)$.

How? We shall choose $N_0, \bar{e}_\alpha, \bar{c}_\alpha, \bar{d}_\alpha, q_\alpha$ by induction on α satisfying the relevant conditions.

In the induction step, first N_α exists as it should just be $\prec M_{\mathbf{x}}$ and include $< \lambda$ specific elements and has to be of cardinality $< \lambda$. Second, A_α exists, if $\alpha \in S_\ell$ it can be any member of $\mathscr{S}_\ell$ satisfying $< \lambda$ requirements, each such requirement is satisfied by a set of A's from $\mathscr{D}_I + \mathscr{S}_\ell$ which is a λ-complete filter.

Third, $\bar{e}_\alpha$ exists by the definition of $\lambda = \mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ and choice of $\bar{\psi}$, more exactly by $\otimes_1(e)$.

Fourth, $\bar{c}_\alpha{}^\frown\bar{d}_\alpha$ exists as $M_{\mathbf{x}}$ is κ-saturated and $\kappa > \lambda$; so we are done carrying the induction.

Let $u_1 = \{i \in u_{\mathbf{x}} : \kappa_{\mathbf{x},i} < \lambda\}$ and $u_2 = u_{\mathbf{x}} \backslash u_1$. For each $\alpha < \lambda$ by 2.22(5) we choose a function $h = h_\alpha \in \Pi\{\kappa_{\mathbf{x},i} : i \in u_{\mathbf{x}}\}$ such that $(B_{\mathbf{x}} + N_\alpha + \bar{e}_\alpha + \bar{d}_{\mathbf{x}} + \bar{c}_{\mathbf{x}}, \bar{\mathbf{I}}_{\mathbf{x},u_2,h})$ is a $(\bar{\mu}, \theta)$-set recalling $\bar{\mathbf{I}}_{\mathbf{x},u_2,h} = \langle \mathbf{I}_{\mathbf{x},i,h(\kappa_i)} : i \in u_2\rangle$ and $\bar{\mathbf{I}}_{\mathbf{x},i,h(\kappa_i)} = \langle \bar{a}_{i,\alpha} : \alpha \in [h(\kappa_i), \kappa_{\mathbf{x},i})\rangle$.

So

$\boxplus_2$ $\langle \bar{a}_{\mathbf{x},\partial,\beta} : \beta \in I_{\mathbf{x},\partial,h_\alpha(\partial)}\rangle$ is indiscernible over $B_{\mathbf{x}} + N_\alpha + \bar{c}_{\mathbf{x}} + \bar{d}_{\mathbf{x}} + \bar{e}_\alpha + \cup\{\bar{a}_{\mathbf{x},\sigma,\alpha} : \sigma \in \mathfrak{a}_{\mathbf{x}} \backslash \{\partial\}$ and $\alpha \in [h_\alpha(\sigma), \sigma)\}$ for every $\partial \in \mathfrak{a}_{\mathbf{x}}$.

Note that

$\boxplus_3$ we can replace $\langle (N_\alpha, \bar{c}_\alpha, \bar{d}_\alpha, \bar{e}_\alpha) : \alpha < \lambda\rangle$ by $\langle ((\cup\{N_{f(\beta)+1} : \beta < 1 + \alpha\} \cup N_\alpha), \bar{c}_{f(\alpha)}, \bar{d}_{f(\alpha)}, \bar{e}_{f(\alpha)}) : \alpha < \lambda\rangle$ when $f : \lambda \to \lambda$ is increasing (so $\alpha \le f(\alpha)$) and $\ell \in \{1, 2, 3\} \wedge \alpha \in S_\ell \Rightarrow f(\alpha) \in S_\ell$.

Hence recalling noting $\Pi(a_{\mathbf{x}} \backslash \lambda^+)$ is λ^+-directed and $\mathrm{cf}(\Pi(\mathfrak{a} \cap \lambda))) < \lambda$ by $\otimes_5$, for some $h_* \in \Pi\mathfrak{a}$ we have $\ell \le 2 \Rightarrow \lambda = \sup\{\alpha \in S_\ell : h_\alpha \restriction (\mathfrak{a} \backslash \{\lambda\}) \le h_*\}$ and $h_*(\lambda) = 0$ hence there is $S' \subseteq \lambda$ such that for every $\alpha < \lambda$ there is $\beta \in S'$ such that $\text{otp}(S' \cap \beta) = \alpha \wedge \bigwedge\limits_\ell (\alpha \in S_\ell) \equiv \beta \in S_\ell)$ hene shrinking $\langle a_\alpha : \alpha < \lambda \rangle$ by a subsequence and as we can replace h_α by any bigger function in $\Pi \mathfrak{a}_{\mathbf{x}}$, without loss of generality

$\boxplus_4$ (a) $h_\alpha \restriction \mathfrak{a}_{\mathbf{x}} \backslash \{\lambda\} = h_*$, i.e. is constant

(b) $\langle h_\alpha(\alpha) : \alpha < \lambda \rangle$ is increasing

$\boxplus_5$ let $B_* = B^+_{\mathbf{x},h_*}$ recalling $h_* \in \Pi(\mathfrak{a} \backslash \{\lambda\})$ and[51] $B^*_\alpha = B_{\mathbf{x},h_\alpha}$ for $\alpha < \lambda$ so $\alpha < \beta < \lambda \Rightarrow B_* \subseteq B^*_\beta \subseteq B^*_\alpha$.

Also without loss of generality $\langle e_{\alpha,0} : \alpha < \lambda \rangle$ is with no repetitions.

Stage C: Let $N^*_\lambda \prec M_{\mathbf{x}}$ be of cardinality $< \kappa$ such that $B^+_{\mathbf{x}} \cup N_\lambda \subseteq N^*_\lambda$. We choose N^+_λ expanding N^*_λ such that $P_0^{N_\lambda} = |N_\lambda|, P_1^{N^+_\lambda} = \{e_{\alpha,0} : \alpha < \lambda\}, P_2^{N^+_\lambda} = \{(a, e_{\alpha,0}) : a \in N_\alpha, \alpha < \lambda\}$ and $F_i^{N^+_\lambda}(e_{\alpha,0}) = e_{\alpha,i}$ for $i < \theta, P^{N^+_\lambda}_{3+\ell} = \{e_{\alpha,0} : \alpha \in S_\ell\}$ for $\ell = 0,1,2$ so N^+_λ and the vocabulary $\tau(N^+_\lambda)$ are well defined.

We shall choose an increasing sequence $\langle \alpha_\varepsilon = \alpha(\varepsilon) : \varepsilon < \lambda \rangle$ enumerating in increasing order a thin enough club of λ.

We shall prove in this stage that there are $N^\oplus_\lambda$ and $\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle$ such that:

$\boxplus_6$ (a) $N^\oplus_\lambda$ is an elementary extension of N^+_λ

(b) $\bar{e}^*_\varepsilon \in {}^\theta\{a : N^\oplus_\lambda \models P_2(a, e_{\alpha(\varepsilon+1),0})\}$

(c) $e^*_{\varepsilon,i} = F_i^{N^\oplus_\lambda}(e^*_{\varepsilon,0})$ for $i < \theta, \varepsilon < \lambda$

(d) $e^*_{\varepsilon,0} \in P_4^{N^\oplus_\lambda}$ and $\neg P_2^{N^\oplus_\lambda}(e^*_{\varepsilon,0}, e_{\alpha(\varepsilon),0})$

(e) $\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle$ is an indiscernible sequence in $N^\oplus_\lambda \restriction \tau_T$

$(f)(\alpha)$ $\bar{e}^*_\varepsilon(\varepsilon < \lambda), \bar{e}_\alpha(\alpha \in S_1)$ realize the same $\mathbb{L}(\tau_T)$-type over B_*

(β) if $\beta < \lambda$ then all $\bar{e}^*_\varepsilon$ such that $\alpha(\varepsilon) < \beta$ and $\bar{e}_\alpha$ such that $\alpha \in S_1 \cap \beta$ realize the same $\mathbb{L}(\tau_T)$-type over B^*_β

(γ) if $\alpha \leq \alpha_\varepsilon, \varepsilon < \lambda$ and $\alpha \in S_0$ then $\bar{e}_\alpha \char`^ \bar{e}^*_\varepsilon$ realizes q_0

(δ) if $\varepsilon < \lambda, \alpha_{\varepsilon+1} \leq \alpha$ and $\alpha \in S_2$ then $\bar{e}^*_\varepsilon \char`^ \bar{e}_\alpha$ realizes q_1.

First note:

$\boxplus_{6.1}$ there is an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ of limit ordinals $< \lambda$ such that: for every $n < \omega$, finite[52] $\Delta \subseteq \cup\{\Gamma_{(\theta)_n} : m < \omega\} = \{\varphi : \varphi = \varphi(\bar{x}_{\bar{e}_0}, \bar{x}_{\bar{e}_1}, \dots, \bar{x}_{\bar{e}_{m-1}})$ and $m < \omega, \varphi \in \mathbb{L}(\tau_T)\}$ and for every $0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_n$ we can find $\beta_\ell \in [\alpha_{\varepsilon_\ell}, \alpha_{\varepsilon_{\ell+1}}) \cap S_1$ for $\ell < n$ such that $\langle \bar{e}_{\beta_0}, \bar{e}_{\beta_1}, \dots, \bar{e}_{\beta_{n-1}} \rangle$ is a Δ-indiscernible sequence (in $M_{\mathbf{x}}$).

[Why? For each such pair (Δ, n) define a game $\Game_{\Delta,n}$ with n moves, in the m-th move the antagonist chooses an ordinal $\beta_m < \lambda$ which is $> \sup\{\gamma_k : k < m\}$ and the protagonist chooses $\gamma_m \in [\beta_m, \lambda) \cap S_1$. In the end of a play the protagonist wins the play when $\langle \bar{e}_{\gamma_0}, \dots, \bar{e}_{\gamma_{n-1}} \rangle$ is a Δ-indiscernible sequence. This game is determined so we choose a winning strategy $\mathbf{st}_{\Delta,n}$ for the winner. Let $E = \{\delta < \kappa : \alpha < 1 + \delta \Rightarrow h_\alpha(\alpha) + 1 < \delta$ and δ is closed under $\mathbf{st}_{\Delta,n}$ for every pair (Δ, n) as

[51]the difference between B_* and B^*_α is concerning $\mathbf{I}_{\mathbf{x},\lambda}$

[52]Alternatively use finite $u \subseteq \ell g(\bar{e}_0), \Delta$ finite $\subseteq \mathbb{L}(\tau_T)$ and get $\langle \bar{e}_{\beta_0} \restriction u, \dots, \bar{e}_{\beta_1} \restriction u)$ is Δ-indiscernible.

above}. As the number of pairs (Δ, n) as above is $\leq \theta < \lambda = \mathrm{cf}(\lambda)$, clearly E is a club of κ and let $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ list E in increasing order.

It is enough to prove that the protagonist wins every such game $\Game_{\Delta,n}$. Now for each pair (Δ, n) if this fails the sequence $\langle \bar{e}_{\alpha(k)+1} : k < \omega \rangle$ has an infinite subsequence $\langle \bar{e}_{k(i)} : i < n \rangle$ of length n which is Δ-indiscernible, hence the protagonist wins in least in one play of $\Game_{\Delta,n}$ in which the antagonist uses the strategy $\mathbf{st}_{\Delta,n}$, i.e. when he chooses $\gamma_m = \alpha_{k(m)}$, it is legal by the choice of E, so $\mathbf{st}_{\Delta,n}$ is not a winning strategy for the antagonist hence it is for the protagonist $\Game_{n,\Delta}$. So easily $\bar{\alpha}$ is as required in $\boxplus_{5.1}$.].

Now let N^+_λ be a $\|N_\lambda\|^+$-saturated elementary extension of N^+_λ, by $\boxplus_{5.1}$ we can find in it a sequence $\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle$ as promised in $\boxplus_6$. Note that clause (f) of $\boxplus_4$ can be gotten by thinning the sequence $\langle e^*_\varepsilon : \varepsilon < \lambda \rangle$.

<u>Stage D</u>: There is $N^\oplus_\lambda$ such that

$\boxplus_7$ (a) $N^\oplus_\lambda$ has cardinality λ (by the LST theorem)

(b) $N^\oplus_\lambda {\restriction} \tau_T \prec M_{\mathbf{x}}$ (by renaming, possible as $M_{\mathbf{x}}$ is κ-saturated while $\kappa > \lambda = \|N^\oplus_\lambda\|$)

(c) if $\bar{b} \in {}^{\omega >}(N_\lambda), \varepsilon < \lambda, \vartheta \in \mathbb{L}(\tau_T)$ and $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap S_1 \Rightarrow$ $\mathfrak{C} \models \vartheta[\bar{e}_\alpha, \bar{b}, \bar{c}_{\mathbf{x}}]$ then $\models \vartheta[\bar{e}^*_\varepsilon, \bar{b}, \bar{c}_{\mathbf{x}}]$.

[Why? As $\mathrm{tp}(\bar{c}_{\mathbf{x}}, M_{\mathbf{x}})$ is finitely satisfiable in B^*_0 and $B^*_0 \subseteq M_{\mathbf{x}}, |B^*_0| < \kappa, M_{\mathbf{x}}$ is κ-saturated.]

(d) let $N'_\varepsilon = M_{\mathbf{x}} {\restriction} \{a \in N^\oplus_\lambda : N^\oplus_\lambda \models P_2(a, e^*_{\varepsilon,0})\}$ for any $\varepsilon < \lambda$.

Now recall

$\odot_1$ $\mathfrak{C} \models$ "$\vartheta_{\varphi^{\mathbf{t}}_0}[\bar{c}_{\mathbf{x}}, \bar{e}_{\alpha(\varepsilon)}, \bar{b}]$" when $\mathbf{t} \in \{0,1\}, \bar{b} \in {}^{\ell g(\bar{y}_0)}(N_{\alpha(\varepsilon)})$ and $\mathfrak{C} \models \varphi^{\mathbf{t}}_0[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{b}]$

[Why? By the choice of $\bar{e}_{\alpha(\varepsilon)}$ and of $\vartheta_{\varphi^{\mathbf{t}}_0}$ recalling $\psi_{\varphi^{\mathbf{t}}_0} = \psi_{\varphi_*}$ by $\otimes_3$.]

$\odot_2$ $\mathfrak{C} \models \varphi_1[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}_{\alpha(\varepsilon)}.]$

[Why? As $\varphi_1 = \psi_{\varphi_0}$, see $\otimes_3(c)$ and the choice of the $\bar{e}_{\alpha(\varepsilon)}$.]

$\odot_3$ $\mathfrak{C} \models$ "$\varphi_{\ell+1}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}_\alpha] \wedge \vartheta_{\varphi_\ell}[\bar{c}_{\mathbf{x}}, \bar{e}_\alpha, \bar{e}_{\alpha(\varepsilon)}]$" when $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$ and $\ell \in \{1,2\}$.

[Why? As $\varphi_{\ell+1} = \psi_{\varphi_\ell}$, see $\otimes_3(d)$ and $\odot_2$.]

$\odot_4$ $q_0 = \mathrm{tp}_{\Delta_0}(\bar{e}_{\alpha(\varepsilon)} {}^\frown \bar{e}_\alpha, B^*_{\alpha_{\varepsilon+1}})$ when $\alpha_\varepsilon \in S_0$ and $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap S_1$.

[Why? See clause (ℓ) of $\boxplus_1$ above, recall that q_0 is from the application of Theorem 4.1 in the beginning of Stage B.]

$\odot_5$ $q_0 {\restriction} B^*_\beta = \mathrm{tp}_{\Delta_0}(\bar{e}_{\alpha(\varepsilon)} {}^\frown \bar{e}^*_\varepsilon, B^*_\beta)$ when $\alpha_{\varepsilon+1} \leq \beta < \lambda$.

[Why? By $\odot_4$ and the choice of $\bar{e}^*_\varepsilon$, i.e. $\boxplus_6(f)(\gamma)$.]

$\odot_6$ $\mathfrak{C} \models \vartheta_{\varphi_1}[\bar{c}_{\mathbf{x}}, \bar{e}^*_\varepsilon, \bar{e}_{\alpha(\varepsilon)}]$ iff $\vartheta_{\varphi_1}[\bar{c}_{\mathbf{x}}, \bar{e}_\alpha, \bar{e}_{\alpha(\varepsilon)}]$ when $\varepsilon < \lambda \wedge \alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap S_1$; recalling $\alpha_\varepsilon \in S_0$ being a limit ordinal.

[Why? By $\boxplus_6(f)(\beta)$ and as $\vartheta_{\varphi_1}(\bar c_{\mathbf x}, \bar x_{[\theta]}, \bar e_{\alpha(\varepsilon)})$ is a Δ_0-formula, see by $\otimes_4(a)$ recalling $\mathrm{tp}(\bar c_{\mathbf x}, M_{\mathbf x})$ is finitely satisfiable in B^*_β as $B^*_\beta = B^+_{\mathbf x, h_\beta}$.]

$\odot_7$ $\mathfrak C \models \varphi_2[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e^*_\varepsilon]$.

[Why? Clearly $\varphi_3 = \psi_{\varphi_2}$, see $\otimes_3(e)$ and ϑ_{φ_2} are well defined and by $\odot_3$ we know that $\mathfrak C \models \varphi_2[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e_\alpha]$ for every $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$. Let $\beta \in S_2 \backslash \alpha(\varepsilon+1)$. Again by $\odot_3$ we have $\mathfrak C \models$ "$\psi_{\varphi_2}[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e_\beta] \wedge \vartheta_{\varphi_2}[\bar c_{\mathbf x}, \bar e_\beta, \bar e_\alpha]$" for $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$.

But by $\boxplus_1(m)$ for every $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap S_1$ we have $q_1 = \mathrm{tp}_{\Delta_1}(\bar e_\alpha{}^\frown \bar e_\beta, B_*)$. So for every $\bar c \in {}^{\ell g(\bar c_{\mathbf x})}(B^*_\beta)$ we have $\mathfrak C \models$ "$\vartheta_{\varphi_2}[\bar c, \bar e_\beta, \bar e_\alpha]$" iff $\vartheta_{\varphi_2}(\bar c, \bar x''_{[\theta]}, \bar x'_{[\theta]}) \in q_1$. Hence for every $\bar c \in {}^{\ell g(\bar c_{\mathbf x})}(B^*_\beta)$ we have $\mathfrak C \models$ "$\vartheta_{\varphi_2}[\bar c, \bar e_\beta, \bar e^*_\varepsilon]$" iff $\vartheta_{\varphi_2}(\bar c, \bar x'_{[\theta]}, \bar x''_{[\theta]}) \in q_1$. As this holds for every such $\bar c$ and $\mathrm{tp}(\bar c_{\mathbf x}, M_{\mathbf x})$ is finitely satisfiable in B_*, clearly $\mathfrak C \models$ "$\vartheta_{\varphi_2}[\bar c_{\mathbf x}, \bar e_\beta, \bar e_\alpha] \equiv \vartheta_{\varphi_2}[\bar c_{\mathbf x}, \bar e_\beta, \bar e^*_\varepsilon]$" for every $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap S_1$.

By the conclusions of the last two paragraphs $\mathfrak C \models \vartheta_{\varphi_2}[\bar c_{\mathbf x}, \bar e_\beta, \bar e^*_\varepsilon]$ and by the conclusion of the first of them $\mathfrak C \models \psi_{\varphi_2}[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e_\beta]$. Together recalling the definition of ϑ_{φ_2} we get $\mathfrak C \models \varphi_2[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e^*_\varepsilon]$, i.e. we are done proving $\odot_7$.]

$\odot_8$ $\mathfrak C \models$ "$\vartheta_{\varphi_1}[\bar c_{\mathbf x}, \bar e^*_\varepsilon, \bar e_{\alpha(\varepsilon)}]$".

[Why? Note $\mathfrak C \models$ "$\vartheta_{\varphi_1}[\bar c_{\mathbf x}, \bar e_\alpha, \bar e_{\alpha(\varepsilon)}]$" for $\alpha \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap S_1$. Let $\mathbf I = \{\bar c' \in {}^{\ell g(\bar c[\mathbf x])}(B_*) : \mathfrak C \models$ "$\vartheta_{\varphi_1}[\bar c', \bar e_\alpha, \bar e_{\alpha(\varepsilon)}]$" but clearly $\mathbf I = \{\bar c' \in {}^{\ell g(\bar c[\mathbf x])}(B_*) : \vartheta_{\varphi_2}[\bar c_{\mathbf x}, \bar x'_{[\theta]}, \bar x''_{[\theta]}) \in q_0\}$ so does not depend on α hence, second, $\mathbf I = \{\bar c' \in {}^{\ell g(\bar c[\mathbf x])}(B_{\mathbf x}) : \mathfrak C \models \vartheta_{\varphi_1}[\bar c'], \bar e^*_\varepsilon, \bar e_{\alpha(\varepsilon)}]\}$. As $\mathrm{tp}(\bar c_{\mathbf x}, M_{\mathbf x})$ is finitely satisfiable in B_*, we get $\mathfrak C \models \vartheta_{\varphi_1}[\bar c_{\mathbf x}, \bar e^*_\varepsilon, \bar e_{\alpha(\varepsilon)}]$.]

$\odot_9$ $\mathfrak C \models$ "$\psi_{\varphi_1}[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e^*_\varepsilon]$", $\mathfrak C \models$ "$\varphi_0^{\mathbf t[\bar b]}[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar b]$" and when $\bar b \in {}^{\ell g(\bar y)}(N_{\alpha(\varepsilon)})$ and $\mathbf t[\bar b]$ is chosen such that $\mathfrak C \models \vartheta_{\varphi_0^{\mathbf t[\bar b]}}[\bar c_{\mathbf x}, \bar e^*_\varepsilon, \bar b]$.

[Why? Recalling $\psi_{\varphi_1} = \varphi_2$, note that $\mathfrak C \models \psi_{\varphi_1}[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar e^*_\varepsilon]$ by $\odot_7$, i.e. the first conclusion of $\odot_9$ holds. By $\odot_7$ we have $\mathfrak C \models \vartheta_{\varphi_1}[\bar c_{\mathbf x}, \bar e^*_\varepsilon, \bar e_{\alpha(\varepsilon)}]$ which means that $\psi_{\varphi_1}(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e^*_\varepsilon) \vdash \varphi_1(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e_{\alpha(\varepsilon)})$. But $\varphi_1 = \psi_{\varphi_0} = \psi_{\neg\varphi_0}$ so by $\otimes_1(d)+\otimes_3(a)+\otimes_3(b)$ we have $\varphi_1(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e_{\alpha(\varepsilon)}) \vdash \varphi_0^{\mathbf t}(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar b)$.

As $\vdash$ is transitive we have $\psi_{\varphi_1}(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e^*_\varepsilon) \vdash \varphi_0^{\mathbf t}(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar b)^{\mathbf t}$ which by $\odot_7$ means $\mathfrak C \models \vartheta_{\varphi_0}[\bar c_{\mathbf x}, \bar e^*_\varepsilon, \bar b]$, i.e. the second conclusion of $\odot_9$ so we are done.]

Stage E: By the choice of $\varphi_* = \varphi_0$ and letting $\psi = \psi_{\varphi_*}$ and of the set A see $\otimes_2$ we can find

$(*)_1$ an ultrafilter D on ${}^{\ell g(\bar z)}(N_\lambda)$ such that for every $\bar e' \in {}^\theta(M_{\mathbf x})$ and ψ' satisfying $\psi'(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e') \in \mathrm{tp}(\bar d_{\mathbf x}, \bar c_{\mathbf x} + M_{\mathbf x})$ the[53] set $\{\bar b \in {}^{\ell g(\bar z)}(N_\lambda) : \mathfrak C \models (\exists \bar x_{\bar d})(\psi'(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e') \wedge \varphi_*(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar b)) \wedge (\exists \bar x_{\bar d})(\psi'(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar e') \wedge \neg\varphi_*(\bar x_{\bar d}, \bar c_{\mathbf x}, \bar b))\}$ belongs to D;

this is as in the proof of **??** or 2.15(2), i.e. [She15, 2.10=tp25.36]

$(*)_2$ For $\bar b \in {}^{\ell g(\bar y)}(N_\lambda)$ let $\varepsilon(\bar b) = \min\{\varepsilon < \lambda : \bar b \subseteq N_{\alpha(\varepsilon)}\}$ and

$(*)_3$ let $\mathbf t(*)$ be such that $\{\bar b \in {}^{\ell g(\bar y)}(N_\lambda) : \mathbf t[\bar b] = \mathbf t(*)\} \in D$ recalling $\mathbf t[\bar b] \in \{0, 1\}$ is such that $\mathfrak C \models \varphi_0^{\mathbf t[\bar b]}[\bar d_{\mathbf x}, \bar c_{\mathbf x}, \bar b]$.

Note that

[53] can use "$\{b \in {}^{\ell g(\bar z)}(A) : \ldots\}$"

$(*)_4$ for every $\varepsilon < \lambda$ the set $\{\bar{b} \in {}^{\ell g(\bar{y})}(N_\lambda) : \varepsilon(\bar{b}) \geq \varepsilon\}$ belongs to D.

[Why? Otherwise $\psi(\bar{x}_{\bar{d}}, \bar{c}, \bar{e}_\varepsilon)$ contradicts the choice of D.]

We use ultrapower to get $(\bar{b}', \bar{e}')$ in $\mathfrak{C}$ realizing $p(\bar{y}, \bar{x}_{[\theta]}) = \{\vartheta(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{y}, \bar{x}_{[\theta]}, \bar{a}) : \vartheta(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}, \bar{x}_{[\theta]}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{a} \in {}^{\ell g(\bar{z})}(M_{\mathbf{x}})$ and $\{\bar{b} \in {}^{\ell g(\bar{z})}(N_\lambda) : \mathfrak{C} \models \vartheta[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{b}, \bar{e}^*_{\varepsilon(\bar{b})}, \bar{a}]\} \in D\}\}$.

Now

$(*)_5$ $\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle\hat{\ }\langle \bar{e}' \rangle$ is an indiscernible sequence.

[Why? By $(*)_6$ below.]

$(*)_6$ $\bar{e}'$ realizes $\mathrm{Av}(\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle, M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$.

[Why? As $\langle \bar{e}^*_\varepsilon : \varepsilon < \kappa \rangle$ is an indiscernible sequence (and T is dependent), the average is well defined. Now recall $(*)_4$ and the choice of $\bar{e}'$.]

$(*)_7$ $\mathrm{tp}(\bar{d}_{\mathbf{x}}, M_{\mathbf{x}} + \bar{c}_{\mathbf{x}}), \mathrm{tp}(\bar{b}', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$ are not weakly orthogonal.

[Why? By the choice of $\bar{b}'$ and of $\mathscr{D}$, i.e. as witnessed by φ_*.]

But (an important point for Claim 5.3) we need a more effective version of $(*)_7$.

Let $p_1(\bar{x}_{\bar{d}}) = \mathrm{tp}(\bar{d}_{\mathbf{x}}, M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$ and $p_2(\bar{y}) = \mathrm{tp}(\bar{b}', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}}), p_3(\bar{x}_{[\theta]}) = \mathrm{tp}(\bar{e}', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$

$(*)_7^+$ $p_1(\bar{x}_{\bar{d}}) \cup p_2(\bar{y}) \cup \{\varphi_*^{\mathbf{t}}(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{y})\}$ is consistent for $\mathbf{t} = 0, 1$.

[Why? By the choice of the ultrafilter D and of the sequence $\bar{b}'$.]

$(*)_8$ $\mathfrak{C} \models \psi_{\varphi_*}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}']$ and $\mathfrak{C} \models \vartheta_{\varphi_*^{\mathbf{t}(*)}}[\bar{c}_{\mathbf{x}}, \bar{e}', \bar{b}']$.

[Why? Because by $(*)_4$ and $\odot_8$ for every $\bar{b} \in {}^{\ell g(\bar{y})}(N_\lambda)$ we have $\mathfrak{C} \models \psi_{\varphi_1}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}_{\varepsilon(\bar{b})}]$ and $\mathfrak{C} \models \vartheta_{\varphi_*^{\mathbf{t}[\bar{b}]}}[\bar{c}_{\mathbf{x}}, \bar{e}_{\varepsilon(\bar{b})}, \bar{b}]$.]

$(*)_9$ $p_1(\bar{x}_{\bar{d}}) \cup p_3(\bar{x}_{[\theta]}) \cup \{\pm\psi_{\varphi_*}(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{x}_{[\theta]})\}$ are consistent.

[Why? First, clearly $\mathfrak{C} \models \psi_{\varphi_*}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}']$ by $(*)_8$ hence $p_1(\bar{x}_{\bar{d}}) \cup p_3(\bar{x}_{[\theta]}) \cup \{\psi_{\varphi_*}(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{x}_{[\theta]})\}$ being realized by $\bar{d}_{\mathbf{x}}\hat{\ }\bar{e}'$ is consistent.

By $(*)_7^+$ for some $\bar{d}'$ realizing $p_1(\bar{x}_{\bar{d}})$ and $\bar{b}''$ realizing $p_2(\bar{y})$ recalling $\mathbf{t}(*)$ is from $(*)_3$ we have $\mathfrak{C} \models \neg\varphi_*^{\mathbf{t}(*)}[\bar{d}', \bar{c}_{\mathbf{x}}, \bar{b}'']$; as $p_1(\bar{x}_{\bar{d}}) = \mathrm{tp}(\bar{d}_{\mathbf{x}}, M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$ without loss of generality $\bar{d}' = \bar{d}_{\mathbf{x}}$. As $\mathrm{tp}(\bar{b}'', \bar{c}_{\mathbf{x}} + M_{\mathbf{x}}) = p_2(\bar{y}) = \mathrm{tp}(\bar{b}', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$ for some $\bar{e}''$ we have $\mathrm{tp}(\bar{b}''\hat{\ }\bar{e}'', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}}) = \mathrm{tp}(\bar{b}'\hat{\ }\bar{e}', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$.

Note that $\mathfrak{C} \models \vartheta_{\varphi_*^{\mathbf{t}(*)}}[\bar{c}_{\mathbf{x}}, \bar{e}', \bar{b}']$ hence by $(*)_8$ we have $\mathfrak{C} \models \vartheta_{\varphi_*^{\mathbf{t}(*)}}[\bar{c}_{\mathbf{x}}, \bar{e}'', \bar{b}'']$.

Now if $\mathfrak{C} \models \psi_{\varphi_*}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}'']$ then by the definition of $\vartheta_{\varphi_*, \mathbf{t}}$, see $\odot_9$ and the last sentence, $\mathfrak{C} \models \varphi_*^{\mathbf{t}(*)}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{b}'']$ but $\bar{d}_{\mathbf{x}}\hat{\ }\bar{e}''$ realizes $p_1(\bar{x}_{\bar{d}}) \cup p_3(\bar{x}_{[\theta]})$ we have a contradiction to the choice of $\bar{b}''$ hence $\mathfrak{C} \models \neg\psi_{\varphi_*}[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}'']$ thus finishing the proof of $(*)_9$.]

As $p_3(\bar{x}_{[\theta]})$ was defined as $\mathrm{tp}(\bar{e}', M_{\mathbf{x}} + \bar{c}_{\mathbf{x}})$ by $(*)_6 + (*)_9$ we are done. $\square_{5.1}$

We shall not use 5.1 as stated but a variant which the proof gives (as mentioned in the proof).

Claim 5.3. *Assume* $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}$, $\mathscr{P} \subseteq \{u \subseteq v_{\mathbf{x}} : u \cap u_{\mathbf{x}}$ *is finite*$\}$ *is* $\subseteq$*-directed with union* $v_{\mathbf{x}}$ *and* $\kappa > \lambda = \mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ *and* $i \in v_{\mathbf{x}} \backslash u_{\mathbf{x}} \Rightarrow \lambda \geq \beth_\omega(|B_i| + \theta)$ *and* λ *is regular, this is similar to 5.1 but omitting the assumption "*$u_{\mathbf{x}}$ *is finite". We still can find* $(\bar{\psi}, A, u, \varphi_*)$ *such that*

(a) $\bar{\psi}$ *illuminates* $(\mathbf{x}, \lambda, \Gamma^1_{\mathbf{x}})$

(b) $A \subseteq M_{\mathbf{x}}$ *is of cardinality* $\lambda, u \in \mathscr{P}$ *and* $\varphi_* = \varphi_*(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c},u}, \bar{y})$ *such that no* $\psi'(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x},u}, \bar{e}') \in \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ *solve* $(\mathbf{x}, \varphi, A)$ *[i.e. for no finite* $u \subseteq \ell g(\bar{c}_{\mathbf{x}})$ *and* $\psi' = \psi'(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c},u}, \bar{z})$ *and* $\bar{e}' \in {}^{\ell g(\bar{z})}(M_{\mathbf{x}})$ *do we have* $\mathfrak{C} \models$ *"*$\psi'[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x},u}, \bar{e}']$*" and* $\psi'(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x},u}, \bar{e}') \vdash \{\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}) : \bar{b} \in {}^{\ell g(\bar{y})}A$ *and* $\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}) \in \mathrm{tp}(\bar{d}_{\mathbf{x}}, A + \bar{c}_{\mathbf{x}})\}$*]; let* $\psi_{\varphi_*} = \psi_{\varphi_*}(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}_{[\theta]}) \equiv \psi_{\varphi_*}(\bar{x}_{\bar{d}}, \bar{x}_c, \bar{x}_{[v]}), v \subseteq \theta$ *finite*

(c) *we get the result of 5.1 with* $\bar{c}_{\mathbf{x}}$ *replaced by* $\bar{c}_{\mathbf{x},u} = \bar{c}_{\mathbf{x},u} {\restriction} u = \langle \bar{c}_{\mathbf{x},i} : i \in u \rangle$*, i.e.:*
   - $(*)$ *there is an indiscernible sequence* $\mathbf{I} = \langle \bar{b}_\alpha : \alpha < \lambda \rangle$ *in* $M_{\mathbf{x}}, \ell g(\bar{b}_\alpha) = \ell g(\bar{y})$ *and* $\bar{b}^0, \bar{b}^1 \in {}^v\mathfrak{C}$ *realizing* $\mathrm{Av}(\mathbf{I}, M_{\mathbf{x}} + \bar{c}_{\mathbf{x},u})$ *and* $\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x},u}, \bar{b}^{\mathbf{t}}]^{\mathbf{t}}$ *for* $\mathbf{t} = 0, 1$.

Proof. We can find $\bar{\psi}, \varphi_*, A$ as in $\otimes_1$ in the proof of 5.1, so $\bar{\psi}$ is as in clause (a) there. Then we find $\varphi_\ell, \vartheta'_\ell$ as in $\oplus_3 + \oplus_4$ in the proof of 5.1, Stage A. Next let $u_* \in \mathscr{P}$ be such that $i \in u_*$ <u>iff</u> $x_{\bar{c}_{\mathbf{x},i}}$ is not dummy in φ_0 or in φ_1 or in φ_2 or in φ_3. Now use the proof of 5.1 from Stage B on, however not on $\mathbf{x}$ but $\mathbf{x}_{[u_*]}$, see Definition 2.6(10). $\square_{5.3}$

Conclusion 5.4. *Assume* T *is countable,* $\theta = \aleph_0, \mu$ *strong limit of uncountable cofinality and* $\mu \leq \kappa = \mathrm{cf}(\kappa) < \mu^{+\omega}$. <u>*Then*</u> *for every* $\mathbf{m} \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ *with* $u_{\mathbf{m}}$ *finite there is* $\mathbf{n} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ *such that* $\mathbf{m} \leq_1 \mathbf{n}$.

Remark 5.5. If we assume $\mathrm{cf}(\mu) = \aleph_0$ and $\kappa = \mu^+$, then we can get a weaker version of density of $\mathrm{tK}_{\kappa,\mu,\theta}$.

Proof. Without loss of generality $\ell g(\bar{d}) = \omega$.

Let $\langle \varphi_n(\bar{x}_{[\omega+\omega]}, \bar{y}_{[\omega]}, \bar{z}_n) : n < \omega \rangle$ list all formulas of such form, each appearing infinitely many times. Without loss of generality $\varphi_n = \varphi_n(\bar{x}_{[w_n]}, \bar{y}_{[n]}, \bar{z}_n), w_n := [0, n) \cup [\omega, \omega + n)$. We choose $\mathbf{m}_n$ by induction on $n < \omega$ such that:

$\boxplus$ (a) $\mathbf{m}_n \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ and $u(\mathbf{m}_n)$ is finite, moreover $\in [n, \omega)$; we may add $v(\mathbf{m}_n)$ finite (as we can assume $v_{\mathbf{m}}$ finite)

(b) $\mathbf{m}_n = \mathbf{m}$

(c) if $n = m + 1$ then $\mathbf{m}_m \leq_1 \mathbf{m}_{n+1}$ and $r_{\mathbf{m}_n}$ is complete

(d) if $n = m + 1$ and there is $\mathbf{m}' \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ satisfying $(\mathbf{m}_m \leq_1 \mathbf{n} \wedge (\mathbf{m}'$ is $(\varphi_m(\bar{x}_{[w_m]}, \bar{y}_{[m]}, \bar{z}_m), i)$ active for some $i \in v(\mathbf{m}') \backslash v(\mathbf{m}_m))$ <u>then</u> $\mathbf{m}_n$ satisfies this

(e) if $n = m + 1$ and the assumption in clause (d) fails, but there is $\mathbf{m}' \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ satisfying $\mathbf{m}_n \leq_1 \mathbf{m}'$ and $\varphi_m \in \Gamma^2_{\bar{\psi}[\mathbf{m}']}$ <u>then</u> $\mathbf{m}_{n+1}$ satisfies this.

We can carry the induction for clauses (d) + (e) because if there is such $\mathbf{m}'$ we can find $\mathbf{m}''$ such that $\mathbf{m}_m \leq_1 \mathbf{m}'' \leq_1 \mathbf{m}'$ such that $u[\mathbf{m}'']$ is finite, and the demand "$r_{\mathbf{m}_{m+1}}$ is complete" is not a problem by 4.10(1A).

Having carried the induction let $\mathbf{n} = \lim\langle \mathbf{m}_n : n < \omega\rangle$, we have to show that $\mathbf{n} \in \mathrm{rK}^{\oplus}_{\kappa,\mu,\theta}$ and more. If $\Gamma_{\bar\psi[\mathbf{n}]} = \Gamma^2_{\mathbf{x}[\mathbf{n}]}$ we shall be done by 3.23, so toward contradiction assume $\varphi = \varphi(\bar x_{\bar d[\mathbf{n}]}, \bar x_{\bar c[\mathbf{n}]}, \bar y) \in \Gamma^2_{\mathbf{x}[\mathbf{n}]} \backslash \Gamma_{\bar\psi[\mathbf{n}]}$, let k be such that $\varphi = \varphi_k$ hence $u = \{n : \varphi_n = \varphi\}$ is infinite. By clause (d) and 2.14(2), i.e. [She15] the set $\{m : \varphi_m = \varphi_k$ and the assumption in $\boxplus(d)$ holds$\}$ is finite. So choose m such that $\varphi_m = \varphi_k$ but the assumption of $\boxplus(d)$ fails. By 5.1 more exactly 5.3, the assumption of $\boxplus(e)$ holds; why? the point is that $\mathrm{ntr}_{\mathrm{lc}}(\mathbf{x}_{\mathbf{n}}) = \{\mu^+, \mu^{+2}, \dots, \mu^{+n}\}$. So the conclusion of $\boxplus(e)$ holds, contradiction.

Lastly, $\mathbf{n} = \cup\{\mathbf{m}_n : n < \omega\}$ is well defined and by Claim 3.23, using (c)$'$ there, $\mathbf{m} = \mathbf{m}_0 \le_1 \mathbf{n} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$. $\square_{5.4}$

* * *

Discussion 5.6. We may like to cover every $\kappa = \kappa^{<\kappa} \ge \beth_\omega$; at least and/or when for countable T as in 4.12, G.C.H. holds). For this, we are still left with the case $\mathrm{cf}(\mu) = \aleph_0$, for this we have to redo some previous definitions and claims, so this is presently delayed.

Conclusion 5.7. *Assume G.C.H. and T is countable and $\theta = \aleph_0$ and μ is strong limit of cofinality $> \aleph_0$ and $\kappa = \mathrm{cf}(\kappa) \in (\mu, \mu^{+\omega})$.*
1) For every κ-saturated M of cardinality κ and $\bar d \in {}^{\theta^+>}\mathfrak{C}$ there is $\mathbf{x} \in \mathrm{tK}_{\kappa,\kappa,\theta}$ with $\bar d \trianglelefteq \bar d_{\mathbf{x}}$.
2) Hence $M \in \mathrm{EC}_{\kappa,\kappa}(T) \Rightarrow |\mathbf{S}^\theta(M)/\equiv_{\mathrm{aut}}| \le \kappa$.
3) If M is a saturated model of T of cardinality κ <u>then</u> $\mathfrak{S}^\theta_{\mathrm{aut}}(M)$ has cardiality $\le \mu$.

Remark 5.8. 1) For vK this is easier.
2) When $\mathrm{cf}(\mu) = \aleph_0$, maybe see more in [S$^+$a].

§ 5(B). **Density of** vK**; Exact recounting of types and** vK**.**

We prove the density of $\mathrm{vK}_{\kappa,\bar\mu,\theta}$. We use 5.22(1) but not 5.22(2)-(6).

Recall that we have difficulties when $\mathrm{ntr}_{\mathrm{lc}}(\mathbf{x})$ was singular. This motive defining relatives in 5.10,5.13 and investigating them. This succeeds but not applicable to rK only to vK.

Convention 5.9. We here tend to use $\varphi \in \Gamma^1_{\mathbf{x}}$ as $\varphi(\bar x_{d,\rho}, \bar x_{\bar c,\varrho}, \bar y)$ where $\rho \in {}^n(\ell g(\bar d_{\mathbf{x}})))$, $\varrho \in {}^m(\ell g(\bar c_{\mathbf{x}}))$ for some n, m.

Definition 5.10. Assume for $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar\mu,\theta}$ and $\varphi = \varphi(\bar x_{\bar d,\rho}, \bar x_{\bar c,\varrho}, \bar y)$, so φ determines ρ, ϱ and $\psi = \psi(\bar x_{d,\rho}, \bar c_{\mathbf{x},\varrho_0}) \in \mathrm{tp}(\bar d_{\mathbf{x},\rho}, \bar c_{\mathbf{x},\varrho_0})$. Below we may omit ψ when $\rho = \langle\rangle = \varrho_0$, the role of ψ in (1) is minor.
1) Let $\mathbf{k}(\varphi, \psi, \mathbf{x})$ be the maximal n such that there is an increasing sequence $\varrho_1 \in {}^n(v_{\mathbf{x}})$ which witness it, which means (note that ψ has a role only via ϱ_0):

- $\ell < \ell g(\varrho) \wedge k < \ell g(\varrho_1) \Rightarrow \varrho(\ell) <_{v_{\mathbf{x}}} \varrho_1(k)$
- $\ell < \ell g(\varrho_0) \wedge k < \ell g(\varrho_1) \Rightarrow \varrho_0(\ell) <_{v_{\mathbf{x}}} \varrho_1(\ell)$
- $\bar c^*_{\ell,0}, \bar c^*_{\ell,1}$ are subsequences of $\bar c_{\mathbf{x},\varrho_1(\ell)}$ realizing the same type over $\bar c_{\mathbf{x},<\varrho_1(\ell)} + M_{\mathbf{x}}$

- $\mathfrak{C} \models$ "$\varphi[\bar{d}_{\mathbf{x},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{c}^*_{\ell,1}] \wedge \neg\varphi[\bar{d}_{\mathbf{x},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{c}^*_{\ell,0}]$.

2) We define $\mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{x})$ as the maximal n such that some η witness it which means; du stands for duplicate:

- η is an increasing sequence in $w_{\mathbf{x}}$ of length $\ell g(\rho)$
- $\ell g(\bar{d}_{\mathbf{x},\rho(\ell)}) = \ell g(\bar{d}_{\mathbf{x},\eta(\ell)})$ for $\ell < \ell g(\rho)$
- $\psi(\bar{x}_{\bar{d},\eta}, \bar{c}_{\mathbf{x},\varrho_0}) \in \mathrm{tp}(\bar{d}_{\mathbf{x},\eta}, \bar{c}_{\mathbf{x},\varrho_0})$
- $\mathrm{tp}_\varphi(\bar{d}_{\mathbf{x},\eta}, \bar{c}_{\mathbf{x},\varrho} \dotplus M_{\mathbf{x}}) = \mathrm{tp}_\varphi(\bar{d}_{\mathbf{x},\rho}, \bar{c}_{\mathbf{x},\varrho} \dotplus M_{\mathbf{x}})$
- $n = \mathbf{k}(\psi(\bar{x}_{\bar{d},\eta}, \bar{c}_{\mathbf{x},\varrho_0}), \varphi(\bar{x}_{\bar{d},\eta}, \bar{x}_{\bar{c},\varrho}, \bar{y}), \mathbf{x})$.

Claim 5.11. *1) In Definition 5.10,* $\mathbf{k}(\varphi, \psi, \mathbf{x})$ *is well defined and* $< \mathrm{ind}(\varphi)$.
2) Also $\mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{x})$ *is well defined and* $< \mathrm{ind}(\varphi)$.
3) If $(\varphi, \psi, \mathbf{x})$ *is as in 5.10 and* $\mathbf{x} \le_1 \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ *then:*

- $(\varphi, \psi, \mathbf{y})$ *is as in 5.10*
- $\mathbf{k}(\varphi, \psi, \mathbf{x}) \le \mathbf{k}(\varphi, \psi, \mathbf{y})$
- $\mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{x}) \le \mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{y})$.

4) If $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ *then there is* $\mathbf{y}$ *such that (see 5.19):*

- $\bullet_1$ $\mathbf{x} \le_1 \mathbf{y} \in \mathrm{pK}_{\kappa,\mu,\theta}$
- $\bullet_2$ *if* $(\varphi, \psi, \mathbf{x})$ *is as in 5.10 and* $\mathbf{y} \le \mathbf{z} \in \mathrm{pK}_{\kappa,\bar{\mu},\varphi}$ *then* $\mathbf{k}(\varphi, \psi, \mathbf{y}) = \mathbf{k}(\varphi, \psi, \mathbf{z})$ *and* $\mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{y}) = \mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{z})$.

5) Like (4) but in $\bullet_2$ *it applies to* (φ, ψ) *such that* $(\varphi, \psi, \mathbf{y})$ *is as in 5.10.*

Proof. 1),2) By the definition of $\mathrm{ind}(\varphi)$ it is always finite as T is dependent, see [She15] or 5.22(1).
3) Read the definition.
4) By parts (1),(2),(3) as

- $(*)_1$ $(\mathbf{K}_{\kappa,\bar{\mu},\theta}, \le_1)$ is a partial order
- $(*)_2$ in this partial order an increasing sequence $\langle \mathbf{x}_i : i < \delta \rangle$ of length $< \theta^+$ has a $\le_1$-upper bound $\mathbf{x}_\delta$; moreover is the union so if $(\varphi, \psi, \mathbf{x}_\delta)$ is as in 5.10 then for some $i < \delta, (\varphi, \psi, \mathbf{x}_i)$ is as in 5.10
- $(*)_3$ for any $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$, there $\le \theta$ relevant pairs (φ, ψ).

5) Similarly using (4). $\square_{5.11}$

Claim 5.12. *If (A) then (B) where*

(A) (a) $\mathbf{x}, \varphi = \varphi(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\rho}, \bar{y}), \psi(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho_0})$ *are as in Definition 5.10*
(b) $n = \mathbf{k}_{\rm du}(\varphi, \psi, \mathbf{x})$ *and* ρ_1 *witness it*
(c) *let* $\varphi' = \varphi(\bar{x}_{\bar{d},\rho_1}, \bar{x}_{\bar{c},\varrho}, \bar{y})$
(d) *let* $\psi' = \psi(\bar{x}_{\bar{d},\rho_1}, \bar{c}_{\mathbf{x},\varrho_0})$ *hence* $\in \mathrm{tp}(\bar{d}_{\mathbf{x},\rho_1}, \bar{c}_{\mathbf{x},\varrho_0})$
(e) *let* ϱ_1 *witness* $\mathbf{k}(\varphi', \psi', \mathbf{x})$ *with* $\langle (c^*_{\ell,0}, \bar{c}^*_{\ell,1}) : \ell < \ell g(\varrho_1) \rangle$ *as in Definition 5.10(1)*
(f) $\psi''(\bar{x}_{\bar{d},\rho_1}, \bar{c}_{\mathbf{x},\varrho_0\char`^\varrho_1}) = \psi''(\bar{x}_{\bar{d},\rho_1}, \bar{c}_{\mathbf{x},\varrho_0\char`^\varrho_1}) \wedge \bigwedge\limits_{\ell < \ell g(\varrho_1)} (\varphi(\bar{x}_{\bar{d},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{c}^*_{\ell,1}) \wedge$
$\neg\varphi(\bar{x}_{\bar{d},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{c}^*_{\ell,0}))$

(*B*) (*a*) $(\varphi', \psi'', \mathbf{x})$ *are as in 5.10*

(*b*) $\mathbf{k}_{\rm du}(\varphi', \psi'', \mathbf{x}) = 0$.

Proof. Straightforward. $\square_{5.12}$

Definition 5.13. 1) For $\mathbf{x} \in {\rm pK}_{\kappa,\bar{\mu},\theta}$ and $\varphi_* = \varphi_*(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$ and $\psi_* = \psi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}_*) \in {\rm tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ let ${\rm ntr}_w(\varphi_*, \psi_*, \mathbf{x})$ be the maximal λ such that: if $A \subseteq M_{\mathbf{x}}$ and $|A| < \lambda$ <u>then</u> for some finite $p \subseteq {\rm tp}_{\pm\varphi_*}(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ we have $p \cup \{\psi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, b_*)\} \vdash {\rm tp}_{\pm\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus A)$.
2) For $\mathbf{x} \in {\rm pK}_{\kappa,\bar{\mu},\theta}$ let ${\rm ntr}_w(\mathbf{x}) = \min\{{\rm ntr}_w(\varphi_*, \psi_*, \mathbf{x}) : \varphi_*, \psi_* \text{ as above}\}$.

Claim 5.14. *1) For* $\mathbf{x}, \varphi_*, \psi_*$ *as in Definition 5.13(1) the cardinal* ${\rm ntr}_w(\varphi_*, \psi_*, \mathbf{x})$ *is a regular (infinite) cardinal.*
2) For $\mathbf{x} \in {\rm pK}_{\kappa,\bar{\mu},\theta}$ *the cardinal* ${\rm ntr}_w(\mathbf{x})$ *is a regular (infinite) cardinal.*
3) If $\lambda := {\rm ntr}_w(\varphi_*, \psi_*, \mathbf{x})$ *is* $> \aleph_0$ <u>*then*</u> *for some* m *we can replace "*$p \subseteq \ldots$ *finite" by: for some fix* n *and* $\eta \in {}^n 2$ *we have "*$p \subseteq {\rm tp}_{\pm\varphi}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + M_{\mathbf{x}})$ *has the form* $\{\varphi(\bar{x}, \bar{a}_\ell)^{{\rm if}(\eta(\ell))} : \ell < n\}$.

Proof. 1) Toward contradiction assume $\lambda = {\rm ntr}_w(\varphi_*, \psi_*, \mathbf{x})$ is singular and $A \subseteq M_{\mathbf{x}}$ has cardinality λ. We shall prove that for some finite $p \subseteq {\rm tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ we have $p \cup \{\psi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}})\} \vdash {\rm tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$, this suffices.

Let $\langle A_\varepsilon : \varepsilon < {\rm cf}(\lambda)\rangle$ be a $\subseteq$-increasing sequence of subsets of A with union A with each A_ε having cardinality $< \lambda$. For each $\varepsilon < {\rm cf}(\lambda)$ there is a finite $p_\varepsilon \in {\rm tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus (M_{\mathbf{x}})$ such that $p_\varepsilon \cup \{\psi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}})\} \vdash {\rm tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus A_\varepsilon)$. As p_ε is finite also $B_\varepsilon := {\rm Dom}(p_\varepsilon)$ is finite hence the cardinality of $B := \cup\{B_\varepsilon : \varepsilon < {\rm cf}(\lambda)\}$ is $\le {\rm cf}(\lambda)$. As $B \subseteq M_{\mathbf{x}}$ there is a finite $q \subseteq {\rm tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ such that $q \cup \{\psi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}})\} \vdash {\rm tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + M_{\mathbf{x}})$.

Now q is as required.
2) Follows from part (1). $\square_{5.14}$

The following is a replacement of 5.1 of §(5A).

The Crucial Claim 5.15. *If (A) then (B)* <u>*where*</u>*:*

(*A*) (*a*) $\lambda < \kappa$ *is regular* $\ge \mu$

(*b*) $\mathbf{x} \in {\rm pK}_{\kappa,\bar{\mu},\theta}$ *and* $\iota_{\mathbf{x}} = 2$

(*c*) $\varphi_* = \varphi_*(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c},\varrho}, \bar{y})$

(*d*) $\psi = \psi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{z})$

(*e*) $\psi_*(\bar{x}_{\bar{d}}) = \psi(\bar{x}_{\bar{d}}, \bar{c}, \bar{b}) \in {\rm tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$

(*f*) $u_{\mathbf{x}}$ *is finite, but see 5.17*

(*g*) $\lambda = {\rm cf}(\lambda) = \min\{|A| : A \subseteq M_{\mathbf{x}}$ *and there is no finite* $p \subseteq {\rm tp}_{\pm\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ *such that* $p \cup \{\psi_*(\bar{x}_{\bar{d}})\} \vdash {\rm tp}_{\pm\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus A)$

(*h*) *if* $A \subseteq M_{\mathbf{x}}, |A| < \lambda$ <u>*then*</u> *there is* $\bar{b}_A \subseteq M$ *such that* $\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}_A) \in {\rm tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ *and* $\{\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}_A)\} \cup \{\psi_*(\bar{x}_{\bar{d}})\} \vdash {\rm tp}_{\pm\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus A)$

(*B*) *there are* $\mathbf{I}, \bar{d}'$ *such that*

(*a*) $\mathbf{I} = \langle \bar{a}_{\alpha,0} {}^\frown \bar{a}_{\alpha,1} : \alpha < \lambda\rangle$ *is an indiscernible sequence in* $M_{\mathbf{x}}$

(*b*) $\ell g(\bar{a}_{\alpha,\ell}) = \ell g(\bar{y})$

(c) $\bar a_{\lambda,0}{}^\frown \bar a_{\lambda,1}$ *realizes* $\mathrm{Av}(\mathbf{I}, \bar c_{\mathbf{x}} + M_{\mathbf{x}})$

(d) $\mathrm{tp}(\bar a_{\lambda,0}, \bar c_{\mathbf{x}} \dot{+} M_{\mathbf{x}}) = \mathrm{tp}(\bar a_{\lambda,1}, \bar c_{\mathbf{x}} \dot{+} M_{\mathbf{x}})$

(e) $\bar d'$ *realizes* $\{\psi_*(\bar x_{\bar d})\} \cup \mathrm{tp}_{\pm\varphi_*}(\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}} \dot{+} M_{\mathbf{x}}) \cup \{\varphi_*(\bar x_{\bar d}, \bar a_{\lambda,1}), \neg\varphi_*(\bar x, \bar c_{\mathbf{x}}, \bar a_{\lambda,0})\}$.

Remark 5.16. 1) Note $\psi_*(\bar x_{\bar d})$ correspond to q_* in §(5C), so we can restrict its form if necessary, see §(5C).
2) How will we justify clause (A)(h)?

(a) we can manipulate φ_* such that $\{\varphi_*(M, \bar a) : \bar a\} = \{\neg\varphi_*(M, \bar a) : \bar a\}$ and $\emptyset, {}^{\ell g(\bar x)}(M)$ belongs to it, as in the proof of 8.4

(b) replacing $\varphi(\bar x, \bar y)$ by $\bigwedge\limits_{\ell < m} \varphi(\bar x, \bar y_\ell)$ change little.

3) We could have weakened clause (B)(c) to Δ-types for Δ derived from 5.22, in fact $\Delta = \Lambda$ with $n = \mathbf{k}(\varphi_*, \mathbf{x})$.
4) So 5.22(2)-(6) is what is really required but we do not need it.

Observation 5.17. *We can omit (A)(f) of 5.18.*

Proof. Let $\varphi_* = \varphi(\bar x_{\bar d,\rho}, \bar x_{\bar c,\varrho}, \bar y)$ with ρ, ϱ as in convention 5.9 and work with $\mathbf{y}$ which is like $\mathbf{x}$ but $\bar d_{\mathbf{y}} = \bar d_{\mathbf{x}} {\restriction} \mathrm{Rang}(\rho), \bar c_{\mathbf{y}} = \bar c_{\mathbf{x}} {\restriction} \mathrm{Rang}(\varrho)$.

Now reflect. $\square_{5.17}$

Proof. Proof of 5.15

We repeat the proof of Claim 5.1, making minor changes in Stages (A)-(D) and replacing stage (E) as follows:

Stage (A)-(D):

We omit $\otimes_1(b) - (e)$, using clauses of (A) of the claim when quoted.

In $\otimes_2(b)$ the set A exemplify (A)(g) of the claim

In $\otimes_3$ let $\varphi_\ell = \varphi_*$ for $\ell = 0, 1, 2$ (or just omit and replace φ_ℓ by φ_* when used, justified by clause (A)(h) hence $\bar e_\alpha, \bar e^*_\varepsilon \in {}^{\ell g(\bar y)})(M_{\mathbf{x}})$

In $\otimes_4(a)$ add "$+\otimes_1$".

We replace $\bar x_{[\theta]}$ by $\bar x_{\ell g(\bar y)}$ or $\bar y$ recalling $\varphi_* = \varphi(x_{\bar d}, \bar x_{\bar c}, \bar y)$.

Stage E: Let $M^+ \prec \mathfrak{C}$ be such that $\bar d_{\mathbf{x}} + \bar c_{\mathbf{x}} + M_{\mathbf{x}} \subseteq M^+$ and let $I = \{(\bar b, \varepsilon, \zeta, \bar d) : \bar b \in {}^{\ell g(\bar y)}(N_\lambda), \varepsilon < \zeta < \lambda$ and $\bar d \in M^+\}$.

For every $\xi < \lambda$ and $p \in \mathscr{P} = \{p : p$ is finite and $p \subseteq \mathrm{tp}_{\pm\varphi_*}(\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}} \dot{+} M_{\mathbf{x}})\}$ we let $I_{p,\xi}$ be the set of $(\bar b, \varepsilon, \zeta, \bar d)) \in I$ such that

- $\bar b \in {}^{\ell g(\bar y)}(N_\lambda)$
- $\bar d$ realizes p
- $\varepsilon < \zeta$ are from $[\xi, \lambda)$
- $\mathfrak{C} \models \psi_*[\bar d]$
- $\mathfrak{C} \models \varphi_*[\bar d, \bar c_{\mathbf{x}}, \bar e^*_\varepsilon]$
- $\mathfrak{C} \models \neg\varphi_*[\bar d, \bar c_{\mathbf{x}}, \bar e^*_\zeta]$.

Now note that

$(*)_1$ if $p_1 \subseteq p_2 \subseteq \mathrm{tp}_{\pm\varphi_*}(\bar d_{\mathbf{x}}, \bar c_{\mathbf{x}} \dot{+} M_{\mathbf{x}})$ are finite and $\xi_1 < \xi_2 < \lambda$ then $I_{p_2,\xi_2} \subseteq I_{p_1,\xi_1}$.

[Why? Read the definition.]

$(*)_2$ if $(p,\xi) \in \mathscr{P} \times \lambda$ then $I_{p,\xi} \neq \emptyset$.

[Why? As $\models \varphi[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{e}^*_\xi]$ clearly $p_1 := p \cup \{\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}^*_\varepsilon)\}$ belongs to $\mathscr{P}$, hence by clause (A)(g) there is $\bar{b} \in {}^{\ell g(\bar{y})}(N_\lambda)$ such that $p_1(x_{\bar{d}}) \cup \{\psi_*(\bar{x}_{\bar{d}})\} \cup \{\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b})^{\mathrm{if}(\mathbf{t})}\}$ is consistent for $\mathbf{t} = 0,1$. Hence recalling $\mathfrak{C} \models \varphi_*[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{b}]^{\mathrm{if}(\mathbf{t}(\bar{b}_j))}$ there is $\bar{d}$ in $\mathfrak{C}$ realizing $p_1(\bar{x}_{\bar{d}}) \cup \{\psi_*(\bar{x}_{\bar{d}})\}$ and $\mathfrak{C} \models \varphi_*[\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, \bar{b}]^{\mathrm{if}(1-\mathbf{t}(\bar{b})}$. Next choose $\zeta < \lambda$ such that $\zeta > \varepsilon$ and $\bar{b} \in N_\zeta$. Now $\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{e}^*_\zeta) \vdash \varphi_*[\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{b}]^{\mathrm{if}(\mathbf{t}(\bar{b}))}$ so necessarily $\mathfrak{C} \models$ "$\neg\varphi_*[\bar{d}, \bar{c}_{\mathbf{x}}, \bar{e}^*_\zeta]^{\mathrm{if}(\mathbf{t}[\bar{b}])}$".

Clearly $(\bar{b}, \varepsilon, \zeta, \bar{d}) \in I_{p,\xi}$ so $I_{p,\xi} \neq \emptyset$ as promised.]

$(*)_3$ choose an ultrafilter $\mathscr{D}$ on I such that $(p,\xi) \in \mathscr{P} \times \lambda \Rightarrow I_{p,\xi} \in \mathscr{D}$.

[Why? As $I_{p,\xi} \subseteq I$ using $(*)_1$ and $(*)_2$ above.]

$(*)_4$ $p(\bar{y}, \bar{y}', \bar{y}'', \bar{x}_{\bar{d}})$ is the following complete type over $\bar{c}_{\mathbf{x}} + M_{\mathbf{x}}$; where $\bar{y}', \bar{y}''$ has length $\ell g(\bar{y})$:
$p(\bar{y}, \bar{y}', \bar{y}'', \bar{x}_{\bar{d}}) = \{\vartheta(\bar{c}_{\mathbf{x}}, \bar{y}, \bar{y}', \bar{y}'', x_{\bar{d}}, \bar{e}) : \vartheta = \vartheta(\bar{x}_{\bar{c}[\mathbf{x}]}, \bar{y}, \bar{y}', \bar{y}'', \bar{x}_{\bar{d}}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{e} \in {}^{\ell g(\bar{z})}(M_{\mathbf{x}})$ and the set $\{(\bar{b}, \varepsilon, \zeta, \bar{d}) \in I : \mathfrak{C} \models \vartheta[\bar{c}_{\mathbf{x}}, \bar{b}, \bar{e}^*_\varepsilon, \bar{e}^*_\zeta, \bar{d}, \bar{e}]\}$ belongs to $\mathscr{D}\}$.

[Why? As $\mathscr{D}$ is an ultrafilter on I.]

$(*)_5$ choose $(\bar{b}', \bar{a}_0, \bar{a}_1, \bar{d}')$ in $\mathfrak{C}$ realizing $p(\bar{y}, \bar{y}', \bar{y}'', \bar{x}_{\bar{d}})$.

Now note

$(*)_6$ (a) $\bar{a}_1$ realizes $\mathrm{Av}(\langle \bar{e}^*_\varepsilon : \varepsilon < \lambda \rangle, \bar{c}_{\mathbf{x}} + M_{\mathbf{x}})$
(b) $\bar{a}_0$ realizes $\mathrm{Av}(\langle e^*_\varepsilon : \varepsilon < \lambda \rangle, \bar{a}_0 + \bar{c}_{\mathbf{x}} + M_{\mathbf{x}})$
(c) $\bar{a}_0{}^\frown \bar{a}_1$ realizes $\mathrm{Av}(\langle \bar{e}^*_{2\varepsilon}{}^\frown \bar{e}^*_{2\varepsilon+1} : \varepsilon < \lambda, \bar{c}_{\mathbf{x}} + M_{\mathbf{x}} \rangle)$.

[Why? Think.]

$(*)_7$ (a) $\bar{d}'$ realizes $\mathrm{tp}_{\varphi_*}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$
(b) $\bar{d}'$ realizes $\varphi_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{a}_0)$ and $\neg\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{a}_1)$.

So clearly we are done. $\square_{5.15}$

Claim 5.18. *Assume* $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}, \mu \geq \beth_\omega, \varphi_* = \varphi(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho}, \bar{y})$ *and*[54] $\psi_* = \psi_*(\bar{x}_{d,\rho}, \bar{x}_{\bar{c},\varrho_0}) \in \mathrm{tp}(\bar{d}_{\mathbf{x},\rho}, \bar{c}_{\mathbf{x},\varrho_0})$ *and* $\mathbf{k}(\varphi_*, \psi_*, \mathbf{x}) = 0$. *If* $\lambda := \mathrm{ntr}_w(\varphi_*, \psi_*, \mathbf{x})$ *is* $< \kappa$ *<u>then</u> there is* $\mathbf{y}$ *such that* $\mathbf{x} \leq_1 \mathbf{y} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ *and* $0 < \mathbf{k}(\varphi_*, \psi_*, \mathbf{y})$.

Proof. Let $A_* \subseteq M_{\mathbf{x}}$ witnessing $\mathrm{ntr}_w(\varphi_*, \psi_*, \mathbf{x}) = \lambda$.

Let $\bar{A}_* = \langle A_\varepsilon : \varepsilon < \lambda \rangle$, etc.

<u>Case 1</u>: $\lambda < \mu$.

As in 2.14 that is [She15, 2.8=tp25.33] and see Definition [She15, 2.6=tp25.32] but here we elaborate.

$\boxplus_1$ Let J be the set of pairs (q, Γ) such that:
(a) $q = q(\bar{x}_{\bar{d},\rho}) \subseteq \mathrm{tp}_{\pm\varphi_*}(\bar{d}_{\mathbf{x},\rho}, \bar{c}_{\mathbf{x},\varrho} \dotplus M)$ is finite

---

[54] May add parameters from $M_{\mathbf{x}}$, but can use trivial members of $\bar{c}_{\mathbf{x}}$, i.e. $\bar{c}_{\mathbf{x},i} \subseteq M_{\mathbf{x}}$.

(b) $\Gamma = \Gamma(\bar{y})$ is a finite subset of $\Lambda = \{\vartheta(\bar{y},\bar{c}) : \vartheta(\bar{y},\bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}(M_{\mathbf{x}})\}$

$\boxplus_2$ for a pair $(q,\Gamma) \in J$ we say $(\bar{c}_0,\bar{c}_1)$ does A_*-exemplifies (q,Γ) <u>when</u> :

(a) $\bar{c}_0,\bar{c}_1 \in {}^{\ell g(\bar{y})}(A_*)$

(b) $\mathfrak{C} \models$ "$\vartheta[\bar{c}_0] \equiv \vartheta[\bar{c}_1]$" when $\vartheta(\bar{y}) \in \Gamma(\bar{y})$

(c) $\{\psi(\bar{x}_{\bar{d}})\} \cup q(\bar{x}_{\bar{d},\eta}) \cup \{\varphi(\bar{x}_{\bar{d},\rho},\bar{c}_1), \neg\varphi(\bar{x}_{\bar{d},\rho},\bar{c}_0)\}$ is consistent

$\boxplus_3$ the family $\{\{(\bar{c}_0,\bar{c}_1) : (\bar{c}_0,\bar{c}_1)$ does A_*-exemplifies $(q,\Gamma)\} : (q,\Gamma) \in J\}$ has the finite intersection property.

[Why does $\boxplus_3$ hold? Otherwise we can find $(q_\ell,\Gamma_\ell) \in J$ for $\ell < n$ such that no $(\bar{c}_0,\bar{c}_1)$ does A_* exemplify (p,Γ_ℓ) for every $\ell < n$. Define the two-place relation E on ${}^{\ell g(\bar{y})}(A_*)$:

$\odot_{3.1}$ $\bar{c}_0 E \bar{c}_1$ iff $\bar{c}_0,\bar{c}_1 \in {}^{\ell g(\bar{y})}(A_*)$ and $\mathfrak{C} \models$ "$\vartheta[\bar{c}_0] \equiv \vartheta[\bar{c}_1]$" for every $\vartheta(\bar{y}) \in \bigcup\limits_{\ell<n} \Gamma_\ell$

clearly E is an equivalence relation with finitely many equivalence classes. Let $\langle \bar{c}^*_\ell : \ell < \ell(*)\rangle$ be a set of representatives and let $q_* = (\bigcup\limits_{\ell<n} q_\ell) \cup \{\varphi(\bar{x}_{\bar{d},\rho},\bar{c}^*_\ell)^{\mathbf{t}} : \mathbf{t} \in \{0,1\}$ and $\mathfrak{C} \models \varphi[\bar{d}_{\mathbf{x},\rho},\bar{c}^*_\ell]^{\mathbf{t}}\}$.

So

$\odot_{3.2}$ q_* is a finite subset of $\mathrm{tp}_{\pm\varphi}(\bar{d}_{\mathbf{x},\rho},\bar{c}_{\mathbf{x},\varrho} \dot{+} M_{\mathbf{x}})$ and

$\odot_{3.3}$ $q_* \vdash \mathrm{tp}_{\pm\varphi}(\bar{d}_{\mathbf{x},\rho},\bar{c}_{\mathbf{x},\varrho} \dot{+} A_*)$.

But $\odot_{3.2}$ contradicts the choice of A_* so $\boxplus_3$ holds indeed.]

So

$\boxplus_4$ there is an ultrafilter on ${}^{2\ell g(\bar{y})}(A_*)$ extending the family from $\boxplus_3$.

Choose such an ultrafilter D.

Let $(\bar{c}'_0,\bar{c}'_1)$ realizes $\mathrm{Av}(D,\bar{d}_{\mathbf{x}} + \bar{c}_{\mathbf{x}} + M_{\mathbf{x}})$, so clearly

$\boxplus_5$ the following set of formulas is finitely satisfiable in $\mathfrak{C}$:

$\{\psi_*(\bar{x}_{\bar{d}})\} \cup \mathrm{tp}_{\pm\varphi_*}(\bar{d}_{\mathbf{x},\rho},\bar{x}_{\bar{c},\varrho} \dot{+} M_{\mathbf{x}}) \cup \{\varphi_*(\bar{x}_{\bar{d},\rho},\bar{c}_{\mathbf{x},\varrho},\bar{c}'_1), \neg\varphi_*(\bar{x}_{\bar{d},\rho},\bar{c}_{\mathbf{x},\varrho},\bar{c}'_\rho)\}$.

So let $\bar{d}'$ realize the type from $\boxplus_5$ and define $\mathbf{y} \in \mathrm{pK}_{\kappa,\mu,\theta}$ by

$\boxplus_6$ (a) $M_{\mathbf{y}} = M_{\mathbf{x}}$

(b) $w_{\mathbf{y}} = w_{\mathbf{x}} + \{t_*\}$

(c) $\bar{d}_{\mathbf{y}} \restriction w_{\mathbf{x}} = \bar{d}_{\mathbf{x}}$ and $\bar{d}_{\mathbf{y},t_*} = \bar{d}'$

(d) $v_{\mathbf{y}} = v_{\mathbf{x}} + \{s_*\}$ and $u_{\mathbf{y}} = u_{\mathbf{x}}$

(e) $\bar{c}_{\mathbf{y}} \restriction v_{\mathbf{x}} = \bar{c}_{\mathbf{x}}$ and $\bar{c}_{\mathbf{y},s_*} = \bar{c}'_1 {}^\frown \bar{c}'_1$

(f) $\mathbf{I}_{\mathbf{y}} = \mathbf{I}_{\mathbf{x}}$

(g) $\bar{B}_{\mathbf{y},s}$ is equal to $B_{\mathbf{x},s}$ if $s \in v_{\mathbf{x}} \backslash u_{\mathbf{x}}$ and is equal to A_* if $s = s_*$.

Clearly $\mathbf{y}$ is as required.

<u>Case 2</u>: $\lambda \geq \mu$ is singular.

Impossible by 5.14.

<u>Case 3</u>: $\lambda \geq \mu$ regular

$\oplus_1$ without loss of generality

(a) for some $\bar{c}_1, \bar{c}_0 \in {}^{\ell g(\bar{y})}(M_{\mathbf{x}})$ we have
$\mathfrak{C} \models$ "$(\forall \bar{x}_{\bar{d}})[\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{c}_1) \wedge \neg\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{c}_0)]$"

(b) for every $\bar{c}_1 \in {}^{\ell g(\bar{y})}(M_{\mathbf{x}})$ for some $\bar{c}_0 \in {}^{\ell g(\bar{y})}(M_{\mathbf{x}})$ we have
$\mathfrak{C} \models$ "$(\forall \bar{x}_{\bar{d}})[\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{c}_1) \equiv \neg\varphi_*(\bar{x}_{\bar{d}}, \bar{c}_{\mathbf{x}}, \bar{c}_2)]$".

[Why? We can use $\varphi'(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}\char`^\langle y_0, y_1, y_2\rangle)) = [y_0 = y_2 \rightarrow \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, y)] \wedge [y_0 \neq y_2 \wedge y_1 = y_2 \rightarrow \neg\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, y)]$ so if $B \subseteq M_{\mathbf{x}}, |B| \geq 2$ we can use for y_0, y_1, y_2 members of B; see more in the proof of 8.4.]

$\oplus_2$ (a) let n_* be as m in 5.14(3)

(b) let $\varphi_{**} = \varphi_{**}(\bar{x}_{\bar{d},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{y}_{**})$ where $\bar{y} = \bar{y}_0\char`^ \ldots \char`^\bar{y}_{n(*)-1}, \ell g(\bar{y}_\ell) = \ell g(\bar{y})$ and $\varphi_{**} = \bigwedge\limits_{\ell < n(*)} \varphi_*(\bar{x}_{\bar{d},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{y}_\ell)$.

Now

$\oplus_3$ $\lambda = \mathrm{ntr}_w(\varphi_{**}, \psi, \mathbf{x})$.

[Why? Think.]

Now we shall use 5.15 + 5.17 for φ_{**}, ψ getting $d', \bar{a}_0, \bar{a}_1, \mathbf{I} = \langle(\bar{a}_{\alpha,0}, \bar{a}_{\alpha,1}) : \alpha < \lambda\rangle$.

Why this suffice? We choose $\mathbf{y}$ by

$\oplus_4$ (a) $M_{\mathbf{y}} = M_{\mathbf{x}}$

(b) $\bar{d}_{\mathbf{y}} = \bar{d}_{\mathbf{x}}\char`^\langle \bar{d}'\rangle$ i.e. $w_{\mathbf{y}} = w_{\mathbf{x}} \cup \{s\}, w_{\mathbf{x}}, p <_{\mathbf{y}} s, \bar{d}_{\mathbf{y},s} = \bar{d}'$.

(c) $\bar{c}_{\mathbf{y}} = \bar{c}_{\mathbf{x}}\char`^(\bar{a}_0\char`^\bar{a}_1)$, i.e. $\bar{a}_0\char`^\bar{a}_1 = \bar{c}_{\mathbf{y},t}, v_{\mathbf{y}} = v_{\mathbf{x}} + \{t\}$

(d) $\mathbf{I}_{\mathbf{y},t} = \mathbf{I}$.

This is this possible? We just have to check that the relevant condition in 5.15, i.e. the clauses in (A) holds which is straight. $\square_{5.18}$

Conclusion 5.19. *1) For every $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}$ <u>there</u> is $\mathbf{y}$ such that $\mathbf{x} \leq_1 \mathbf{y} \in \mathrm{pK}_{\kappa,\mu,\theta}$ and: if $\varphi_*(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho}, \bar{y}) \in \Gamma^1_{\mathbf{x}}, \psi_*(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho}) \in \mathrm{tp}(\bar{d}_{\mathbf{x},\rho}\char`^\bar{c}_{\mathbf{x},\varrho}, \emptyset)$ and $\mathbf{y} \leq_1 \mathbf{z} \in \mathrm{pK}_{\kappa,\mu,\theta}$ then $\mathbf{k}_{\mathrm{du}}(\varphi_*, \psi_*, \mathbf{z}) = \mathbf{k}_{\mathrm{du}}(\varphi_*, \psi_*, \mathbf{y})$.*
2) Above $\mathbf{y} \in \mathrm{uK}_{\kappa,\mu,\theta}$, see Definition 3.6(3C).

Proof. By 5.15, recalling 5.12, as in 5.14(4). $\square_{5.19}$

Conclusion 5.20. *If $\kappa > \mu \geq \beth_\omega + \theta^+, \theta \geq |T|$ <u>then</u> $\mathrm{vK}^{\otimes}_{\kappa,\mu,\theta}$ is $\leq_1$-dense in $\mathrm{sK}^{\oplus}_{\kappa,\mu,\theta}$. Moreover, if $\mathbf{m} \in \mathrm{sK}^{\oplus}_{\kappa,\mu,\theta}$ then for some $\mathbf{n}$ we have $\mathbf{m} \leq_1 \mathbf{n} \in \mathrm{uK}^{\otimes}_{\kappa,\mu,\theta}$.*

Proof. Assume $\mathbf{m} \in \mathrm{sK}^{\oplus}_{\kappa,\mu,\theta}$ we apply 5.11(5) to $\mathbf{x}$ getting $\mathbf{y}$ as there. By 5.18 we have

$(*)_1$ if $\varphi = \varphi(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho}, \bar{y})$ and $\psi = \psi(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho_0}) \in \mathrm{tp}(\bar{d}_{\mathbf{x},\rho}, \bar{c}_{\mathbf{x},\rho_0})$ and $\mathbf{k}_{\mathrm{du}}(\varphi, \psi, \mathbf{y}) = 0$ <u>then</u> $\mathrm{ntr}_w(\varphi, \psi, \mathbf{y}) \geq \kappa$.

Clearly we can find $\mathbf{n}$ such that

$(*)_2$ (a) $\mathbf{m} \leq_1 \mathbf{n} \in \mathrm{sK}^{\oplus}_{\kappa,\mu,\theta}$

(b) $\mathbf{x}_{\mathbf{n}} = \mathbf{y}$

(c) if φ, ψ are as in $(*)_1$ then $\varphi \in \Gamma^3_{\bar{\psi}_{\mathbf{n}}}$.

By 5.12 + 5.15 clearly $\mathbf{n} \in \mathrm{vK}^{\oplus}_{\kappa,\mu,\theta}$.

The "moreover" is proved similarly. $\square_{5.20}$

Theorem 5.21. *The recounting theorem* *Assume* $\kappa = \kappa^{<\kappa} = \aleph_\alpha = \mu + \alpha \geq \mu \geq \beth_\omega + \theta^+, \theta \geq |T|$.

Then for any $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ *the cardinality of* $\mathbf{S}^\theta(M)/\equiv_{\mathrm{aut}}$ *is* $\leq 2^{<\mu} + |\alpha|^{\theta+|T|}$.

Proof. By 5.20 and 3.13(2). $\square_{5.21}$

§ 5(C). Exact recounting of types and vK.

The following analysis look more carefully at decomposition and $\varphi \in \Gamma^1_{\mathbf{x}}$: eventually it was not used in proving the density of vK.

Here we use $\mathrm{ind}(\varphi)$.

Definition 5.22. 1) For $\varphi = \varphi(\bar{x}, \bar{y}', \bar{y})$ let $\mathrm{ind}(\varphi) = \mathrm{ind}_T(\varphi) = \min\{n :$ the set $\{\varphi(\bar{x}_\eta, \bar{y}', \bar{y}_k)^{[\eta(k)]} : \eta \in {}^n 2$ and $k < n\}$ is inconsistent with $T\}$; compare with 2.6(5), 2.15(1).

2) Above if $\bar{y}'$ is the empty sequence we may omit it; we may ignore the case $\mathrm{ind}(\varphi) = 1$; it is always ≥ 1.

3) For $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}$ and $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$ and $k < \mathrm{ind}_T(\varphi)$ let $\Lambda^1_{\mathbf{x},\varphi,k} = \{\psi : \psi = \psi(\bar{y}_k, \bar{y}^+_0{}^\frown \ldots {}^\frown \bar{y}^+_{k-1}, y^+_{k+1}{}^\frown \ldots {}^\frown \bar{y}^+_{\mathrm{ind}(\varphi)-1}; \bar{x}_{\bar{c}})$ and for each $\ell \in \mathrm{ind}(\varphi) \backslash \{k\}, \bar{y}_{\ell,0}$ or $\bar{y}_{\ell,1}$ is a dummy in $\psi\}$ where we fix $y^+_m = \bar{y}_{m,0}{}^\frown \bar{y}_{m,1}, \ell g(\bar{y}_{m,0}) = \ell g(\bar{y}_m) = \ell g(\bar{y}_{m,1})$ and in $\bar{y}_k{}^\frown \bar{y}^+_0{}^\frown \ldots {}^\frown \bar{y}^+_{\mathrm{ind}(\varphi)-1}$ there is no repetitions.

4) $\Lambda^0_{\mathbf{x},\varphi,k} = \{\psi_{\mathbf{x},\varphi,\eta,\nu,k} : \psi_{\mathbf{x},\varphi,\eta,\nu,k} = \psi_{\mathbf{x},\varphi,\eta,\nu,k}(\bar{y}_k, \bar{y}^+_0{}^\frown \ldots {}^\frown \bar{y}^+_{k-1}, \bar{y}^+_{k+1}{}^\frown \ldots {}^\frown \bar{y}^+_{\mathrm{ind}(\varphi)-1}; \bar{x}_{\bar{c}}) = (\exists \bar{x}_{\bar{d}})(\bigwedge\limits_{m<\mathrm{ind}(\varphi), m\neq k} (\varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_{m,\eta(m)})^{[\nu(m)]} \wedge \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}_k)^{[\nu(k)]}))$ where $\eta, \nu \in {}^{\mathrm{ind}(\varphi)}2\}$.

5) Let $\Omega^0_{\mathbf{x},\varphi,k,\bar{c}_*} = \{\psi_{\mathbf{x},\varphi,\eta,\nu,k}(\bar{y}, \bar{c}_*, \bar{e}, \bar{c}_{\mathbf{x}}) : \bar{e} = \bar{e}_{k+1}{}^\frown \ldots {}^\frown \bar{e}_{\mathrm{ind}(\varphi)-1}$ and $\bar{e}_m \in {}^{2\ell g(\bar{y})}(M_{\mathbf{x}})$ for each $m\}$ where: $\bar{c}_* = \bar{c}^*_{0,0}{}^\frown \bar{c}^*_{0,1}{}^\frown \ldots {}^\frown \bar{c}^*_{k-1,0}{}^\frown c^*_{k-1,1}, \ell g(\bar{c}^*_{\ell,0}) = \ell g(\bar{c}^*_{\ell,1}) = \ell g(\bar{y})$.

6) For $\Lambda \subseteq \Lambda^1_{\mathbf{x},\varphi,k}$ and we let $\Omega^1_{\mathbf{x},\varphi,\Lambda,k,\bar{c}_*} = \{\psi(\bar{y}, \bar{c}_*, \bar{e}, \bar{c}_{\mathbf{x}}) : \psi = \psi(\bar{y}, \bar{y}^+_0{}^\frown \ldots {}^\frown \bar{y}^+_{k-1}, \bar{y}^+_{k+1}{}^\frown \ldots {}^\frown \bar{y}^+_{\mathrm{ind}(\varphi)-1}, \bar{x}_{\bar{c}}) \in \Lambda$ and $\bar{e} = \bar{e}_{k+1}{}^\frown \ldots {}^\frown \bar{e}_{\mathrm{ind}(\varphi)}$ and $\bar{e}_m \in {}^{2\ell g(\bar{y})}(M_{\mathbf{x}})$ for each $m\}$.

A relative of 2.14(1) = [She15, 2.8=tp25.33] imitating vK is

Definition 5.23. Let $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ be normal[55] and $\varphi = \varphi(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{y}) \in \Gamma^1_{\mathbf{x}}$, really $\varphi = \varphi(\bar{x}_{\bar{d},\rho}, \bar{x}_{\bar{c},\varrho}, \bar{y})$ for some $\rho \in {}^{\omega>}(\ell g(\bar{d}_{\mathbf{x}})$ and $\varrho \in {}^{\omega>}(\ell g(\bar{c}_{\mathbf{x}}))$ and $n = \mathrm{ind}(\varphi)$, see 5.22(1).

We call $\mathbf{w}$ an $(\mathbf{x}, \varphi)$-witness when $\mathbf{w} = \langle (\bar{c}_{k,0}, \bar{c}_{k,1}) : \ell < n\rangle = \langle \bar{c}_{\mathbf{w},k,0}, \bar{c}_{\mathbf{w},k,1} : k < \mathbf{n}_{\mathbf{w}}\rangle$, their concatanation is denoted by $\bar{c}_{\mathbf{w}}$ and there is ρ_1 exemplifying it such that

- (a) let $\bar{c}_k = \bar{c}_{k,0}{}^\frown \bar{c}_{k,1}$ and $\bar{c}_{<k} = (\bar{c}_0, \ldots, \bar{c}_{k-1})$
- (b) $\bar{c}_{k,0}, \bar{c}_{k,1}$ are finite subsequences of some $\bar{c}_{\mathbf{x},i(k)}$ with $\langle i(k) : k < n\rangle$ increasing and $\varrho(\ell) < i(k)$ for $\ell < \ell g(\varrho), k < n$

[55]This indicates we may forget $\bar{c}_{\mathbf{x}}$ and instead have a set of sequences some $\bar{d}_i$'s which function as $\bar{c}_i$'s, so we have B_η or $\mathbf{I}_\eta$ but even if $\ell g(\eta_\ell) = \eta_\ell + 1, \eta_1(n_1) = \eta_2(n_2)$ we still may have $B_{\eta_1} \neq B_{\eta_2}$, etc.

(c) $\bar{c}_{k,0}, \bar{c}_{k,1}$ satisfies the same formulas from $\Omega^0_{\mathbf{x},\varphi,k,\bar{c}_{<k}}$

(d) $\rho_1 \in {}^{\ell g(\rho)}\ell g(\bar{d}_{\mathbf{x}})$ and

(e) $\bar{d}_{\mathbf{x},\rho}$ and $\bar{d}_{\mathbf{x},\rho_1}$ satisfies the same formulas from $\{\varphi(\bar{x}_{\bar{d},\rho}, \bar{c}_{\mathbf{x},\varrho}, \bar{b}) : \bar{b} \in {}^{\ell g(\bar{y})}(M_{\mathbf{x}})\}$

(f) $\bar{d}_{\mathbf{x},\rho_1}$ realizes $q_{\mathbf{w}} := \{\varphi(\bar{x}_{\bar{d},\rho_1}, \bar{c}_{\mathbf{x},\varrho}), \bar{c}_{k,1}) \wedge \neg\varphi(\bar{x}_{\bar{d},\rho_1}, \bar{c}_{\mathbf{x},\varrho}, \bar{c}_{k,0}) : k < n\}$

(g) (nec?) $\mathrm{tp}(\bar{c}_{i(k)}, \bar{c}_{<i(k)} + M_{\mathbf{x}})$ is[56] finitely satisfiable in $M_{\mathbf{x}}$.

Observation 5.24. *Above in Definition 5.23, $\ell g(\mathbf{w}) < \mathrm{ind}_T(\varphi)$.*

Definition 5.25. In Definition 5.23
0) We say $\mathbf{w}$ is a maximal $(\mathbf{x}, \varphi)$-witness <u>when</u> it is an $(\mathbf{x}, \varphi)$-witness and there is no $(\mathbf{x}, \varphi)$-witness $\mathbf{w}_1$ such that $\mathbf{w} \triangleleft \mathbf{w}_1$.
1) We say $\mathbf{w}$ is a successful $(\mathbf{x}, \varphi)$-witness <u>when</u> it is an $(\mathbf{x}, \varphi)$-witness and for every $(\mathbf{x}_1, \mathbf{w}_1)$ satisfying $\mathbf{x} \leq_1 \mathbf{x}_1 \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{w}_1$ an $(\mathbf{x}_1, \varphi)$-witness $\mathbf{w} \trianglelefteq \mathbf{w}_1$ we have $\mathbf{w} = \mathbf{w}_1$.
2) We say $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ is full for $(\kappa, \bar{\mu}, \theta)$ <u>when</u> $\mathbf{x}$ is normal and for every $\varphi \in \Gamma^1_{\mathbf{x}}$ there is a successful $(\mathbf{x}, \varphi)$-witness.

Remark 5.26. In Definition 5.25 we may consider "every maximal $(\mathbf{x}, \varphi)$-witness is successful".

Definition 5.27. 1) Let $\mathrm{ntr}_{\mathrm{v}}(\varphi(\bar{x}_{\bar{d},\eta}, \bar{x}_{\bar{c},\varrho}, \bar{y}), \mathbf{w}, \mathbf{x})$ where $\mathbf{x} \in \mathrm{pK}_{\kappa,\bar{\mu},\theta}$ and $\mathbf{w}$ is a $\varphi(\bar{x}_{\bar{d},\eta}, \bar{x}_{\bar{c},\varrho}, \bar{y})$-witness, be the minimal cardinal λ such that, recalling $q_{\mathbf{w}}$ is from Definition 5.23(f);

(∗) for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \lambda$ there is a finite $q_A = q_A(\bar{x}_{\bar{d},\eta}) \subseteq \mathrm{tp}_{\pm\varphi}(\bar{d}_{\mathbf{x},\eta}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ such that $q_A(\bar{x}_{\bar{d},\eta}) \cup q_{\mathbf{w}} \vdash \mathrm{tp}_{\varphi}(\bar{d}_{\mathbf{x},\eta}, \bar{c}_{\mathbf{x}} \dotplus A)$.

2) Let $\mathrm{ntr}_{\mathrm{v}}(\mathbf{x}) = \min\{\mathrm{ntr}_{\mathrm{v}}(\varphi(\bar{x}_{\bar{d},\eta}, \bar{x}_{\bar{c},\varrho}, \bar{y}), \mathbf{w}, \mathbf{x}) : \varphi = \varphi(x_{\bar{d},\eta}, \bar{x}_{\bar{c},\varrho}, \bar{y})$ and $\mathbf{w}$ is a maximal $(\mathbf{x}, \varphi)$-witness$\}$. It is regular (see case 2 in the proof of **??** below and we can replace finite by "of cardinality $< n_*$" if $\lambda > \aleph_0$, see case 3 there.

Discussion 5.28. 1) The point is that looking for $q \subseteq \mathrm{tp}_{\varphi}(\bar{d}_{\mathbf{x},\eta}, \bar{c}_{\mathbf{x}} \dotplus M_{\mathbf{x}})$ enables us to deal with singular $\mathrm{ntr}_{\mathrm{v}}(\varphi)$.
2) Do we really have to change $\bar{d}_{\mathbf{x},\eta}$ to $\bar{d}_{\mathbf{x},\rho}$ in the definition of $\mathrm{ntr}_{\varphi_v}(\mathbf{x}, \varphi, \mathbf{w})$? when we succeed, i.e. is it κ?

Part is a finite subset q of $\mathrm{tp}^{\pm}_{\varphi}(\bar{d}_{\mathbf{x},\eta}, M_{\mathbf{x}})$ so η, ρ are not distinguished. But we have $q_{\mathbf{w}}$ is a $\pm\varphi$-type on $\bar{c}^{\mathbf{w}}_{k,\ell}(k < n_{\mathbf{w}}, \ell = 0, 1\}$ and $\mathrm{tp}(\bar{c}_{\mathbf{w}}, M_{\mathbf{x}})$ is definable.

[56] First, we can use just tp_{Δ_k} for Δ_k large enough. Second, does clause (g) follows from the earlier ones?

§ 6. Indiscernibles

Hypothesis 6.1. T dependent.

Theorem 6.2. *Assume $\kappa = \kappa^{<\kappa} > \mu = \beth_\omega + \theta, \theta \geq |T|$ and $M \in \mathrm{EC}_{\kappa,\kappa}(T)$.*

If $\varepsilon < \theta^+$ and $p \in \mathbf{S}^\varepsilon(M)$ then there is an indiscernible sequence $\mathbf{I} = \langle \bar{a}_\alpha : \alpha < \kappa \rangle$ of ε-tuples from M, i.e. $\bar{a}_\alpha \in {}^\varepsilon M$ for $\alpha < \kappa$ such that $p = \mathrm{Av}(\mathbf{I}, M)$.

Proof. Let $\bar{d}$ realize p hence for some $\mathbf{x} \in \mathrm{rK}_{\kappa,\kappa,\theta}$ we have $\bar{d}_{\mathbf{x}} = \bar{d}, \bar{c}_{\mathbf{x}} = \langle \rangle, v_{\mathbf{x}} = \emptyset$. Let $\mathbf{m} = (\mathbf{x}, \langle \rangle, \langle \rangle)$, it $\in \mathrm{rK}^{\oplus}_{\kappa,\kappa,\theta}$. By the density of $\mathrm{uK}^{\oplus}_{\kappa,\mu,\theta}$ there is $\mathbf{n} \in \mathrm{uK}^{\otimes}_{\kappa,\mu,\theta}$ such that $\mathbf{m} \leq_1 \mathbf{n}$, hence $\bar{d} \trianglelefteq \bar{d}_{\mathbf{x}[\mathbf{m}]}$. By 6.3 below we are done. $\square_{6.2}$

§ 6(A). Indiscernibility and materializing m.

Claim 6.3. *1) Assume $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r) \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $M_{\mathbf{x}}$ has cardinality κ, then for some $\langle \bar{c}_\alpha {}^\frown \bar{d}_\alpha : \alpha \leq \kappa + \omega \rangle$ we have:*

- (a) *$\bar{c}_\alpha, \bar{d}_\alpha$ are from $M_{\mathbf{x}}$ and $\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}} {}^\frown \bar{c}_\alpha {}^\frown \bar{d}_\alpha$ realizes r for $\alpha < \kappa$*
- (b) *$\mathbf{I} = \langle \bar{c}_\alpha {}^\frown \bar{d}_\alpha : \alpha \leq \kappa + \omega \rangle$ is an indiscernible sequence and $(\bar{c}_\kappa, \bar{d}_\kappa) = (\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}})$*
- (c) *$\mathrm{tp}(M, \cup\{\bar{c}_\alpha {}^\frown \bar{d}_\alpha : \alpha < \kappa\} + \bar{c}_\kappa + \ldots + \bar{c}_{\kappa+n_*}) \vdash \mathrm{tp}(M, \cup\{\bar{d}_\alpha : \alpha \leq \kappa + n_*\})$*
- (d) *if $A \subseteq M_{\mathbf{x}}$ is finite[57] and $\alpha < \kappa$ is large enough then $\mathrm{tp}(\bar{c}_\alpha {}^\frown d_\alpha, A) = \mathrm{tp}(\bar{c}_{\mathbf{x}} {}^\frown \bar{d}_{\mathbf{x}}, A)$ and $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{d}_\alpha) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + A + \bar{c}_\alpha {}^\frown \bar{d}_\alpha)$ according to $\bar{\psi}$.*

2) For $(\mathbf{m}, \mathbf{w}) \in \mathrm{vK}^{\otimes}_{\kappa,\bar{\mu},\theta}$ similarly but replace clause (d) by

- (d)′ *if $A \subseteq M_{\mathbf{x}}$ is finite and $\alpha < \kappa$ is large enough then $(\bar{c}_\alpha, \bar{d}_\alpha)$ solve $(\mathbf{m}, A + \cup\{\bar{c}_\beta {}^\frown \bar{d}_\beta : \beta < \alpha\}$ in the $\mathrm{rK}^{\oplus}_{\kappa,\bar{\mu},\theta}$-sense-see Definition 3.3(f).*

Definition 6.4. 1) We say an indiscernible sequence $\mathbf{I} = \langle \bar{c}_s {}^\frown \bar{d}_s : s \in I \rangle$ materialize $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ when in the linear order I there is no last element and for some $\bar{c}_n, \bar{d}_n$ for $n < \omega$ the sequence $\langle \bar{c}_s {}^\frown \bar{d}_s : s \in I + \omega \rangle$ satisfies (a)-(d) of Claim 6.3, and $(\bar{c}_n, \bar{d}_n)$ here standing for $\bar{c}_{\kappa+n}, \bar{c}_{\kappa+n}$ there.
1A) Similarly for $\mathrm{vK}^{\otimes}_{\kappa,\bar{\mu},\theta}$.
2) $\mathbf{I}$ is also said to materialize $\mathbf{x}$ when this holds for some $\mathbf{m}$ with $\mathbf{x} = \mathbf{x}_{\mathbf{m}}$.
3) We say that D is the ultrafilter of $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ or just $\mathbf{m} \in \mathrm{tK}^{\otimes}_{\kappa,\mu,\theta}$ and we may (see 6.9(3)), $D = D_{\mathbf{m}}$, when $D \in \mathrm{uf}(\bar{c}_{\mathbf{x}[\mathbf{m}]} {}^\frown \bar{d}_{\mathbf{x}[\mathbf{m}]}, M_{\mathbf{x}[\mathbf{m}]})$ satisfies: for every $A \subseteq M_{\mathbf{m}[\mathbf{x}]}$ of cardinality $< \kappa$ and sequence $\bar{c}' {}^\frown \bar{d}'$ from $M_{\mathbf{m}[\mathbf{x}]}$, the sequence realizes $\mathrm{tp}(D, A)$ iff $\bar{c}' {}^\frown \bar{d}'$ realizes $\mathrm{tp}(\bar{c}_{\mathbf{x}[\mathbf{m}]} {}^\frown \bar{d}_{\mathbf{x}[\mathbf{m}]}, A)$ and $\bar{c}_{\mathbf{x}[\mathbf{m}]} {}^\frown \bar{d}_{\mathbf{x}[\mathbf{m}]} {}^\frown \bar{c}' {}^\frown \bar{d}'$ realizes $r_{\mathbf{m}}$, recalling Definition 1.19(7).

Proof. Proof of 6.3 1) Let $\langle a_\alpha : \alpha < \kappa \rangle$ list $M_{\mathbf{x}}$ and choose $(\bar{c}_\alpha, \bar{d}_\alpha)$ in M by induction on $\alpha < \kappa$ which solves $(\mathbf{x}, \bar{\psi}, r)$ over $A_\alpha := \{a_\beta + \bar{c}_\beta + \bar{d}_\beta : \beta < \alpha\} \cup B^+_{\mathbf{x}}$, see clause (f) Definition 3.3.

Next, let $(\bar{c}_\kappa, \bar{d}_\kappa) = (\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}})$. By 3.14 for each $\alpha < \kappa$ the sequence $\langle \bar{c}_\beta {}^\frown \bar{d}_\beta : \beta \in [\alpha, \kappa] \rangle$ is indiscernible over A_α and choose $(\bar{c}_{\kappa+n}, \bar{d}_{\kappa+n})$ for $n \in [1, \omega)$ such that $\langle \bar{c}_\beta {}^\frown \bar{d}_\beta : \beta \in [\alpha, \kappa + \omega) \rangle$ is an indiscernible sequence over A_α for every $\alpha < \kappa$, possible by compactness, so clauses (a),(b),(d) of 6.3(1) hold.

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[57] we can say of cardinality $< \kappa$, but for 6.4 sake we use this form

We are left with clause (c). By clause (d) we have $\mathrm{tp}(\bar d_{\mathbf x}, c_{\mathbf x}, \bar d_\alpha + \bar c_\alpha) \vdash \mathrm{tp}(\bar d_{\mathbf x}, \bar c_{\mathbf x} + \bar d_\alpha + \bar c_\alpha + A_\alpha)$. Now for stationarily many $\alpha < \kappa$ we have $\mathrm{tp}(\bar d_{\mathbf x}, \bar c_{\mathbf x} + \bar d_\alpha + \bar c_\alpha) \vdash \mathrm{tp}(\bar d_{\mathbf x}, \bar c_{\mathbf x} + \bar d_\alpha + \bar c_\alpha + A_\alpha + \sum\limits_{n<\omega} \bar c_{\kappa+n})$, otherwise by Fodor Lemma we get contradiction to 2.14(2). So by indiscernibility we get, for $n < \omega, \beta < \kappa+n$ that $\mathrm{tp}(\bar d_{\kappa+n+1}, c_{\kappa+n} + \bar d_\beta + \bar c_\beta) \vdash \mathrm{tp}(\bar d_{\kappa+n}, c_{\kappa+n} + \sum\limits_{i<\beta} \bar c_i{}^\frown \bar d_i + A_{\beta,\kappa} + \sum\limits_{m \geq n} \bar c_{\kappa+n})$.

Hence for $n < \omega$ we have $\mathrm{tp}(M_\kappa, \sum\limits_{\alpha<\kappa+n} \bar c_\alpha{}^\frown \bar d_\alpha + \sum\limits_{n<\omega} c_{\kappa+m}) \vdash \mathrm{tp}(M_\kappa, \sum\limits_{\alpha \leq \kappa+n} \bar c_\alpha{}^\frown d_\alpha + \sum\limits_{m<\omega} \bar c_{\kappa+n})$ hence we get the desired conclusion.

2) Similarly. $\square_{6.3}$

A variant of 6.3

Claim 6.5. *If* $\mathbf m = (\mathbf x, \bar\psi, r) \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ *with* $M_{\mathbf x}$ *of cardinality* κ *and* $I_2 = I_1 \times \mathbb Z$ *ordered lexicographically of course,* I_1 *is a saturated model of* $\mathrm{Th}(\mathbb Q, <)$ *of cardinality* κ, *then we can find* $\langle \bar c_s{}^\frown \bar d_s : s \in I_2 + \{\kappa\}\rangle$ *such that:*

- (a) $(\bar c_\kappa, \bar d_\kappa) = (\bar c_{\mathbf x}, \bar d_{\mathbf x})$
- (b) $\langle \bar c_s{}^\frown \bar d_s : s \in I_2 + \{\kappa\}\rangle$ *is an indiscernible sequence*
- (c) $M_{\mathbf x}$ *is* $|T|^+$*-atomic over* $\cup\{\bar c_s{}^\frown \bar d_s : s \in I\}$
- (c)′ *if* $J_2 = J_1 \times \mathbb Z$, *where* J_1 *(is a linear order which) extends* I_1 *and* $\bar c_s, \bar d_s$ *for* $s \in J_2 \backslash I_2$ *are such that* $\langle \bar c_s{}^\frown \bar d_s : s \in J\rangle$ *is indiscernible, then* $\mathrm{tp}(M, \cup\{\bar c_s{}^\frown \bar d_s\} : s \in I\}) \vdash \mathrm{tp}(M, \cup\{\bar c_s{}^\frown \bar d_s : s \in J\})$
- (d) *if* $s \in I_2$ *then* $\bar d_{\mathbf x}{}^\frown \bar c_{\mathbf x}{}^\frown \bar d_s{}^\frown \bar c_s$ *realizes* r *and for every* $A \in [M_{\mathbf x}]^{<\kappa}$ *for every large enough* $t \in I_2$ *we have* $\mathrm{tp}(\bar d_{\mathbf x}, \bar c_{\mathbf x} + \bar d_t + c_t) + \mathrm{tp}(d_{\mathbf x}, \bar c_{\mathbf x} + \bar d_t + c_t + \sum\limits_{s<t} \bar c_s{}^\frown \bar d_s + A)$.

Remark 6.6. 1) In 6.5 we cannot use I_2 a saturated model of $\mathrm{Th}(\mathbb Q, <)$ as then some $b \in M_{\mathbf x}$ may induce a cut with both cofinalities $> |T|$.
2) In 6.5 we can replace $\mathbb Z$ by any linear order with at least two elements but $< \lambda$.
3) Note that if $\mathbf m \in \mathrm{tK}^{\oplus}_{\kappa,\bar\mu,\theta} \subseteq \mathrm{vK}^{\oplus}_{\kappa,\bar\mu,\theta}$, then also $(\mathbf m, \mathbf w) \in \mathrm{vK}^{\otimes}_{\kappa,\bar\mu,\theta}$ for $\mathbf w$ the "identity" on $\Gamma^2_{\mathbf x_{\mathbf m}}$, see Definition 3.6(4C).

Proof. Let $\langle \bar a_\alpha : \alpha < \kappa\rangle$ list the finite sequences of $M_{\mathbf x}$ each appearing stationarily many times.

Let $\langle t_\alpha : \alpha < \kappa\rangle$ list the elements of I_1 without repetitions and for technical reasons $\langle t_n : n < \omega\rangle$ is increasing.

Now we choose $J_{1,\alpha}, J_{2,\alpha}, \langle \bar c_s{}^\frown \bar d_s : s \in J_{2,\alpha}\rangle$ by induction on $\alpha < \lambda$ such that

- (a) $J_{1,\alpha}$ is a subset of I_1 of cardinality $< \lambda, \subseteq$-increasing continuous
- (b) $J_{2,\alpha} \subseteq J_{1,\alpha} \times \mathbb Z$ ordered lexicographically and contains $J_{1,\alpha} \times \{0\}$
- (c) $\{t_\beta : \beta < \alpha\} \subseteq J_{1,\alpha}$ and $\{t_\beta : \beta < \alpha\} \times \mathbb Z \subseteq J_{2,\alpha}$
- (d) $\mathbf I_\alpha = \langle \bar c_t{}^\frown \bar d_t : t \in J_{2,\alpha}\rangle$ is indiscernible for $\alpha \geq \omega$
- (e) $(\bar c_t, \bar d_t)$ solves $(\mathbf x, \bar\psi, \cup\{\bar c_s{}^\frown \bar d_s : s <_{J_{2,\alpha}} t\}$ for each $t \in J_{2,\alpha}$
- (f) if $\alpha < \omega$ or $\alpha = 1 \bmod 3$ let $\beta(\alpha) < \lambda$ be minimal such that $t_{\beta(\alpha)}$ is $<_I$-above $J_{1,\alpha}$ then $\bar c_{(t_{\beta(\alpha)},0)}{}^\frown \bar d_{t_{(\beta(\alpha)},0)}$ solves $(\mathbf x, \bar\psi, r, A_\alpha)$ where $A_\alpha := \cup\{\bar c_t{}^\frown \bar d_t : t \in J_{2,\alpha} \cup \{(t_{\beta(\alpha)}, m) : m < n\}\}$ and $J_{1,\alpha+1} = J_{1,\alpha} \cup \{t_{\beta(\alpha)}\}, J_{2,\alpha+1} = J_{2,\alpha} \cup \{(t_{\beta(\alpha)}, n) : n \in \mathbb N\}$

(g) if $\alpha = 2 \mod 3, \alpha \ge \omega$ and let $J_{1,\alpha+1} = J_{1,\alpha} \cup \{t_\gamma : \beta \le \alpha, t_\beta < t_{\beta(\alpha)}\}, J_{2,\alpha+1} = J_{1,\alpha+1} \times \mathbb{Z}$ and choose $\bar{a}_t$ for $t \in J_{2,\alpha+1} \backslash J_{2,\alpha}$ (such that (d) + (e) holds)

(h) if $\alpha = 3\beta \ge \omega$, then we choose $J_{1,\alpha+1}, J_{2,\alpha+1}, \langle(\bar{c}_s, \bar{d}_s) : s \in J_{2,\alpha+1} \backslash J_{2,\alpha}\rangle$ such that[58], if possible, for some finite $I \subseteq I_1 \backslash J_{1,\alpha}$ we have $J_{1,\alpha+1} = J_{1,\alpha} \cup I, J_{2,\alpha} = J_{1,\alpha} \times \mathbb{Z}$ and defining $I^\alpha \in K_p$ as $(I \times \mathbb{Z}, P_s)_{s \in I}, P_s = \{s\} \times \mathbb{Z}$, the sequence $\langle \bar{c}_s \hat{\ } \bar{d}_s : s \in I^\alpha\rangle$ is not indiscernible over $\bar{a}_\beta$.

It is easy to carry the induction.

The main point is to verify clause (c) hence (c)$'$. By [She04, 3.4] or see §(1C), if $\bar{a} \in {}^n(M_{\mathbf{x}})$ and $\varphi = \varphi(\bar{x}_{\bar{d}[\mathbf{x}]}, \bar{x}_{\bar{c}[\mathbf{x}]}, \bar{z}_{[n]})$ <u>then</u> there is an expansion of I_2 to $I_2^+ = (I_2^+, P_0, \ldots, P_n)$ each P_ℓ a (non-empty) convex subset of I_2 such that $\langle \bar{c}_s \hat{\ } d_s : s \in I_2\rangle$ is $\{\varphi\}$-indiscernible over $\bar{a}$.

Without loss of generality if $t \in I_1, \ell \le n$ and $(\{t\} \times \mathbb{Z}) \cap P_\ell \ne \emptyset \wedge (\{t\} \times \mathbb{Z}) \backslash P_\ell \ne \emptyset$ <u>then</u> $P_\ell \subseteq \{t\} \times \mathbb{Z}$ and let $I^1_{\bar{a}}$ be the set of such t's. Let $\alpha < \kappa$ be such that $\ell \le n \Rightarrow P_\ell \cap J_{2,\alpha} \ne \emptyset$, without loss of generality $J_{2,\alpha} = J_{1,\alpha} \times \mathbb{Z}, \alpha = \omega\alpha$.

By clause (h) of the construction we get that $\mathrm{tp}_\varphi(\bar{a}, \{\bar{c}_s \hat{\ } \bar{d}_s : s \in I^1_{\bar{a}} \times \mathbb{Z}\}) \vdash \mathrm{tp}_\varphi(\bar{a}, \{\bar{c}_s \hat{\ } \bar{d}_s : s \in I_2\})$, treating $\bar{c}_s \hat{\ } \bar{d}_s$ are singletons, of course.

As this holds for any such φ we are done. $\square_{6.5}$

Observation 6.7. *1) If $\mathbf{I} = \langle \bar{c}_s \hat{\ } \bar{d}_s : s \in I\rangle$ materializes $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ <u>then</u> we can replace $\mathbf{I}$ by $\mathbf{I} \restriction J$ for any $J \subseteq I$ cofinal in I and $\mathrm{cf}(I) \ge \kappa$.*
2) If $\|M_{\mathbf{x}}\| = \kappa$ then $\mathrm{cf}(I) = |I|$ is necessarily κ.

Remark 6.8. Recall that if T is stable (or just $\mathbf{I}$ is an indiscernible set), necesssarily we get that $\bar{d}_s$ is algebraic over $\bar{c}_s$.

Proof. Straightforward. $\square_{6.7}$

Claim 6.9. *1) If $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r) \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ or $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r, \mathbf{u}) \in \mathrm{vK}^{\otimes}_{\kappa,\mu,\theta}$, <u>then</u> any two materializations $\mathbf{I}_1, \mathbf{I}_2$ of $\mathbf{m}$ are equivalent, see Definition 1.36(5).*
2) If $\mathbf{x} \in \mathrm{uK}^{\otimes}_{\kappa,\bar{\mu},\theta}$ or $\mathbf{x} \in \mathrm{vK}^{\otimes}_{\kappa,\bar{\mu},\theta}$ and $M_{\mathbf{x}}$ has cardinality κ, the number of materializations of $\mathbf{x}$ up to equivalence is $\le 2^\theta$.
3) If $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$ <u>then</u> there is one and only one $D = D_{\mathbf{m}}$, the ultrafilter of $\mathbf{m}$, see 6.4(3).

Proof. 1) Suppose $\mathbf{I}_\ell = \langle \bar{c}_{\ell,s} \hat{\ } \bar{d}_{\ell,s} : s \in I_\ell\rangle$ is a materialization of $\mathbf{m}$ and $\bar{c}^\ell_n \hat{\ } \bar{d}^\ell_n$ be as in Definition 6.4, or see 6.3, for $\ell = 1, 2$. We can replace I_ℓ by any cofinal sequence hence without loss of generality $\mathrm{otp}(I_\ell) = \kappa_\ell = \mathrm{cf}(\kappa_\ell)$, so by 6.7 $\kappa_\ell \ge \kappa > |T|$. Without loss of generality $\kappa_1 \le \kappa_2$, now we let $I_\ell = \{t_{\ell,\varepsilon} : \varepsilon < \kappa_\ell\}$ with $t_{\alpha,\varepsilon}$ being $<_{I_\ell}$-increasing with ε.

First assume $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\mu,\theta}$, so for every $\alpha < \kappa_1$ for some $h_1(\alpha) < \kappa_2$ we have:

$(*)_\alpha$ $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{d}_{2,t_2,\beta}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{d}_{2,t_2,\beta} + \{\bar{c}_{1,t_1,\varepsilon} \hat{\ } \bar{d}_{1,t_1,\varepsilon} : \varepsilon < \alpha\})$ if $\beta \in [h_1(\alpha), \kappa_2)$.

<u>Case 1</u>: $\kappa_1 < \kappa_2$

Then $\beta(*) = \sup\{h_1(\alpha) : \alpha < \kappa_1\}$ is $< \kappa_2$, so applying $(*)_\alpha$ for every $\alpha < \kappa_1$, for $\beta = \beta(*)$ we get that $\bar{d}_{2,t_2,\beta(*)+1} \hat{\ } \bar{c}_{2,t_2,\beta(*)+1}$ realizes $\mathrm{tp}(\bar{d}_{\mathbf{x}} \hat{\ } \bar{c}_{\mathbf{x}} \cup \{\bar{c}_{1,t_1,\beta} \hat{\ } \bar{d}_{1,t_1,\beta} : \beta < \alpha\}$ which is realized in $M_{\mathbf{x}}$ so we get contradiction.

[58] $J_{2,\alpha+1}$ has an infinite end segment included in $J_{2,\alpha}$

Case 2: $\kappa_1 = \kappa_2$

So $h_2 : \kappa_2 \to \kappa_1$ can be defined similarly and let $E = \{\delta < \kappa_1 : \delta$ a limit ordinal such that $\alpha < \delta \Rightarrow h_1(\alpha) < \delta \wedge h_2(\alpha) < \delta\}$, it is a club of κ_1.

Now for any $h : E \to \{1, 2\}$ the sequence $\mathbf{I}_h = \langle \bar{a}_{t_{h(\alpha),\alpha}} : \alpha \in E\rangle$ is an indiscernible sequence by Claim 3.14 or 3.16.

So all the $\mathbf{I}_h$'s and $\mathbf{I}_1, \mathbf{I}_2$ are equivalent. Second, assume $(\mathbf{m}, \mathbf{u}) \in \mathrm{vK}^{\otimes}_{\kappa,\bar{\mu},\theta}$, easy too. The case of vK is similar.

2) For $\mathbf{x} \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ the number of pairs $(\bar{\psi}, r)$ such that $(\mathbf{x}, \bar{\psi}, r) \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ is $\le 2^\theta$, and now apply part (1). Similarly, if $\mathbf{x} \in \mathrm{vK}_{\kappa,\bar{\mu},\theta}$ then the number of triples $(\bar{\psi}, r, \mathbf{u})$ such that $(\mathbf{x}, \bar{\psi}, r, \mathbf{u}) \in \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\kappa}$.

3) E.g. force by Levy$(\kappa, \|M_{\mathbf{x}}\|)$ and use absoluteness. $\square_{6.9}$

Definition 6.10. Assume $M \in \mathrm{EC}_{\kappa,\kappa}(T)$ and $p \in \mathbf{S}^\sigma(M)$.

Let

(a) $\mathbb{I}_p = \{\mathbf{I} : \mathbf{I}$ is an (endless) indiscernible sequence in M with $\mathrm{Av}(\mathbf{I}, M) = p\}$

(b) $\mathbb{I}^\chi_p = \{\mathbf{I} \in \mathbb{I}_p : \mathbf{I}$ has length $\chi\}$

(c) $\mathbb{I}^*_p = \mathbb{I}^\kappa_p$.

Definition 6.11. Assume $M \prec \mathfrak{C}, \mathbf{m} = (\mathbf{x}, \bar{\psi}, r) \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ or $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r, \mathbf{u}) \in \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ and $\gamma < \ell g(\bar{d}_{\mathbf{x}})$ and $p = p(\bar{x}) \in \mathbf{S}^\gamma(M_{\mathbf{x}})$. We say that $\mathbf{I}$ materializes the quadruple $(p, \mathbf{x}, \bar{\psi}, r)$ or $(p, \mathbf{x}, \bar{\psi}, r, \mathbf{w})$ or in $(p, \mathbf{m})$ when:

(a) $\mathbf{I} = \langle \bar{b}_s{}^\frown\bar{c}_s{}^\frown\bar{d}_s : s \in I\rangle$ is an indiscernible sequence in $M_{\mathbf{x}}$

(b) $\langle \bar{c}_s{}^\frown\bar{d}_s : s \in I\rangle$ materialize $\mathbf{m} = (\mathbf{x}, \bar{\psi}, r)$

(c) $\langle \bar{b}_s : s \in I\rangle \in \mathbb{I}_p$

(d) Case 1: $\mathbf{m} \in \mathrm{tK}^{\oplus}_{\kappa,\bar{\mu},\theta}$ for every finite $A \subseteq M_{\mathbf{x}}$ for every large enough $s \in I$ we have $\mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} + \bar{d}_s) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}} \dotplus (A + \bar{b}_s + \bar{c}_s + \bar{d}_s))$ according to $\bar{\psi}$

Case 2: $\mathbf{m} \in \mathrm{vK}^{\oplus}_{\kappa,\bar{\mu},\theta}$: for every finite $A \subseteq M_{\mathbf{x}}$ for every large enough $s \in I$, the pair (c_s, d_s) solves $(\mathbf{m}, A + \bar{b}_s)$.

Claim 6.12. *Assume $M, \mathbf{m}, \gamma, p$ are as in Definition 6.11 and $\|M_{\mathbf{x}}\| = \kappa$.*

If $\mathbf{I}_* = \langle \bar{b}^*_\alpha : \alpha < \kappa\rangle \in \mathbb{I}^*_p$, then there is $\mathbf{I} = \langle \bar{b}_\alpha{}^\frown\bar{c}_\alpha{}^\frown\bar{d}_\alpha : \alpha < \kappa\rangle$ which materialize $(p, \mathbf{x}, \bar{\psi}, r)$ such that the sequences $\mathbf{I}_*$ and $\langle \bar{b}_\alpha : \alpha < \kappa\rangle$ are equivalent (even are equal on a stationary set of indices).

Proof. Let $\langle a_\alpha : \alpha < \kappa\rangle$ list the members of $M_{\mathbf{x}}$. Now repeat the proof of 6.3 before choosing $(\bar{c}_\alpha, \bar{d}_\alpha)$ in stage α, choose minimal $\gamma(\alpha) < \lambda$ such that $\bar{b}_{\gamma(\alpha)}$ realizes $\mathrm{Av}(\mathbf{I}_*, A'_\alpha)$ where $A'_\alpha := \cup\{\langle a_\beta\rangle{}^\frown\bar{b}_{\gamma(\beta)}{}^\frown\bar{c}_\beta{}^\frown\bar{d}_\beta : \beta < \alpha\}$ and choose $(\bar{c}_\alpha, \bar{d}_\alpha)$ as a solution of $(\mathbf{x}, \bar{\psi}, r)$ over $A'_\alpha + \bar{b}_{\gamma(\alpha)}$.

As $\langle \bar{b}_\alpha : \alpha < \kappa\rangle, \langle \bar{c}_\alpha{}^\frown\bar{d}_\alpha : \alpha < \kappa\rangle$ are indiscernible sets, for some type r_α we have $(\forall^\kappa \beta < \kappa)(\forall^\kappa \gamma < \beta)[\mathrm{tp}(\bar{b}_\beta{}^\frown\bar{c}_\gamma{}^\frown\bar{d}_\gamma, \bar{c}_{\mathbf{x}} + \bar{d}_{\mathbf{x}} + A'_\alpha) = r_\alpha]$, and clearly $\bar{b}_{\gamma(\alpha)}{}^\frown\bar{c}_\alpha{}^\frown\bar{d}_\alpha$ realizes the type and r_α increases with α.

So again by 3.14 or 3.16, the sequence $\langle (\bar{b}_{\gamma(\alpha)}, \bar{c}_\alpha, \bar{d}_\alpha) : \alpha < \lambda\rangle$ is indiscernible and also the rest should be clear. $\square_{6.12}$

Discussion 6.13. Recall 1.21(4). If we replace the type by its ω-th iteration, see [She80], i.e. if $\langle \bar{d}_n : n < \omega\rangle$ is an indiscernible sequence witnessing $D \in \mathrm{uf}(\mathrm{tp}(\bar{a}, M))$ then $\mathrm{tp}(a, \bar{d}_0{}^\frown\bar{d}_1{}^\frown \ldots {}^\frown D)$ determine D.

Definition 6.14. 1) For $M \prec \mathfrak{C}_T$, ultrafilter D on ${}^\zeta M$ and $\mathbf{I}_D = \langle \bar{a}_n : n < \omega \rangle$ based on D (see Definition 1.19(6)), let $\mathbf{T}_D$ be the set of sequences $\langle (A_s, \bar{a}_s, \Delta_s) : s \in u \rangle$ such that:

- (a) u is an inverted tree with root being maximal
- (b) $A_s \subseteq M$ finite, increases with $s \in u$
- (c) $\bar{a}_s \in {}^\zeta M$
- (d) finite $\Delta_s \subseteq \Gamma_\zeta$ which is $\subseteq$-increasing with $s \in u$ recalling $\Gamma_\zeta = \{\varphi(\bar{x}_0, \ldots, \bar{x}_n; \bar{y}) : \bar{x}_\ell = \langle x_{\ell,\varepsilon} : \varepsilon < \zeta \rangle$ and $y = \langle y_\ell : \ell < n \rangle$ for some $n\}$
- (e) $\langle \bar{a}_t \rangle\hat{\ }\mathbf{I}_D$ is Δ_t-indiscernible over $\cup\{A_s \cup (\bar{a}_s {\restriction} w) : s <_u t$ and $w \subseteq \gamma$ is the finite set of places not dummy in $\Delta_s\}$.

2) For $\mathfrak{n} \in \mathbf{T}_D$ let $\mathfrak{n} = \langle A_{\mathfrak{n},s}, \bar{a}_{\mathfrak{n},s}, \Delta_{\mathfrak{n},n}) : s \in u_{\mathfrak{n}} \rangle, u[\mathfrak{n}] = u_{\mathfrak{n}}$ and let $\max(\mathfrak{n})$ be the $\leq_u$-maximal member (= root) of $u_{\mathfrak{n}}$ and $(A_{\mathfrak{n}}, \bar{a}_{\mathfrak{n}}, \Delta_{\mathfrak{n}}) = (A_{\mathfrak{n},\max(\mathfrak{n})}, \bar{a}_{\mathfrak{n},\max(\mathfrak{n})}, \Delta_{\mathfrak{n},\max(\mathfrak{n})})$. Lastly, $\bar{a}_{\mathfrak{n}}$ is $\bar{a}_{\mathfrak{n},\max(\mathfrak{n})}$.
4) If $\mathfrak{n} \in \mathbf{T}_D$ and $s \in u_{\mathfrak{n}}$ let $u_{\mathfrak{n}} {\restriction} (\leq s)$ be $u {\restriction} \{s_1 \in u_{\mathfrak{n}} : s_1 \leq_{u_{\mathfrak{n}}} s_1\}$ as a partial order and let $\mathfrak{n} {\restriction} (\leq s)$ be $\langle (A_{\mathfrak{n},s_1}, \bar{a}_{\mathfrak{n},s_1}, \Delta_{\mathfrak{n},s_1}) : s_1 \in u {\restriction} (\leq s) \rangle$.
5) We say $\mathbf{w} = \langle \mathfrak{n}_t : t \in I \rangle$ is a witness for D <u>when</u>:

- (a) I is a directed partial order
- (b) $\mathbf{n}_t \in \mathbf{T}_D$ for every $t \in I$
- (c) if $t_1 <_I t_2$ then for some $s \in u[\mathfrak{n}_{t_2}]$ we have $\mathfrak{n}_{t_1} = \mathfrak{n}_{t_2} {\restriction} (\leq s)$.

6) In part (5) let $(A_t, \bar{a}_t, \Delta_t) = (A_{\mathfrak{x},t}, \bar{a}_{\mathfrak{x},t}, \Delta_{\mathfrak{x},t})$ denote $(A_{\mathfrak{n}_t,\max(\mathfrak{n}_t)}, \bar{a}_{\mathfrak{n}_t,\max(\mathfrak{n}_t)}, \Delta_{\mathfrak{n}_t,\max(\mathfrak{n}_t)})$.

Claim 6.15. *1) If D is an ultrafilter on ${}^\zeta M$ <u>then</u> there is a witness $\mathfrak{x} = \langle \mathfrak{n}_t : t \in I \rangle$ for D, let $\mathfrak{n}[t] = \mathfrak{n}_t$.*
2) If $\Delta \subseteq \Gamma_\zeta(\tau_T)$ is finite and $A \subseteq M$ is finite <u>then</u> for some $t_0 \in I$, if $k < \omega$ and $t_0 <_I \ldots < t_k$ then $\langle \bar{a}_{\mathfrak{n}[t_\ell]} : \ell \leq k \rangle$ is Δ-indiscernible over A.

Proof. See [She04, §1] or an exercise. $\square_{6.15}$

§ 6(B). Indiscernible existence from bounded directionality.

We affirm here the conjecture from §(1C) for the case $k = 1$, for dependent theory T of bounded directionality. We state the more informative version (see Defintion 1.46(1)).

Claim 6.16. <u>*The Strong Indiscernible Existence Theorem*</u> *1) Let T be of finite directionality, see Definition 1.23. Assume $\kappa = \mathrm{cf}(\kappa) > \theta = |\gamma| + |T|, \bar{d}_\alpha \in {}^\gamma\mathfrak{C}$ for $\alpha \in \kappa$ and $\langle \bar{d}_\alpha : \alpha < \kappa \rangle$ is a type-increasing sequence, see 6.18(0) below, <u>then</u> for some $I \in K_{q,\theta}$ expanding $(\kappa, <)$ the sequence $\langle \bar{d}_s : s \in I \rangle$ is mod club locally-indiscernible, see Definition 1.46.*
2) Let T be of bounded directionality. <u>Then</u> we get a similar result for $I \in K_{\mathrm{reg},\theta}$, see Definition 6.17 below.

Proof. 1) By 6.20 + 6.21(2) below.
2) By 6.20 + 6.25 below. $\square_{??}$

Definition 6.17. 1) $K_{\mathrm{reg},\zeta}$ is the class of structures $I = (I, <_I, P^I_i, F^I_j)_{i<\theta, j<\theta}$ such that $(I,<)$ is a linear order, P^I_i a unary relation and F^I_j is a unary function such that $F^I_j(t) \le_I t$; reg stands for regressive.
2) Assume κ is regular uncountable and $I \in K_{\mathrm{reg},\theta}$ expand $(\kappa,<)$. We say the sequence $\langle \bar d_\alpha : \alpha \in I\rangle$ is mod club locally indiscernible <u>when</u> ($\bar d_s \in {}^\zeta\mathfrak{C}, \zeta < \theta$ and) for some club E of κ, for every $\eta \in {}^\theta 2, \nu \in {}^\theta(\kappa+1)$ and finite $\Delta \subseteq \mathbb{L}(\tau_T)$, we have: if $S_{\eta,\nu} = \{\alpha \in E : I \models P_i(\alpha)^{[\eta(i)]}$ for every $i \in u$ and $F^I_j(\alpha) = \nu(j) \vee (F^I_j(\alpha) = \alpha \wedge \nu(j) = \kappa)$ for every $j \in v\}$ is unbounded in κ then $\langle \bar d_\alpha : \alpha \in S_{\eta,\nu}\rangle$ is Δ-indiscernible.

Definition 6.18. 0) We say that $\langle \bar d_\alpha : \alpha < \beta\rangle$ is type-increasing over B <u>when</u> $\mathrm{tp}(\bar d_\beta, \cup\{\bar d_\alpha : \alpha < \beta\} \cup B)$ is $\subseteq$-increasing with α; if $B = \emptyset$ we may omit it.
1) Let $\mathrm{aK}_{\lambda,\kappa,\theta}$ be the class of $\mathbf{x}$ consisting of

- (a) $\bar M = \langle M_\alpha : \alpha \le \kappa\rangle$, which is $\prec$-increasing, $\alpha < \kappa \Rightarrow \|M_\alpha\| < \lambda$
- (b) $\mathbf{I} = \langle \bar d_\alpha : \alpha \le \kappa\rangle$ and $\bar d = \bar d_\kappa$ is of length $< \theta^+$
- (c) $\bar d_\alpha \in {}^{\ell g(\bar d)}(M_\kappa)$ realizes $\mathrm{tp}(\bar d, M_\alpha)$
- (d) $M_\kappa = \cup\{M_\alpha : \alpha < \kappa\}$
- (e) $\bar d_{\mathbf{x}} = \bar d_\kappa$.

2) Let $\mathrm{eK}_{\lambda,\kappa,\theta}$, be the class of $\mathbf{x}$ consisting of

- (a) $\bar M$ as above
- (b) $\bar{\mathbf{I}} = \langle \mathbf{I}_\alpha : \alpha < \kappa\rangle$ and $\bar d = \bar d_{\mathbf{x}}$
- (c) each $\bar d' \in \mathbf{I}_\alpha$ belongs to M_κ and realizes $\mathrm{tp}(\bar d, M_\alpha)$
- (d) $M_\kappa = \cup\{M_\alpha : \alpha < \kappa\}$.

3) Let $\mathrm{aK}_{\kappa,\theta} = \mathrm{aK}_{\kappa,\kappa,\theta}$ and $\mathrm{eK}_{\kappa,\theta} = \mathrm{eK}_{\kappa,\kappa,\theta}$.

Observation 6.19. *If $\mathbf{x} \in \mathrm{aK}_{\lambda,\kappa,\theta}$ <u>then</u> for a unique $\mathbf{y} \in \mathrm{eK}_{\lambda,\kappa,\theta}$ we have $M_{\mathbf{y},\alpha} = M_{\mathbf{x},\alpha}$ for $\alpha \le \kappa$ and $\mathbf{I}_{\mathbf{y},\alpha} = \{\bar a_{\mathbf{x},\alpha}\}$.*

Claim 6.20. *If $\langle \bar d_\alpha : \alpha \le \kappa\rangle$ is type increasing and $\kappa = \mathrm{cf}(\kappa) > |T|$, <u>then</u> there is $\mathbf{x} \in \mathrm{aK}_{\kappa,\theta}$ such that for a club of $\alpha < \kappa$ we have $\bar d_{\mathbf{x},\alpha} = \bar d_\alpha$.*

Proof. Let $C_\alpha = \cup\{\bar d_\beta : \beta < \alpha\}$ for $\alpha \le \kappa+1$. We can find a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ such that

- (a) $\mathrm{tp}(a_\alpha, A_\alpha)$, where $A_\alpha := C_\kappa \cup \{a_\beta : \beta < \alpha\})$, does not split over some $B_\alpha \subseteq A_\alpha$ of cardinality $\le |T|$;
- (b) every finite type over $A_\alpha, \alpha < \kappa$ is realized by some $a_\beta, \beta < \kappa$.

This is possible by [She90, III,7.5,pg.140] or see [She04, 4.24=np4.10]. So A_κ is well defined and is the universe of a model $M \prec \mathfrak{C}$.

As κ is regular, for every α for some $\beta_\alpha < \kappa$ we have: the type $\mathrm{tp}(\bar d_\beta, A'_\alpha)$ where $A'_\beta = \cup\{\bar d_\gamma{}^\frown\langle a_\gamma\rangle : \gamma < \beta\})$ is the same for all $\beta \in [\beta_\alpha, \kappa)$, just consider the definition of non-splitting.

Hence without loss of generality this holds for $\beta \in [\beta_\alpha, \kappa+1)$, too. Clearly $\mathscr{U} := \{\delta < \kappa : A'_\delta$ is universe of an elementary submodel of $M\}$ is a club of κ.

Define $\mathbf{x}$ by letting $M_{\mathbf{x},\kappa} = M$ and for $\alpha < \kappa$ letting $M_{\mathbf{x},\alpha} = M{\restriction}A'_{\min}(\mathscr{U}\backslash\alpha)$ and $\bar d_\alpha = \bar d_{\min(\mathscr{U}\backslash\alpha)}$. Clearly $\mathbf{x}$ is as required. $\square_{6.20}$

Claim 6.21. *Assume T is of finite directionality. Assume $\mathbf{x} \in \mathrm{aK}_{<\lambda,\kappa,\theta}, \kappa = \mathrm{cf}(\kappa) > \theta \geq |T|$ and $\zeta = \ell g(\bar{d}_{\mathbf{x}})$.*
1) If $\kappa > 2^\theta$ <u>then</u> for some club $\mathscr{U}$ of κ and partition $\langle S_i : i < 2^\theta \rangle$ of $\mathscr{U}$, letting $I = (\mathscr{U}, <, S_i)_{i<2^\theta}$ the sequence $\langle \bar{d}_{\mathbf{x},\alpha} : \alpha \in I \rangle$ is an indiscernible sequence.
2) If $\Delta \subseteq \Gamma_\zeta := \{\varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}; \bar{y}) : \bar{x}_\ell = \langle x_{\ell,\varepsilon} : \varepsilon < \zeta \rangle$ and $\bar{y} = \langle y_\ell : \ell < n \rangle$ for some $n\}$ is finite <u>then</u> for some club $\mathscr{U}_\Delta$ of κ and finite partition $\langle P_{\Delta,\ell} : \ell < \ell_\Delta \rangle$ of $\mathscr{U}_\Delta$ we have: $\langle \bar{d}_{\mathbf{x},\alpha} : \alpha \in (\mathscr{U}_\Delta, <, P_{\Delta,\ell})_{\ell<\ell_\Delta} \rangle$ is Δ-indiscernible in the sense of 6.23 below.

Before we prove, similarly:

Claim 6.22. *Assume T is of finite directionality. As in 6.21 for $\mathrm{eK}_{\kappa,\theta}$.*

In full: assume $\mathbf{x} \in \mathrm{eK}_{<\lambda,\kappa,\theta}, \kappa = \mathrm{cf}(\kappa) > \theta \subseteq |T|, \zeta$ and finite $\Delta \subseteq \Gamma$ where Γ is as in 6.21(2), there are functions $F_n : \cup\{\mathbf{I}_\alpha : \alpha < \kappa\} \to \kappa$ for $n, n_\Delta(*), n_\Delta$ large enough (i.e. $\varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}; \bar{y}) \in \Delta \Rightarrow n < n(*)$ such that if $\varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}; \bar{y}) \in \Delta, m < n$ and $\kappa > \alpha_0^\iota > \ldots > \alpha_{n-1}^\iota \geq \gamma$ and for $\iota = 1,2$ and $\bar{d}_\ell^\iota \in I_{\alpha_\ell^\iota}$ for $\ell < m, \iota = 1,2$ and $k < m \Rightarrow \alpha_k^\iota > F_{m-k}(\alpha_{k(1)}^\iota, \ldots, \alpha_{m-1}^\iota)$ and $\bar{b} \in {}^\zeta(M_{\mathbf{x},\gamma}), \bar{d}^*$ and $\bar{d}_m^*, \ldots, \bar{d}_{n-1}^* \in {}^\zeta(M_{\mathbf{x},\gamma})$ <u>then</u>

$$\mathfrak{C} \models \varphi[\bar{d}_{\alpha_0^1}, \ldots, \bar{d}_{\alpha_{m-1}^1}, \bar{d}_m^*, \ldots, \bar{d}_{n-1}^*, \bar{b}]$$

<u>iff</u>

$$\mathfrak{C} \models \varphi[\bar{d}_{\alpha_0^2}, \ldots, \bar{d}_{\alpha_{m-1}^2}, \bar{d}_m^*, \ldots, \bar{d}_{n-1}^*, \bar{b}].$$

Definition 6.23. For Γ as in 6.21(2) and $\Delta \subseteq \Gamma$ we say $\langle \bar{d}_\alpha : \alpha < \alpha(*) \rangle$ is Δ-indiscernible over A <u>when</u> if $m \leq n, \alpha(*) > \alpha_0 > \ldots > \alpha_{m-1}$ and $\alpha(*) > \beta_0 > \ldots > \beta_{m-1}$ and $\bar{b} \in {}^{\ell g(\bar{y})}A$ and $\bar{d}_\ell^* \in {}^\zeta A$ for $\ell = m, \ldots, n-1$ then

$$\mathfrak{C} \models \varphi[\bar{d}'_{\alpha_0}, \ldots, \bar{d}'_{\alpha_{m-1}}, \bar{d}_m^*, \ldots, \bar{d}_{n-1}^*; \bar{b}]$$

<u>iff</u>

$$\mathfrak{C} \models \varphi[\bar{d}'_{\beta_0}, \ldots, \bar{d}'_{\beta_{m-1}}, \bar{d}_m^*, \ldots, \bar{d}_{n-1}^*].$$

Discussion 6.24. Even for singletons we cannot replace "finite" by one in 6.21, because even for $T = \mathrm{Th}(\mathbb{Q}, <)$, a cut has two cofinalities in general.

Claim 6.25. *Let T be of bounded directionality.*
1) Assume $\mathbf{x} \in \mathrm{aK}_{\kappa,\theta}$ and $\ell g(\bar{d}_\alpha) = \zeta$ and $\Delta \subseteq \Gamma_\zeta$ is finite. <u>Then</u> we can find a club E of κ and a regressive function f on E such that for every $\gamma < \kappa$ the sequence $\langle \bar{d}_\alpha : \alpha \in E$ and $f(\alpha) = \gamma \rangle$ is Δ-indiscernible or is empty.
2) Parallel for 6.22.

Proof. <u>Proof of 6.21</u> 1) It follows from part (2) as if $\mathscr{U}, \langle P_{\Delta,i} : i < \ell_\Delta, \Delta \subseteq \Gamma$ finite$\rangle$ is as gotten there, we let $E = \{(\alpha, \beta) : \alpha, \beta \in \mathscr{U}$ and $\alpha \in P_{\Delta,i} \Leftrightarrow \beta \in P_{\Delta,i}$ for every finite $\Delta \subseteq \mathbb{L}(\tau_T)$ and $\langle P_i : i < i(*) \leq 2^\theta \rangle$ list the E-equivalence classes then $(\mathscr{U}, <, P_i)_{i<i(*)}$ is as required.
2) We prove here also 6.12(2). We call $\Delta \subseteq \Gamma_\zeta$ cyclically closed <u>when</u>: if $\varphi(\bar{x}_0, \ldots, \bar{x}_{n-1}; \bar{y}) \in \Delta$ then some $\varphi'(\bar{x}_0, \ldots, \bar{x}_{n-1}; \bar{y}) \in \Delta$ is equivalent to $\varphi(\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_0; \bar{y})$. Clearly it suffices to deal only with cyclically closed Δ's.

Let $\mathbb{D}_\Delta = \{D \cap \mathrm{Def}^\zeta_\Delta(M_{\mathbf{x}}) : D \in \mathrm{uf}(\mathrm{tp}(\bar d, M_{\mathbf{x},\kappa})\}$ says $\langle D_{\Delta,i} : i < \lambda_\Delta\rangle$ list it. So by 1.23 and 1.24 we have:

Case 1: T of finite directionality. Then $\mathbb{D}_\Delta$ is finite so λ_D is finite.

Case 2: Not Case 1 but T of bounded directionality. Then $\lambda_D \le \kappa$.

Let $\bar d_\alpha = \bar d_{\mathbf{x},\alpha}, \bar d = d_{\mathbf{x},\kappa}$. Now fix $\zeta = \ell g(\bar d_{\mathbf{x}}), \Delta \subseteq \Gamma_\zeta, \Delta$ finite (cyclically closed). For each $i < \lambda_\Delta$ choose $D_i \in \mathrm{uf}(\mathrm{tp}(\bar d, M_{\mathbf{x},\kappa}))$ such that $D_i \cap \mathrm{Def}^\zeta_\Delta(M_{\mathbf{x}}) = D_{\Delta,i}$. Let γ_* be $\kappa + \omega$ if T is of finite dimensionality and $\kappa + \kappa$ otherwise.

Let $J = ([\kappa,\gamma_*), <, P^J_i)_{i<\lambda_\Delta} \in K_{p,\lambda_\Delta}$ be such that each P^J_i is unbounded. For $\gamma \in [\kappa,\gamma_*)$ let $\mathbf{i}(\gamma)$ be such that γ belongs to $P^J_{\mathbf{i}(\gamma)}$ and let $J_\gamma = J{\restriction}\{\gamma\}$. Let $J_{\alpha,i} = (\{\alpha\}, <, P^{J_{\alpha,i}}_j)_{j<\lambda_\Delta} \in K_{p,\lambda_\Delta}$ be such that $P^{J_{\alpha,i}}_i = \{\alpha\}$.

We can choose $\bar d_\gamma$ for $\gamma \in [\kappa,\gamma_*)$ so redefining $\bar d_{\mathbf{x}}$ such that $\mathrm{tp}(\bar d_\gamma, \cup\{\bar d_\gamma : \gamma \in [\gamma,\gamma_*)\} \cup M_{\mathbf{x},\kappa})$ is equal to $\mathrm{Av}(D_{\mathbf{i}(\beta)}, \cup\{\bar d_\beta : \beta \in (\gamma,\gamma_*)\} \cup M_{\mathbf{x},\kappa})$. How? For any finite $u \subseteq [\kappa,\gamma_*)$ we can use downward induction and now use general compactness.

For $i < \lambda_\Delta$, let $S_i = \{\alpha < \kappa$: the sequence $\langle \bar d_\varepsilon : \varepsilon \in J_{\alpha,i} + J\rangle$ is Δ-indiscernible over $M_{\mathbf{x},\alpha}$ and for simplicity $\alpha \notin \cup\{S_j : j < i\}\}$.

For $\alpha < \kappa$ let $\mathbf{i}(\alpha) = i \Leftrightarrow \alpha \in S_i$ and let $J_\alpha = J_{\alpha,\mathbf{i}(\alpha)}$, so $\mathbf{i}(\alpha)$ may be undefined. Now

$\boxplus_1$ if $i < \lambda_\Delta$ and $\gamma < \kappa$ then the sequence $\langle d_\alpha : \alpha \in S_i \backslash \gamma\rangle\char`^\langle \bar d_\gamma : \gamma \in [\kappa,\gamma_*)$ and $\mathbf{i}(\gamma) = i\rangle$ is Δ-indiscernible over $M_{\mathbf{x},\gamma}$.

Moreover

$\boxplus_2$ Assume $\gamma < \kappa$, let $\mathscr{W}_{\Delta,\gamma} = \bigcup\limits_{i<\lambda_\Delta} S_i \cup [\kappa,\gamma_*)\backslash\gamma, J_{\Delta,\gamma} = \Sigma\{J_\alpha : \alpha \in \mathscr{W}_{\Delta,\gamma}\} + J \in K_{p,\lambda_\Delta}$ let $J_\Delta = J_{\Delta,0}, I_{\Delta,\gamma} = \Sigma\{J_\alpha : \alpha \in \mathscr{W}_{\Delta,\gamma}\}$ and $I_\Delta = I_{\Delta,0}$. Then the sequence $\langle \bar d_\alpha : \alpha \in J_{\Delta,\gamma}\rangle$ is Δ-indiscernible over $M_{\mathbf{x},\gamma}$.

[Why? Without loss of generality consider only $\gamma \in \mathscr{W}_{\Delta,0}$. Let $\varphi = \varphi(\bar x_0,\dots,\bar x_{n-1};\bar y) \in \Delta$. We now prove by induction on m, the statement for m simultaneously for all $\gamma < \kappa$. That is

$(*)$ if $n \ge m$ and $\varphi(\bar x_0,\dots,\bar x_{n-1},\bar y) \in \Delta$ and $J_\Delta \models \alpha > \alpha^\iota_0 > \dots > \alpha^\iota_{m-1} \ge \gamma$ for $\iota = 1,2$ and $k < m \wedge i < \ell_\Delta \Rightarrow [\alpha^1_k \in P^{J_\Delta}_i \leftrightarrow \alpha^2_k \in P^{J_\Delta}_i]$ and $\bar b \in {}^{\ell g(\bar y)}(M_{\mathbf{x},\gamma})$ and $\bar d^*_m,\dots,\bar d^*_{n-1} \in {}^\zeta(M_{\mathbf{x},\gamma})$ then $\mathfrak{C} \models \varphi[\bar d_{\alpha_0},\dots,\bar d_{\alpha^1_{m-1}},\bar d^*_m,\dots,\bar d^*_{n-1},\bar b^*]$ iff $\mathfrak{C} \models \varphi[\bar d_{\alpha^2_0},\dots,\bar d_{\alpha^2_{m-1}},\bar d^*_m,\dots,\bar d^*_{n-1},\bar b]$.

We prove this by induction on $|\{\alpha^\iota_k : k < m$ and $\iota = 1,2\} \cap \kappa|$. If it is zero this should be clear by the use of ultrafilters. If not, let (ι,k) be such that $\alpha^\iota_k \notin \kappa$ and $\iota + 2k$ maximal.

Let $\beta \in [\kappa,\gamma_*)$ be such that $\mathbf{i}(\alpha^\iota_k) = \mathbf{i}(\beta)$. Easily $\langle \bar d_\beta : \beta \in J\rangle$ is indiscernible over $M_{\mathbf{x}}$ so without loss of generality $\{\alpha^{\iota(1)}_{k(1)} : \iota(1) \in \{1,2\}$ and $k(1) < m\} \cap J$ is[59] disjoint to $[\kappa,\beta+1)$. But now note that replacing α^ι_k by β does not change the truth value. So $\boxplus_2$ hence $\boxplus_1$ indeed holds.]

Clearly $\boxplus_1 + \boxplus_2$ are nice but will say nothing if, e.g. each S_i is empty.

$\boxplus_3$ the set $S := \kappa\backslash \cup \{S_i\backslash i : i < \lambda_\Delta\}$ is non-stationary.

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[59] would be easier if we choose J with no first member

[Why? Toward contradiction assume S is a stationary subset of κ. For each $\delta \in S$ and $i < \lambda_\Delta$ we know $\delta \notin S_i$, hence there are $n_{\delta,i} = n(\delta,i)$ and formula $\varphi_{\delta,i}(\bar{x}_0,\dots,\bar{x}_{n_{\delta,i}-1};\bar{y}_{\delta,i}) \in \Delta$ and $\bar{d}^*_{\delta,i,1},\dots,\bar{d}^*_{\delta,i,n-1} \in \cup\{\bar{d}_\gamma : \gamma \in [\kappa,\gamma_*)\}$ and $\bar{b}_{\delta,i} \in {}^{\ell g(\bar{y}_{\delta,i})}(M_{\mathbf{x},\delta})$ such that $\mathfrak{C} \models \neg\varphi_{\delta,i}[\bar{d}_\delta,\bar{d}^*_{\delta,i,1},\dots,\bar{d}^*_{\delta,i,n-1},\bar{b}_{\delta,i}]$ but $\varphi^*_{\delta,i}(\bar{x}_0,\bar{d}^*_{\delta,i,1},\dots,\bar{d}^*_{\delta,i,n-1};\bar{b}_{\delta,i}) \in \mathrm{Av}(D_i,\cup\{\bar{d}_\gamma : \gamma \in [\kappa,\delta_*)\} + M_{\mathbf{x},\delta})$. Only finitely many of the members of $\bar{d}^*_{\delta,i,\ell}$ matter say $\bar{d}^*_{\delta,i,\ell}{\restriction}v_{\delta,i,\ell}, v_{\delta,i,\ell}$ finite.

<u>Case 1</u>: λ_Δ is finite

Let $C_\delta = \cup\{\mathrm{Rang}(\bar{d}^*_{\delta,i,\ell}{\restriction}v_{\delta,i,\ell} : \ell < n_{\delta,i}$ and $i < \lambda_\Delta\} \cup \{\mathrm{Rang}(\bar{b}_{\delta,i,\ell})) : i < \ell_\Delta\}$ so it is finite.

Also $C_\delta \subseteq \cup\{\bar{d}_\gamma : \gamma \in [\kappa,\gamma_*)\} \cup M_{\mathbf{x},\delta}$ and $\cup\{\bar{d}_\gamma : \gamma \in [\kappa,\gamma_*)\}$ has cardinality $\le |T|$.

Hence by Fodor lemma for some $C_* \subseteq M_{\mathbf{x}}$ the set $S' = \{\delta \in S : C_\delta = C_*\}$ is a stationary subset of κ. The number of possibilities for $\langle(n_{\delta,i},\varphi_{\delta,i})\rangle{}^\frown\langle\bar{d}_{\delta,i,\ell}{\restriction}v_{\delta,i,\ell} : i < \lambda_\Delta, \ell < n\rangle$ is $\le |T|$ and the number of possible $\langle\bar{b}_{\delta,i} : i < \lambda_\Delta\rangle$ is finite so for some stationary $S'' \subseteq S'$ for every $\delta \in S''$ we have $n_{\delta,i} = n_{*,i}, \varphi_{\delta,i} = \varphi_{*,i,v_{\delta,i,\ell}} = v_{*,i,\ell}, \bar{d}_{\delta,i,\ell}{\restriction}v_{\delta,i,\ell} = \bar{d}_{*,i,\ell}, \bar{b}_{\delta,i} = \bar{b}_{*,i}$.

Let $\mathscr{D}$ be an ultrafilter on κ to which κ and every club of κ belong as well as S''.

Let $D' = \{\mathbf{I} \subseteq {}^\zeta(M_{\mathbf{x},\kappa})$: the set $\{\alpha < \kappa : \bar{d}_\alpha \in \mathbf{I}\}$ belongs to $\mathscr{D}\}$, clearly D' is an ultrafilter on ${}^\zeta(M_{\mathbf{x},\kappa})$. As $\mathrm{tp}(\bar{d},M_{\mathbf{x},\kappa}) = \cup\{\mathrm{tp}(\bar{d}_\alpha,M_{\mathbf{x},\alpha}) : \alpha < \kappa\}$ clearly $D' \in \mathrm{uf}(\mathrm{tp}(\bar{d}_{\mathbf{x}},M_{\mathbf{x},\kappa}))$ or pedantically $D' \cap \mathrm{Def}({}^\zeta(M_{\mathbf{x}})) \in \mathrm{uf}(\mathrm{tp}(\bar{d}_{\mathbf{x}},M_{\mathbf{x},\kappa}))$ hence we can find $i < \lambda_\Delta$ such that $\mathscr{D}_i \cap \mathrm{Def}_\Delta(M_{\mathbf{x},\kappa}) = \mathscr{D}' \cap \mathrm{Def}(M_{\mathbf{x},\kappa})$. But this is a contradiction to $\{\bar{d}_\delta : \delta \in S''\} \in \mathscr{D}$ and the choice of $\varphi_{\delta,i}(\bar{x}_0,\bar{d}^*_{*,i,1},\dots,\bar{d}^*_{*,i,n_{*,i}-1},\bar{b}_{*,i})$.

<u>Case 2</u>: λ_Δ is infinite.

For $\alpha < \kappa + 1$ let $M^+_{\mathbf{x},\alpha}$ be $(M_{\mathbf{x},\alpha})_{[\bar{d}]}$, also $E := \{\delta < \kappa : M^+_{\mathbf{x},\delta} \prec M^+_{\mathbf{x},\kappa}\}$ is a club of κ. For each $\delta \in E$ choose $D_\delta \in \mathrm{uf}(\mathrm{tp}(\bar{d}_{\mathbf{x}},M_{\mathbf{x},\delta}))$ and choose $\bar{d}_{\delta,n} \in {}^\zeta\mathfrak{C}$ for $n < \omega$ such that $\bar{d}_{\delta,n}$ realizes $\mathrm{Av}(D_\delta,\cup\{\bar{d}_{\delta,m} : m \in (n,\omega) + M_{\mathbf{x},\delta}\}$. As T has bounded directionality for each $\varphi = \varphi(\bar{x}_0,\dots,\bar{x}_{n(\varphi)-1};\bar{y}_\varphi) \in \mathbb{L}(\tau_T)$ and $\delta \in E$ there are formulas $\psi_\delta(\bar{y}_\varphi,\bar{z}_{\varphi,\delta}) \in \mathbb{L}(\tau_{M^+_{\mathbf{x},\kappa}})$ and $\bar{c}_{\varphi,\delta} \in {}^{\ell g(\bar{z}_{\varphi,\delta})}(M_{\mathbf{x},\delta})$ such that

$\odot$ for $\bar{b} \in {}^m(M_{\mathbf{x},\delta})$ we have: $\mathfrak{C} \models \varphi[\bar{d}_{\delta,0},\dots,\bar{d}_{\delta,n-1},\bar{b}]$ <u>iff</u> $M_{\mathbf{x},\delta} \models \psi_{\varphi,\delta}[\bar{b},\bar{c}_{\varphi,\delta}]$.

Note that if: $\varphi(\bar{x}_0,\dots,\bar{x}_{n_\varphi-1};\bar{y}) \in \Delta$ and $m < n_\varphi$ <u>then</u> $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{x}_m{}^\frown\dots{}^\frown\bar{x}_{n(\varphi)-1}{}^\frown\bar{y}) \in \Delta$.

For transparency without loss of generality τ_T is countable, $\zeta < \omega$ let $\langle\Delta_n : n < \omega\rangle$ be $\subseteq$-increasing with union Γ_ζ and $\Delta_0 = \Delta$ and each Δ_n finite. For induction on n, for some stationary $S'_n \subseteq S \cap E$ we have $\delta \in S_n \wedge \varphi \in \Delta_n \Rightarrow \psi_{\varphi_\delta}(\bar{y}_\varphi,\bar{z}_{\varphi,\delta}) = \psi_{\varphi,*}(\bar{y}_\varphi,\bar{z}_{\varphi,*})$ and $m < n \Rightarrow S_n \subseteq S_m$. Let $\mathscr{D}$ be a uniform ultrafilter on κ such that $n < \omega \Rightarrow S_n \in \mathscr{D}_*$, let $\langle\bar{d}_{*,n} : n < \omega\rangle$ realize $\mathrm{Av}(\mathscr{D}_*,\langle\langle\bar{d}_{\delta,n} : n < \omega\rangle : \delta \in S\rangle, M_{\mathbf{x}})$. Easily $\langle\bar{d}_{*,n} : n < \omega\rangle$ is indiscernible over $M_{\mathbf{x}}$, each $\bar{d}_{*,n}$ realizes $\mathrm{tp}(\bar{d}_{\mathbf{x}},M_{\mathbf{x}})$ and $\mathrm{tp}(\bar{d}_{*,n},\cup\{\bar{d}_{*,m}, m \in (n,\omega)\} + M_{\mathbf{x}})$ is finitely satisfiable in $M_{\mathbf{x}}$, so it is based on some $D \in \mathrm{uf}(\mathrm{tp}(\bar{d}_{\mathbf{x}},M_{\mathbf{x}}))$, so for some $i < \lambda_\Delta$ we have $D_i \cap \mathrm{Def}^\zeta_\Delta(M_{\mathbf{x}}) = D \cap \mathrm{Def}^\zeta_\Delta(M_{\mathbf{x}})$. We easily get a contradiction to the choice of S as disjoint to $(S_i \backslash (i+1))$ and S_1. $\square_{6.21}$

§ 7. Applications

§ 7(A). The generic pair conjecture/On uniqueness of (κ, σ)-limit models.

We now return to the (κ, σ)-limit model conjecture and generic pair conjecture for κ.

We shall not deal with the first, only represent it. The second, the generic pair conjecture was solved in [She15] for $\kappa > |T|$ measurable. Here we solve it for $\kappa = \kappa^{<\kappa} > |T| + \beth_\omega$, it is the case $\xi = 1$ in Definition 7.1.

Note that even under GCH the picture is somewhat cumbersome when: $\kappa = \chi^+ = 2^\chi > |T| + \beth_\omega$ and χ strong limit singular. It is natural to restrict ourselves to $S^{\kappa^+}_{\rm gd}$ (see [She79]). We may still like to deal with $|T| < \kappa < \beth_\omega$.

Presently, the proof is complete only for $\xi = 1$, i.e. the generic pair conjecture.

Now we rephrase the conjecture; the use of $2^\lambda = \lambda^+$ (in addition to $\lambda = \lambda^{<\lambda}$) is for transparency only as an equivalent version without it is absolute under forcing with Levy$(\lambda^+, 2^\lambda)$, see §1.

Definition 7.1. 1) We say that T satisfies the uniqueness of limit models above μ <u>when</u> for any μ-complete forcing notion $\mathbb{Q}$ in $\mathbf{V}^{\mathbb{Q}}$ and $\xi < \lambda$ we have $(A) \Rightarrow (B)_\xi$, see below. Omitting μ means $\mu = |T| + \beth_\omega$.
2) For regular $\lambda > |T|$ and ordinal $\xi < \lambda$ we say that T satisfies the (λ, ξ)-limit uniqueness <u>when</u> for every λ-complete forcing notion $\mathbb{Q}$ such that $\mathbf{V}^{\mathbb{Q}} \models$ "$\lambda = \lambda^{<\lambda} \wedge 2^\lambda = \lambda^+$" clause $(B)_\xi$ holds.
3) We can add above "for the trivial $\mathbb{Q}$" or other restrictions. Instead "for the trivial $\mathbb{Q}$" we may say "presently"
<u>where</u>

- (A) (a) $\lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+ \geq \mu$
  - (b) density for $\mathrm{vK}^{\otimes}_{\lambda,\lambda,\theta}$ holds for every $\theta < \lambda$, see §5
  - (c) $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is a $\prec$-increasing continuous chain of models of cardinality λ with union M, a saturated model of cardinality λ^+
- $(B)_\xi$ for some club $\mathscr{U}$ of λ^+, if $\langle \alpha_{\ell,\varepsilon} : \varepsilon \leq \xi \rangle$ is an increasing continuous sequence of ordinals from $\mathscr{U}$ for $\ell = 1, 2$ such that $[\varepsilon < \xi$ non-limit $\Rightarrow \alpha_{\ell,\varepsilon}$ of cofinality $\lambda]$ <u>then</u> there is an isomorphism π from $M_{\alpha_{1,\xi}}$ onto $M_{\alpha_{2,\xi}}$ mapping $M_{\alpha_{1,\varepsilon}}$ onto $M_{\alpha_{2,\varepsilon}}$ for every $\varepsilon \leq \xi$.

We now translate the relevant questions represented in §0 to this definition.

Observation 7.2. *Assume T is dependent.*
0) If $|T| < \lambda = \lambda^{<\lambda}$, <u>then</u> T has $(\lambda, 0)$-uniqueness (even for the trivial forcing).
1) Assume $|T| < \lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+$. <u>Then</u> T has $(\lambda, 1)$-uniqueness, for trivial forcing <u>iff</u> T satisfies the generic pair conjecture <u>iff</u> in $(B)_1$ of 7.1 above, if $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ are all from $\mathscr{U}$ and has cofinality λ then $(M_{\beta_1}, M_{\alpha_1}) \approx (M_{\beta_2}, M_{\alpha_2})$.
2) Assume $|T| < \lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+$ and $\sigma = \mathrm{cf}(\sigma) \in [\aleph_0, \lambda]$. Then T has uniqueness of (λ, σ)-model <u>iff</u> T has (λ, σ)-limit-uniqueness for the trivial forcing.

Theorem 7.3. *T satisfies the generic pair conjecture for λ __when__ $\lambda = \lambda^{<\lambda} > |T|^+ + \beth_\omega^+$.*

Remark 7.4. This is closed to the proof from [She15] as we could restrict ourselves to $\mathbf{x}$ with $u_{\mathbf{x}} = \emptyset$.

Proof. By older works, we can assume T is dependent. Without loss of generality $2^\lambda = \lambda^+$ by absoluteness, see [She14a].

So let $\langle M_\alpha : \alpha < \lambda^+\rangle$ be given, $M = \cup\{M_\alpha : \alpha < \lambda^+\}$. Let E be the set of limit $\delta < \lambda^+$ such that:

$\circledast_\delta$ (a) for every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ the model M_β is saturated

(b) if $\alpha < \beta < \delta, \zeta < \lambda, \{\bar b_1, \bar b_2\} \subseteq {}^\zeta(M_\beta)$ and there is an automorphism g of M such that $g(M_\alpha) = M_\alpha, g(\bar b_1) = (\bar b_2)$ __then__ there is such g mapping M_δ onto itself.

So

$(*)_0$ (a) E is a club of λ^+

(b) if $\alpha \in E$ has cofinality λ __then__ M_α is saturated.

[Why? As M is saturated and $\lambda = \lambda^{<\lambda}$.]

$(*)_1$ if $\alpha < \lambda^+$ and M_α is saturated and $\mathbf{m}_1, \mathbf{m}_2 \in \mathrm{vK}^{\otimes}_{\lambda,\lambda,<\lambda}$ satisfies $M_{\mathbf{m}_1} = M_\alpha = M_{\mathbf{m}_2}$ and $\mathbf{m}_1 \le_1 \mathbf{m}_2$ and $\bar c_{\mathbf{m}_1}{}^\frown \bar d_{\mathbf{m}_1}$ is from M __then__ there is an automorphism g of M over $B^+_{\mathbf{m}_2}$ mapping M_α onto itself such that $g(\bar c_{\mathbf{m}_1}{}^\frown \bar d_{\mathbf{m}_1}) = \bar c_{\mathbf{m}_1}{}^\frown \bar d_{\mathbf{m}_1}$.

[Why? See uniqueness of $M_{[\mathbf{x}]}$ in 3.10, see Definition 2.6(6).]

Fix $\alpha_1 < \beta_2, \alpha_2 < \beta_2$, all from E and of cofinality λ and we have to prove just that $(M_{\beta_1}, M_{\alpha_1}) \cong (M_{\beta_2}, M_{\alpha_2})$. Let AP be the set of triples $(\mathbf{m}_1, \mathbf{m}_2)$ satisfying:

$(*)_2$ (a) $\mathbf{m}_\ell \in \mathrm{vK}^{\otimes}_{\lambda,\lambda,<\lambda}$, and $r_{\mathbf{m}_\ell}$ is complete

(b) $M_{\mathbf{x}[\mathbf{m}_\ell]} = M_{\alpha_\ell}$

(c) $\bar c_{\mathbf{x}[\mathbf{m}_\ell]}{}^\frown \bar d_{\mathbf{x}[\mathbf{m}_\ell]} \subseteq M_{\beta_\ell}$

(d) g is an elementary mapping with domain $B^+_{\mathbf{x}[\mathbf{m}_\ell]} + \bar c_{\mathbf{x}[\mathbf{m}_2]} + \bar d_{\mathbf{x}[\mathbf{m}_1]}$

(e) g maps $\mathbf{m}_1$ onto $\mathbf{m}_2$.

Let the two place relation $\le_{\mathrm{AP}}$ on AP be

$(*)_3$ $(\mathbf{m}_1, \mathbf{m}_2, g) \le_{\mathrm{AP}} (\mathbf{n}_1, \mathbf{n}_2, h)$ __iff__ both triples are from AP, and $g \subseteq h$ and $\mathbf{m}_1 \le_1 \mathbf{n}_1, \mathbf{m}_2 \le_1 \mathbf{n}_2$.

Now

$(*)_4$ $\mathrm{AP} \ne \emptyset$.

[Why? Use $\mathbf{m}_\ell$ which is empty except $M_{\mathbf{m}_\ell} = M_{\alpha_\ell}$, see 3.7(3).]

$(*)_5$ if the sequence $\langle(\mathbf{m}_{1,\varepsilon}, \mathbf{m}_{2,\varepsilon}, g_\varepsilon) : \varepsilon < \zeta\rangle$ is $\le_{\mathrm{AP}}$-increasing and ζ is a limit ordinal $< \lambda$ __then__ this sequence has a $\le_{\mathrm{AP}}$-lub, its union $(\mathbf{m}_{1,\zeta}, \mathbf{m}_{2,\zeta}, g_\zeta)$, i.e.

(a) $\mathbf{x}_{\mathbf{m}_{\ell,\zeta}} = \cup\{\mathbf{x}_{\mathbf{m}_{\ell,\varepsilon}} : \varepsilon < \zeta\}$ for $\ell = 1, 2$

(b) similarly for $\bar{\psi}_{\mathbf{m}_{\ell,\varepsilon}}$

(c) similarly for $\mathrm{r}_{\mathbf{m}_{\ell,\varepsilon}}$

(d) $g_\zeta = \cup\{g_\varepsilon : \varepsilon < \zeta\}$.

[Why? See 3.26.]

The main point is

$(*)_6$ if $(\mathbf{m}_1, \mathbf{m}_2, g) \in \mathrm{AP}$ and $\ell \in \{1,2\}$ and $A \subseteq M_{\beta_\ell}$ has cardinality $< \lambda$ <u>then</u> for some $(\mathbf{n}_1, \mathbf{n}_2, h) \in \mathrm{AP}$ which is $\leq_{\mathrm{AP}}$-above $(\mathbf{m}_1, \mathbf{m}_2, g)$ we have $A \subseteq B_{\mathbf{n}_\ell} + \bar{c}_{\mathbf{n}_\ell} + \bar{d}_{\mathbf{n}_\ell}$.

[Why? By symmetry we can assume $\ell = 1$. Now trivially we can find $\mathbf{x} \in \mathrm{pK}_{\lambda,\lambda,<\lambda}$ such that $\mathbf{x}_{\mathbf{m}_1} \leq_1 \mathbf{x}$ and $A \subseteq \mathrm{Rang}(\bar{d}_{\mathbf{x}[\mathbf{m}_1]})$. By 5.20 there is $\mathbf{n}'_1 \in \mathrm{rK}^{\oplus}_{\lambda,\lambda,<\lambda}$ such that $\mathbf{m}_1 \leq_1 \mathbf{n}'_1$ and $\mathbf{x} \leq_1 \mathbf{x}_{\mathbf{n}'_1}$.

Let $C_\ell = \bar{d}_{\mathbf{x}[\bar{\mathbf{m}}_\ell]} + \bar{c}_{\mathbf{x}[\mathbf{m}_\ell]} + B^+_{\mathbf{x}[\mathbf{m}_\ell]}$. Now recall that by 3.10, the model $(M_{\alpha_1})_{[C_1]}$ is $(\lambda, \mathbf{D}_\ell)$-sequence homogeneous and moreover g induces a mapping from $\mathbf{D}_1$ onto $\mathbf{D}_2$, because g maps $\mathbf{m}_1$ to $\mathbf{m}_2$. So there is an isomorphism f from M_{α_1} onto M_{α_2} such that $f \cup g$ is an elementary mapping (of $\mathfrak{C}$), hence it can be extended to an automorphism f^+ of M. Now $(\mathbf{n}_1, f^+(\mathbf{n}_1), f^+{\restriction}(B_{\mathbf{n}_1} + \bar{c}_{\mathbf{n}_1} + \bar{d}_{\mathbf{n}_1}))$ is almost as required but $f(\bar{c}_{\mathbf{n}_1}{}^\frown \bar{d}_{\mathbf{n}_1})$ is $\subseteq M$ rather than $\subseteq M_{\beta_2}$. But $\beta_2 \in E$ hence the definition of E we can finish.]

Now by $(*)_4$+$(*)_5$+$(*)_6$ we can find a $\leq_{\mathrm{AP}}$-increasing sequence $\langle(\mathbf{m}_{1,\varepsilon}, \mathbf{m}_{2,\varepsilon}, g_\varepsilon) : \varepsilon < \lambda\rangle$ such that: for any $A_1 \subseteq M_{\beta_1}, A_2 \subseteq M_{\beta_2}$ of cardinality $< \lambda$ for some $\varepsilon < \lambda$ we have $A_\ell \subseteq B_{\mathbf{m}_{\ell,\varepsilon}} + \bar{c}_{\mathbf{m}_{\ell,\varepsilon}} + \bar{d}_{\mathbf{m}_{\ell,\varepsilon}}$ for $\ell = 1, 2$.

So $g_\lambda = \cup\{g_\varepsilon : \varepsilon < \lambda\}$ is an isomorphism as required. $\square_{7.3}$

Discussion 7.5. 1) So we know that $T_2 = \mathrm{Th}(M_{\alpha_0}, M_{\alpha_1})$ for every $\alpha_0 < \alpha_1$ of cofinality λ from $\mathscr{U}$, is a complete theory and does not depend in (α_0, α_1) and even on λ. But we may like to understand it better, see Kaplan-Shelah [KS14b].

2) Still $M_{[\alpha_0,\alpha_1]} = (M_{\alpha_0}, M_{\alpha_1})$ is close to being sequence-homogeneous. So this leads us to deal with dependent finite diagrams $\mathbf{D}$. Because if we like to deal with (λ, ζ)-uniqueness we have to look at $(M_{\alpha_0}, M_{\alpha_1})$ for any $\bar{a} \in {}^{\lambda>}(M_{\lambda^+})$.

* * *

§ 7(B). Criterion for saturativity.

Claim 7.6. *Assume $\sigma > \mu = (2^{|T|})^+ + \beth^+_\omega$.*
Then M is σ-saturated <u>iff</u>

(a) *M is μ-saturated*

(b) *if $\kappa \in [\mu, \sigma)$ and $\langle a_\alpha : \alpha < \kappa\rangle$ is an indiscernible sequence in M <u>then</u> for some $a \in M$ the sequence $\langle a_\alpha : \alpha < \kappa\rangle{}^\frown\langle a\rangle$ is indiscernible*

(c) *if $\kappa \in [\mu, \sigma)$ is regular, $\langle a_s : s \in I_1 + I_2\rangle$ is an indiscernible sequence in M where $I_1 \cong (\kappa, <), I_2 \cong (\alpha, >)$ for some $\alpha \leq \kappa + 1$ <u>then</u> for some $a \in M$ the sequence $\langle a_s : s \in I_1\rangle{}^\frown\langle a\rangle{}^\frown\langle a_t : t \in I_2\rangle$ is an indiscernible sequence.*

Proof. The "only if" implication is obvious. For the "if" direction assume (a),(b),(c) and we prove that M is κ^+-saturated by induction on $\kappa \in [\mu, \sigma)$; clearly this suffices. By clause (a) of the assumption the model M is μ-saturated. So by the induction hypothesis M is κ-saturated. Let $A_* \subseteq M$ be of cardinality $\kappa, p_* \in \mathbf{S}(A_*)$ and we should prove that p_* is realized in M. Let $\mathbf{x} \in \mathrm{pK}_{\kappa,\mu,\theta}, \theta = |T|, \bar{d}_{\mathbf{x}} = \langle d \rangle$ where d realizes $p_*, M_{\mathbf{x}} = M, v_{\mathbf{x}} = 0$.

Now

$\boxplus_1$ if $m = 1, \mathbf{I} = \langle \bar{a}_\alpha : \alpha < \kappa \rangle$ is an indiscernible sequence of m-tuples from M and $A \subseteq M$ have cardinality $\leq \kappa$ <u>then</u> the type $\mathrm{Av}(\mathbf{I}, A)$ is realized in M.

[Why? Choose $b_\alpha \in M$ for $\alpha < \kappa$ such that $A \subseteq \{b_\alpha : \alpha < \kappa\}$ and let $A_\alpha = \cup\{\bar{a}_\beta{}^\frown\langle b_\beta \rangle : \beta < \alpha\}$ for $\alpha \leq \kappa$. Let $\{\varphi_\varepsilon(x, \bar{c}_\varepsilon) : \varepsilon < \kappa\}$ list the type $q = \mathrm{Av}(\mathbf{I}, A_\kappa)$ and for $\bar{a} \in {}^m\mathfrak{C}$ define $\varepsilon(\bar{a})$ as $\min\{\varepsilon \leq \kappa$: if $\varepsilon < \kappa$ then $\mathfrak{C} \models \neg\varphi_\varepsilon[\bar{a}, \bar{c}_\varepsilon]\}$. We try to choose $\bar{a}'_\alpha, \varepsilon_\alpha$ by induction on $\alpha < \kappa$ such that

$(*)$ (a) $\bar{a}'_\alpha$ realizes $p_\alpha := \mathrm{Av}(\mathbf{I}, \cup\{\bar{a}'_\beta : \beta < \alpha\})$

(b) if α is even then $\varepsilon(\bar{a}')$ is minimal, i.e. $\varepsilon(\bar{a}') \leq \varepsilon(\bar{a}'')$ whenever $\bar{a}''_\alpha$ realizes $\mathrm{Av}(\mathbf{I}, \cup\{\bar{a}'_\beta : \beta < \alpha\})$

(c) if α is odd then $\mathfrak{C} \models \varphi[\bar{a}'_\alpha, \bar{c}_{\varepsilon(\bar{a}'_{\alpha-1})}]$.

We can choose $\bar{a}'_\alpha$ satisfying clause (a) as p_α is an m-type in M of cardinality $< \kappa$ and M and is κ-saturated.

If α is even it follows trivially that we can satisfy clause (b), too. If α is odd, and $\varepsilon(\bar{a}'_{\alpha-1}) = \kappa$ then $\bar{a}'_{\alpha-1}$ is as required, i.e. so we are done proving $\boxplus_1$, so assume $\varepsilon(\bar{a}'_{\alpha-1}) < \kappa$ hence also $p_\alpha \cup \{\varphi_{\varepsilon(\bar{a}'_\alpha)}(\bar{x}, \bar{c}_{\varepsilon(\bar{a}'_{\alpha-1})})\}$ is well defined and being a subset of q it is an m-type in M hence is realized in M, and any $\bar{a}'_\alpha$ realizing it is O.K.

Having carried the definition, clearly $\langle \bar{a}_\alpha : \alpha < \kappa \rangle$ is an indiscernible sequence; also by clause (b) of the theorem there is $\bar{a}'_\kappa \in {}^mM$ such that $\langle \bar{a}'_\alpha : \alpha \leq \kappa \rangle$ is an indiscernible sequence. If $\bar{a}'_\kappa$ realizes q we are done, if not choose $\varepsilon < \kappa$ such that $\mathfrak{C} \models \neg\varphi_\varepsilon[\bar{a}'_\kappa, \bar{c}_\varepsilon]$. So for every even $\alpha < \kappa, \bar{a}'_\kappa$ satisfies clause (a) hence $\varepsilon(\bar{a}'_\alpha) \leq \varepsilon$. So for some $\zeta \leq \varepsilon$ the set $\{\alpha < \kappa : \alpha$ even and $\varepsilon(\bar{a}'_\alpha) = \zeta\}$ is infinite. But by $(*)(b) + (c)$ this is a contradiction to "$\langle \bar{a}'_\alpha : \alpha < \kappa \rangle$ is an indiscernible sequence" from the beginning of the paragraph and "T is dependent". So $\boxplus_1$ holds.]

$\boxplus_2$ if $B \subseteq A \subseteq M, |B| < \kappa, |A| \leq \kappa, m = 1$ and p is an m-type over A which is finitely satisfiable in B, <u>then</u> p is realized in M.

[Why? Let D be an ultrafilter on ${}^mB$ such that $p \subseteq \mathrm{Av}(D, A)$. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be $\subseteq$-increasing with union A such that $|A_\alpha| \leq |\alpha|$. Choose $\bar{a}_\alpha \in {}^mM$ by induction on $\alpha < \kappa$ such that $\bar{a}_\alpha$ realizes $\mathrm{Av}(D, \{\bar{a}_\beta : \beta < \alpha\} \cup A_\alpha \cup B)$. So $\mathbf{I} = \langle \bar{a}_\alpha : \alpha < \kappa \rangle$ is an indiscernible sequence ([She90, I,§1] or see [She04, §1]) and $p \subseteq \mathrm{Av}(\mathbf{I}, A)$, hence by $\boxplus_1$ we are done proving $\boxplus_2$.]

$\oplus_1$ if κ is singular, p_* is realized in M.

[Why? Let $\langle A^*_\varepsilon : \varepsilon < \mathrm{cf}(\kappa) \rangle$ be $\subseteq$-increasing with union A_* such that $|A^*_\varepsilon| < \kappa$ for $\varepsilon < \kappa$. As M is κ-saturated for each $\varepsilon < \kappa$ there is $a_\varepsilon \in M$ realizing $p_* {\restriction} A_\varepsilon$. Let $B = \{a_\varepsilon : \varepsilon < \mathrm{cf}(\kappa)\}$ and let D be an ultrafilter on B such that $\varepsilon < \kappa \Rightarrow \{a_\zeta : \zeta \in (\varepsilon, \mathrm{cf}(\kappa)\} \in D$. Clearly $p_* \subseteq \mathrm{Av}(D, A_*)$ hence by $\boxplus_2$ we are done.]

$\oplus_2$ if κ is regular, <u>then</u> p_* is realized in M.

[Why? Let $\langle A^*_\alpha : \alpha < \kappa\rangle$ be $\subseteq$-increasing with union A_* such that $|A^*_\alpha| < \kappa$ for $\alpha < \kappa$. Let $\mathbf{m} = (\mathbf{y}, \bar\psi, r, \mathbf{u}) \in \mathrm{vK}^{\otimes}_{\kappa,\mu,\theta}$ be such that $\mathbf{x} \le_1 \mathbf{y}$, exists by 5.20. We can choose $(\bar d_\alpha, \bar c_\alpha)$ by induction on $\alpha < \kappa$ such that it solves $(\mathbf{m}, A_\alpha \cup \{\bar d_\beta{}^\frown \bar c_\beta : \beta < \alpha\} \cup B^+_{\mathbf{y}})$. By 3.14 the sequence $\langle \bar c_\alpha{}^\frown \bar d_\alpha : \alpha < \kappa\rangle$ is an indiscernible sequence over $B^+_{\mathbf{y}}$.

Let $d'_\alpha = d_{\alpha,0}$ so $\mathbf{I} = \langle d'_\alpha : \alpha < \kappa\rangle$ is an indiscernible sequence and d'_α realizes $p_* {\restriction} A_\alpha$. Hence $\mathrm{Av}(\mathbf{I}, A_*)$ is equal to p_*. So by $\boxplus_1$ we are done.]

By $\oplus_1 + \oplus_2$ we are done. $\square_{7.6}$

Another result of interest is (compare with 6.2)

Conclusion 7.7. *Assume* $\mathrm{vK}_{\kappa,\bar\mu,\theta}$ *is dense and* $\varepsilon < \theta^+$.

If M is a κ-saturated model <u>*then*</u> *for any* $p \in \mathbf{S}^\varepsilon(M)$ *there is a κ-complete filter on* ${}^\varepsilon|M|$ *which is an ultrafilter when restricted to* $\mathrm{Def}_\varepsilon(M)$*, see Definition 1.19.*

§ 8. Concluding Remark

Another relative of 4.6 is

Claim 8.1. *1) Assume $\varphi_n(\bar{x}_{[\zeta]}, \bar{y}_n)$ is a formula for $n < n_*$. If $\mathscr{D}$ is a filter on I and $\bar{a}_t \in {}^\zeta\mathfrak{C}$ for $t \in I$, <u>then</u> there is $\bar{\mathscr{S}}$ such that*

- *(a) for some $k_*, \bar{\mathscr{S}} = \langle \mathscr{S}_k : k < k_* \rangle$ is a partition of I*
- *(b) $\mathscr{S}_k \in \mathscr{D}^+$*
- *(c) if $\ell < n_*$ and $\bar{b} \in {}^{\ell g(\bar{y}_\ell)}\mathfrak{C}$ <u>then</u> for some truth value $\mathbf{t}$ and $k < k_*$ we have $\{t \in \mathscr{S}_k : \mathfrak{C} \models \varphi_\ell[\bar{a}_t, \bar{b}]^{\text{if}(\mathbf{t})}\} = \mathscr{S}_k \mod \mathscr{D}$.*

1A) Above we find $\bar{S}$ such that

- *(a), (b) are as there and*
- *(c) if $\bar{b}_n \in {}^{\ell g(\bar{y}_\ell)}\mathfrak{C}$ for $n < n_*$ <u>then</u> for some k we have*
  - *for each $n < n_*$ for some truth value $\mathbf{t}$, the set $\{t \in \mathscr{S}_k : \mathfrak{C} \models \varphi_n[\bar{a}_t, \bar{b}_n]^{\text{if}(\mathbf{t})}\}$ is $= \mathscr{S}_k \mod \mathscr{D}$.*

2) Assume $\mathscr{D}_\ell$ is a filter on I_ℓ for $\ell = 0, 1$ and $C \subseteq \mathfrak{C}_T, \Delta \subseteq \mathbb{L}(\tau_T)$ are finite and $\bar{a}_{\ell,t} \in {}^{m(i)}\mathfrak{C}$ for $t \in I_\ell, \ell < 2$. <u>Then</u> we can find $\mathscr{S}_\ell \in D^+_\ell$ for $\ell < 2$ such that for some q we have $(\forall^{\mathscr{D}_0} s_0 \in \mathscr{S}_0)(\forall^{\mathscr{D}_1} s_1 \in \mathscr{S}_1)[q = \text{tp}_\Delta(\bar{a}_{0,s_0} \hat{\ } \bar{a}_{1,s_1}, C)]$.
3) Like part (2) for $\langle (I_\ell, \mathscr{D}_\ell) : \ell < n_* \rangle$.

Proof. 1) We try to choose $n_\ell, \bar{b}_\ell, \bar{\mathscr{S}}_\ell$ by induction on $\ell \in \mathbb{N}$ such that

$\boxplus$
- (a) $n_\ell < n_*$
- (b) $\bar{b}_\ell$ has length $\ell g(\bar{y}_{n_\ell})$
- (c) $\bar{\mathscr{S}}_\ell = \langle \mathscr{S}_\eta : \eta \in {}^{\ell+1}2 \rangle$ is a partition of I
- (d) $\mathscr{S}_\eta \in \mathscr{D}^+$ for $\eta \in {}^\ell 2$
- (e) $\mathscr{S}_\eta = \{t \in I$: if $k < \ell g(\eta)$ then $\mathfrak{C} \models$ "$\varphi_{n_k}[\bar{a}_t, \bar{b}_k]^{\text{if}(\eta(k))}$"$\}$.

We stipulate $\bar{\mathscr{S}}_{-1} = \langle \mathscr{S}_{<>} \rangle, \mathscr{S}_{<>} = I$.

If we succeed, we get a contradiction to "T is dependent". Arriving to ℓ, clearly $\bar{\mathscr{S}}_{\ell-1}$ has been defined, and if we cannot choose $n_\ell, \bar{b}_\ell$ are required, the conclusion of part (1) holds.
1A) Similarly; e.g. without loss of generality ζ is finite from a failure we get that for every k we can find $A, |A| \le (\Sigma\{\ell g(\bar{y}_n) : n < n_*\} \times k, |\mathbf{S}^\zeta_{\{\varphi_n : n < n_*\}}(A)| \ge 2^k$, contradiction to "$T$ dependent" (see **??**(b)).
2) Let $\Delta = \{\varphi^1_n(\bar{x}_{[m(0)]}, \bar{y}_{[m(1)]}, \bar{z}_n) : n < n_*\}$ and $\Phi = \{\varphi^1_n(\bar{x}_{[m(0)]}, \bar{y}_{[m(1)]}, \bar{c}_k) : n < n_*, \bar{c} \in {}^{\ell g(\bar{z}_n)}C)$, it is finite and clearly it suffices to deal with one pair $(\varphi^1_n, \bar{c})$, as we can replace $\mathscr{D}_\ell$ by $\mathscr{D}_\ell + \mathscr{S}$ when $\mathscr{S} \in \mathscr{D}^+_\ell$ and let $\varphi^2_n = \varphi^1_n(\bar{y}_{[m(1)]}, \bar{x}_{[m(0)]}, \bar{z}_n) = \varphi^1_n(\bar{y}_{[m(0)]}, \bar{y}_{[m(1)]}, \bar{z}_n)$. We apply part (1) with $m(n), 1, \varphi^2_n, \langle \bar{a}_t \hat{\ } \bar{c} : t \in I_\ell \rangle, \mathscr{D}_1$ here standing for $\zeta, n_*, \varphi_n, \langle \bar{a}_t : t \in I \rangle, \mathscr{D}$ there and get $\bar{\mathscr{S}}_1 = \langle \mathscr{S}_{1,k} : k < k_* \rangle$ as there. We define a funtion $h : I_0 \to \{0, \dots, k_* - 1\}$, by $h(s) = \min\{k : (\forall^{\mathscr{D}_1} t \in I_1)\varphi^2_n(\bar{a}_{1,t}, \bar{a}_{0,s}, \bar{c})$ or $(\forall^{\mathscr{D}_1} t \in I_1)(\neg\varphi^2_n(\bar{a}_{1,t}, \bar{a}_{0,s}, \bar{c}))\}$. By the choice of $\bar{\mathscr{S}}$, this is a well defined function. Clearly for some k and $\mathbf{t}$, the set $\mathscr{S}_0 := \{s \in I_1 : h(s) = k$ and $(\forall^{\mathscr{D}_1} t \in I_1)[\varphi^2_n(\bar{a}_{1,1}, \bar{a}_{0,s}, \bar{c})^{\text{if}(\mathbf{t})}]\}$ belongs to $\mathscr{D}^+_0$ and let $\mathscr{S}_1 = \mathscr{S}_{1,k}$, clearly we are done. $\square_{8.1}$

Here we look again at decomposition as in [She15], i.e. with $u_{\mathbf{x}} = \emptyset$.

Claim 8.2. *Assume $\Delta = \{\varphi_*(\bar{x},\bar{y}), \neg\varphi_*(\bar{x},\bar{y})\}, m = \ell g(\bar{x})$ and $m < \omega$ and $n_* = \mathrm{ind}(\varphi_*)$. For any $A (\subseteq \mathfrak{C})$ and $p \in \mathbf{S}^m_\Delta(A)$ is consistent with $r_*(\bar{x})$ and $\mu > \aleph_0$ we can find the following objects:*

(A) (a) $\bar{d} \in {}^m\mathfrak{C}$ *realizing* p

(b) $n_1 < \mathrm{ind}(\varphi(\bar{x},\bar{y}))$

(c) $A_n \subseteq A$ *has cardinality* $< \mu$ *for* $n < n_1$

(d) $\bar{b}_{n,0}, \bar{b}_{n,1} \in {}^{\ell g(\bar{y})}\mathfrak{C}$ *for* $n < n_1$

(e) D_n *is an ultrafilter on* ${}^{\ell g(\bar{y})}(A_n)$

$(f)(\alpha)$ $\bar{b}_{n,0}\hat{\ }\bar{b}_{n,1}$ *realizes* $\mathrm{Av}(D_n, \{\bar{b}_{k,0}, \bar{b}_{k,1} : k < n\} + A$

(β) *if* $\ell \le n$ *then* $\bar{b}_{\ell,0}, \bar{b}_\ell$ *hence realizes the same type over* $\{b_{k,\iota} : k \le n, k \ne \ell, \iota < 2\} + A$

(γ) *if* $\eta, \nu \in {}^{n+1}2$, $\langle \bar{b}_{0,\eta(0)}, \bar{b}_{1,\eta(1)}, \ldots, \bar{b}_{n,\eta(n)}\rangle, \langle \bar{b}_{0,\nu(0)}, b_{1,\nu(1)}, \ldots\rangle$ *realizes the same type over* A

(g) $p \cup r^*_{n_1}$ *is consistence where* $r^*_k(\bar{x}) = \{\varphi(\bar{x}, \bar{b}_{\ell,1}) \equiv \neg\varphi(\bar{x},\bar{b}_{\ell,0}) : \ell < k\}$

(h) $\bar{d}$ *realizes the type from (g)*

(B) (a) *if* $q \subseteq p$ *has cardinality* $< \mu$ *then for some finite* $r \subseteq p$ *we have* $r \cup r_* \vdash q$

(b) *for some* n_2 *depending on* p *only, we can demand* $|r| = n_2$ *so* $\{r_* \cup r : r \subseteq p, |r| = n_2\}$ *is a* μ*-directed partial ordered by* $r_1 \le r_2 \Leftrightarrow (r_2 \vdash r_1)$.

Remark 8.3. 1) This is a relative of a claim from [She15]. We lose not fixing $\bar{d}$ a priori but can use e.g. finite Δ.
2) We can chose D_n such that if $B \supseteq A_n$ and $\bar{b}'_0\hat{\ }\bar{b}_1$ realizes $\mathrm{Av}(D_n, B)$ then $\mathrm{tp}(\bar{b}_0, B) = \mathrm{tp}(\bar{b}_1, B)$, moreover the two natural projections of D_n to an ultrafilter on ${}^{\ell g(\bar{y})}(A_n)$ are equivalent.
3) If we are analyzing $\mathrm{tp}(\bar{d}, A)$ and already have $\bar{c}$ as in decompositions, w can work in $\mathfrak{C}^*_{\bar{c}} = (C, c_\alpha)_{\alpha < \ell g(\bar{c})}$ and use $\varphi' = \varphi(\bar{x}_{\bar{d}}, \bar{c}, \bar{y}), A' = A$ and apply the claim.
4) This may be used in §5.

Proof. We try to choose $(A_n, D_n, \bar{b}_{n,0}, \bar{b}_{n,1})$ by induction on n such that

$\boxplus$ clauses $(c), (d), (e), (f)(\alpha), (\beta)$ of (A) of the assumption holds as well as (g) of (A), i.e. $p \cup r^*_{n+1}$ is consistent where r^*_{n+1} is from clause (A)(g).

Note that $p \cup r^*_0$ is consistent by an assumption.

Case 1: We can carry the induction for $n < \mathrm{ind}(\varphi)$.
We get a contradiction to the definition of $\mathrm{ind}(-)$ as in [She15].

Case 2: We are stuck in n_1 (i.e. cannot choose for n_1)

$\oplus_1$ clause (B)(a) holds.

Why? Toward contradiction, let $q(\bar{x}) \subseteq p(\bar{x})$ be of cardinality $< \mu$ be a counterexample so let $q(\bar{x}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \mu_*\}$ where $\mu_* = |q(\bar{x})|$.
For any finite $r \subseteq p$ let

$$\mathscr{U}_r = \{\alpha < \mu_* : r(\bar{x}) \cup r^*_{n_1} \nVdash \text{“}\varphi_\alpha(\bar{x}, \bar{b}_\alpha)\text{” and } r(\bar{x}) \cup r^*_{n_1} \nVdash \text{“}\neg\varphi_\alpha(\bar{x}, \bar{b}_\alpha)\text{”}\}$$

$$\mathscr{U}^1_r = \{(\alpha,\beta) \in \mu_* : r(\bar{x}) \cup r^*_{n_1} \cup \{\varphi_\beta(\bar{x},\bar{b}_\beta), \neg\varphi_\alpha(\bar{x},\bar{b}_\alpha)\} \text{ is consistent}\}.$$

Clearly

$(*)_1$ $\mathscr{U}_r \neq \emptyset$.

[Why? Otherwise recalling $r \subseteq p, r$ is as promised in (B) of the claim.]

So $\varphi_\alpha(\bar{x},\bar{b}_\alpha) = \varphi_*(\bar{x},\bar{b}_\alpha)^{\text{if}(\mathbf{t}(\alpha)}$ for some truth value $\mathbf{t}(\alpha)$.

Let $\bar{y}_\ell = \langle y_{\ell,k} : k < \ell g(\bar{y})\rangle$.

Let

$(*)_2$ $\Gamma_* = \{\psi(\bar{y}_1,\bar{c}) \equiv \psi(\bar{y}_2,\bar{c}) : \psi = \psi(\bar{y},\bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}(\Sigma\{b_{n,\iota} : n < n_1, \iota < 2\} + A)\}$.

Now

$(*)_3$ for any finite $\Gamma \subseteq \Gamma_*$ let
$\mathscr{U}^2_\Gamma = \{(\alpha,\beta) \in \mu_* \times \mu_* : (\bar{b}_\alpha,\bar{b}_\beta)$ realizes Γ and $\mathbf{t}(\alpha) = \mathbf{t}(\beta)\}$.

Now

$(*)_4$ if $\Gamma \subseteq \Gamma_*$ is finite then $\mathscr{U}^2_\Gamma$ is an equivalence relation on μ_* with $\leq 2^{|\Gamma|+1}$ equivalence class.

[Why? By inspection.]

$(*)_5$ if $r(\bar{x}) \subseteq p(\bar{x})$ and $\Gamma \subseteq \Gamma_*$ are finite then $\mathscr{U}^1_r \cap \mathscr{U}^2_\Gamma \neq \emptyset$.

[Why? As $\mathscr{U}^2_\Gamma$ has $\leq 2^{|\Gamma|+1}$ equivalence classes, we can find a sequence $\langle\alpha(j) : j < 2^{|\Gamma|+1}\rangle$ of ordinals $< \mu_*$ represent all the $\mathscr{U}^2_\Gamma$-equivalence classes. Let $r_1(\bar{x}) = r(\bar{x}) \cup \{\varphi_{\alpha_j}(\bar{x},\bar{b}_{\alpha(j)} : j < 2^{|\Gamma|+1}\}$ as $q(\bar{x}) \subseteq p(\bar{x})$, necessarily $r_1(\bar{x})$ is a subset of $p(\bar{x})$ and of course it is finite. So $\mathscr{U}_{r_1} \neq \emptyset$ and choose $\beta \in \mathscr{U}_{r_1}$ and let $j < 2^{|\Gamma|}$ be such that α_j, β are $\mathscr{U}^r_\Gamma$-equivalent. Recalling $\varphi_*(\bar{x},\bar{b}_{\alpha(j)})^{\text{if}(\mathbf{t}(\alpha,j))} \in p$ so in particular $r_1(\bar{x}) \cup \{\varphi_*(\bar{x}),\bar{b}_{\alpha(j)})^{\text{if}(1-\mathbf{t}(\alpha(j)))}\}$ is consistent.

Let $\bar{d}'$ realize it then the pairs $(\alpha(j),\beta),(\beta,\alpha(j))$ belongs to $\mathscr{U}^2_p$ and at least one of them belongs to $\mathscr{U}^1_r$. So $(*)_5$ holds indeed.]

$(*)_6$ If $r_1, r_2 \subseteq p(\bar{x})$ and $\Gamma_1,\Gamma_2 \subseteq \Gamma_*$ are finite then $\mathscr{U}^1_{r_1\cup r_2} \cap \mathscr{U}^2_{\Gamma_1\cup\Gamma_2} \subseteq (\mathscr{U}^1_{r_1} \cap \mathscr{U}^2_{\Gamma_1}) \cap (\mathscr{U}^1_{r_2} \cap \mathscr{U}^2_{\Gamma_2})$.

[Why? By inspection.]

By $(*)_5 + (*)_6$ clearly

$(*)_7$ there is an ultrafilter $\mathscr{D}_{n_1}$ on $\mu_* \times \mu_*$ such that:
if $r(\bar{x}) \subseteq p(\bar{x})$ is finite and $\Gamma \subseteq \Gamma_*$ is finite then $\mathscr{U}^1_{r(\bar{x})} \cap \mathscr{U}^2_\Gamma \in \mathscr{D}_{n_1}$

$(*)_8$ let $\mathbf{t}$ be such that $\{(\alpha,\beta) \in \mu_* \times \mu_* : \mathbf{t}(\alpha) = \mathbf{t}(\beta)$ is equal to $\mathbf{t}\}$ belongs to D.

[Why well defined? As $\mathscr{U}^2_\emptyset \in \mathscr{D}_{n_1}$ by $(*)_7$ and see $(*)_3$.]

Let $\bar b_{n_1,0}, \bar b_{n_1,1} \in {}^{\ell g(\bar y)}\mathfrak{C}$ be such that $\bar b_{n_1,0}\hat{}\bar b_{n_1,1}$ realize $\mathrm{Av}(\mathscr{D}_{n_1}, \langle \bar b_\alpha \hat{} \bar b_\beta : (\alpha,\beta) \in \mu_* \times \mu_*\rangle, \Sigma\{\bar b_{k,\iota} : k < n_1 \text{ and } \iota < 2\} + A)$. Clearly $\mathscr{D}_{n_1}$ satisfies clause (A)(e).

Let $A_{n_1} := \cup\{\mathrm{Rang}(\bar b_\alpha) : \alpha < \mu_*\}$ so clause (A)(i) holds. Now $(\bar b_{n_1,0}, \bar b_{n_1+1})$ satisfies clauses (A)(d) and (A)(f)$(\alpha),(\beta)$ and $r^*_{n_*+1}$ is well defined.

Lastly, concerning clause (e), the set $p(\bar x) \cup r^*_{n_1+1}$ is well defined and consistent because for any finite $r(\bar x) \subseteq p(\bar x)$, for the $\mathscr{D}_{n_1}$-majority of $(\alpha,\beta) \in \mu_* \times \mu_*, p(\bar x) \cup r^*_{n_1} \cup \{\varphi_*(\bar x, \bar a_\beta)^{\mathrm{if}(\mathbf{t})}, \neg\varphi_*(\bar x, \bar a_\alpha)^{\iff(\mathbf{t})}\}$ is inconsistent, contradiction to D assumptions. So indeed $(A_{n_1}, D'_n, \bar b_{n_1,0}, \bar b_{n_1,1})$ are as required.

Contradiction to the case assumption so really to "$\oplus_1$ fail". So indeed $\oplus_1$, i.e. clause $(B)(a)$ holds.

$\oplus_2$ choose $\bar d \in {}^n\mathfrak{C}$ realizing $p(\bar x) \cup r^*_{n_1+1}$ so clauses (A)(a),(b) hold.

[Why possible? As $p(\bar x) \cup r^*_{n_1+1}$ is consistent by the induction assumption, i.e. clause (A)(g), see above.]

$\oplus_3$ clause (A)(f)(γ) holds.

$\oplus_4$ clause (B)(b) holds.

[Why? Otherwise for every n there is $q_n(\bar x) \subseteq p(\bar x)$ of cardinality $< \mu$ for which in clause (B)(a) there is no $r(\bar x) \subseteq p(\bar x)$ with n elements such that $r(\bar x) \cup r^*_{n_1}(\bar x) \vdash q_n(\bar x)$. Still there is a finite $r_n(\bar x) \subseteq p(x)$ such that $r_n(\bar x) \cup r^*_{n_1}(\bar x) \vdash q_n(\bar x)$. Let $q(\bar x) = \cup\{q_n(\bar x) : n \in \mathbb{N}\}$, by (B)(a) there is a finite $r(\bar x) \subseteq p(\bar x)$ such that $r(\bar x) \cup r^*_{n_1} \vdash q(\bar x)$; let $n = |r(\bar x)|$ and we get a contradiction to the choice of $q_n(\bar x)$.

Together by $\oplus_1 - \oplus_4$ and the induction hypothesis $\boxplus$ we are done. $\square_{8.2}$

Claim 8.4. *Assume $\Delta \subseteq \{\varphi : \varphi = \varphi(\bar x_{[m]}, \bar y) \in \mathbb{L}(\tau_T)\}$ is finite and closed under negation (well we stipulate $\neg\neg\varphi = \varphi$). Then 8.2 holds.*

Proof. We may repeat the proof. Alternatively we can in [She90, Ch.II] manipulate Δ to one formula φ_*, i.e. let $\Delta = \{\varphi_\ell(\bar x, \bar y_\ell) : \ell < n_*\}$ and we can consider only A with at least two members. Let $\ell g(\bar y_\ell) = k_\ell, (\forall \ell < n_*)(k_\ell \leq k_0)$ let

$$\begin{aligned}\varphi_*(\bar x, \bar y_0\hat{}\langle z_0, z_1, z_2, \quad z_{2n_*+1}) = &\bigwedge_{\ell<n_*} (z_{2n_*+1} = z_\ell \wedge \bigwedge_{k<\ell} z_{2n_*+1} \neq z_\ell \rightarrow \varphi_\ell(\bar x, \bar y_0 {\restriction} k_\ell))\\ &\wedge \bigwedge_{\ell<n_*} (z_{2n_*+1} = z_{n_*+\ell} \wedge \bigwedge_{k<n_*+\ell} z_{2n_*+1} \neq z_k \rightarrow \neg\varphi_\ell(\bar x, \bar y_\ell {\restriction} k_\ell))\\ &\wedge (\bigvee_{\ell<2n_*+1} z_{2n_*+1} = z_\ell).\end{aligned}$$

So

$(*)_1$ for any $\bar c \in {}^{(k_0+2n_*+2)}A$ one of the following cases occurs:
- (a) for some $\ell < n_*$ and $\bar b \triangleleft \bar c$ and truth value $\mathbf{t}$ we have $(\forall \bar x)[\varphi_*(\bar x, \bar c) \equiv \varphi_\ell(\bar x, \bar b)^{\mathrm{if}(\mathbf{t})}]$
- (b) $(\forall \bar x)\varphi(\bar x, \bar c)$
- (c) $(\forall \bar x)(\neg\varphi(\bar x, \bar c)$

$(*)_2$ if $a^*_0 \neq a^*_1, \ell < n_*$ and $b \in {}^{\ell g(\bar y)}\mathfrak{C}$ and $\mathbf{t}$ a truth value <u>then</u> for some $\bar c \subseteq (\mathrm{Rang}(\bar b) \cup \{a_0, a_1\})$ we have $(\forall \bar x)[\varphi_*(\bar x, \bar c) \equiv \varphi_\ell(\bar x, \bar b)^{\mathrm{if}(\mathbf{t})}]$

$(*)_3$ if $a_0 \neq a_1$ then for some $\bar c_0, \bar c_1 \in {}^{2n_*+2)}\{a_0, a_1\}$ we have $\mathfrak{C} \models (\forall \bar x)(\varphi(\bar x, \bar c_1)) \wedge (\forall \bar x)(\neg\varphi(\bar x, \bar c_0))$.

$\square_{8.4}$

References

[BB00] John T. Baldwin and Michael Benedikt, *Stability theory, permutations of indiscernibles, and embedded finite model theory*, Transactions of the American Mathematical Society **352** (2000), 4937–4969.

[CK73] Chen C. Chang and H. Jerome Keisler, *Model theory*, Studies in Logic and the Foundation of Math., vol. 73, North–Holland Publishing Co., Amsterdam, 1973.

[EHMR84] Paul Erdős, Andras Hajnal, A. Maté, and Richard Rado, *Combinatorial set theory: Partition relations for cardinals*, Studies in Logic and the Foundation of Math., vol. 106, North–Holland Publ. Co, Amsterdam, 1984.

[EM56] Andrzej Ehrenfeucht and Andrzej Mostowski, *Models of axiomatic theories admitting automorphisms*, Fundamenta Mathematicae **43** (1956), 50–68.

[HS82] Leo A. Harrington and Saharon Shelah, *Counting equivalence classes for co-κ-Souslin equivalence relations*, Logic Colloquium '80 (Prague, 1980), Stud. Logic Foundations Math., vol. 108, North-Holland, Amsterdam-New York, 1982, pp. 147–152. MR 673790

[KS92] Menachem Kojman and Saharon Shelah, *Nonexistence of universal orders in many cardinals*, J. Symbolic Logic **57** (1992), no. 3, 875–891, arXiv: math/9209201. MR 1187454

[KS14a] Itay Kaplan and Saharon Shelah, *A dependent theory with few indiscernibles*, Israel J. Math. **202** (2014), no. 1, 59–103, arXiv: 1010.0388. MR 3265314

[KS14b] ______, *Examples in dependent theories*, J. Symb. Log. **79** (2014), no. 2, 585–619, arXiv: 1009.5420. MR 3224981

[Mor65] Michael Morley, *Categoricity in power*, Transaction of the American Mathematical Society **114** (1965), 514–538.

[Poi00] Bruno Poizat, *A course in model theory. an introduction to contemporary mathematical logic*, Universitext, Springer-Verlag, New York, 2000, Translated from the French by Moses Klein and revised by the author.

[S^+a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F973.

[S^+b] ______, *Tba*, In preparation. Preliminary number: Sh:F1124.

[Shea] Saharon Shelah, *Categoricity of an abstract elementary class in two successive cardinals, revisited*, Ch. 6 of [Sh:i].

[Sheb] ______, *Stability theory for a model*, Ch. V of [Sh:i].

[Shec] ______, *Universal Classes: the heart of the matter*, Ch. V (F) of [Sh:i].

[She69a] ______, *Note on a min-max problem of Leo Moser*, J. Combinatorial Theory **6** (1969), 298–300. MR 241312

[She69b] ______, *Stable theories*, Israel J. Math. **7** (1969), 187–202. MR 0253889

[She71] ______, *The number of non-isomorphic models of an unstable first-order theory* , Israel J. Math. **9** (1971), 473–487. MR 0278926

[She78] ______, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam-New York, 1978. MR 513226

[She79] ______, *On successors of singular cardinals*, Logic Colloquium '78 (Mons, 1978), Stud. Logic Foundations Math., vol. 97, North-Holland, Amsterdam-New York, 1979, pp. 357–380. MR 567680

[She80] ______, *Simple unstable theories*, Ann. Math. Logic **19** (1980), no. 3, 177–203. MR 595012

[She84] ______, *On co-κ-Souslin relations*, Israel J. Math. **47** (1984), no. 2-3, 139–153. MR 738165

[She90] ______, *Classification theory and the number of nonisomorphic models*, second ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990. MR 1083551

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She00a] ______, *On what I do not understand (and have something to say), model theory*, Math. Japon. **51** (2000), no. 2, 329–377, arXiv: math/9910158. MR 1747306

[She00b] ______, *The generalized continuum hypothesis revisited*, Israel J. Math. **116** (2000), 285–321, arXiv: math/9809200. MR 1759410

[She04] ______, *Classification theory for elementary classes with the dependence property—a modest beginning*, Sci. Math. Jpn. **59** (2004), no. 2, 265–316, arXiv: math/0009056. MR 2062198

[She06] ______, *More on the revised GCH and the black box*, Ann. Pure Appl. Logic **140** (2006), no. 1-3, 133–160, arXiv: math/0406482. MR 2224056

[She09] ______, *Dependent first order theories, continued*, Israel J. Math. **173** (2009), 1–60, arXiv: math/0406440. MR 2570659

[She11] ______, *No limit model in inaccessibles*, Models, logics, and higher-dimensional categories, CRM Proc. Lecture Notes, vol. 53, Amer. Math. Soc., Providence, RI, 2011, arXiv: 0705.4131, pp. 277–290. MR 2867976

[She14a] ______, *Dependent T and existence of limit models*, Tbilisi Math. J. **7** (2014), no. 1, 99–128, arXiv: math/0609636. MR 3313049

[She14b] ______, *Strongly dependent theories*, Israel J. Math. **204** (2014), no. 1, 1–83, arXiv: math/0504197. MR 3273451

[She15] ______, *Dependent theories and the generic pair conjecture*, Commun. Contemp. Math. **17** (2015), no. 1, 1550004, 64, arXiv: math/0702292. MR 3291978

[She17] ______, *Definable groups for dependent and 2-dependent theories*, Sarajevo J. Math. **13(25)** (2017), no. 1, 3–25, arXiv: math/0703045. MR 3666349

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