

PSEUDO PCF
SH955

SAHARON SHELAH

ABSTRACT. We continue our investigation on pcf with weak forms of the axiom of choice. Characteristically, we assume $DC + \mathcal{P}(Y)$ when looking at $\prod_{s \in Y} \delta_s$. We get more parallels of pcf theorems.

Date: May 2, 2014.

1991 *Mathematics Subject Classification.* MSC 2010: Primary 03E04, 03E25.

Key words and phrases. set theory, pcf, axiom of choice.

The author thanks Alice Leonhardt for the beautiful typing. First Typed - 08/July/28; revised version of F728. Proofread in January 2013, after proof-reading for the Journal. Partially supported by the Israel Science Foundation (Grant No. 710/07). Part of this work was done during the author's visit to Mittag-Leffler Institute, Djursholm, Sweden. He thanks the Institute for hospitality and support, Publication 955.

Anotated Content

§0 Introduction, pg.3

§1 On pseudo true cofinality, pg.5

[We continue [She12, §5] to try to generalize the pcf theory for \aleph_1 -complete filters D on Y assuming only $DC + AC_{\mathcal{P}(Y)}$. So this is similar to [She82, ChXII]. We suggest to replace cofinality by pseudo cofinality. In particular we get the existence of a sequence of generators, get a bound to $\text{Reg} \cap \text{pp}(\mu) \setminus \mu_0$, the size of $\text{Reg} \cap \mu \setminus \mu_0$ using a no-hole claim and existence of lub (unlike [She]).

§2 Composition and generating sequences for pseudo pcf, pg.16

[We deal with pseudo true cofinality of $\prod_{i \in Z} \prod_{j \in Y_i} \lambda_{i,j}$, also with the degenerated case in which each $\langle \lambda_{i,j} : j \in Y_i \rangle$ is constant. We then use it to clarify the state of generating sequences; see 2.1, 2.2, 2.4, 2.6, 2.12, 2.13.]

§3 Measuring Reduced products, pg.27

§(3A) On $\text{ps-T}_D(g)$, pg.27

[We get that several measures of ${}^\kappa \mu / D$ are essentially equal.]

§(3B) Depth of reduced powers of ordinals, pg.31

[Using the independence property for a sequence of filters we can bound the relevant depth. This generalizes [She00] or really [She02, §3].]

§(3C) Bounds on the Depth, pg.37

[We start by basic properties dealing with the No-Hole Claim (1.13(1)) and dependence on $\langle |\alpha_s| : s \in Y \rangle / D$ only (3.23). We give a bound for $\lambda^{+\alpha(1)} / D$ (in Theorem 3.24, 3.26).]

§ 0. INTRODUCTION

In the first section we deal with generalizing the pcf theory in the direction started in [She12, §5] trying to understand the pseudo true cofinality of small products of regular cardinals. The difference with earlier works is that here we assume $\text{AC}_{\mathcal{U}}$ for any set \mathcal{U} of power $\leq |\mathcal{P}(\mathcal{P}(Y))|$ or, actually working harder, just $\leq |\mathcal{P}(Y)|$ when analyzing $\prod_{t \in Y} \alpha_t$, whereas in [She97] we assumed $\text{AC}_{\sup\{\alpha_t : t \in Y\}}$ and in [She] we have (in addition to $\text{AC}_{\mathcal{P}(\mathcal{P}(Y))}$) assumptions like “ $[\sup\{\alpha_t : t \in Y\}]^{\aleph_0}$ is well ordered”. In [She12, §1-§4] we assume only $\text{AC}_{<\mu} + \text{DC}$ and consider \aleph_1 -complete filters on μ but in the characteristic case μ is a limit of measurable cardinals.

Note that generally in this work, though we try occasionally not to use DC, it will not be a real loss to assume it all the time. More specifically, we prove the existence of a minimal \aleph_1 -complete filter D on Y such that $\lambda = \text{ps-tcf}(\Pi \bar{\alpha}, <_D)$ assuming $\text{AC}_{\mathcal{P}(Y)}$ and (of course) DC and α_t of large enough cofinality. We then prove the existence of one generator, that is, of $X \subseteq Y$ such that $J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] = J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] + X$, see 1.6 and even (in 1.8) the parallel of the existence of a $<_{D_1}$ -lub for an $<_D$ -increasing sequence $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$, generalize the no-hole claim in 1.13, and give a bound on pp for non-fix points (in 1.11).

In §2 we further investigate true cofinality. In Claim 2.2, assuming AC_λ and D an \aleph_1 -complete filter on Y , we start from $\text{ps-tcf}(\Pi \bar{\alpha}, <_D)$, dividing by $\text{eq}(\bar{\alpha}) = \{(s, t) : \alpha_s = \alpha_t\}$. We also prove the composition Theorem 2.6: it tells us when $\text{ps-tcf}(\prod_i \text{ps-tcf}(\prod_j \lambda_{i,j}, <_{D_i}), <_E)$ is equal to $\text{ps-tcf}(\prod_{(i,j)} \lambda_{i,j}, <_D)$.

We then prove the pcf closure conclusion: giving a sufficient condition for the operation $\text{ps-pcf}_{\aleph_1\text{-comp}}$ to be idempotent. Lastly, we revisit the generating sequence.

In §(3A) we measure $\prod_{t \in Y} g(t)$ modulo a filter D on Y for $g \in {}^Y(\text{Ord} \setminus \{0\})$ in three ways and show they are almost equal in 3.2. The price is that we replace (true) cofinality by pseudo (true) cofinality, which is inevitable. We try to sort out the “almost equal” in 3.5 - 3.7.

In §(3B) we prove a relative of [She02, §3]; again dealing with depth (instead of rank as in [She12]) adding some information even under ZFC. Assuming that the sequence $\langle D_n : n < \omega \rangle$ of filters has the independence property (IND), see Definition 3.12, with D_n a filter on Y_n we can bound the depth of $(\langle Y_n \rangle \zeta, <_{D_n})$ by ζ , for every ζ for many n 's, see 3.13. Of course, we can generalize this to $\langle D_s : s \in S \rangle$. This is incomparable with the results of [She12, §4]. See a continuation of [She] in [She16].

Note that the assumptions like $\text{IND}(\bar{D})$ are complementary to ones used in [She] to get considerable information. Our original hope was to arrive to a dichotomy. The first possibility will say that one of the versions of an axiom suggested in [She] holds, which means “for some suitable algebra”, there is no independent ω -sequence; in this case [She] tells us much. The second possibility will be a case of IND, and then we try to show that there is a rank system in the sense of [She12]. But presently for this we need too much choice. The dichotomy we succeed to prove is with small o-Depth in one side, the results of [She] on the other side. It would be better to have ps-o-Depth in the first side.

Question 0.1. [DC + $\text{AC}_{\mathcal{P}(Y)}$]

Assume

- (a) $\bar{\alpha} \in {}^Y \text{Ord}$
 - (b) $\text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y))$ for every $t \in Y$
 - (c) $\lambda_t \in \text{pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ for $t \in Z$, in fact, $\lambda_t = \text{ps-tcf}(\Pi\bar{\alpha}, \langle D_t \rangle)$, D_t is an \aleph_1 -complete filter on Y
 - (d) $\lambda = \text{ps-tcf}_{\aleph_1\text{-comp}}(\langle \lambda_t : t \in Z \rangle)$
 - (e) (a possible help) $X_t \in D_t, \langle X_t : t \in Y \rangle$ are pairwise disjoint.
- (A) Now does $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$? (See 2.6.)
- (B) Can we say something on D_λ from [She12, 5.9] improved in 1.3?

Question 0.2. How well can we generalize the RGCH, see [She00] and [She06]; the above may be relevant; see [She12] and here in §(3C).

Recall

- Notation 0.3.* 1) For any set X let $\text{hrtg}(X) = \min\{\alpha : \alpha \text{ an ordinal such that there is no function from } X \text{ onto } \alpha\}$.
- 2) $A \leq_{\text{qu}} B$ means that either $A = \emptyset$ or there is a function from A onto B .

Central in this work is

Definition 0.4. For a quasi order P we say P has pseudo-true-cofinality λ or “ λ is the pseudo true cofinality of P ” when λ is a regular cardinal and λ is a pseudo true cofinality of P which means that there is a sequence \mathcal{F} such that:

- (a) $\mathcal{F} = \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$
- (b) $\mathcal{F}_\alpha \subseteq P$
- (c) if $\alpha_1 < \alpha_2, p_1 \in \mathcal{F}_{\alpha_1}$ and $p_2 \in \mathcal{F}_{\alpha_2}$ then $p_1 \leq_P p_2$
- (d) if $q \in \mathcal{F}$ then for some $\alpha < \lambda$ and $p \in \mathcal{F}_\alpha$ we have $q <_P p$
- (e) $\lambda = \sup\{\alpha < \lambda : \mathcal{F}_\alpha \neq \emptyset\}$.

We may consider replacing AC_A by more refined version, $\text{AC}_{A,B}$ defined below (e.g. in 1.1, 2.6) but we have not dealt with it systematically.

- Definition 0.5.** 1) $\text{AC}_{A,B}$ means: if $\langle X_a : a \in A \rangle$ is a sequence of non-empty sets then there is a sequence $\langle Y_a : a \in A \rangle$ such that $Y_a \subseteq X_a$ is not empty and $Y_a \leq_{\text{qu}} B$.
- 2) $\text{AC}_{A, < \kappa}, \text{AC}_{A, \leq B}$ are defined similarly but $|Y_a| < \kappa, |Y_a| \leq |B|$ respectively in the end.

- Observation 0.6.** 1) We have AC_A iff $\text{AC}_{A,1}$.
- 2) $\text{AC}_{A,B}$ fails if $B = \emptyset$.
- 3) If $\text{AC}_{A,B}$ and $|A_1| \leq |A|$ and $B \leq_{\text{qu}} B_1$ then AC_{A_1, B_1} .

§ 1. ON PSEUDO TRUE COFINALITY

We continue [She12, §5].

Below we improve [She12, 5.19] by omitting DC from the assumptions but first we observe

Claim 1.1. *Assume AC_Z .*

1) *We have $\theta \geq \text{hrtg}(Z)$ when $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ and $\theta \in \text{ps-pcf}(\Pi\bar{\alpha})$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Z)$.*

2) *We have $\text{cf}(\text{rk}_D(\bar{\alpha})) \geq \text{hrtg}(Z)$ when $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Z)$.*

Remark 1.2. We can weaken the assumption $\text{cf}(\alpha_t) \geq \text{hrtg}(Z)$ by using the ideal of small cofinality, $\text{cf} - \text{id}_\theta(\bar{\alpha})$, see [She16, 1.1=Lc2]. This can be done systematically in this work.

Proof. 1) If we have AC_α for every $\alpha < \text{hrtg}(Z)$ then we can use [She12, 5.7(4)] but we do not assume this. In general let D be a filter on Y such that $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, exists as we are assuming $\theta \in \text{ps-pcf}(\Pi\bar{\alpha})$. Let $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha < \theta \rangle$ witness $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, i.e. as in [She12, 5.6(2)] or see 0.4 here; note $t \in Y \Rightarrow \alpha_t > 0$, as we are assuming $\mathcal{F}_\alpha \subseteq \Pi\bar{\alpha}$ for some $\alpha < \theta$; also if $\Pi\bar{\alpha}$ is non-empty then we can assume $\mathcal{F}_\alpha \neq \emptyset$ for every $\alpha < \theta$.

Toward contradiction assume $\theta < \text{hrtg}(Z)$. As $\theta < \text{hrtg}(Z)$, there is a function h from Z onto θ , so the sequence $\langle \mathcal{F}_{h(z)} : z \in Z \rangle$ is well defined. As we are assuming AC_Z , there is a sequence $\langle f_z : z \in Z \rangle$ such that $f_z \in \mathcal{F}_{h(z)}$ for $z \in Z$. Now define $g \in {}^Y(\text{Ord})$ by $g(s) = \cup\{f_z(s) : z \in Z\}$; clearly g exists and $g \leq \bar{\alpha}$. But for each $s \in Y$, the set $\{f_z(s) : z \in Z\}$ is a subset of α_s of cardinality $\leq \theta < \text{hrtg}(Z)$ hence $< \text{cf}(\alpha_s)$ hence $g(s) < \alpha_s$. Together $g \in \Pi\bar{\alpha}$ is a $<_D$ -upper bound of $\cup\{\mathcal{F}_\varepsilon : \varepsilon < \theta\}$, contradiction to the choice of $\bar{\mathcal{F}}$.

2) Otherwise let $\theta = \text{cf}(\text{rk}_D(\bar{\alpha}))$ so $\theta < \text{hrtg}(Z)$, $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ be increasing with limit $\text{rk}_D(\bar{\alpha})$ and again let g be a function from Z onto θ . As AC_Z holds, we can find $\langle f_z : z \in Z \rangle$ such that for every $z \in Z$ we have $\text{rk}_D(f_z) \geq \alpha_{h(z)}$ and $f_z <_D \bar{\alpha}$ and without loss of generality $f_z \in \Pi\bar{\alpha}$. Let $f \in \Pi\bar{\alpha}$ be defined by $f(t) = \sup\{f_{h(z)}(t) : z \in Z\}$ so $\text{rk}_D(f) \geq \sup\{\alpha_z : z \in Z\} = \text{rk}_D(\bar{\alpha}) > \text{rk}_D(f)$, contradiction. $\square_{1.1}$

Theorem 1.3. *The Canonical Filter Theorem Assume $AC_{\mathcal{P}(Y)}$.*

Assume $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in {}^Y\text{Ord}$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y))$ and $\partial \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ hence is a regular cardinal. Then there is $D = D_{\bar{\alpha}}^\partial$, an \aleph_1 -complete filter on Y such that $\partial = \text{ps-tcf}(\Pi\bar{\alpha}/D)$ and $D \subseteq D'$ for any other such $D' \in \text{Fil}_{\aleph_1}^1(D)$.

Remark 1.4. 1) By [She12, 5.9] there are some such ∂ if DC holds.

2) We work more to use just $AC_{\mathcal{P}(Y)}$ and not more.

3) If $\kappa > \aleph_0$ we can replace “ \aleph_1 -complete” by “ κ -complete”.

4) If we waive “ ∂ regular” so just ∂ , an ordinal, is a pseudo true cofinality of $(\Pi\bar{\alpha}, <_D)$ for $D \in \mathbb{D} \subseteq \text{Fil}_{\aleph_1}^1(Y)$, exemplified by $\bar{\mathcal{F}}^D, \mathbb{D} \neq \emptyset$ the proof gives some $\partial', \text{cf}(\partial') = \text{cf}(\partial)$ and $\bar{\mathcal{F}}$ witnessing $(\Pi\bar{\alpha}, <_{D_*})$ has pseudo true cofinality ∂' where $D_* = \cap\{D : D \in \mathbb{D}\}$ for \mathbb{D} as below.

Proof. Note that by 1.1

$$\boxplus_1 \partial \geq \text{hrtg}(\mathcal{P}(Y)).$$

Let

$$\begin{aligned} \boxplus_2 (a) \quad \mathbb{D} &= \{D : D \text{ is an } \aleph_1\text{-complete filter on } Y \text{ such that } (\Pi\bar{\alpha}/D) \text{ has} \\ &\quad \text{pseudo true cofinality } \partial\}, \\ (b) \quad D_* &= \cap\{D : D \in \mathbb{D}\}. \end{aligned}$$

Now obviously

$$\begin{aligned} \boxplus_3 (a) \quad \mathbb{D} &\text{ is non-empty} \\ (b) \quad D_* &\text{ is an } \aleph_1\text{-complete filter on } Y. \end{aligned}$$

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : (Y \setminus A) \notin D\}$ and let $\mathcal{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$, equivalently $\mathcal{P}_* = \{A \subseteq Y : A \neq \emptyset \text{ mod } D \text{ for some } D \in \mathbb{D}\}$. As $\text{AC}_{\mathcal{P}(Y)}$ holds also $\text{AC}_{\mathcal{P}_*}$ holds hence we can find $\langle D_A : A \in \mathcal{P}_* \rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathcal{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathcal{P}_*\}$, clearly

$$\begin{aligned} \boxplus_4 (a) \quad D_* &= \cap\{D : D \in \mathbb{D}_*\} \\ (b) \quad \mathbb{D}_* &\subseteq \mathbb{D} \text{ is non-empty.} \end{aligned}$$

As $\text{AC}_{\mathcal{P}_*}$ holds clearly

$$(*)_1 \text{ we can choose } \langle \tilde{\mathcal{F}}^A : A \in \mathcal{P}_* \rangle \text{ such that } \tilde{\mathcal{F}}^A \text{ exemplifies } D_A \in \mathbb{D} \text{ as in [She12, 5.17,(1),(2)], so in particular } \tilde{\mathcal{F}}^A \text{ is } \aleph_0\text{-continuous and without loss of generality } \mathcal{F}_\alpha^A \neq \emptyset, \mathcal{F}_\alpha^A \subseteq \Pi\bar{\alpha} \text{ for every } \alpha < \partial.$$

For each $\beta < \partial$ let

$$\begin{aligned} (*)_2 \quad \mathbf{F}_\beta^1 &= \{\bar{f} = \langle f_A : A \in \mathcal{P}_* \rangle : \bar{f} \text{ satisfies } A \in \mathcal{P}_* \Rightarrow f_A \in \mathcal{F}_\beta^A\} \\ (*)_3 \quad \text{for } \bar{f} \in \mathbf{F}_\beta^1 &\text{ let } \sup\{f_A : A \in \mathcal{P}_*\} \text{ be the function } f \in {}^Y\text{Ord defined by} \\ &\quad f(y) = \sup\{f_A(y) : A \in \mathcal{P}_*\} \\ (*)_4 \quad \mathcal{F}_\beta^1 &= \{\sup\{f_A : A \in \mathcal{P}_*\} : \bar{f} = \langle f_A : A \in \mathcal{P}_* \rangle \text{ belongs to } \mathbf{F}_\beta^1\}. \end{aligned}$$

Now

$$\begin{aligned} (*)_5 (a) \quad \langle \mathcal{F}_\beta^1 : \beta < \partial \rangle &\text{ is well defined, i.e. exist} \\ (b) \quad \mathcal{F}_\beta^1 &\subseteq \Pi\bar{\alpha}. \end{aligned}$$

[Why? Clause (a) holds by the definitions, clause (b) holds as $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y))$.]

$$(*)_6 \quad \mathcal{F}_\beta^1 \neq \emptyset \text{ for } \beta < \partial.$$

[Why? As for $\beta < \lambda$, the sequence $\langle \tilde{\mathcal{F}}_\beta^A : A \in \mathcal{P}_* \rangle$ is well defined (as $\langle \tilde{\mathcal{F}}^A : A \in \mathcal{P}_* \rangle$ is) and $A \in \mathcal{P}_* \Rightarrow \mathcal{F}_\beta^A \neq \emptyset$, so we can use $\text{AC}_{\mathcal{P}(Y)}$ to deduce $\mathcal{F}_\beta^1 \neq \emptyset$.]

Define

$$\begin{aligned} (*)_7 (a) \quad \text{for } f \in \Pi\bar{\alpha} \text{ and } A \in \mathcal{P}_* &\text{ let} \\ &\quad \beta_A(f) = \min\{\beta < \partial : f < g \text{ mod } D_A \text{ for every } g \in \mathcal{F}_\beta^A\} \\ (b) \quad \text{for } f \in \Pi\bar{\alpha} &\text{ let } \beta(f) = \sup\{\beta_A(f) : A \in \mathcal{P}_*\}. \end{aligned}$$

Now

$$(*)_8 (a) \quad \text{for } A \in \mathcal{P}_* \text{ and } f \in \Pi\bar{\alpha}, \text{ the ordinal } \beta_A(f) < \partial \text{ is well defined}$$

(b) for $f \in \Pi\bar{\alpha}$ the sequence $\langle \beta_A(f) : A \in \mathcal{P}_* \rangle$ is well defined.

[Why? Clause (a) holds because $\langle \mathcal{F}_\gamma^A : \gamma < \partial \rangle$ is cofinal in $(\Pi, \bar{\alpha}, <_{D_A})$, clause (b) holds by $(*)_7(a)$.]

- (*)₉ (a) for $f \in \Pi\bar{\alpha}$ the ordinal $\beta(f)$ is well defined and $< \partial$
 (b) if $f \leq g$ are from $\Pi\bar{\alpha}$ then $\beta(f) \leq \beta(g)$.

[Why? For clause (a), first, $\beta(f)$ is well defined and $\leq \partial$ by $(*)_8$ and the definition of $\beta(f)$ in $(*)_7(b)$. Second, recalling that ∂ is regular $\geq \text{hrtg}(\mathcal{P}(Y)) \geq \text{hrtg}(\mathcal{P}_*)$ clearly $\beta(f) < \partial$. Clause (b) is obvious.]

Now

- (*)₁₀ (a) if $A \in \mathcal{P}_*, \gamma < \partial$ and $f \in \mathcal{F}_\gamma^A$ then $\beta_A(f) > \gamma$
 (b) if $\gamma < \partial$ and $f \in \mathcal{F}_\gamma^1$ then $\beta(f) > \gamma$.

[Why? Clause (a) holds because $\beta < \gamma \wedge g \in \mathcal{F}_\beta^A \Rightarrow g < f \text{ mod } D_A$ and $\beta = \gamma \Rightarrow f \in \mathcal{F}_\gamma^A \wedge f \not< f \text{ mod } D_A$. Clause (b) holds because for some $\langle f_B : B \in \mathcal{P}_* \rangle \in \Pi\{\mathcal{F}_\gamma^B : B \in \mathcal{P}_*\}$ we have $f = \sup\{f_B : B \in \mathcal{P}_*\}$ hence $B \in \mathcal{P}_* \Rightarrow f_B \leq f$ hence in particular $f_A \leq f$; now recalling $\beta(f_A) > \gamma$ by clause (a) it follows that $\beta(f) > \gamma$.]

- (*)₁₁ (a) for $\xi < \partial$ let $\gamma_\xi = \min\{\beta(f) : f \in \mathcal{F}_\xi^1\}$
 (b) for $\xi < \partial$ let $\mathcal{F}_\xi^2 = \{f \in \mathcal{F}_\xi^1 : \beta(f) = \gamma_\xi\}$
 (*)₁₂ (a) $\langle (\gamma_\xi, \mathcal{F}_\xi^2) : \xi < \partial \rangle$ is well defined, i.e. exists
 (b) if $\xi < \partial$ then $\xi < \gamma_\xi < \partial$.

[Why? γ_ξ is the minimum of a set of ordinals which is non-empty by $(*)_6$ and $\subseteq \partial$, by $(*)_9(a)$, and all members are $> \gamma$ by $(*)_{10}(b)$.]

- (*)₁₃ for $\xi < \partial$ we have $\mathcal{F}_\xi^2 \subseteq \Pi\bar{\alpha}$ and $\mathcal{F}_\xi^2 \neq \emptyset$.

[Why? By $(*)_{11}$ as $\mathcal{F}_\xi^1 \neq \emptyset$ and $\mathcal{F}_\xi^1 \subseteq \Pi\bar{\alpha}$.]

- (*)₁₄ we try to define $\beta_\varepsilon < \partial$ by induction on the ordinal $\varepsilon < \partial$
 $\varepsilon = 0$: $\beta_\varepsilon = 0$
 ε limit: $\beta_\varepsilon = \cup\{\beta_\zeta : \zeta < \varepsilon\}$
 $\varepsilon = \zeta + 1$: $\beta_\varepsilon = \gamma_{\beta_\zeta}$
 (*)₁₅ (a) if $\varepsilon < \partial$ then $\beta_\varepsilon < \partial$ is well defined $\geq \varepsilon$
 (b) if $\zeta < \varepsilon$ is well defined then $\beta_\zeta < \beta_\varepsilon$.

[Why? Clause (a) holds as ∂ is a regular cardinal so the case ε limit is O.K., the case $\varepsilon = \zeta + 1$ holds by $(*)_{12}(b)$. As for clause (b) we prove this by induction on ε ; for $\varepsilon = 0$ this is empty, for ε a limit ordinal use the induction hypothesis and the choice of β_ε in $(*)_{14}$ and for $\varepsilon = \zeta + 1$, clearly by $(*)_{12}(b)$ and the choice of γ_ε in $(*)_{14}$ we have $\beta_\xi < \beta_\varepsilon$ and use the induction hypothesis.]

- (*)₁₆ if $f \in \Pi\bar{\alpha}$, then for some $g \in \cup\{\mathcal{F}_{\beta_\varepsilon}^2 : \varepsilon < \partial\}$ we have $f < g \text{ mod } D_*$.

[Why? Recall that $\beta_A(f)$ for $A \in \mathcal{P}_*$ and $\beta(f)$ are well defined ordinals $< \partial$ and $A \in \mathcal{P}_* \Rightarrow \beta_A(f) \leq \beta(f)$. Now let $\zeta < \partial$ be such that $\beta(f) < \beta_\zeta$, exists as we can prove by induction on ε (using $(*)_{15}(b)$) that $\beta_\varepsilon \geq \varepsilon$. As \mathcal{F}^A is $<_{D_A}$ -increasing for $A \in \mathcal{P}_*$ clearly $A \in \mathcal{P}_* \wedge g \in \mathcal{F}_{\beta_\zeta}^A \Rightarrow f < g \text{ mod } D_A$. So by the definition of $\mathcal{F}_{\beta_\zeta}^1$ we have $A \in \mathcal{P}_* \wedge g \in \mathcal{F}_{\beta_\zeta}^1 \Rightarrow f < g \text{ mod } D_A$ hence $g \in \mathcal{F}_{\beta_\zeta}^1 \Rightarrow f < g \text{ mod } D_*$. As $\mathcal{F}_{\beta_\zeta}^2 \subseteq \mathcal{F}_{\beta_\zeta}^1$ we are done.]

$(*)_{17}$ if $\zeta < \xi < \partial$ and $f \in \mathcal{F}_\zeta^2$ and $g \in \mathcal{F}_\xi$ then $f < g \text{ mod } D_*$.

[Why? As in the proof of $(*)_{16}$ but now $\beta(f) = \gamma_\zeta$.]

Together by $(*)_{13} + (*)_{16} + (*)_{17}$ the sequence $\langle \mathcal{F}_{\beta_\varepsilon}^2 : \varepsilon < \partial \rangle$ is as required. $\square_{1.3}$

A central definition here is

Definition 1.5. 1) For $\bar{\alpha} \in {}^Y \text{Ord}$ let $J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] = \{X \subseteq Y : \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha} \upharpoonright X) \subseteq \lambda\}$. So for $X \subseteq Y, X \notin J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ iff there is an \aleph_1 -complete filter D on Y such that $X \neq \emptyset \text{ mod } D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is well defined $\geq \lambda$ iff there is an \aleph_1 -complete filter D on Y such that $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is well defined $\geq \lambda$ and $X \in D$.
2) $J_{<\lambda}^{\aleph_1\text{-comp}}$ is $J_{<\lambda^+}^{\aleph_1\text{-comp}}$ and we can use a set \mathfrak{a} of ordinals instead of $\bar{\alpha}$.

Claim 1.6. *The Generator Existence Claim*

Let $\bar{\alpha} \in {}^Y(\text{Ord} \setminus \{0\})$.

- 1) $J_{<\lambda}^{\aleph_1\text{-comp}}(\bar{\alpha})$ is an \aleph_1 -complete ideal on Y for any cardinal λ except that it may be $\mathcal{P}(Y)$.
- 2) $[\text{AC}_{\mathcal{P}(Y)}]$ Assume $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y))$. If $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ then for some $X \subseteq Y$ we have

$$(A) J_{<\lambda^+}^{\aleph_1\text{-comp}}[\bar{\alpha}] = J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] + X$$

$$(B) \lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_{J_{=\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]}) \text{ where } J_{=\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] := J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] + (Y \setminus X)$$

$$(C) \lambda \notin \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha} \upharpoonright (Y \setminus X)).$$

Remark 1.7. 1) Recall that if $\text{AC}_{\mathcal{P}(Y)}$ then without loss of generality AC_{\aleph_0} holds. Why? Otherwise by $\text{AC}_{\mathcal{P}(Y)}$ we have Y is well ordered and AC_Y hence $|Y| = n$ for some $n < \omega$ and in this case our claims are obvious, e.g. 1.6(2), 1.8.

2) Note that $J_{=\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ is a well defined ideal in 1.6(2)(B) though X is not uniquely determined.

3) Note that if $\theta = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ and $X \in D^+$ then $\theta = \text{ps-tcf}(\Pi(\bar{\alpha} \upharpoonright X), <_{(D+X) \cap \mathcal{P}(X)})$.

Proof. 1) Clearly $J_{<\lambda}^{\aleph_1\text{-comp}}(\bar{\alpha})$ is a \subseteq -downward closed subset of $\mathcal{P}(Y)$. If the desired conclusion fails, then we can find a sequence $\langle A_n : n < \omega \rangle$ of members of $J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ such that their union $A := \cup\{A_n : n < \omega\}$ does not belong to it. As $A \notin J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$, by the definition there is an \aleph_1 -complete filter D on Y such that $A \neq \emptyset \text{ mod } D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is well defined, so let it be $\mu = \text{cf}(\mu) \geq \lambda$ and let $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ exemplify it.

As D is \aleph_1 -complete and $A = \cup\{A_n : n < \omega\} \neq \emptyset \text{ mod } D$ necessarily for some $n, A_n \neq \emptyset \text{ mod } D$ but then D witness $A_n \notin J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$, contradiction.

2) Recall λ is a regular cardinal by [She12, 5.8(0)] and $\lambda \geq \text{hrtg}(\mathcal{P}(Y))$ by 1.1.

Let $D = D_\lambda^\alpha$ be as in [She12, 5.19] when DC holds, and as in 1.3 in general, i.e. $\Pi\bar{\alpha}/D$ has pseudo true cofinality λ and D contains any other such \aleph_1 -complete

filter on Y . Now if $X \in D^+$ then $\lambda = \text{ps-tcf}_{\aleph_1\text{-comp}}(\bar{\alpha} \upharpoonright X, \langle_{(D+X) \cap \mathcal{P}(X)} \rangle)$ hence $X \notin J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$, so

$$(*)_1 \quad X \in J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] \Rightarrow X = \emptyset \text{ mod } D.$$

A major point is

$$(*)_2 \quad \text{some } X \in D \text{ belongs to } J_{<\lambda^+}^{\aleph_1\text{-comp}}[\bar{\alpha}].$$

Why $(*)_2$? The proof will take awhile; assume that not, we have $\text{AC}_{\mathcal{P}(Y)}$ hence AC_D , so we can find $\langle (\mathcal{F}^X, D_X, \lambda_X) : X \in D \rangle$ such that:

- (a) λ_X is a regular cardinal $\geq \lambda^+$, i.e. $> \lambda$
- (b) D_X is an \aleph_1 -complete filter on Y such that $X \in D_X$ and $\lambda_X = \text{ps-tcf}(\Pi\bar{\alpha}, \langle_{D_X} \rangle)$
- (c) $\mathcal{F}^X = \langle \mathcal{F}_\alpha^X : \alpha < \lambda_X \rangle$ exemplifies that $\lambda_X = \text{ps-tcf}(\Pi\bar{\alpha}, \langle_{D_X} \rangle)$
- (d) moreover \mathcal{F}^X is as in [She12, 5.17(2)], that is, it is \aleph_0 -continuous and $\alpha < \lambda_X \Rightarrow \mathcal{F}_\alpha^X \neq \emptyset$.

Let

$$(e) \quad D_1^* = \{A \subseteq Y : \text{for some } X_1 \in D \text{ we have } X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X\}.$$

Clearly

$$(f) \quad D_1^* \text{ is an } \aleph_1\text{-complete filter on } Y \text{ extending } D.$$

[Why? First, clearly $D_1^* \subseteq \mathcal{P}(Y)$ and $\emptyset \notin D_1^*$ as $X \in D \Rightarrow \emptyset \notin D_X$. Second, if $A \in D$ then $X \in D \wedge X \subseteq A \Rightarrow A \in D_X$ by clause (b) hence choosing $X_1 = A$ the demand for “ $A \in D_1^*$ ” holds so indeed $D \subseteq D_1^*$. Third, assume $\bar{A} = \langle A_n : n < \omega \rangle$ and “ $A_n \in D_1^*$ ” for $n < \omega$, then for each A_n there is a witness $X_n \in D$, so by AC_{\aleph_0} , recalling 1.7, there is an ω -sequence $\langle X_n : n < \omega \rangle$ with X_n witnessing $A_n \in D_1^*$. Then $X = \bigcap \{X_n : n < \omega\}$ belongs to D and witness that $A := \bigcap \{A_n : n < \omega\} \in D_1^*$ because every D_X is \aleph_1 -complete. Fourth, if $A \subseteq B \subseteq Y$ and $A \in D_1^*$, then some X_1 witness $A \in D_1^*$, i.e. $X \in D \wedge X \subseteq X_1 \Rightarrow A \in D_X$; but then X_1 witness also $B \in D_1^*$.]

- (g) assume $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ is \langle_D -increasing in $\Pi\bar{\alpha}$, i.e. $\alpha < \lambda \Rightarrow \mathcal{F}_\alpha \subseteq \Pi\bar{\alpha}$ and $\alpha_1 < \alpha_2 \wedge f_1 \in \mathcal{F}_{\alpha_1} \wedge f_2 \in \mathcal{F}_{\alpha_2} \Rightarrow f_1 <_D f_2$ and $\mathcal{F}_\alpha \neq \emptyset$ for every or at least unboundedly many $\alpha < \lambda$ then $\bigcup_{\alpha < \lambda} \mathcal{F}_\alpha$ has a common $\langle_{D_1^*}$ -upper bound.

[Why? For each $X \in D$ recall $(\Pi\bar{\alpha}, \langle_{D_X} \rangle)$ has true cofinality λ_X which is regular $> \lambda$ hence by [She12, 5.7(1A)] is pseudo λ^+ -directed hence there is a common \langle_{D_X} -upper bounded h_X of $\bigcup \{\mathcal{F}_\alpha : \alpha < \lambda\}$. As we have $\text{AC}_{\mathcal{P}(Y)}$ we can find a sequence $\langle h_X : X \in D \rangle$ with each h_X as above. Define $h \in \Pi\bar{\alpha}$ by $h(t) = \sup \{h_X(t) : X \in D\}$, it belongs to $\Pi\bar{\alpha}$ as we are assuming $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathcal{P}(Y)) \geq \text{hrtg}(D)$. So $h \in \Pi\bar{\alpha}$ is a \langle_{D_X} -upper bound of $\bigcup \{\mathcal{F}_\alpha : \alpha < \lambda\}$ for every $X \in D$, hence by the choice of D_1^* it is a $\langle_{D_1^*}$ -upper bound of $\bigcup \{\mathcal{F}_\alpha : \alpha < \lambda\}$.]

But by the choice of D in the beginning of the proof we have $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, \langle_D \rangle)$ so there is a sequence $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda \rangle$ witnessing it. By clause (f) we have $D \subseteq D_1^*$ so clearly $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda \rangle$ is also $\langle_{D_1^*}$ -increasing hence we can apply clause (g) to

the sequence $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda \rangle$ and got a $<_{D_1^*}$ -upper bound $f \in \Pi\bar{\alpha}$, contradiction to the choice of $\langle \hat{\mathcal{F}}_\alpha : \alpha < \lambda \rangle$ recalling 0.4(d) because $D \subseteq D_1^*$, contradiction. So $(*)_2$ really holds.

Choose X as in $(*)_2$, now

$$(*)_3 \quad D = \text{dual}(J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] + (Y \setminus X)).$$

[Why? The inclusion \supseteq holds by $(*)_1$ and $(*)_2$, i.e. the choice of X as a member of D . Now for every $Z \subseteq X$ which does not belong to $J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$, by the definition of $J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ there is an \aleph_1 -complete filter D_Z on Y to which Z belongs such that $\theta := \text{ps-cf}(\Pi\bar{\alpha}, <_{D_Z})$ is well defined and $\geq \lambda$. But $\theta \geq \lambda^+$ is impossible as we know that $Z \subseteq X \in J_{<\lambda^+}^{\aleph_1\text{-comp}}[\bar{\alpha}]$, so necessarily $\theta = \lambda$, hence by the choice of D by using 1.3 we have $D \subseteq D_Z$, hence $Z \neq \emptyset \pmod D$. Together we are done.]

$$(*)_4 \quad \lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_{J_{=\lambda}^{\aleph_1\text{-comp}}}), \text{ see clause (B) of the conclusion of 1.6(2).}$$

[Why? By $(*)_3$, the choice of $J_{=\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ and as $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ by the choice of D .]

$$(*)_5 \quad \lambda \notin \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha} \upharpoonright (Y \setminus X)).$$

[Why? Otherwise there is an \aleph_1 -complete filter D' on Y such that $Y \setminus X \in D'$ and $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_{D'})$. But this contradicts the choice of D by using 1.3.]

So X is as required in the desired conclusion of 1.6(2): clause (B) by $(*)_4$, clause (C) by $(*)_5$ and clause (A) follows. Note that the notation $J_{=\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ is justified, as if X' satisfies the requirements on X then $X' = X \pmod J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]$. $\square_{1.6}$

Conclusion 1.8. [AC $_{\mathcal{P}(Y)}$] Assume $\bar{\alpha} \in {}^Y\text{Ord}$ and each α_t a limit ordinal of cofinality $\geq \text{hrtg}(\mathcal{P}(Y))$ and $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ is not empty.

1) If $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\text{Fil}_{\aleph_1}^1(Y))$ then there is a function h such that:

- ₁ the domain of h is $\mathcal{P}(Y)$
- ₂ $\text{Rang}(h)$ includes $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ and is included in $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}) \cup \{0\} \cup \{\mu : \mu = \sup(\mu \cap \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}))\}$, also $\text{Rang}(h)$ includes $\{\text{cf}(\alpha_t) : t \in Y\}$, but see •₅
- ₃ $A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B)$ and $h(A) = 0 \Leftrightarrow A = \emptyset$
- ₄ $h(A) = \min\{\lambda : A \in J_{\leq\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]\}$
- ₅ if $h(A) = \lambda$ and $\text{cf}(\lambda) > \aleph_0$ then λ is regular and $\lambda \in \text{ps-tcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$, i.e. for some \aleph_1 -complete filter D on Y we have $A \in D$ and $\text{ps-tcf}(\Pi\bar{\alpha}, <_D) = \lambda$
- ₆ the set $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ has cardinality $< \text{hrtg}(\mathcal{P}(Y))$
- ₇ if $h(A) = \lambda$ and $\text{cf}(\lambda) = \aleph_0$ then we can find a sequence $\langle A_n : n < \omega \rangle$ such that $A = \cup\{A_n : n < \omega\}$ and $h(A_n) < \lambda$ for $n < \omega$
- ₈ $J_{<\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] = \{A \subseteq Y : h(A) < \lambda\}$ when $\text{cf}(\lambda) > \aleph_0$
- ₉ if $\text{cf}(\text{otp}(\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}))) > \aleph_0$ then $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ has a last member.

2) Without the extra assumption of part (1), still there is h such that:

- ₁ h is a function with domain $\mathcal{P}(Y)$

- ₂ the range of h is $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}) \cup \{0\} \cup \{\mu : \mu = \sup(\mu \cap \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})) \text{ and } \text{cf}(\mu) = \aleph_0 \text{ or just } \text{cf}(\mu) < \text{hrtg}(\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})) \text{ and } J_{<\mu}^{\aleph_1\text{-comp}}[\bar{\alpha}] \neq \cup\{J_{<\chi}^{\aleph_1\text{-comp}}[\bar{\alpha}] : \chi < \mu\}\}$
- ₃ $A \subseteq B \subseteq Y \Rightarrow h(A) \leq h(B)$ and $h(A) = 0 \Leftrightarrow A = \emptyset$
- ₄ $h(A) = \min\{\lambda : A \in J_{\leq\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]\}$
- ₅ if $h(A) = \lambda$ and $\text{cf}(\lambda) \geq \text{hrtg}(\text{ps-pcf}_{\aleph_1\text{-comp}}[\bar{\alpha}])$ then $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$, i.e. there is an \aleph_1 -complete filter D on Y such that $(\Pi\bar{\alpha}, <_D)$ has true cofinality λ
- ₆ as above
- ₇ as above
- ₈ as above.

3) The set $\mathfrak{c} := \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{a})$ satisfies $\mathfrak{c} \leq_{\text{qu}} \mathcal{P}(Y)$. If also AC_α holds for $\alpha < \text{hrtg}(\mathcal{P}(Y))$ or just $\text{AC}_{\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})}$ then we can find a sequence $\langle X_\lambda : \lambda \in \mathfrak{c} \rangle$ of subsets of Y such that for every cardinality μ , $J_{<\mu}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ is the \aleph_1 -complete ideal on Y generated by $\{X_\lambda : \lambda < \mu \text{ and } \lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})\}$.

Proof. 1) Let $\Theta = \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$. We define the function h from $\mathcal{P}(Y)$ into Θ^+ which is defined as the closure of $\Theta \cup \{0\}$, i.e. $\Theta \cup \{\mu : \mu = \sup(\mu \cap \Theta)\}$, by $h(X) = \text{Min}\{\lambda \in \Theta^+ : X \in J_{\leq\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}]\}$. It is well defined as $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ is a set, that is as $\mu_* = \text{hrtg}(\Pi\bar{\alpha})$ is well defined and so $J^{\aleph_1\text{-comp}}[\bar{\alpha}] = \mathcal{P}(Y)$ (see [She12, 5.8(2)]), non-empty by an assumption and $J_{\leq\lambda}^{\aleph_1\text{-comp}}[\bar{\alpha}] = \mathcal{P}(Y)$ when $\lambda \geq \sup(\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}))$. This function h , its range is included in Θ^+ , but $\text{otp}(\Theta^+) \leq \text{otp}(\Theta) + 1$; also clearly •₁ of the conclusion holds. Also if $\lambda \in \Theta$ and X is as in 1.6(2) then $h(X) = \lambda$; so h is a function from $\mathcal{P}(Y)$ into Θ^+ and its range include Θ hence $|\Theta| < \text{hrtg}(\mathcal{P}(Y))$ so •₂ first clause holds; the second clause of •₂ holds as trivially $h(\emptyset) = 0$ and the definition of Θ^+ and the third clause by $t \in Y \Rightarrow h(\{t\}) = \text{cf}(\alpha_t)$ holds. Now first by 1.1 we have $\theta \in \Theta \Rightarrow \theta \geq \text{hrtg}(\mathcal{P}(Y))$, hence $\theta \in \Theta \Rightarrow \theta > \sup(\Theta \cap \theta)$ so the range of h is as required in •₂.

Second, if $\lambda \in \Theta^+$ and $\text{cf}(\lambda) = \aleph_0$ then clearly $\lambda \in \Theta^+ \setminus \Theta$ and we can find an increasing sequence $\langle \lambda_n : n < \omega \rangle$ of members of $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ with limit λ . For each n there is $X_n \in J_{\leq\lambda_n}^{\aleph_1\text{-comp}}[\bar{\alpha}] \setminus J_{<\lambda_n}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ by 1.6(2), but AC_{\aleph_0} holds, see 1.7 hence such a sequence $\langle X_n : n < \omega \rangle$ exists. Easily $A := \cup\{X_n : n < \omega\} \in \mathcal{P}(Y)$ satisfies $h(A) = \lambda$ hence $\lambda \in \text{Rang}(h)$. Third, if $\lambda = \sup(\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}))$ and $\text{cf}(\lambda) > \aleph_0$, then $\bigcup_{\mu < \lambda} J_{<\mu}[\bar{\alpha}] \neq \mathcal{P}(Y)$ because Y does not belong to the union while $J_{<\lambda^+}(\bar{\alpha}) = \mathcal{P}(Y)$ so $h(Y) = \lambda$.

Fourth, assume $\lambda = h(A)$, $\lambda \notin \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ and $\text{cf}(\lambda) > \aleph_0$, we can find $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$, an increasing sequence with limit λ , but by the definition of h necessarily $\lambda \cap \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ is an unbounded subset of λ so without loss of generality all are members of $\text{ps-pcf}_{\aleph_1\text{-comp}}(\Pi\bar{\alpha})$. Now $\langle J_i := J_{<\lambda_i}^{\aleph_1\text{-comp}}[\bar{\alpha}] : i < \text{cf}(\lambda) \rangle$ is a \subseteq -increasing sequence of \aleph_1 -complete ideals on Y , no choice is needed, and by our present assumption $\aleph_0 < \text{cf}(\lambda)$ hence the union $J = \cup\{J_i : i < \text{cf}(\lambda)\}$ is an \aleph_1 -complete ideal on Y and obviously $A \notin J$. So also $D_1 = \text{dual}(J) + A$ is an \aleph_1 -complete filter hence by [She12, 5.9] (recalling the extra assumption $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\text{Fil}_{\aleph_1}^1(Y))$)

for some \aleph_1 -complete filter D_2 extending D_1 we have $\mu = \text{ps-tcf}(\Pi\alpha, <_{D_2})$ is well defined, so by 1.6(2) we have some $D_2 \cap J_{\leq \mu}^{\aleph_1\text{-comp}}[\bar{\alpha}] \neq \emptyset$ but $\emptyset = D_2 \cap J_i = D_2 \cap J_{< \lambda_i}^{\aleph_1\text{-comp}}[\bar{\alpha}]$ hence $\mu \geq \lambda_i$. Hence $\mu \geq \lambda_i$ for every $i < \text{cf}(\lambda)$ but λ is singular so $\mu > \lambda$ and $\mu \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$. Hence $\chi := \min(\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}) \setminus \lambda)$ is well defined and $J_{< \chi}^{\aleph_1\text{-comp}}[\bar{\alpha}] = J$ trivially $\chi \geq \lambda$, but as χ is regular while λ is singular clearly $\chi > \lambda$. But as $h(A) = \lambda < \chi$ we get that $A \in J_{< \chi}^{\aleph_1\text{-comp}}[\bar{\alpha}]$, contradiction to the definition of h .

So we have proved \bullet_5 , the fifth clause of the conclusion. The other clauses follow from the properties of h .

2) Similar proof.

3) We define a function g with domain $\mathcal{P}(Y)$ by $g(A) = \min\{\lambda : A \in J_{< \lambda^+}[\bar{\alpha}]\}$. This function is well defined as if $\lambda = \text{hrtg}(\Pi\bar{\alpha})$ then $A \subseteq Y \Rightarrow A \in J_{< \lambda}[\bar{\alpha}]$; and the cardinals are well ordered. Also $\mathfrak{c} \subseteq \text{Rang}(h)$ because if $\lambda \in \mathfrak{c}$, then by 1.6(2) we are done recalling that we are assuming $\text{AC}_{\mathcal{P}(Y)}$.

So clearly $\mathfrak{c} \leq_{\text{qu}} \mathcal{P}(Y)$ so as \mathfrak{c} is a set of cardinals, clearly $\text{otp}(\mathfrak{c}) < \text{hrtg}(\mathcal{P}(Y))$ hence $|\mathfrak{c}| < \text{hrtg}(\mathcal{P}(Y))$.

For the second sentence in 1.8(3) by the last sentence it suffices to assume $\text{AC}_{\mathfrak{c}}$. For $\lambda \in \mathfrak{c}$ let $\mathcal{P}_\lambda = \{X \subseteq Y : X \text{ as in 1.6(2)}\}$, so $\mathcal{P}_\lambda \neq \emptyset$. By $\text{AC}_{\mathfrak{c}}$ there is a sequence $\langle X_\lambda : \lambda \in \mathfrak{c} \rangle \in \prod_{\lambda \in \mathfrak{c}} \mathcal{P}_\lambda$. For $\lambda \in \mathfrak{c}$, let J_λ^* be the \aleph_1 -complete ideal on Y generated by $\{X_\mu : \mu \in \mathfrak{c} \cap \lambda\}$, so by the definitions of \mathcal{P}_λ we have $\mu < \lambda \wedge \mu \in \mathfrak{c} \Rightarrow X_\mu \in J_{\leq \mu}[\bar{\alpha}] \subseteq J_{< \lambda}[\bar{\alpha}]$, also $J_{< \lambda}[\bar{\alpha}]$ is \aleph_1 -complete hence $\lambda \in \mathfrak{c} \Rightarrow J_\lambda^* \subseteq J_{< \lambda}[\bar{\alpha}]$.

If for every λ equality holds we are done, otherwise there is a minimal counterexample and use 1.6(2). $\square_{1.8}$

Definition 1.9. Assume $\text{cf}(\mu) < \text{hrtg}(Y)$ and μ is singular of uncountable cofinality limit of regulars. We let

- (a) $\text{pp}_Y^*(\mu) = \sup\{\lambda : \text{ for some } \bar{\alpha}, D \text{ we have}$
 - (a) $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$,
 - (b) D is an \aleph_1 -complete filter on Y
 - (c) $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$, each α_t regular
 - (d) $\mu = \lim_D \bar{\alpha}\}$
- (b) $\text{pp}_Y^+(\mu) = \sup\{\lambda^+ : \lambda \text{ as above}\}$.
- (c) similarly $\text{pp}_{\kappa\text{-comp}, Y}^*(\mu), \text{pp}_{\kappa\text{-comp}, Y}^+(\mu)$ restricting ourselves to κ -complete filters D ; similarly for other properties
- (d) we can replace Y by an \aleph_1 -complete filter D on Y , this means we fix D but not $\bar{\alpha}$ above.

Remark 1.10. 1) of course, if we consider sets Y such that AC_Y may fail, it is natural to omit the regularity demands, so $\bar{\alpha}$ is just a sequence of ordinals.

2) We may use $\bar{\alpha}$ a sequence of cardinals, not necessarily regular; see §3.

Conclusion 1.11. $[\text{DC} + \text{AC}_{\mathcal{P}(Y)}]$ Assume $\theta = \text{hrtg}(\mathcal{P}(Y)) < \mu$, μ is as in Definition 1.9, $\mu_0 < \mu$ and $\bar{\alpha} \in {}^Y(\text{Reg} \cap \mu_0^+) \wedge \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}) \neq \emptyset \Rightarrow \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}) \subseteq \mu$. If $\sigma = |\text{Reg} \cap \mu \setminus \mu_0| < \mu$ and $\kappa = |\text{Reg} \cap \text{pp}_Y^+(\mu) \setminus \mu_0|$ then $\kappa < \text{hrtg}(\theta \times {}^Y\sigma)$.

Remark 1.12. In the ZFC parallel the assumption on $\mu_0 < \mu$ is not necessary.

Proof. Obvious by Definition [She12, 5.6] noting Conclusion 1.8 above and 1.13 below. That is, letting $\Xi := \text{Reg} \cap \text{pp}_Y^+(\mu) \setminus \mu_0$ so $|\Xi| = \kappa$ and $\Lambda = \text{Reg} \cap \mu \setminus \mu_0$, for every $\bar{\alpha} \in {}^Y \Lambda$ by Definition 1.9 the set $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ is a subset of $\text{Reg} \cap \text{pp}_Y^+(\mu) \setminus \mu_0$, and by claim 1.8 it is a set of cardinality $< \text{hrtg}(\mathcal{P}(Y))$. By Definition 1.9 and Claim 1.13 below we have $\Xi = \cup \{\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}) : \bar{\alpha} \in {}^Y \Lambda\}$. Clearly there is a function h with domain $\text{hrtg}(\mathcal{P}(Y)) \times {}^Y \sigma$ such that $\varepsilon < \text{hrtg}(\mathcal{P}(Y)) \wedge \bar{\alpha} \in {}^Y \sigma \Rightarrow (h(\varepsilon, \bar{\alpha}))$ is the ε -th member of $\text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$ if there is one, $\min(\Lambda)$ otherwise). So h is a function from $\text{hrtg}(\mathcal{P}(Y)) \times {}^Y \sigma$ onto a set including Ξ which has cardinality κ , so we are done. $\square_{1.11}$

Claim 1.13. *The No Hole Claim[DC]*

1) If $\bar{\alpha} \in {}^Y \text{Ord}$ and $\lambda_2 \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha})$, for transparency $t \in Y \Rightarrow \alpha_t > 0$ and $\text{hrtg}(\mathcal{P}(Y)) \leq \lambda_1 = \text{cf}(\lambda_1) < \lambda_2$, then for some $\bar{\alpha}' \in \Pi \bar{\alpha}$ we have $\lambda_1 = \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar{\alpha}')$.

2) In part (1), if in addition AC_Y then without loss of generality $\bar{\alpha}' \in {}^Y \text{Reg}$.

3) If in addition $\text{AC}_{\mathcal{P}(Y)} + \text{AC}_{<\kappa}$ then even witnessed by the same filter (on Y).

Proof. 1) Let D be an \aleph_1 -complete filter on Y such that $\lambda_2 = \text{ps-tcf}(\Pi \bar{\alpha}, <_D)$, let $\langle \mathcal{F}_\alpha : \alpha < \lambda_2 \rangle$ exemplify this.

First assume $\text{hrtg}(\text{Fil}_{\aleph_1}^1(Y)) \leq \lambda_1$, clearly $f \in \mathcal{F}_\alpha \Rightarrow \text{rk}_D(f) \geq \alpha$ for every $\alpha < \lambda_2$, hence in particular for $\alpha = \lambda_1$ hence there is $f \in {}^Y \text{Ord}$ such that $\text{rk}_D(f) = \lambda_1$ and now use [She12, 5.9] but there we change the filter D , (extend it), so is O.K. for part (1). In general, i.e. without the extra assumption $\text{hrtg}(\text{Fil}_{\aleph_2}^1(Y)) \leq \lambda_1$, use 1.14(1),(2) below.

2) Easy, too.

3) Similarly using 1.14(3) below. $\square_{1.13}$

Claim 1.14. *Assume $D \in \text{Fil}_{\aleph_1}^1(Y)$, $\kappa > \aleph_0$, $\mathcal{F}_\alpha \subseteq {}^Y \text{Ord}$ non-empty for $\alpha < \delta$ and $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha < \delta \rangle$ is $<_D$ -increasing, δ a limit ordinal.*

1) [DC] *There is $f^* \in \Pi \bar{\alpha}$ which satisfies $f \in \cup \{\mathcal{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_D f^*$ but there is no such $f^{**} \in \Pi \bar{\alpha}$ satisfying $f^{**} <_D f$.*

2) [AC $_{<\kappa}$] *For f^* as above, let $D_1 = D_{f^*, \bar{\mathcal{F}}} := \{Y \setminus A : A = \emptyset \text{ mod } D \text{ or } A \in D^+ \text{ and there is } f^{**} \in {}^Y \text{Ord} \text{ such that } f^{**} <_{D+A} f^* \text{ and } f \in \cup \{\mathcal{F}_\alpha : \alpha < \lambda_1\} \Rightarrow f <_{D+A} f^{**}\}$. Now D_1 is a κ -complete filter and $\emptyset \notin D_1$, D_1 extends D and if $\text{cf}(\delta) \geq \text{hrtg}(\mathcal{P}(Y))$ then $\langle \mathcal{F}_\alpha : \alpha < \delta \rangle$ witness that f^* is a $<_{D_1}$ -exact upper bound of $\bar{\mathcal{F}}$ hence $(\prod_{y \in Y} f^*(y), <_{D_1})$ has pseudo-true-cofinality $\text{cf}(\delta)$.*

3) [DC + AC $_{<\kappa}$ + AC $_{\mathcal{P}(Y)}$]

If $\text{cf}(\delta) \geq \text{hrtg}(\mathcal{P}(Y))$ then there is $f' \in {}^Y \text{Ord}$ which is an $<_D$ -exact upper bound of $\bar{\mathcal{F}}$, i.e. $f <_D f' \Rightarrow (\exists \alpha < \delta)(\exists g \in \mathcal{F}_\alpha)[f < g \text{ mod } D]$ and $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \Rightarrow f <_D f'$.

Proof. 1) If not then by DC we can find $\bar{f} = \langle f_n : n < \omega \rangle$ such that:

- (a) $f_n \in {}^Y \text{Ord}$
- (b) $f_{n+1} < f_n \text{ mod } D$
- (c) if $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha$ and $n < \omega$ then $f < f_n \text{ mod } D$.

So $A_n = \{t \in Y : f_{n+1}(t) < f_n(t)\} \in D$ hence $\cap \{A_n : n < \omega\} \in D$, contradiction.

2) First, clearly $D_1 \subseteq \mathcal{P}(Y)$ and by the assumption $\emptyset \notin D_1$. Second, if f^{**} witness $A \in D_1$ and $A \subseteq B \subseteq Y$ then f^{**} witness $B \in D_1$.

Third, we prove D_1 is closed under intersection of $< \kappa$ members, so assume $\zeta < \kappa$ and $\bar{A} = \langle A_\varepsilon : \varepsilon < \zeta \rangle$ is a sequence of members of D_1 . Let $A := \bigcap \{A_\varepsilon : \varepsilon < \zeta\}$, $B_\varepsilon = Y \setminus A_\varepsilon$ for $\varepsilon < \zeta$ and $B'_\varepsilon = B_\varepsilon \setminus \bigcup \{B_\xi : \xi < \varepsilon\}$ and $B = \bigcup \{B'_\varepsilon : \varepsilon < \zeta\}$. Clearly $B = Y \setminus A$, $A \subseteq Y$ and $\langle B'_\varepsilon : \varepsilon < \zeta \rangle$ is a sequence of pairwise disjoint subsets of Y with union B . But AC_ζ holds and $\varepsilon < \zeta \Rightarrow A_\varepsilon \in D_1$ hence we can find $\langle f_\varepsilon^{**} : \varepsilon < \zeta \rangle$ such that $f_\varepsilon^{**} \in {}^Y \text{Ord}$ and if $A_\varepsilon \notin D$ then f_ε^{**} witness $A_\varepsilon \in D_1$. Let $f^{**} \in {}^Y \text{Ord}$ be defined by $f^{**}(t) = f_\varepsilon^{**}(t)$ if $t \in B'_\varepsilon$ or $\varepsilon = 0 \wedge t \in Y \setminus B$; easily $B'_\varepsilon \in D^+ \wedge f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \Rightarrow f < f_\varepsilon^{**} = f^{**} \text{ mod } (D + B'_\varepsilon)$ but $B = \bigcup \{B'_\varepsilon : \varepsilon < \zeta\}$ and D is κ -complete hence $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \Rightarrow f < f^{**} \text{ mod } (D + B)$. So as $A = Y \setminus B$ clearly f^{**} witness $A = \bigcap_{\varepsilon < \zeta} A_\varepsilon \in D_1$ so D_1 is indeed κ -complete.

Lastly, assume $\text{cf}(\delta) \geq \text{hrtg}(\mathcal{P}(Y))$ and we shall show that f^* is an exact upper bound of $\bar{\mathcal{F}}$ modulo D_1 . So assume $f^{**} \in {}^Y \text{Ord}$ and $f^{**} < f^* \text{ mod } D_1$ and we shall prove that there are $\alpha < \delta$ and $f \in \mathcal{F}_\alpha$ such that $f^{**} \leq f \text{ mod } D_1$.

Let $\mathcal{A} = \{A \in D_1^+ : \text{there is } f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \text{ such that } f^{**} \leq f \text{ mod } (D + A)\}$, yes, not $D_1!$

Case 1: For every $B \in D_1^+$ there is $A \in \mathcal{A}$, $A \subseteq B$.

For every $A \in \mathcal{A}$ let $\alpha_A = \min\{\beta : \text{there is } f \in \mathcal{F}_\beta \text{ such that } f^{**} \leq f \text{ mod } (D + A)\}$.

So the sequence $\langle \alpha_A : A \in \mathcal{A} \rangle$ is well defined.

Let $\alpha(*) = \sup\{\alpha_A + 1 : A \in \mathcal{A}\}$, it is $< \delta$ as $\text{cf}(\delta) \geq \text{hrtg}(\mathcal{P}(Y)) \geq \text{hrtg}(\mathcal{A})$.

Choose $f \in \mathcal{F}_{\alpha(*)}$ and let $B_f := \{t \in Y : f^{**}(t) > f(t)\}$. Now if $A \in \mathcal{A}$ (so $A \in D_2^+$) and $f' \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha$ witness this (i.e. $f^{**} \leq f' \text{ mod } (D + A)$); without loss of generality $f' \in \mathcal{F}_{\alpha_A}$ hence $f' < f \text{ mod } D$ recalling $\alpha_A < \alpha(*)$, then $A \not\subseteq B_f$ as otherwise $f^{**} \leq f' < f < f^{**} \text{ mod } (D + A)$. So B_f contains no $A \in \mathcal{A}$ hence necessarily B_f is $= \emptyset \text{ mod } D_1$ by the case assumption; this means that $f^{**} \leq f \text{ mod } D_1$. So recalling $f \in \mathcal{F}_{\alpha(*)} \subseteq \bigcup_{\alpha < \delta} \mathcal{F}_\alpha$, we have “ f is as required” thus finishing the proof of “ f^* is an exact upper bound of $\bar{\mathcal{F}} \text{ mod } D$ ”.

Case 2: $B \in D_1^+$ and there is no $A \in \mathcal{A}$ such that $A \subseteq B$.

For $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha$ let $B_f = \{t \in B : f(t) < f^{**}(t)\}$ and for $\alpha < \delta$ we define $\mathcal{B}_\alpha = \{B_f : f \in \mathcal{F}_\alpha\}$ and we define a partial function h from $\mathcal{P}(Y)$ into δ by $h(A) = \sup\{\alpha < \delta : A \in \mathcal{B}_\alpha\}$. As $\text{cf}(\delta) \geq \text{hrtg}(\mathcal{P}(Y))$ necessarily $\alpha(*) = \sup(\delta \cap \text{Rang}(h))$ is $< \delta$. Choose $g \in \mathcal{F}_{\alpha(*)+1}$, hence $u := \{\alpha : \alpha \in [\alpha(*), \delta]\}$ and $B_g \in \mathcal{B}_\alpha$ is an unbounded subset of δ .

Let $A = B \cap B_g$, now if $A \in D^+$ then $\alpha \in u \Rightarrow \bigvee_{f \in \mathcal{F}_\alpha} f < f^{**} \text{ mod } (D + A)$ but $\bar{\mathcal{F}}$ is $<_D$ -increasing and $\delta = \sup(u)$ hence $f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \Rightarrow f < f^{**} \text{ mod } (D + A)$ hence by the definition of D_1 , f^{**} witness that $Y \setminus A \in D_1$, hence $A = \emptyset \text{ mod } D_1$. As $B \in D_1^+$ and $A = B \cap B_g$ it follows that $B \setminus B_g \in D_1^+$ and by the choice of \mathcal{A} the set $B \setminus B_g$ belongs to \mathcal{A} . But $B \setminus B_g \subseteq B$ by its definition so we get a contradiction to the case assumption.

3) By [She12, 5.12] without loss of generality $\bar{\mathcal{F}}$ is \aleph_0 -continuous. For every $A \in D^+$ the assumptions hold even if we replace D by $D + A$ and so there are D_1, f^* as in part (2), we are allowed to use part (1) as we have DC and part (2) as we have $\text{AC}_{<\kappa}$. As we are assuming $\text{AC}_{\mathcal{P}(Y)}$ there is a sequence $\langle (D_A, f_A) : A \in D^+ \rangle$ such that:

- (*)₁ (a) D_A is a κ -complete filter extending $D + A$
- (b) $f_A \in {}^Y\text{Ord}$ is a $<_{D_A}$ -exact upper bound of $\bar{\mathcal{F}}$.

Recall $|A| \leq_{\text{qu}} |B|$ is defined as: A is empty or there is a function from B onto A . Of course, this implies $\text{hrtg}(A) \leq \text{hrtg}(B)$.

Let $\mathcal{U} = \langle \mathcal{U}_t : t \in Y \rangle$ be defined by $\mathcal{U}_t = \{f_A(t) : A \in D^+\} \cup \{\sup\{f(t) : f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha\}\}$ hence $t \in Y \Rightarrow 0 < |\mathcal{U}_t| \leq_{\text{qu}} \mathcal{P}(Y)$ even uniformly so there is a sequence $\langle h_t : t \in Y \rangle$ such that h_t is a function from $\mathcal{P}(Y)$ onto \mathcal{U}_t hence $|\prod_{t \in Y} \mathcal{U}_t| \leq_{\text{qu}} \mathcal{P}(Y) \times Y \leq_{\text{qu}} \mathcal{P}(Y \times Y)$ but $\text{AC}_{\mathcal{P}(Y)}$ holds hence Y can be well ordered however without loss of generality Y is infinite hence $|Y \times Y| = Y$, so $|\prod_{t \in Y} \mathcal{U}_t| \leq_{\text{qu}} |\mathcal{P}(Y)|$.

Let $\mathcal{G} = \{g : g \in \prod_{t \in Y} \mathcal{U}_t \text{ and not for every } f \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha \text{ do we have } f < g \text{ mod } D\}$, so $|\mathcal{G}| \leq |\prod_{t \in Y} \mathcal{U}_t| \leq_{\text{qu}} |\mathcal{P}(Y \times Y)| = |\mathcal{P}(Y)|$ hence $\text{hrtg}(\mathcal{G}) \leq \text{hrtg}(\mathcal{P}(Y)) \leq \text{cf}(\delta)$.

Now for every $g \in \mathcal{G}$ the sequence $\langle \{t \in Y : g(t) \leq f(t)\} : f \in \bigcup_{\beta < \alpha} \mathcal{F}_\beta \rangle$ is a \subseteq -increasing sequence of subsets of $\mathcal{P}(Y)$, but $\text{hrtg}(\mathcal{P}(Y)) \leq \text{cf}(\delta)$ hence the sequence is eventually constant and let $\alpha(g) < \delta$ be the minimal α such that

$$(*)_g \ (\forall \beta)[\alpha \leq \beta < \delta \Rightarrow \{t \in Y : g(t) \leq f(t)\} : f \in \bigcup_{\gamma < \beta} \mathcal{F}_\gamma = \{t \in Y : g(t) \leq f(t)\} : f \in \bigcup_{\gamma < \alpha} \mathcal{F}_\gamma].$$

But recalling $\text{hrtg}(\mathcal{G}) \leq \text{cf}(\delta)$, the ordinal $\alpha(*) := \sup\{\alpha(g) : g \in \mathcal{G}\}$ is $< \delta$. Now choose $f^* \in \mathcal{F}_{\alpha(*)+1}$ and define $g^* \in \prod_{t \in Y} \mathcal{U}_t$ by $g^*(t) = \min(\mathcal{U}_t \setminus f^*(t))$, well defined as $\sup\{f(t) : t \in \bigcup_{\alpha < \delta} \mathcal{F}_\alpha\} \in \mathcal{U}_t$. It is easy to check that g^* is as required. $\square_{1.14}$

Observation 1.15. 1) Let D be a filter on Y .

If D is κ -complete for every κ then for every $f \in {}^Y\text{Ord}$ and $A \in D^+$ there is $B \subseteq A$ from D^+ such that $f|_B$ is constant.

2) If $\bar{\alpha} = \langle \alpha_s : s \in Y \rangle$ and $X_\varepsilon \subseteq Y$ for $\varepsilon < \alpha < \kappa$ and $X = \bigcup_{\varepsilon} X_\varepsilon$ then $\text{ps} - \text{pcf}_{\kappa\text{-comp}}(\bar{\alpha} \upharpoonright X) = \bigcup_{\varepsilon} \text{ps} - \text{pcf}_{\kappa\text{-comp}}(\bar{\alpha} \upharpoonright X_\varepsilon)$.

Remark 1.16. 1) Note that 1.15(1) is not empty; its assumptions hold when Y is an infinite set such that: for every $X \subseteq Y$, $|X| < \kappa \vee |Y \setminus X| < \kappa$ and $D = \{X \subseteq Y : |Y \setminus X| \not\leq \kappa\}$.

Proof. Straightforward. $\square_{1.15}$

§ 2. COMPOSITION AND GENERATING SEQUENCE FOR PSEUDO PCF

How much choice suffice to show $\lambda = \text{ps-tcf}(\prod_{(i,j) \in Y} \lambda_{i,j}/D)$ when λ_i is the pseudo true equality of $(\prod_{j \in Y_i} \lambda_{i,j}, <_{D_i})$ for $i \in Z$ where $Z = \{i : (i,j) \in Y\}$ and $Y_i = \{(i,j) : i \in Z, j \in Y_i\}$ and $\lambda = \text{ps-tcf}(\prod_{i \in Z} \lambda_i, <_E)$? This is 2.6, the parallel of [She94, Ch.II,1.10,pg.12].

Claim 2.1. *If \boxplus below holds then for some partition (Y_1, Y_2) of Y and club E of λ we have*

- ⊕ (a) *if $Y_1 \in D^+$ and $f, g \in \cup\{\mathcal{F}_\alpha : \alpha \geq \min(E)\}$ then $f = g \text{ mod}(D+Y_1)$*
- (b) *if $Y_2 \in D^+$ then $\langle \mathcal{F}_\alpha : \alpha \in E \rangle$ is $<_{D+Y_2}$ -increasing*

where

- ⊕ (a) *λ is regular $\geq \text{hrtg}(\mathcal{P}(Y))$*
- (b) *$\mathcal{F}_\alpha \subseteq {}^Y \text{Ord}$ for $\alpha < \lambda$ is non-empty*
- (c) *D is an \aleph_1 -complete filter on Y*
- (d) *if $\alpha_1 < \alpha_2 < \lambda$ and $f_\ell \in \mathcal{F}_{\alpha_\ell}$ for $\ell = 1, 2$ then $f_1 \leq f_2 \text{ mod } D$.*

Proof. For $Z \in D^+$ let

- (*)₁ (a) $S_Z = \{(\alpha, \beta) : \alpha \leq \beta < \lambda \text{ and for some } f \in \mathcal{F}_\alpha \text{ and } g \in \mathcal{F}_\beta \text{ we have } f < g \text{ mod } (D+Z)\}$
- (b) $S_Z^+ = \{(\alpha, \beta) : \alpha \leq \beta < \lambda \text{ and for every } f \in \mathcal{F}_\alpha \text{ and } g \in \mathcal{F}_\beta \text{ we have } f < g \text{ mod } (D+Z)\}$.

Note

- (*)₂ (a) *if $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ and $(\alpha_2, \alpha_3) \in S_Z$ then $(\alpha_1, \alpha_4) \in S_Z$*
- (b) *similarly for S_Z^+*
- (c) *if $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ and $(\alpha_1 \neq \alpha_2) \wedge (\alpha_3 \neq \alpha_4)$ and $(\alpha_2, \alpha_3) \in S_Z$ then $(\alpha_1, \alpha_4) \in S_Z^+$*
- (d) $S_Z \subseteq S_Z^+$.

[Why? By the definitions.]

Let

- (*)₃ $J := \{Z \subseteq Y : Z \in \text{dual}(D) \text{ or } Z \in D^+ \text{ and } (\forall \alpha < \lambda)(\exists \beta)((\alpha, \beta) \in S_Z^+)\}$.

Next

- (*)₄ (a) *J is an \aleph_1 -complete ideal on Y*
- (b) *if D is κ -complete then J is κ -complete¹*
- (c) $J = \{Z \subseteq Y : Z \in \text{dual}(D) \text{ or } Z \in D^+ \text{ and } (\forall \alpha < \lambda)(\exists \beta)((\alpha, \beta) \in S_Z)\}$.

[Why? For clauses (a),(b) check and for clause (c) recall (*)₂(c).]

Let

¹not used; note that AC_κ holds in the non-trivial case as $\text{AC}_{\mathcal{P}(Y)}$ holds, see 1.15

- (*)₅ (a) for $Z \in J^+$ let $\alpha(Z) = \min\{\alpha < \lambda: \text{for no } \beta \in (\alpha, \lambda) \text{ do we have } (\alpha, \beta) \in S_Z\}$
- (b) $\alpha(*) = \sup\{\alpha_Z : Z \in J^+\}$
- (*)₆ (a) for $Z \in J^+$ we have $\alpha(Z) < \lambda$
- (b) $\alpha(*) < \lambda$.

[Why? Clause (a) by the definition of the ideal J , and clause (b) as $\lambda = \text{cf}(\lambda) \geq \text{hrtg}(\mathcal{P}(Y))$.]

Let

- (*)₇ (a) for $Z \in D^+$ let $f_Z : \lambda \rightarrow \lambda + 1$ be defined by $f_Z(\alpha) = \text{Min}\{\beta : (\alpha, \beta) \in S_Z^+ \text{ or } \beta = \lambda\}$
- (b) $f_* : \lambda \rightarrow \lambda$ be defined by: $f_*(\alpha) = \sup\{f_Z(\alpha) : Z \in D^+ \cap J\}$
- (c) $E = \{\delta : \delta \text{ a limit ordinal } < \lambda \text{ such that } \alpha < \delta \Rightarrow f_*(\alpha) < \delta\} \setminus \alpha(*)$.

Hence

- (*)₈ (a) if $Z \in D^+ \cap J$ then f_Z is indeed a function from λ to λ
- (b) f_* is indeed a function from λ to λ
- (c) f_* is non-decreasing
- (d) E is a club of λ .

[Why? Clause (a) by the definition of J and of f_* and clause (b) as $\lambda = \text{cf}(\lambda) \geq \text{hrtg}(\mathcal{P}(Y))$ and clause (c) by (*)₂ and clause (d) follows from (b)+(c).]

- (*)₉ Let $\alpha_0 = \min(E)$, $\alpha_1 = \min(E \setminus (\alpha_0 + 1))$ choose $f_0 \in \mathcal{F}_{\alpha_0}$, $f_1 \in \mathcal{F}_{\alpha_1}$ and let $Y_1 = \{y \in Y : f_0(y) = f_1(y)\}$ and $Y_2 = Y \setminus Y_1$
- (*)₁₀ (Y_1, Y_2, E) are as required.

[Why? Think.]

□_{2.1}

Claim 2.2. *We have $\lambda = \text{ps} - \text{tcf}(\Pi\bar{\alpha}_1, <_{D_1}) = \text{ps} - \text{tcf}(\Pi\bar{\alpha}, <_D)$, this means also that one of them is well defined iff the other is, when*

- (a) $\bar{\alpha} \in {}^Y\text{Ord}$ and $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Y)$
- (b) E is the equivalence relation on Y such that $sEt \Leftrightarrow \alpha_s = \alpha_t$
- (c) D is a filter on X
- (d) $Y_1 = Y/E$
- (e) $D_1 = \{Z \subseteq Y/E : \cup\{X : X \in Z\} \in D\}$, so a filter on Y_1
- (f) $\bar{\alpha}_1 = \langle \alpha_{1, y_1} : y_1 \in Y_1 \rangle$ where $y_1 = y/E \Rightarrow \alpha_{1, y_1} = \alpha_y$.

Remark 2.3. We can for the “only if” direction in 2.2 weaken the demand on $\text{cf}(\alpha_t)$ to $\text{cf}(\alpha_t) \geq \text{hrtg}(t/E)$.

Proof. The claim means

- (*) $\lambda = \text{ps} - \text{tcf}(\Pi\bar{\alpha}_1, <_{D_1})$ if and only if $\lambda = \text{ps} - \text{tcf}(\Pi\bar{\alpha}_2, <_{D_2})$.

First “only if” direction holds by 2.4.

Second, for the “if direction”, assume that $\text{ps} - \text{pcf}(\Pi\bar{\alpha}_1, <_{D_1})$ is well defined and call it λ_1 . Let $\langle \mathcal{F}_{1,\alpha} : \alpha < \lambda \rangle$ witness this, for $f \in \mathcal{F}_{1,\alpha}$ let $f^{[0]} \in {}^Y \text{Ord}$ be defined by $f^{[0]}(s) = f(s/E)$ and let $\mathcal{F}_\alpha = \{f^{[0]} : f \in \mathcal{F}_{1,\alpha}\}$. It is easy to check that $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ witness $\lambda_1 = \text{ps} - \text{tcf}(\Pi\bar{\alpha}, <_D)$ recalling $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(Y)$ by clause (d), so we have proved also the “if” implication. $\square_{2.2}$

By the following claims we do not really lose by using $\mathfrak{a} \subseteq \text{Reg}$ instead $\bar{\alpha} \in {}^Y \text{Ord}$ as by 2.5 below, without loss of generality $\alpha_t = \text{cf}(\alpha_t)$ (when AC_Y) and by 2.2.

Claim 2.4. *Assume $\bar{\alpha} \in {}^Y \text{Ord}$, $D \in \text{Fil}(Y)$ and $\lambda = \text{ps-pcf}(\Pi\bar{\alpha}, <_D)$ so λ is regular, and $y \in Y \Rightarrow \alpha_y < \lambda$.*

If $\langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$ witness $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ and $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \text{hrtg}(Y)$ and $\lambda \geq \text{hrtg}(Y)$ then for some e :

- (a) $e \in \text{eq}(Y) = \{e : e \text{ an equivalence relation on } Y\}$
- (b) the sequence $\mathcal{F}_e = \langle \mathcal{F}_{e,\alpha} : \alpha < \lambda \rangle$ witness $\text{ps-tcf}(\langle \alpha_{y/e} : y \in Y/e \rangle, D/e)$ where
- (c) $\alpha_{y/e} = \alpha_y$, $D/e = \{A/e : A \in D\}$ where $A/e = \{y/e : y \in A\}$ and $\mathcal{F}_{e,\alpha} = \{f^{[*]} : f \in \mathcal{F}_\alpha\}$, $f^{[*]} : Y/e \rightarrow \text{Ord}$ is defined by $f^{[*]}(t/e) = \sup\{f(s) : s \in t/e\}$; noting $\text{hrtg}(Y/e) \leq \text{hrtg}(Y)$
- (d) $e = \{(s_1, s_2) : \alpha_{s_1} = \alpha_{s_2}\}$.

Proof. Let $e = \text{eq}(\bar{\alpha}) = \{(y_1, y_2) : y_1 \in Y, y_2 \in Y \text{ and } \alpha_{y_1} = \alpha_{y_2}\}$. For each $f \in \Pi\bar{\alpha}$ let the function $f^{[*]} \in \Pi\bar{\alpha}$ be defined by $f^{[*]}(y) = \sup\{f(z) : z \in y/e\}$. Clearly $f^{[*]}$ is a function from $\prod_{y \in Y} (\alpha_y + 1)$ and it belongs to $\Pi\bar{\alpha}$ as $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \text{hrtg}(Y) \geq \text{hrtg}(y/E)$. Let $H : \lambda \rightarrow \lambda$ be: $H(\alpha) = \min\{\beta < \lambda : \beta > \alpha \text{ and there are } f_1 \in \mathcal{F}_\alpha \text{ and } f_2 \in \mathcal{F}_\beta \text{ such that } f_1^{[*]} < f_2 \text{ mod } D\}$, well defined as \mathcal{F} is cofinal in $(\Pi\bar{\alpha}, <_D)$. We choose $\alpha_i < \lambda$ by induction on i by: $\alpha_i = \cup\{H(\alpha_j) + 1 : j < i\}$. So $\alpha_0 = 0$ and $\langle \alpha_i : i < \lambda \rangle$ is increasing continuous. Let $\mathcal{F}'_i = \{f^{[*]} : f \in \mathcal{F}_{\alpha_i}\}$ and there is $g \in \mathcal{F}_{H(\alpha_i)} = \mathcal{F}_{\alpha_{i+1}-1}$ such that $f^{[*]} < g \text{ mod } D$.

So

$$(*)_1 \quad \mathcal{F}'_i \subseteq \{f \in \Pi\bar{\alpha} : \text{eq}(\bar{\alpha}) \text{ refine } \text{eq}(f)\}.$$

[By the choice of \mathcal{F}'_i and of e]

$$(*)_2 \quad \mathcal{F}'_i \text{ is non-empty.}$$

[Why? By the choice of $H(\alpha_i)$.]

$$(*)_3 \quad \text{if } i(1) < i(2) < \lambda \text{ and } h_\ell \in \mathcal{F}'_{i_\ell} \text{ for } \ell = 1, 2 \text{ then } h_1 < h_2 \text{ mod } D.$$

[Why? For $\ell = 1, 2$ let $g_\ell \in \mathcal{F}_{H(\alpha_{i(\ell)})}$ be such that $h_\ell = f_\ell^{[*]} < g_\ell \text{ mod } D$, exists by the definition of $\mathcal{F}'_{i(\ell)}$. But $H(\alpha_{i(1)}) < \alpha_{i(1)+1} \leq \alpha_{i(2)}$ hence $g_1 \leq f_2 \text{ mod } D$ so together $h_1 = f_1^{[*]} < g_1 \leq f_2 \leq f_2^{[*]} = h_2 \text{ mod } D$ hence we are done.]

$$(*)_4 \quad \bigcup_{i < \lambda} \mathcal{F}'_i \text{ is cofinal in } (\Pi\bar{\alpha}, <_D).$$

[Easy, too.]

Lastly, let $\mathcal{F}_i^+ = \{f/e : e \in \mathcal{F}_i'\}$ where $f/e \in {}^Y\text{Ord}$, is defined by $(f/e)(y/e) = f(y)$, clearly well defined. $\square_{2.4}$

Claim 2.5. *Assume AC_Y and $\bar{\alpha}_\ell = \langle \alpha_y^\ell : y \in Y \rangle \in {}^Y\text{Ord}$ for $\ell = 1, 2$. If $y \in Y \Rightarrow \text{cf}(\alpha_y^1) = \text{cf}(\alpha_y^2)$ then $\lambda = \text{ps-tcf}(\prod_{(i,j) \in Y} \bar{\alpha}_1, <_D)$ iff $\lambda = \text{ps-tcf}(\prod_{(i,j) \in Y} \bar{\alpha}_2, <_D)$.*

Proof. Straightforward. $\square_{2.5}$

Now we come to the heart of the matter

Theorem 2.6. *The Composition Theorem [Assume AC_Z and $\kappa \geq \aleph_0$]*

We have $\lambda = \text{ps-tcf}(\prod_{(i,j) \in Y} \lambda_{i,j}, <_D)$ and D is a κ -complete filter on Y when:

- (a) E is a κ -complete filter on Z
- (b) $\langle \lambda_i : i \in Z \rangle$ is a sequence of regular cardinals
- (c) $\lambda = \text{ps-tcf}(\prod_{i \in Z} \lambda_i, <_E)$
- (d) $\bar{Y} = \langle Y_i : i \in Z \rangle$
- (e) $\bar{D} = \langle D_i : i \in Z \rangle$
- (f) D_i is a κ -complete filter on Y_i
- (g) $\bar{\lambda} = \langle \lambda_{i,j} : i \in Z, j \in Y_i \rangle$ is a sequence of regular cardinals (or just limit ordinals)
- (h) $\lambda_i = \text{ps-tcf}(\prod_{j \in Y_i} \lambda_{i,j}, <_{D_i})$
- (i) $Y = \{(i, j) : j \in Y_i \text{ and } i \in Z\}$
- (j) $D = \{A \subseteq Y : \text{for some } B \in E \text{ we have } i \in B \Rightarrow \{j : (i, j) \in A\} \in D_i\}$.

Proof.

(*)₀ D is a κ -complete filter on Y .

[Why? Straightforward (and do not need any choice).]

Let $\langle \mathcal{F}_{i,\alpha} : \alpha < \lambda_i, i \in Z \rangle$ be such that

- (*)₁ (a) $\bar{\mathcal{F}}_i = \langle \mathcal{F}_{i,\alpha} : \alpha < \lambda_i \rangle$ witness $\lambda_i = \text{ps-tcf}(\prod_{j \in Y_i} \lambda_{i,j}, <_{D_i})$
- (b) $\mathcal{F}_{i,\alpha} \neq \emptyset$.

[Why? Exists by clause (h) of the assumption and AC_Z , for clause (b) recall [She12, 5.6].]

By clause (c) of the assumption let $\bar{\mathcal{G}}$ be such that

- (*)₂ (a) $\bar{\mathcal{G}} = \langle \mathcal{G}_\beta : \beta < \lambda \rangle$ witness $\lambda = \text{ps-tcf}(\prod_{i \in Z} \lambda_i, <_E)$
- (b) $\mathcal{G}_\beta \neq \emptyset$ for $\beta < \lambda$.

Now for $\beta < \lambda$ let

- (*)₃ $\mathcal{F}_\beta := \{f : f \in \prod_{(i,j) \in Y} \lambda_{i,j} \text{ and for some } g \in \mathcal{G}_\beta \text{ and } \bar{h} = \langle h_i : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i,g(i)} \text{ we have } (i, j) \in Y \Rightarrow f((i, j)) = h_i(j)\}$
- (*)₄ the sequence $\langle \mathcal{F}_\beta : \beta < \lambda \rangle$ is well defined (so exists).

[Why? Obviously.]

(*)₅ if $\beta_1 < \beta_2$, $f_1 \in \mathcal{F}_{\beta_1}$ and $f_2 \in \mathcal{F}_{\beta_2}$ then $f_1 <_D f_2$.

[Why? Let $g_\ell \in \mathcal{G}_{\beta_\ell}$ and $\bar{h}_\ell = \langle h_i^\ell : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i, g_\ell(i)}$, witness $f_\ell \in \mathcal{F}_{\beta_\ell}$ for $\ell = 1, 2$. As $\beta_1 < \beta_2$ by (*)₂ we have $B := \{i \in Z : g_1(i) < g_2(i)\} \in E$. For each $i \in B$ we know that $g_1(i) < g_2(i) < \lambda_i$ and so $h_i^1 \in \mathcal{F}_{i, g_1(i)}$, $h_i^2 \in \mathcal{F}_{i, g_2(i)}$; hence recalling the choice of $\langle \mathcal{F}_{i, \alpha} : \alpha < \lambda_i \rangle$, see (*)₁, we have $A_i \in D_i$ where for every $i \in Z$ we let $A_i := \{j \in Y_i : h_i^1(j) < h_i^2(j)\}$. As \bar{h}_1, \bar{h}_2 exists clearly $\langle A_i : i \in Z \rangle$ exist hence $A = \{(i, j) : i \in B \text{ and } j \in A_i\}$ is a well defined subset of Y and it belongs to D by the definition of D .

Lastly, $(i, j) \in A \Rightarrow f_1((i, j)) < f_2((i, j))$, shown above; so by the definition of D we are done.]

(*)₆ for every $\beta < \lambda$ the set \mathcal{F}_β is non-empty.

[Why? Recall $\mathcal{G}_\beta \neq \emptyset$ by (*)₂(b) and let $g \in \mathcal{G}_\beta$. As $\langle \mathcal{F}_{i, g(i)} : i \in Z \rangle$ is a sequence of non-empty sets (recalling (*)₂(b)), and we are assuming AC_Z there is a sequence $\langle h_i : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i, g(i)}$. Let f be the function with domain Y defined by $f((i, j)) = h_i(j)$; so g, \bar{h} witness $f \in \mathcal{F}_\beta$, so $\mathcal{F}_\beta \neq \emptyset$ as required.]

(*)₇ if $f_* \in \prod_{(i, j) \in Y} \lambda_{i, j}$ then for some $\beta < \lambda$ and $f \in \mathcal{F}_\beta$ we have $f_* < f \text{ mod } D$.

[Why? We define $\bar{f} = \langle f_i^* : i \in Z \rangle$ as follows: f_i^* is the function with domain Y_i such that

$$j \in Y_i \Rightarrow f_i^*(j) = f((i, j)).$$

Clearly \bar{f} is well defined and for each i , $f_i^* \in \prod_{j \in Y_i} \lambda_{i, j}$ hence by (*)₁(a) for some $\alpha < \lambda_i$ and $h \in \mathcal{F}_{i, \alpha}$ we have $f_i^* < h \text{ mod } D_i$ and let α_i be the first such α so $\langle \alpha_i : i \in Z \rangle$ exists.

By the choice of $\langle \mathcal{G}_\beta : \beta < \lambda \rangle$ there are $\beta < \lambda$ and $g \in \mathcal{G}_\beta$ such that $\langle \alpha_i : i \in Z \rangle < g \text{ mod } E$ hence $A := \{i \in Z : \alpha_i < g(i)\}$ belongs to E . So $\langle \mathcal{F}_{i, g(i)} : i \in Z \rangle$ is a (well defined) sequence of non-empty sets hence recalling AC_Z there is a sequence $\bar{h} = \langle h_i : i \in Z \rangle \in \prod_{i \in Z} \mathcal{F}_{i, g(i)}$. By the property of $\langle \mathcal{F}_{i, \alpha} : \alpha < \lambda_i \rangle$ and the choice of h_i recalling the definition of A , we have $i \in A \Rightarrow f_i^* < h_i \text{ mod } D_i$, exists as $\langle h_i : i \in Z \rangle$ exist.

Lastly, let $f \in \prod_{(i, j) \in Y} \lambda_{i, j}$ be defined by $f((i, j)) = h_i(j)$. Easily g, \bar{h} witness that $f \in \mathcal{F}_\beta$, and by the definition of D , recalling $A \in E$ and the choice of \bar{h} we have $f_* < f \text{ mod } D$, so we are done.]

Together we are done proving the theorem. □_{2.6}

Conclusion 2.7. The pcf closure conclusion Assume $\text{AC}_{\mathcal{P}(\mathbf{a})}$. We have $\mathfrak{c} = \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{c})$ when:

- (a) \mathbf{a} a set of regular cardinals, non-empty
- (b) $\text{hrtg}(\mathcal{P}(\mathbf{a})) \leq \min(\mathbf{a})$
- (c) $\mathfrak{c} = \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{a})$.

Proof. Note that \mathfrak{c} is non-empty because $\mathfrak{a} \subseteq \mathfrak{c}$.

Assume $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{c})$, hence there is E an \aleph_1 -complete filter on \mathfrak{c} such that $\lambda = \text{ps-tcf}(\Pi\mathfrak{c}, <_E)$. As we have $\text{AC}_{\mathcal{P}(\mathfrak{a})}$ by 1.3 (as the D there is unique) there is a sequence $\langle D_\theta : \theta \in \mathfrak{c} \rangle$, D_θ an \aleph_1 -complete filter on \mathfrak{a} such that $\theta = \text{ps-tcf}(\Pi\mathfrak{a}, <_{D_\theta})$, also by 1.8 there is a function h from $\mathcal{P}(\mathfrak{a})$ onto \mathfrak{c} , let $E_1 = \{S \subseteq \mathcal{P}(\mathfrak{a}) : \{\theta \in \mathfrak{c} : h^{-1}\{\theta\} \subseteq S\} \in E\}$. By claim 2.2, the “if” direction with $\mathcal{P}(Y)$ here standing for Y there, we have $\lambda = \text{ps-tcf}(\Pi\{h(\mathfrak{b}) : \mathfrak{b} \in \mathcal{P}(\mathfrak{a})\}, <_{E_1})$ and E_1 is an \aleph_1 -complete filter on $\mathcal{P}(\mathfrak{a})$.

Now we apply Theorem 2.6 with $E_1, \langle D_{h(\mathfrak{b})} : \mathfrak{b} \in \mathcal{P}(\mathfrak{a}) \rangle, \lambda, \langle h(\mathfrak{b}) : \mathfrak{b} \in \mathcal{P}(\mathfrak{a}) \rangle, \langle \theta : \theta \in \mathfrak{a} \rangle$ here standing for $E, \langle D_i : i \in Z \rangle, \lambda, \langle \lambda_i : i \in Z \rangle, \langle \lambda_{i,j} : j \in Y_i \rangle$ for every $j \in Z$ (constant here). We get a filter D_1 on $Y = \{(\mathfrak{b}, \theta) : \mathfrak{b} \in \mathcal{P}(\mathfrak{a}), \theta \in \mathfrak{a}\}$ such that $\lambda = \text{ps-tcf}(\Pi\{\theta : (\mathfrak{b}, \theta) \in Y\}, <_{D_1})$.

Now $|Y| = |\mathcal{P}(\mathfrak{a})|$ as \mathfrak{a} can be well ordered (hence $\aleph_0 \leq |\mathfrak{a}|$ or \mathfrak{a} finite and all is trivial) so applying 2.2 again we get an \aleph_1 -complete filter D on \mathfrak{a} such that $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_D)$, so we are done. $\square_{2.7}$

Definition 2.8. Let a set \mathfrak{a} of regular cardinals.

1) We say $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \rangle$ is a generating sequence for \mathfrak{a} when:

- (α) $\mathfrak{b}_\lambda \subseteq \mathfrak{a} \subseteq \mathfrak{c} \subseteq \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{a})$
- (β) $J_{<\lambda^+}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda$ for every $\lambda \in \mathfrak{c}$, hence for every cardinal λ we have $J_{<\lambda}[\mathfrak{a}]$ is the \aleph_1 -complete ideal on \mathfrak{a} generated by $\{\mathfrak{b}_\theta : \theta \in \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{a})$ and $\theta < \lambda\}$.

2) We say $\bar{\mathcal{F}}$ is a witness for $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \subseteq \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{a}) \rangle$ when:

- (α) $\bar{\mathcal{F}} = \langle \bar{\mathcal{F}}_\lambda : \lambda \in \mathfrak{c} \rangle$
- (β) $\bar{\mathcal{F}}_\lambda = \langle \mathcal{F}_{\lambda,\alpha} : \alpha < \lambda \rangle$ witness $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_{=\lambda}[\mathfrak{a}]})$.

3) Above $\bar{\mathfrak{b}}$ is closed when $\mathfrak{b}_\lambda = \mathfrak{a} \cap \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{b}_\lambda)$; if \mathfrak{a} is not mentioned it means $\mathfrak{a} = \mathfrak{c}$.

3A) Above $\bar{\mathfrak{b}}$ is smooth when $\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda$.

4) We say above $\bar{\mathfrak{b}}$ is full when $\mathfrak{c} = \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{a})$.

Remark 2.9. 1) Note that 1.8 gives sufficient conditions for the existence of $\bar{\mathfrak{b}}$ as in 2.8(1) which is full.

2) Of course, Definition 2.8 is interesting particularly when $\mathfrak{a} = \text{ps-pcf}_{\aleph_1\text{-com}}(\mathfrak{a})$.

Theorem 2.10. Assume $\text{AC}_{\mathfrak{c}}$ and $\text{AC}_{\mathcal{P}(\mathfrak{a})}$. Then $\mathfrak{c} = \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{c})$ has a full closed generating sequence for \aleph_1 -complete filters (see below) when:

- (a) \mathfrak{a} is a set of regular cardinals
- (b) $\text{hrtg}(\mathcal{P}(\mathfrak{a})) < \min(\mathfrak{a})$
- (c) $\mathfrak{c} = \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathfrak{a})$.

Proof. Proof of 2.10

$$(*)_1 \quad \mathfrak{c} = \text{ps-pcf}_{\aleph_1\text{-com}}(\mathfrak{c}).$$

[Why? By 2.7 using $\text{AC}_{\mathcal{P}(\mathfrak{a})}$.]

$$(*)_2 \quad \text{there is a generating sequence } \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \rangle \text{ for } \mathfrak{a}.$$

[Why? By 1.8(3) using also AC_c .]

(*)₃ let $\mathbf{b}_\lambda^* = \text{ps-pcf}_{\aleph_1\text{-com}}(\mathbf{b}_\lambda)$ for $\lambda \in \mathbf{c}$.

Now

- (*)₄ (a) $\bar{\mathbf{b}}^* = \langle \mathbf{b}_\lambda^* : \lambda \in \mathbf{c} \rangle$ is well defined
- (b) $\mathbf{b}_\lambda \subseteq \mathbf{b}_\lambda^* \subseteq \mathbf{c}$
- (c) $\mathbf{b}_\lambda^* = \text{ps-pcf}_{\aleph_1\text{-com}}(\mathbf{b}_\lambda^*)$
- (d) $\lambda = \max(\mathbf{b}_\lambda^*)$
- (e) $\lambda \notin \text{pcf}(\mathbf{c} \setminus \mathbf{b}_\lambda^*)$.

[Why? First, $\bar{\mathbf{b}}^*$ is well defined as $\bar{\mathbf{b}} = \langle \mathbf{b}_\lambda : \lambda \in \mathbf{c} \rangle$ is well defined. Second, $\mathbf{b}_\lambda \subseteq \mathbf{b}_\lambda^*$ by the choice of \mathbf{b}_λ^* and $\mathbf{b}_\lambda^* \subseteq \mathbf{c}$ as $\mathbf{b}_\lambda \subseteq \mathbf{a}$ hence $\mathbf{b}_\lambda^* = \text{ps-pcf}_{\aleph_1\text{-com}}(\mathbf{b}_\lambda^*) \subseteq \text{ps-pcf}_{\aleph_1\text{-com}}(\mathbf{c}) = \mathbf{c}$, the last equality by 2.7. Third, $\mathbf{b}_\lambda^* = \text{ps-pcf}_{\aleph_1\text{-com}}(\mathbf{b}_\lambda^*)$ by Conclusion 2.7, it is easy to check that its assumption holds recalling $\mathbf{b}_\lambda \subseteq \mathbf{a}$. Fourth, $\lambda \in \mathbf{b}_\lambda^*$ as $J_{=\lambda}[\mathbf{a}]$ witness $\lambda \in \text{ps-pcf}_{\aleph_1\text{-com}}(\mathbf{b}_\lambda) = \mathbf{b}_\lambda^*$ and $\max(\mathbf{b}_\lambda^*) = \lambda$ by (*)₂ recalling Definition 2.8.

Lastly, note that $\text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{a}) = \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{b}_\lambda) \cup \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{a} \setminus \mathbf{b}_\lambda)$ by 1.15(2) hence $\mu \in \mathbf{c} \setminus \mathbf{b}_\lambda^* \Rightarrow \mu \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{a} \setminus \mathbf{b}_\lambda)$; so if $\lambda \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{c} \setminus \mathbf{b}_\lambda^*)$ by 2.7 it follows that $\lambda \in \text{pcf}(\mathbf{a} \setminus \mathbf{b}_\lambda^*)$ which contradict 1.8(3), 1.6(2) so $\lambda \notin \text{ps-pcf}_{\aleph_1\text{-comp}}(\mathbf{c} \setminus \mathbf{b}_\lambda^*)$ that is, clause (e) holds.]

We can now choose $\bar{\mathcal{F}}$ such that

- (*)₅ (a) $\bar{\mathcal{F}} = \langle \bar{\mathcal{F}}_\lambda : \lambda \in \mathbf{c} \rangle$
- (b) $\bar{\mathcal{F}}_\lambda = \langle \mathcal{F}_{\lambda,\alpha} : \alpha < \lambda \rangle$
- (c) $\bar{\mathcal{F}}_\lambda$ witness $\lambda = \text{ps-tcf}(\Pi \mathbf{a}, <_{J_{=\lambda}[\mathbf{a}]})$
- (d) if $\lambda \in \mathbf{a}$, $\alpha < \lambda$ and $f \in \mathcal{F}_{\lambda,\alpha}$ then $f(\lambda) = \alpha$.

[Why? For each λ there is such $\bar{\mathcal{F}}$ as $\lambda = \text{ps-tcf}(\Pi \mathbf{a}, <_{J_{=\lambda}[\mathbf{a}]})$. But we are assuming AC_c and for clause (d) it is easy; in fact it is enough to use $AC_{\mathcal{P}(\mathbf{a})}$ and h as in 2.7, getting $\langle \bar{\mathcal{F}}_\mathbf{b} : \mathbf{b} \in \mathcal{P}(\mathbf{a}) \rangle$, $\bar{\mathcal{F}}_\mathbf{b}$ witness $h(\mathbf{b}) = \text{ps-tcf}(\Pi \mathbf{a}, <_{J_{=\lambda}[\mathbf{a}]})$ and putting $\langle \bar{\mathcal{F}}_\mathbf{b} : \mathbf{b} \in h^{-1}\{\lambda\} \rangle$ together for each $\lambda \in \mathbf{c}$.]

- (*)₆ (a) for $\lambda \in \mathbf{c}$ and $f \in \Pi \mathbf{b}_\lambda$ let $f^{[\lambda]} \in \Pi \mathbf{b}_\lambda^*$ be defined by: $f^{[\lambda]}(\theta) = \min\{\alpha < \lambda : \text{for every } g \in \mathcal{F}_{\theta,\alpha} \text{ we have } f \upharpoonright \mathbf{b}_\lambda \leq (g \upharpoonright \mathbf{b}_\lambda) \text{ mod } J_{=\theta}[\mathbf{b}_\lambda]\}$
- (b) for $\lambda \in \mathbf{c}$ and $\alpha < \lambda$ let $\mathcal{F}_{\lambda,\alpha}^* = \{(f \upharpoonright \mathbf{b}_\lambda)^{[\lambda]} : f \in \mathcal{F}_{\lambda,\alpha}\}$.

Now

- (*)₇ (a) $f^{[\lambda]} \upharpoonright \mathbf{a} \geq f$ for $f \in \Pi \mathbf{b}_\lambda$, $\lambda \in \mathbf{c}$
- (b) $\langle \mathcal{F}_{\lambda,\alpha}^* : \lambda \in \mathbf{c}, \alpha < \lambda \rangle$ is well defined (hence exist)
- (c) $\mathcal{F}_{\lambda,\alpha}^* \subseteq \Pi \mathbf{b}_\lambda^*$.

[Why? Obvious, e.g. for clause (a) note that $\theta \in \mathbf{a} \Rightarrow \{\theta\} \in (J_{=\theta}[\mathbf{b}_\lambda])^+$.]

(*)₈ let J_λ be the \aleph_1 -complete ideal on \mathbf{b}_λ^* generated by $\{\mathbf{b}_\theta^* \cap \mathbf{b}_\lambda^* : \theta \in \mathbf{c} \cap \lambda\}$

(*)₉ $J_\lambda \subseteq J_{<\lambda}^{\aleph_1\text{-comp}}[\mathbf{b}_\lambda^*]$.

[Why? As for $\theta_0, \dots, \theta_n \dots \in \mathfrak{c} \cap \lambda$ by 1.15(2) we have $\text{ps} - \text{pcf}_{\aleph_1 - \text{comp}}(\cup\{\mathfrak{b}_{\theta_n}^* : n < \omega\}) = \cup\{\text{ps} - \text{pcf}_{\aleph_1 - \text{comp}}(\mathfrak{b}_{\theta_n}^*) : n < \omega\} = \cup\{\mathfrak{b}_{\theta_n}^* : n < \omega\} \in J_{<\lambda}^{\aleph_1 - \text{comp}}[\mathfrak{c}]$.

\odot_1 if $\lambda \in \mathfrak{c}$ and $\alpha_1 < \alpha_2 < \lambda$ and $f_\ell \in \mathcal{F}_{\lambda, \alpha_\ell}$ for $\ell = 1, 2$ then $f_1^{[\lambda]} \leq f_2^{[\lambda]} \pmod{J_\lambda}$.

[Why? Let $\mathfrak{a}_* = \{\theta \in \mathfrak{b}_\lambda : f_1(\theta) \geq f_2(\theta)\}$, hence by the assumption on $\langle \mathcal{F}_{\lambda, \alpha} : \alpha < \lambda \rangle$ we have $\mathfrak{a}_* \in J_{<\lambda}^{\aleph_1 - \text{comp}}[\mathfrak{a}]$, hence we can find a sequence $\langle \theta_n : n < \mathfrak{n} \leq \omega \rangle$ such that $\theta_n \in \mathfrak{c} \cap \lambda$ and $\mathfrak{a}_* \subseteq \mathfrak{b}_* := \cup\{\mathfrak{b}_{\theta_n} : n < \mathfrak{n}\}$ hence $\mathfrak{c}_* := \text{ps} - \text{pcf}_{\aleph_1 - \text{com}}(\mathfrak{a}_*) \subseteq \cup\{\mathfrak{b}_{\theta_n}^* : n < \mathfrak{n}\} \in J_\lambda$. So it suffices to prove $f_1^{[\lambda]} \upharpoonright (\mathfrak{b}_\lambda^* \setminus \mathfrak{c}_*) \leq f_2^{[\lambda]} \upharpoonright (\mathfrak{b}_\lambda^* \setminus \mathfrak{c}_*)$, so let $\theta \in \mathfrak{b}_\lambda^* \setminus \bigcup_n \mathfrak{b}_{\theta_n}^*$, by $(*)_4(d)$ we have $\theta \leq \lambda$, let $\alpha := f_2^{[\lambda]}(\theta)$, so by the definition of $f_2^{[\lambda]}(\theta)$ we have $(\forall g \in \mathcal{F}_{\theta, \alpha})(f_2 \upharpoonright \mathfrak{b}_\lambda) \leq (g \upharpoonright \mathfrak{b}_\lambda) \pmod{J_{=\theta}[\mathfrak{b}_\lambda]}$. But $\mathfrak{a}_* \subseteq \bigcup_n \mathfrak{b}_{\theta_n}$ and $n < \omega \Rightarrow \theta \notin \mathfrak{b}_{\theta_n}^* = \text{ps} - \text{pcf}_{\aleph_1 - \text{comp}}(\mathfrak{b}_{\theta_n})$ hence by 1.15(2) we have $\theta \notin \text{ps} - \text{pcf}_{\aleph_1 - \text{comp}}(\bigcup_n \mathfrak{b}_{\theta_n})$ hence $\bigcup_n \mathfrak{b}_{\theta_n} \in J_{<\theta}^{\aleph_1 - \text{comp}}[\mathfrak{b}_\lambda]$ hence $\mathfrak{a}_* \in J_{=\theta}^{\aleph_1 - \text{comp}}[\mathfrak{b}_\lambda]$. So (first inequality by the previous sentence and the choice of \mathfrak{a}_* , second by the earlier sentence)

$$(f_1 \upharpoonright \mathfrak{b}_\lambda) \leq (f_2 \upharpoonright \mathfrak{b}_\lambda) \leq (g \upharpoonright \mathfrak{b}_\lambda) \pmod{J_{=\theta}^{\aleph_1 - \text{comp}}[\mathfrak{b}_\lambda]}$$

hence by the definition of $f_1^{[\lambda]}, f_1^{[\lambda]}(\theta) \leq \alpha = f_2^{[\lambda]}(\theta)$. So we are done.]

\odot_2 if $\lambda \in \mathfrak{c}$ and $g \in \Pi \mathfrak{b}_\lambda^*$ then for some $\alpha < \lambda$ and $f \in \mathcal{F}_{\lambda, \alpha}$ we have $g < f \pmod{J_\lambda}$.

[Why? We choose $\langle h_\theta : \theta \in \mathfrak{b}_\lambda^* \rangle$ such that $h_\theta \in \mathcal{F}_{\theta, g(\theta)}$ for each $\theta \in \mathfrak{b}_\lambda^*$; this is possible as we are assuming $\text{AC}_\mathfrak{c}$ and $\mathfrak{b}_\lambda^* \subseteq \mathfrak{c}$. Let $h_1 \in \Pi \mathfrak{b}_\lambda^*$ be defined by $h_1(\kappa) = \sup\{h_\theta^{[\lambda]}(\kappa) : \kappa \in \mathfrak{b}_\theta \text{ and } \theta \in \mathfrak{b}_\lambda^*\}$ for $\kappa \in \mathfrak{b}_\lambda^*$, the result is $< \kappa$ because the supremum is on $\leq |\mathfrak{b}_\theta|$ ordinals and $\kappa \geq \min(\mathfrak{b}_\lambda^*) \geq \min(\mathfrak{c}) = \min(\mathfrak{a}) \geq \text{hrtg}(\mathcal{P}(\mathfrak{a}))$. Hence there are $\alpha < \lambda$ and $h_2 \in \mathcal{F}_{\lambda, \alpha}$ such that $h_1 \leq h_2 \pmod{J_{=\lambda}[\mathfrak{a}]}$. Now $f := h_2^{[\lambda]} \in \Pi \mathfrak{b}_\lambda^*$ recalling $(*)_7(a)$ is as required, in particular $f \in \mathcal{F}_{\lambda, \alpha}^*$.]

\odot_3 the sequence $\langle \mathcal{F}_{\lambda, \alpha} : \alpha < \lambda \rangle$ witness $\lambda = \text{ps} - \text{tcf}(\Pi \mathfrak{b}_\lambda^*, <_{J_\lambda})$.

[Why? In $(*)_7(b), (c) + \odot_1 + \odot_2$.]

\odot_4 if $\lambda \in \mathfrak{c}$ then $J_{<\lambda} = J_{<\lambda}^{\aleph_1 - \text{comp}}[\mathfrak{b}_\lambda^*]$.

[Why? By $(*)_4, (*)_8, (*)_9$ and \odot_3 .]

So

\odot_5 $\bar{\mathfrak{b}}^* = \langle \mathfrak{b}_\lambda^* : \lambda \in \mathfrak{c} \rangle$ is a generating sequence for \mathfrak{c} .

[Why? By $\odot_4, (*)_8$ recalling that $\lambda \notin \text{ps} - \text{pcf}_{\aleph_1 - \text{comp}}(\mathfrak{c} \setminus \mathfrak{b}_\lambda^*)$ by $(*)_4(e)$.] $\square_{2.10}$

Remark 2.11. Clearly $\bar{\mathfrak{b}}^*$ is full and closed, but what about smooth? Is this necessary for generalizing [She00]?

Discussion 2.12. Naturally the definition now of $\bar{\mathcal{F}}$ as in 2.8(2) for $\Pi \mathfrak{a}$ is more involved where $\bar{\mathcal{F}} = \langle \mathcal{F}_\lambda : \lambda \in \text{ps} - \text{pcf}_{\aleph_1 - \text{com}}(\mathfrak{a}) \rangle$, $\mathcal{F}_\lambda = \langle \mathcal{F}_{\lambda, \alpha} : \alpha < \lambda \rangle$ exemplifies $\text{ps} - \text{tcf}(\Pi \mathfrak{a}, J_{=\lambda}(\mathfrak{a}))$.

Claim 2.13. $[DC + AC_{<\kappa}]$ Assume

- (a) \mathfrak{a} a set of regular cardinals
- (b) κ is regular $> \aleph_0$
- (c) $\mathfrak{c} = \text{ps-pcf}_{\kappa\text{-comp}}(\mathfrak{a})$
- (d) $\min(\mathfrak{a})$ is $\geq \text{hrtg}(\mathcal{P}(\mathfrak{c}))$ or at least $\geq \text{hrtg}(\mathfrak{c})$
- (e) $\bar{\mathcal{F}} = \langle \bar{\mathcal{F}}_\lambda : \lambda \in \mathfrak{c} \rangle, \bar{\mathcal{F}}_\lambda = \langle \mathcal{F}_{\lambda,\alpha} : \alpha < \lambda \rangle$ witness² $\lambda = \text{ps-tcf}(\Pi\mathfrak{a}, <_{=\lambda}^{J^{\kappa\text{-comp}}}$
[\mathfrak{a}]).

Then

- ▯ for every $f \in \Pi\mathfrak{a}$ for some $g \in \Pi\mathfrak{c}$, if $g \leq g_1 \in \Pi\mathfrak{c}$ and $\bar{h} \in \Pi\{\mathcal{F}_{\lambda,g_1(\lambda)} : \lambda \in \mathfrak{c}\}$ then $(\exists \mathfrak{d} \in [\mathfrak{c}]^{<\kappa})(f < \sup\{h_\lambda : \lambda \in \mathfrak{d}\})$.

Proof. Let $f \in \Pi\mathfrak{a}$. For each $\lambda \in \text{ps-pcf}_{\kappa\text{-comp}}(\mathfrak{a})$ let $\alpha_{f,\lambda} = \min\{\alpha < \lambda : f < g \text{ mod } J_{=\lambda}[\mathfrak{a}] \text{ for every } g \in \mathcal{F}_{\lambda,\alpha}\}$, so clearly each α_f is well defined hence $\bar{\alpha} = \langle \alpha_{f,\lambda} : \lambda \in \text{ps-pcf}_{\kappa\text{-comp}}(\mathfrak{a}) \rangle$ exists. So $g = \langle \alpha_{f,\lambda} : \lambda \in \mathfrak{c} \rangle \in \Pi\mathfrak{c}$ is well defined. Assume $g_1 \in \Pi\mathfrak{c}$ and $g \leq g_1$. Let $\langle h_\lambda : \lambda \in \mathfrak{c} \rangle$ be any sequence from $\prod_{\lambda \in \mathfrak{c}} \mathcal{F}_{\lambda,g_1(\lambda)}$, at least one exists when $AC_{\mathfrak{c}}$ holds but this is not needed here. Let $\mathfrak{a}_{f,\lambda} = \{\theta \in \mathfrak{a} : f(\theta) < h_\lambda(\theta)\}$ so $\langle \mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{c} \rangle$ exists and we claim that for some $\mathfrak{d} \in [\mathfrak{c}]^{<\kappa}$ we have $\mathfrak{a} = \cup\{\mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{d}\}$. Otherwise let J be the κ -complete ideal on \mathfrak{a} generated by $\{\mathfrak{a}_{f,\lambda} : \lambda \in \mathfrak{c}\}$, it is a κ -complete ideal. So by [She12, 5.9=r9], applicable by our assumptions, there is a κ -complete ideal J_1 on \mathfrak{a} extending J such that $\lambda_* = \text{ps-tcf}(\Pi\mathfrak{a}, <_{J_1})$ is well defined. So $\lambda_* \in \mathfrak{c}$ and $\mathfrak{a}_{f,\lambda_*} \in J_1$, easy contradiction. $\square_{2.13}$

Claim 2.14. $[AC_{\aleph_0}]$ We can uniformly define³ a \aleph_0 -continuous witness for $\lambda = \text{ps-pcf}_{\kappa\text{-comp}}(\bar{\Pi}\bar{\alpha}, <_D)$ where:

- (a) $\bar{\alpha} \in {}^Y\text{Ord}$
- (b) each α_t is a limit ordinal with $\text{cf}(\alpha_t) \geq \text{hrtg}(S)$
- (c) λ is regular $\geq \text{hrtg}(S)$
- (d) $\bar{\mathcal{F}} = \langle \bar{\mathcal{F}}_a : a \in S \rangle$ satisfies: each $\bar{\mathcal{F}}_a$ is a witness for $\lambda = \text{pcf}_{\kappa\text{-comp}}(\bar{\Pi}\bar{\alpha}, <_D)$
- (e) if $a \in S$ then $\bar{\mathcal{F}}_a$ is \aleph_0 -continuous and $f_1, f_2 \in \bar{\mathcal{F}}_{a,\alpha} \Rightarrow f_1 = f_2 \text{ mod } D$.

Proof.

- (*)₀ $\text{hrtg}(S \times S)$ is $\leq \lambda$ and $\leq \text{cf}(\alpha_t)$ for $t \in Y$.

[Why? As $\lambda, \text{cf}(\alpha_t)$ are regular cardinals.]

For $a, b \in S$ let

- (*)₁ (a) $E_{a,b} = \{\delta < \lambda : \text{if } \alpha < \delta \text{ then for some } \beta \in (\alpha, \delta) \text{ and } f_1 \in \bar{\mathcal{F}}_{a,\alpha}, f_2 \in \bar{\mathcal{F}}_{b,\beta} \text{ we have } f_1 < f_2 \text{ mod } D\}$
- (b) define $g_{a,b} : \lambda \rightarrow \lambda$ by $g_{a,b}(\alpha) = \min\{\beta < \lambda : \text{there are } f_1 \in \bar{\mathcal{F}}_{a,\alpha} \text{ and } f_2 \in \bar{\mathcal{F}}_{b,\beta} \text{ such that } f_1 < f_2 \text{ mod } D\}$
- (*)₂ $g_{a,b}$ is well defined.

²So we are assuming it is well defined, now if $AC_{\mathcal{P}(Y)}$ such $\bar{\mathcal{F}}$ exists.

³Of course, mere existence is already given by the assumptions.

[Why? As $\bar{\mathcal{F}}_b$ is cofinal in $(\Pi\bar{\alpha}, <_D)$.]

(*)₃ $g_{a,b}$ is non-decreasing.

[Why? As $\bar{\mathcal{F}}_a$ is $<_D$ -increasing.]

Hence

(*)₄ $E_{a,b} = \{\delta < \lambda : \delta \text{ a limit ordinal and } (\forall \alpha < \delta)(g_{a,b}(\alpha) < \delta)\}$.

Also

(*)₅ $E_{a,b}$ is a club of λ .

[Why? By its definition, $E_{a,b}$ is a closed subset of λ and it is unbounded as $\text{cf}(\lambda) = \lambda > \aleph_0$, because for every $\alpha < \lambda$ letting $\alpha_0 = \alpha, \alpha_{n+1} = g_{a,b}(\alpha_n) + 1 < \lambda$ clearly $\beta := \cup\{\alpha_n : n < \omega\}$ is $< \lambda$ and $\gamma < \delta \Rightarrow (\exists n)(\gamma < \alpha_n) \Rightarrow (\exists n)(g_{a,b}(\gamma) < \alpha_{n+1})$.]

(*)₆ let $g : \lambda \rightarrow \lambda$ be $g(\alpha) = \sup\{g_{a,b}(\alpha) : a, b \in S\}$

(*)₇ g is a (well defined) non-decreasing function from λ to λ .

[Why? “Non-decreasing trivial”, and it is “into λ ” as $\text{hrtg}(S \times S) \leq \lambda$ recalling (*₀).]

(*)₈ $E = \cap\{E_{a,b} : a, b \in S\} = \{\delta < \lambda : (\forall \alpha < \delta)(g(\alpha) < \delta)\}$ is a club of λ .

[Why? Like (*₇).]

(*)₉ let $E_1 = \{\delta \in E : \text{cf}(\delta) = \aleph_0\}$ so $E_1 \subseteq \lambda = \sup(\lambda)$, $\text{otp}(E_1) = \lambda$

(*)₁₀ for $\delta \in E$ of cofinality \aleph_0 let $\mathcal{F}_\delta = \{\sup\{f_n : n < \omega\} : \text{for some } a \in S \text{ and } \bar{\alpha} = \langle \alpha_n : n < \omega \rangle \text{ increasing of cofinality } \aleph_0 \text{ we have } \langle f_n : n < \omega \rangle \in \prod_n \mathcal{F}_{a, \alpha_n}\}$

(*)₁₁ $\langle \mathcal{F}_\delta : \delta \in E_1 \rangle$ is $<_D$ -increasing cofinal in $(\Pi\bar{\alpha}, <_D)$ in particular $\mathcal{F}_\delta \neq \emptyset$.

[Why? $\mathcal{F}_\delta \neq \emptyset$ as $\delta \in E$, $\text{cf}(\delta) = \aleph_0$ and AC_{\aleph_0} .]

We can correct $\langle \mathcal{F}_\delta : \delta \in E_1 \rangle$ to be \aleph_0 -continuous easily (and as in [She12, §5]). □_{2.14}

Question 2.15. 1) Can we in 2.5 get smoothness?

2) If 2.10 does it suffice to assume $\text{AC}_{\mathcal{P}(a)}$ (and omit AC_a) and we can conclude that $\mathfrak{c} = \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{c})$ has a full closed generating sequence.

We may try to repeat the proof of 2.10, only in the proof of (*₅) we use claim 2.16 below.

Claim 2.16. In 2.10 we can add “ $\bar{\mathfrak{b}}$ is weakly smooth” which means $\theta \in \mathfrak{b}_\lambda \Rightarrow \theta \notin \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{c} \setminus \mathfrak{b}_*)$.

Proof. Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{c} \rangle$ be a full closed generating sequence.

We choose \mathfrak{b}_λ^1 by induction on $\lambda \in \mathfrak{c}$ such that

(*)₁ (a) $J_{\leq \lambda}[\mathfrak{a}] = J_{\leq \lambda}^{\aleph_1\text{-comp}}[\mathfrak{a}] + \mathfrak{b}_\lambda^1$

(b) $\text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{b}_\lambda^1) = \mathfrak{b}_\lambda^1$

(c) $\max(\mathfrak{b}_\lambda^1) = \lambda$

(d) if $\theta \in \mathfrak{b}_\lambda^1$ then $\mathfrak{b}_\lambda^1 \supseteq \mathfrak{b}_\theta \pmod{J_{=\lambda}[\mathfrak{a}]}$, i.e. $\mathfrak{b}_\theta^1 \setminus \mathfrak{b}_\lambda^1 \in J_{< \theta}^{\aleph_1\text{-comp}}[\mathfrak{a}]$.

Arriving to λ let $\mathfrak{d}_\lambda = \{\theta \in \mathfrak{b}_\lambda : \mathfrak{b}_\theta^1 \setminus \mathfrak{b}_\lambda^1 \notin J_{<\theta}^{\aleph_1\text{-comp}}[\mathfrak{a}]\}$, $\mathfrak{d}_\lambda^1 = \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{d}_\lambda)$.

Now

$$(*)_2 \text{ ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{d}_\lambda) \subseteq \mathfrak{b}_\lambda \cap \lambda.$$

[Why? $\subseteq \mathfrak{b}_\lambda$ is obvious; recalling $\mathfrak{b}_\lambda^1 = \text{ps} - \text{pcf}(\mathfrak{b}_\lambda^1 \cap \mathfrak{a})$ because $\bar{\mathfrak{b}}$ is closed. If “ $\not\subseteq \lambda$ ” recall $\mathfrak{d}_\lambda^1 = \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{d}_\lambda)$, now $\mathfrak{d}_\lambda \subseteq \mathfrak{b}_\lambda$ hence $\mathfrak{d}_\lambda^1 \subseteq \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{b}_\lambda) \subseteq \lambda^+$. So the only problematic case is $\lambda \in \mathfrak{d}_\lambda^1 = \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{d}_\lambda)$. But then, $\mathfrak{d}_\lambda \subseteq \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{c} \setminus \mathfrak{b}_\lambda)$ by the definition of \mathfrak{d}_λ hence by the composition theorem we have $\lambda \in \text{ps} - \text{pcf}_{\aleph_1\text{-comp}}(\mathfrak{c} \setminus \mathfrak{b}_\lambda)$, contradicting an assumption on $\bar{\mathfrak{b}}$.]

$$(*)_3 \text{ there is a countable } \mathfrak{e}_\lambda \subseteq \mathfrak{d}_\lambda^1 \text{ such that } \mathfrak{d}_\lambda^1 \subseteq \cup\{\mathfrak{b}_\sigma^1 : \sigma \in \mathfrak{e}_\lambda\}.$$

[Why? Should be clear.]

Lastly, let $\mathfrak{b}_\lambda^1 = \cup\{\mathfrak{b}_\theta^1 : \theta \in \mathfrak{e}_\lambda\} \cup \mathfrak{b}_\lambda$ and check.

□_{2.16}

§ 3. MEASURING REDUCED PRODUCTS

§ 3(A). On $\text{ps-T}_D(g)$.

Now we consider some ways to measure the size of ${}^\kappa\mu/D$ and show that they essentially are equal; see Discussion 3.9 below.

Definition 3.1. Let $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle \in {}^Y\text{Ord}$ be such that $t \in Y \Rightarrow \alpha_t > 0$.

- 1) For D a filter on Y let $\text{ps-T}_D(\bar{\alpha}) = \sup\{\text{hrtg}(\mathbf{F}) : \mathbf{F} \text{ is a family of non-empty subsets of } \Pi\bar{\alpha} \text{ such that for every } \mathcal{F}_1 \neq \mathcal{F}_2 \text{ from } \mathbf{F} \text{ we have } f_1 \in \mathcal{F}_1 \wedge f_2 \in \mathcal{F}_2 \Rightarrow f_1 \neq_D f_2\}$, recalling $f_1 \neq_D f_2$ means $\{t \in Y : f_1(t) \neq f_2(t)\} \in D$.
- 2) Let $\text{ps-T}_{\kappa\text{-comp}}(\bar{\alpha}) = \sup\{\text{hrtg}(\mathbf{F}) : \text{for some } \kappa\text{-complete filter } D \text{ on } Y, \mathbf{F} \text{ is as above for } D\}$.
- 3) If we allow $\alpha_t = 0$ just replace $\Pi\bar{\alpha}$ by $\Pi^*\bar{\alpha} := \{f : f \in \prod_t (\alpha_t + 1) \text{ and } \{t : f(t) = \alpha_t\} = \emptyset \text{ mod } D\}$.

Theorem 3.2. $[\text{DC} + \text{AC}_{\mathcal{P}(Y)}]$ Assume that D is a κ -complete filter on Y and $\kappa > \aleph_0$ and $g \in {}^Y(\text{Ord} \setminus \{0\})$, if g is constantly α we may write α . The following cardinals are equal or at least $\lambda_1, \lambda_2, \lambda_3$ are $\text{Fil}_\kappa^1(D)$ -almost equal which means:

for $\ell_1, \ell_2 \in \{1, 2, 3\}$ we have $\lambda_{\ell_1} \leq_{\text{Fil}_\kappa^1(D)}^{\text{sal}} \lambda_{\ell_2}$ which means if $\alpha < \lambda_{\ell_1}$ then α is included in the union of S sets each of order type $< \lambda_{\ell_2}$:

- (a) $\lambda_1 = \sup\{|\text{rk}_{D_1}(g)|^+ : D_1 \in \text{Fil}_\kappa^1(D)\}$
- (b) $\lambda_2 = \sup\{\lambda^+ : \text{there are } D_1 \in \text{Fil}_\kappa^1(D) \text{ and a } <_{D_1}\text{-increasing sequence } \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle \text{ such that } \mathcal{F}_\alpha \subseteq \prod_{t \in Y} g(t) \text{ is non-empty}\}$
- (c) $\lambda_3 = \sup\{\text{ps-T}_{D_1}(g) : D_1 \in \text{Fil}_\kappa^1(D)\}$.

Remark 3.3. 1) Recall that for D a κ -complete filter on Y we let $\text{Fil}_\kappa^1(D) = \{E : E \text{ is a } \kappa\text{-complete filter on } Y \text{ extending } D\}$.

- 2) The conclusion gives slightly less than equality of $\lambda_1, \lambda_2, \lambda_3$.
- 3) See 3.10(6) below, by it $\lambda_2 = \text{ps-Depth}^+({}^\kappa\mu, <_D)$ recalling 3.10(5).
- 4) We may replace κ -complete by $(\leq Z)$ -complete if $\aleph_0 \leq |Z|$.
- 5) Compare with Definition 3.10.
- 6) Note that those cardinals are $\leq \text{hrtg}(\Pi^*g)$, see 3.1(3).

Proof. Stage A: $\lambda_1 \leq_{\text{Fil}_\kappa^1(D)}^{\text{sal}} \lambda_2, \lambda_3$.

Why? Let $\chi < \lambda_1$, so by clause (a) there is $D_1 \in \text{Fil}_\kappa^1(D)$ such that $\text{rk}_{D_1}(g) \geq \chi$. Let $X_{D_2} = \{\alpha < \chi : \text{some } f \in \prod_{t \in Y} g(t) \text{ satisfies}^4 D_2 = \text{dual}(J[f, D_1]) \text{ and } \alpha = \text{rk}_{D_2}(f)\}$, for any $D_2 \in \text{Fil}_\kappa^1(D_1)$. By [She12, 1.11(5)] we have $\chi = \bigcup\{X_{D_2} : D_2 \in \text{Fil}_\kappa^1(D_1)\}$.

Now

$$\odot D_2 \in \text{Fil}_\kappa^1(D_1) \Rightarrow |\text{otp}(X_{D_2})| < \lambda_2, \lambda_3; \text{ this is enough.}$$

⁴recall $\text{dual}(J[f, D_1]) = \{X \subseteq Y : X \in D_1 \text{ or } \text{rk}_{D_1+(X \setminus Y)}(f) > \text{rk}_{D_1}(f)\}$.

Why does this hold? Letting $\mathcal{F}_{D_2,i} = \{f \in {}^Y\mu : \text{rk}_{D_1}(f) = i \text{ and } J[f, D_1] = \text{dual}(D_2)\}$, by [She12, 1.11(2)] we have: $i < j \wedge i \in X_{D_2} \wedge j \in X_{D_2} \wedge f \in \mathcal{F}_{D_2,i} \wedge g \in \mathcal{F}_{D_2,j} \Rightarrow f < g \pmod{D_2}$ so by the definitions of λ_2, λ_3 we have $\text{otp}(X_{D_2}) < \lambda_2, \lambda_3$.

Stage B: $\lambda_2 \leq_{\text{Fil}_\kappa^1(D)}^{\text{sal}} \lambda_1, \lambda_3$, moreover $\lambda_2 \leq \lambda_1, \lambda_3$.

Why? Let $\chi < \lambda_2$ and let D_1 and $\langle \mathcal{F}_\alpha : \alpha < \chi \rangle$ exemplify $\chi < \lambda_2$. Let $\gamma_\alpha = \min\{\text{rk}_{D_1}(f) : f \in \mathcal{F}_\alpha\}$ so easily $\alpha < \beta < \chi \Rightarrow \gamma_\alpha < \gamma_\beta$ hence $\text{rk}_D(g) \geq \chi$. So $\chi < \lambda_1$ by the definition of λ_1 and as for $\chi < \lambda_3$ this holds by Definition 3.1(2) as $\alpha < \beta \wedge f \in \mathcal{F}_\alpha \wedge g \in \mathcal{F}_\beta \Rightarrow f < g \pmod{D_1} \Rightarrow f \neq g \pmod{D_1}$ as $\chi^+ = \text{hrtg}(\chi) \leq \lambda_3$.

Stage C: $\lambda_3 \leq_{\text{Fil}_\kappa^1(D)}^{\text{sal}} \lambda_1, \lambda_2$.

Why? Let $\chi < \lambda_3$. Let $\langle \mathcal{F}_\alpha : \alpha < \chi \rangle$ exemplify $\chi < \lambda_3$. For each $\alpha < \chi$ let $\mathbf{D}_\alpha = \{\text{dual}(J[f, D]) : f \in \mathcal{F}_\alpha\}$ so a non-empty subset of $\text{Fil}_\kappa^1(Y)$. Now for every $D_1 \in \mathbf{D}_* := \cup\{\mathbf{D}_\alpha : \alpha < \chi\}$ let $X_{D_1} = \{\alpha < \chi : D_1 \in \mathbf{D}_\alpha\}$ and for $\alpha \in X_{D_1}$ let $\zeta_{D_1,\alpha} = \min\{\text{rk}_D(f) : f \in \mathcal{F}_\alpha \text{ and } D_1 = \text{dual}(J[f, D])\}$ and let $\mathcal{F}_{D_1,\alpha} = \{f \in \mathcal{F}_\alpha : D_1 = J[f, D] \text{ and } \text{rk}_{D_1}(f) = \zeta_{D_1,\alpha}\}$ so a non-empty subset of \mathcal{F}_α and clearly $\langle (\zeta_{D_1,\alpha}, \mathcal{F}_{D_1,\alpha}) : \alpha \in X_{D_1} \rangle$ exists.

Now

- (a) $\alpha \mapsto \zeta_{D_1,\alpha}$ is a one-to-one function with domain X_{D_1} for $D_1 \in \mathbf{D}_*$
- (b) $\chi = \cup\{X_{D_1} : D_1 \in \mathbf{D}_*\}$ noting $\mathbf{D}_* \subseteq \text{Fil}_\kappa^1(D)$
- (c) for $D \in \mathbf{D}_*$ if $\alpha < \beta$ are from X_{D_1} and $\zeta_{D_1,\alpha} < \zeta_{D_1,\beta}$, $f \in \mathcal{F}_{D_1,\alpha}$, $g \in \mathcal{F}_{D_1,\beta}$ then $f < g \pmod{D_1}$.

[Why? For clause (a), if $\alpha \neq \beta \in X_{D_1}$, $f \in \mathcal{F}_{D_1,\alpha}$, $g \in \mathcal{F}_{D_1,\beta}$ then $f \neq g \pmod{D}$ hence by [She12, 1.11] we have $\zeta_{D_1,\alpha} \neq \zeta_{D_1,\beta}$. For clause (b), it follows by the choices of $\mathbf{D}_\alpha, X_{D_1}$. Lastly, clause (c) follows by [She12, 1.11(2)].]

Hence (by clause (c))

- (d) $\text{otp}(X_{D_1})$ is $< \lambda_2$ and is $\leq \text{rk}_{D_1}(g)$ for $D_1 \in \cup\{\mathbf{D}_\alpha : \alpha < \chi\} \subseteq \text{Fil}_\kappa^1(D)$.

Together clause (d) shows that $D \in \mathbf{D}_* \Rightarrow |X_D| < \lambda_1, \lambda_2$ so by clause (b), $\lambda_3 \leq_{\text{Fil}_\kappa^1(D)}^{\text{sal}} \lambda_1, \lambda_2$ hence we are done. $\square_{3.2}$

Observation 3.4. If D is a filter on Y and $\bar{\alpha} \in {}^Y(\text{Ord} \setminus \{0\})$ then

$\text{ps} - \mathbf{T}_D(\bar{\alpha}) = \sup\{\lambda^+ : \text{there is a sequence } \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle \text{ such that } \mathcal{F}_\alpha \subseteq \Pi \bar{\alpha}, \mathcal{F}_\alpha \neq \emptyset \text{ and } \alpha \neq \beta \wedge f_1 \in \mathcal{F}_\alpha \wedge f_2 \in \mathcal{F}_\beta \Rightarrow f_1 \neq_D f_2\}$.

Proof. Clearly the new definition gives a cardinal $\leq \text{ps} - \mathbf{T}_D(\bar{\alpha})$. For the other inequality assume $\lambda < \text{ps} - \mathbf{T}_D(\bar{\alpha})$ so there is \mathbf{F} as there such that $\lambda < \text{hrtg}(\mathbf{F})$. As $\lambda < \text{hrtg}(\mathbf{F})$ there is a function h from \mathbf{F} onto λ . For $\alpha < \lambda$ define $\mathcal{F}'_\alpha = \cup\{\mathcal{F} : \mathcal{F} \in \mathbf{F} \text{ and } h(\mathcal{F}) = \alpha\}$. So $\langle \mathcal{F}'_\alpha : \alpha < \lambda \rangle$ exists and is as required. $\square_{3.4}$

Concerning Theorem 3.2 we may wonder “when does λ_1, λ_2 being S -almost equal implies they are equal”. We consider a variant this time for sets (or powers, not just cardinals).

Definition 3.5. 1) We say “the power of \mathcal{U}_1 is S -almost smaller than the power of \mathcal{U}_2 ”, or write $|\mathcal{U}_1| \leq |\mathcal{U}_2| \pmod{S}$ or $|\mathcal{U}_1| \leq_S^{\text{alm}} |\mathcal{U}_2|$ when: we can find a sequence $\langle u_{1,s} : s \in S \rangle$ such that $\mathcal{U}_1 = \cup\{u_{1,s} : s \in S\}$ and $s \in S \Rightarrow |\mathcal{U}_{1,s}| \leq |\mathcal{U}_2|$.

- 2) We say the power $|\mathcal{U}_1|, |\mathcal{U}_2|$ are S -almost equal (or $|\mathcal{U}_1| = |\mathcal{U}_2| \bmod S$ or $|\mathcal{U}_1| =_S^{\text{alm}} |\mathcal{U}_2|$) when $|\mathcal{U}_1| \leq_S^{\text{alm}} |\mathcal{U}_2| \leq_S^{\text{alm}} |\mathcal{U}_2|$.
- 3) Let $|\mathcal{U}_1| \leq_{<S}^{\text{alm}} |\mathcal{U}_2|$ be defined naturally.
- 4) In particular this applies to cardinals.
- 5) Let $|\mathcal{U}_1| <_S^{\text{alm}} |\mathcal{U}_2|$ means there is a sequence $\langle u_{1,s} : s \in S \rangle$ with union \mathcal{U}_1 such that $s \in S \Rightarrow |u_{1,s}| < |\mathcal{U}_2|$.
- 6) Let $|\mathcal{U}_1| \leq_S^{\text{sal}} |\mathcal{U}_2|$ means that if $|\mathcal{U}| < |\mathcal{U}_1|$ then $|\mathcal{U}| <_S^{\text{alm}} |\mathcal{U}_2|$.

Observation 3.6. 1) If $|\mathcal{U}_1| \leq |\mathcal{U}_2|$ and $S \neq \emptyset$ then $|\mathcal{U}_1| \leq_S^{\text{alm}} |\mathcal{U}_2|$.

2) If $\lambda_1 \leq \lambda_2$ and $S \neq \emptyset$ then $\lambda_1 \leq_S^{\text{sal}} \lambda_2$.

3) If $\lambda_2 = \lambda_1^+$ and $\text{cf}(\lambda_2) < \text{hrtg}(S)$ then the power of λ_2 is S -almost smaller than S .

Proof. Immediate. □_{3.6}

Observation 3.7. 1) The cardinals λ_1, λ_2 are equal when $\lambda_1 =_S^{\text{alm}} \lambda_2$ and $\text{cf}(\lambda_1), \text{cf}(\lambda_2) \geq \text{hrtg}(\mathcal{P}(S))$.

2) The cardinals λ_1, λ_2 are equal when $\lambda_1 =_S^{\text{alm}} \lambda_2$ and λ_1, λ_2 are limit cardinals $> \text{hrtg}(\mathcal{P}(S))$.

3) If $\lambda_1 \leq_S^{\text{alm}} \lambda_2$ and $\partial = \text{hrtg}(\mathcal{P}(S))$ then $\lambda_1 \leq_{<\partial}^{\text{alm}} \lambda_2$.

4) If $\lambda_1 \leq_{<\theta}^{\text{alm}} \lambda_2$ and $\text{cf}(\lambda_1) \geq \theta$ then $\lambda_1 \leq \lambda_2$.

5) If $\lambda_1 \leq_{<\theta}^{\text{alm}} \lambda_2$ and $\theta \leq \lambda_2^+$ then $\lambda_1 \leq \lambda_2^+$.

Proof. 1) Otherwise, let $\partial = \text{hrtg}(\mathcal{P}(S))$, without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \leq_{<\partial}^{\text{alm}} \lambda_2$ and by part (4) we have $\lambda_1 \leq \lambda_2$ contradiction.

2) Otherwise letting $\partial = \text{hrtg}(\mathcal{P}(S))$ without loss of generality $\lambda_2 < \lambda_1$ and by part (3) we have $\lambda_1 \leq_{<\partial}^{\text{alm}} \lambda_2$ but $\partial < \lambda_2$ is assume and $\lambda_2^+ < \lambda_1$ as λ_2 is a limit cardinal so together we get contradiction to part (5).

3) If $\langle u_s : S \in S \rangle$ witness $\lambda_1 \leq_S^{\text{alm}} \lambda_2$, let $w = \{\alpha < \lambda_1 : \text{for no } \beta < \alpha \text{ do we have } (\forall s \in S)(\alpha \in u_s \equiv \beta \in u_s)\}$ so clearly $|w| < \text{hrtg}(\mathcal{P}(S)) = \theta$ and for $\alpha \in w$ let $v_\alpha = \{\beta < \lambda_1 : (\forall s \in S)(\alpha \in u_s \equiv \beta \in u_s)\}$ so $\langle v_\alpha : \alpha \in w \rangle$ witness $\lambda_1 \leq_w^{\text{alm}} \lambda_2$ hence $\lambda_1 \leq_{<\theta}^{\text{alm}} \lambda_2$.

4),5) Let $\sigma < \theta$ be such that $\lambda_1 \leq_\sigma^{\text{alm}} \lambda_2$ and let $\langle u_\varepsilon : \varepsilon < \sigma \rangle$ witness $\lambda_1 \leq_\sigma^{\text{alm}} \lambda_2$, that is $|u_\varepsilon| \leq \lambda_2$ for $\varepsilon < \sigma$ and $\cup\{u_\varepsilon : \varepsilon < \sigma\} = \lambda_1$.

For part (4), if $\lambda_2 < \lambda_1$, then we have $\varepsilon < \sigma \Rightarrow |u_\varepsilon| < \lambda_1$, but $\text{cf}(\lambda_1) > \sigma$ hence $|\cup\{u_\varepsilon : \varepsilon < \sigma\}| < \lambda_1$, contradiction.

For part (5) for $\varepsilon < \sigma$, let $u'_\varepsilon = u_\varepsilon \setminus \cup\{u_\zeta : \zeta < \varepsilon\}$ and so $\text{otp}(u'_\varepsilon) \leq \text{otp}(u_\varepsilon) < |u_\varepsilon|^+ \leq \lambda_2^+$ so easily $|\lambda_1| = |\cup\{u_\varepsilon : \varepsilon < \sigma\}| = |\cup\{u'_\varepsilon : \varepsilon < \sigma\}| \leq \sigma \cdot \lambda_2^+ \leq \lambda_2^+ \cdot \lambda_2^+ = \lambda_2^+$. □_{3.7}

Similarly

Observation 3.8. 1) If $\lambda_1 <_S^{\text{alm}} \lambda_2$ and $\partial = \text{hrtg}(\mathcal{P}(S))$ then $\lambda_1 <_{<\partial}^{\text{alm}} \lambda_2$.

2) If $\lambda_1 <_{<\theta}^{\text{alm}} \lambda_2$ and $\text{cf}(\lambda_1) \geq \theta$ then $\lambda_1 < \lambda_2$.

3) If $\lambda_1 <_{<\theta}^{\text{alm}} \lambda_2$ and $\theta \leq \lambda_2^+$ then $\lambda_1 \leq \lambda_2$.

4) If $\lambda_1 \leq_S^{\text{sal}} \lambda_2$ and $\partial = \text{hrtg}(\mathcal{P}(S))$ then $\lambda_1 <_{<\partial}^{\text{sal}} \lambda_2$.

5) If $\lambda_1 \leq_{<\theta}^{\text{sal}} \lambda_2$ and $\partial \leq \lambda_2^+, \theta < \lambda_2$ and $\text{cf}(\lambda_2) \geq \theta$ then $\lambda_1 \leq \lambda_2$.

6) If $\lambda_1 \leq_{<\theta}^{\text{sal}} \lambda_2$ and $\theta \leq \lambda_2^+$ then $\lambda_1 \leq \lambda_2^+$.

Proof. Similar, e.g.

1) Like the proof of 3.7(3). □_{3.8}

Discussion 3.9. 1) We like to measure $({}^Y\mu)/D$ in some ways and show their equivalence, as was done in ZFC. Natural candidates are:

- (A) $\text{pp}_D(\mu)$: say of length of increasing sequence \bar{P} (not \bar{p} !, i.e. sets) ordered by $<_D$
- (B) $\text{pp}_Y^+(\mu) = \sup\{\text{pp}_D^+(\mu) : D \text{ an } \aleph_1\text{-complete filter on } Y\}$
- (C) As in 3.1.

- 2) We may measure ${}^Y\mu$ by considering all ∂ -complete filters.
- 3) We may be more lenient in defining “same cardinality”. E.g.

- (A) we define when sets have similar powers say by divisions to $\mathcal{P}(\mathcal{P}(Y))$ sets we measure $({}^Y\mu)/\approx_{\mathcal{P}(\mathcal{P}(Y))}$ where \approx_B is the following equivalence relation on sets:
 $X \approx_B Y$ when we can find sequences $\langle X_b : b \in B \rangle, \langle Y_b : b \in B \rangle$ such that:
 - (a) $X = \cup\{X_b : b \in B\}$
 - (b) $Y = \cup\{Y_b : b \in B\}$
 - (c) $|X_b| = |Y_b|$
- (B) we may demand more: the $\langle X_b : b \in B \rangle$ are pairwise disjoint and the $\langle Y_b : b \in B \rangle$ are pairwise disjoint
- (C) we may demand less: e.g.
 - (c)' $|X_b| \leq_* |Y_b| \leq_* |X_b|$
and/or
 - (c)* $(\forall b \in B)(\exists c \in B)(|X_b| \leq |Y_c|)$ and
 $(\forall b \in B)(\exists c \in B)(|Y_b| \leq |X_c|)$.

Note that some of the main results of [She] can be expressed this way.

- (D) $\text{rk-sup}_{Y,\partial}(\mu) = \text{rk-sup} \{\text{rk}_D(\mu) : D \text{ is } \partial\text{-complete filters on } Y\}$
- (E) for each non-empty $X \subseteq {}^Y\mu$ let

$$\text{sp}_\alpha^1(X) = \{(D, J) : D \text{ an } \aleph_1\text{-complete filter on } Y, J = J[f, D], \alpha = \text{rk}_D(f) \text{ and } f \in X\}$$

$$\text{sp}_1(X) = \cup\{\text{sp}_\alpha^1(X) : \alpha\}$$

- (F) question: If $\{\text{sp}(X_s) : s \in S\}$ is constant, can we bound J ?
- (G) X, Y are called connected when $\text{sp}(X_1), \text{sp}(X_2)$ are non-disjoint or equal.

4) We hope to prove, at least sometimes $\gamma := \Upsilon({}^Y\mu) \leq \text{pp}_\kappa(\mu)$ that is we like to immitate [She] without the choice axioms on ${}^\omega\mu$. So there is $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ witnessing $\gamma < \Upsilon({}^Y\mu)$. We define $u = u_{\bar{f}} = \{\alpha : \text{there is no } \bar{\beta} \in {}^\omega\alpha \text{ such that } (\forall t \in Y)(f_\alpha(t) \in \{f_{\beta_n}(t) : n < \omega\})\}$. You may say that $u_{\bar{f}}$ is the set of $\alpha < \delta$ such that f_α is “really novel”.

By DC this is O.K., i.e.

$$\boxplus_1 \text{ for every } \alpha < \delta \text{ there is } \bar{\beta} \in {}^\omega(u_{\bar{f}} \cap \alpha) \text{ such that } (\forall t \in Y)(f_\alpha(t)) = \{f_{\beta_n}(t) : n < \omega\}.$$

Next for $\alpha \in u_{\bar{f}}$ we can define $D_{\bar{f},\alpha}$, the \aleph_1 -complete filter on Y generated by $\left\{ \{t \in Y : f_\beta(t) = f_\alpha(t)\} : \beta < \alpha \right\}$. So clearly $\alpha \neq \beta \in u_{\bar{f}} \wedge D_{\bar{f},\alpha} = D_{\bar{f},\beta} \Rightarrow f_\alpha \neq_D f_\beta$. Now for each pair $\bar{D} = (D_1, D_2) \in \text{Fil}_Y^4$ (i.e. for the \aleph_1 -complete case) let $\Lambda_{\bar{f},\bar{D}} = \{\alpha \in u_{\bar{f}} : D_{\bar{f},\alpha} = D_1 \text{ and } J[f_\alpha, D_1]\} = \text{dual}(D_2)$. So γ is the union of $\leq \mathcal{P}(\mathcal{P}(Y))$ -sets (as $|Y| = |Y| \times |Y|$, well ordered).

So

$$(*)_1 \quad \gamma \leq \text{hrtg}(({}^Y\omega \times {}^\omega(\mu))$$

$$(*)_2 \quad u \text{ is the union of } \mathcal{P}(\mathcal{P}(\kappa))\text{-sets each of cardinality } < \text{pp}_{Y,\aleph_1}^+(\mu)$$

$$(I) \quad \text{what about } \text{hrtg}({}^\kappa\mu) < \text{ps-pp}_{Y,\aleph_1}(\mu)?$$

We are given $\langle \mathcal{F}_\alpha : \alpha < \kappa \rangle \neq F_\alpha \neq \emptyset, \mathcal{F}_\alpha \subseteq \mu, \alpha \neq \beta \Rightarrow \mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$.

Easier: looking modulo a fix filter D .

$$(*)_2 \quad \text{for } D \in \text{Fil}_{Y,\aleph_1}, \text{ let } \mathcal{F}_{\alpha,D} = \{f \in \mathcal{F}_\alpha : \neg(\exists g \in \mathcal{F}_\alpha)(g <_D f)\}.$$

Maybe we have somewhere a bound on the size of $\mathcal{F}_{\alpha,D}$.

§ 3(B). Depth of Reduced Power of Ordinals.

Our intention has been to generalize a relative of [She00], but actually we are closed to [She02, §3]. So as there we use IND but unlike [She12] rather than with rank we deal with depth.

Definition 3.10. 1) Let $\text{suc}_X(\alpha)$ be the first ordinal β such that we cannot find a sequence $\langle \mathcal{U}_x : x \in X \rangle$ of subsets of β , each of order type $< \alpha$ such that $\beta = \cup\{\mathcal{U}_x : x \in X\}$.

2) We define $\text{suc}_X^{[\varepsilon]}(\alpha)$ by induction on ε naturally: if $\varepsilon = 0$ it is α , if $\varepsilon = \zeta + 1$ it is $\text{suc}_X(\text{suc}_X^{[\zeta]}(\alpha))$ and if ε is a limit ordinal then it is $\cup\{\text{suc}_X^{[\zeta]}(\alpha) : \zeta < \varepsilon\}$.

3) For a quasi-order P let the pseudo ordinal depth of P , denoted by $\text{ps-o-Depth}(P)$ be $\sup\{\gamma : \text{there is a } <_P\text{-increasing sequence } \langle X_\alpha : \alpha < \gamma \rangle \text{ of non-empty subsets of } P\}$.

4) $\text{o-Depth}(P)$ is defined similarly demanding $|X_\alpha| = 1$ for $\alpha < \gamma$.

5) Omitting the ‘‘ordinal’’ means γ is replaced by $|\gamma|$; similarly in the other variants including omitting the letter o in ps-o-Depth.

6) Let $\text{ps-o-Depth}^+(P) = \sup\{\gamma + 1 : \text{there is an increasing sequence } \langle X_\alpha : \alpha < \gamma \rangle \text{ of non-empty subsets of } P\}$. Similarly for the other variants, e.g. without o we use $|\gamma|^+$ instead of $\gamma + 1$ in the supremum.

7) For D a filter on Y and $\bar{\alpha} \in {}^Y(\text{Ord} \setminus \{0\})$ let $\text{ps-o-Depth}_D^+(\bar{\alpha}) = \text{ps-o-Depth}^+(\Pi\bar{\alpha}, <_D)$. Similarly for the other variants and we may allow $\alpha_t = 0$ as in 3.1(3).

8) Let $\text{ps-o-depth}_D^+(\bar{\alpha})$ be the cardinality of $\text{ps-o-Depth}_D^+(\bar{\alpha})$.

Remark 3.11. Note that 1.14 can be phrased using this definition.

Definition 3.12. 0) We say \mathbf{x} is a filter ω -sequence when $\mathbf{x} = \langle (Y_n, D_n) : n < \omega \rangle = \langle Y_{\mathbf{x},n}, D_{\mathbf{x},n} : n < \omega \rangle$ is such that D_n is a filter on Y_n for each $n < \omega$; we may omit Y_n as it is $\cup\{Y : Y \in D\}$ and may write D if $\bigwedge_n D_n = D$.

1) Let $\text{IND}(\mathbf{x})$, \mathbf{x} has the independence property, mean that for every sequence $\bar{F} = \langle F_{m,n} : m < n < \omega \rangle$ from $\text{alg}(\mathbf{x})$, see below, there is $\bar{t} \in \prod_{n < \omega} Y_n$ such that $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{t} \upharpoonright (m, n])$. Let $\text{NIND}(\mathbf{x})$ be the negation.

2) Let $\text{alg}(\mathbf{x})$ be the set of sequence $\langle F_{n,m} : m < n < \omega \rangle$ such that $F_{m,n} :$

$$\prod_{\ell=m+1}^n Y_\ell \rightarrow \text{dual}(D_n).$$

3) We say \mathbf{x} is κ -complete when each $D_{\mathbf{x},n}$ is a κ -complete filter.

Theorem 3.13. *Assume $\text{IND}(\mathbf{x})$ where $\mathbf{x} = \langle (Y_n, D_n) : n < \omega \rangle$ is as in Definition 3.12, D_n is κ_n -complete, $\kappa_n \geq \aleph_1$.*

1) [DC + AC_{Y_n} for $n < \omega$] For every ordinal ζ , for infinitely many n 's $\text{ps-o-Depth}^{(Y_n)} \zeta, <_{D_n} \leq \zeta$.

2) [DC] For every ordinal ζ for infinitely many n , $\text{o-Depth}^{(Y_n)} \zeta, <_{D_n} \leq \zeta$, equivalently there is no $<_{D_n}$ -increasing sequence of length $\zeta + 1$.

Remark 3.14. 0) Note that the present results are incomparable with [She12, §4] - the loss is using depth instead of rank and possibly using "pseudo".

1) [Assume AC_{\aleph_0}] If for every n we have $\text{rk}_{D_n}(\zeta) > \text{suc}_{\text{Fil}_{\aleph_1}^1(D_n)}(\zeta)$ then for some $D_n^1 \in \text{Fil}_{\aleph_1}^1(Y_n)$ for $n < \omega$ we have $\text{NIND}(\langle (Y_n, D_n^1) : n < \omega \rangle)$. (Why? By [She12, 5.9]). But we do not know much on the D_n^1 's.

2) This theorem applies to e.g. $\zeta = \aleph_\omega, Y_n = \aleph_n, D_n = \text{dual}(J_{\aleph_n}^{\text{bd}})$. So even in ZFC, it tells us things not covered by [She02, §3]. So it also tells us that it is easy by forcing to get, e.g. $\text{NIND}(\langle (\aleph_{n+1}, \text{dual}(J_{\aleph_{n+1}}^{\text{bd}})) : n < \omega \rangle)$, see 3.19. Note that Depth and pcf are closely connected but only for sequences of length $\geq \text{hrtg}(\mathcal{P}(Y))$ and see 3.19 below.

3) If we assume $\text{IND}(\langle (Y_{\eta(n)}, D_{\eta(n)}) : n < \omega \rangle)$ for every increasing $\eta \in {}^\omega \omega$, which is quite reasonable then in Theorem 3.13 we can strengthen the conclusion, replacing "for infinitely many n 's" by "for every $n < \omega$ large enough".

4) Note that 3.13(2) is complimentary to [She].

Observation 3.15. 1) If \mathbf{x} is a filter ω -sequence, \mathbf{x} is \aleph_1 -complete and $n_* < \omega$ and $\text{IND}(\mathbf{x} \upharpoonright [n_*, \omega])$ then $\text{IND}(\mathbf{x})$.

2) If \mathbf{x} is a filter ω -sequence and $\text{IND}(\mathbf{x})$ and $\eta \in {}^\omega \omega$ is increasing and $\mathbf{y} = \langle (Y_{\mathbf{x},\eta(n)}, D_{\mathbf{x},\eta(n)}) : n < \omega \rangle$ then \mathbf{y} is a filter ω -sequence and $\text{IND}(\mathbf{y})$.

Proof. 1) Let $\bar{F} = \langle (F_{n,m} : n < m < \omega) \in \text{alg}(\mathbf{x})$, so $\langle (F_{n,m} : n \in [n_*, \omega] \text{ and } m \in (n, \omega)) \rangle$ belongs to $\text{alg}(\mathbf{x} \upharpoonright [n_*, \omega])$ hence by the assumption "IND($\mathbf{x} \upharpoonright [n_*, \omega]$)" there is $\bar{t} = \langle t_n : n \in [n_*, \omega) \rangle \in \prod_{n \geq n_*} Y_n$ such that $t_n \notin F_{n,m}(\bar{t} \upharpoonright (n, m))$ when

$n_* \leq n < \omega$. Now by downward induction on $n < n_*$ we choose $t_n \in Y_n$ such that $t_n \notin F_{n,m}(\bar{t} \upharpoonright [n+1, m])$ for $m \in [n+1, \omega)$. This is possible as the countable union of members of $\text{dual}(D_n)$ is not equal to Y_n . We can carry the induction and $\langle t_n : n < \omega \rangle$ is as required to verify $\text{IND}(\mathbf{x})$.

2) Let $\bar{F} = \langle (F_{i,j} : i < j < \omega) \in \text{alg}(\mathbf{y})$. For $m < n$ we define $F'_{m,n}$ as the following function from $\prod_{k=m-1}^n Y_{\mathbf{x},k}$ into $\text{dual}(D_{\mathbf{x},m})$ by

- if $i < j, m = \eta(i), n = \eta(j)$ and $\bar{s} = \langle s_k : k \in (m, n] \rangle \in \prod_{k=m+1}^n Y_{\mathbf{x},k}$ then $F'_{m,n}(\bar{s}) = F_{i,j}(\langle s_{\eta(i+k)} : k \in [1, j-i] \rangle)$
- if there are no such i, j then $F_{m,n}$ is constantly \emptyset .

As $\text{IND}(\mathbf{x})$ holds there is $\bar{t} \in \prod_n Y_{\mathbf{x},k}$ such that $m < n \Rightarrow t_m \notin F_{m,n}(\bar{t} \upharpoonright (m, n))$. Now $\bar{t}' = \langle t_{\eta(k)} : k < \omega \rangle \in \prod_n Y_{\mathbf{x},\eta(n)} = \prod_n Y_{\mathbf{y},n}$ is necessarily as required. $\square_{3.15}$

Proof. Proof of Theorem 3.13

We concentrate on proving part (1), part (2) is easier, (i.e. below each $\mathcal{F}_{n,\varepsilon}$ is a singleton hence so is $\mathcal{G}_{m,n,\varepsilon}^1$ so there is no need to use AC_{Y_n}).

Assume this fails. So for some $n_* < \omega$ for every $n \in [n_*, \omega)$ there is a counter-example. As AC_{\aleph_0} holds we can find a sequence $\langle \bar{\mathcal{F}}_n : n \in [n_*, \omega) \rangle$ such that:

- ⊙ for $n \in [n_*, \omega)$
 - (a) $\bar{\mathcal{F}}_n = \langle \mathcal{F}_{n,\varepsilon} : \varepsilon \leq \zeta \rangle$
 - (b) $\mathcal{F}_{n,\varepsilon} \subseteq Y_n \zeta$ is non-empty
 - (c) $\bar{\mathcal{F}}_n$ is a $<_{D_n}$ -increasing sequence of sets, i.e. $\varepsilon_1 < \varepsilon_2 \leq \zeta \wedge f_1 \in \mathcal{F}_{n,\varepsilon_1} \wedge f_2 \in \mathcal{F}_{n,\varepsilon_2} \Rightarrow f_1 <_{D_n} f_2$.

Now by AC_{\aleph_0} we can choose $\langle f_n : n \in [n_*, \omega) \rangle$ such that $f_n \in \mathcal{F}_{n,\zeta}$ for $n \in [n_*, \omega)$.

(*) without loss of generality $n_* = 0$.

[Why? As $\mathbf{x}[n_*, \omega)$ satisfies the assumptions on \mathbf{x} by 3.15(2).]

Now

- ⊞₁ for $m \leq n < \omega$ let $Y_{m,n}^0 = \prod_{\ell=m}^{n-1} Y_\ell$ and for $m, n < \omega$ let $Y_{m,n}^1 := \cup \{Y_{k,n}^0 : k \in [m, n]\}$ so $Y_{m,n}^0 = \emptyset = Y_{m,n}^1$ if $m > n$ and $Y_{m,n}^0 = \{\langle \rangle\} = Y_{m,n}^1$ if $m = n$; so if $\eta \in Y_{m+1,n}^0$ and $s \in Y_m, t \in Y_{n+1}$ we define $\langle s \rangle \hat{\wedge} \eta \in Y_{m,n}^0$ and $\eta \hat{\wedge} \langle t \rangle \in Y_{m+1,n+1}$ naturally
- ⊞₂ for $m \leq n$ let $\mathcal{G}_{m,n}^1$ be the set of functions g such that:
 - (a) g is a function from $Y_{m,n}^1$ into $\zeta + 1$
 - (b) $\langle \rangle \neq \eta \in Y_{m,n}^1 \Rightarrow g(\eta) < \zeta$
 - (c) if $k \in [m, n)$ and $\eta \in Y_{k+1,n}^0$ then the sequence $\langle g(\langle s \rangle \hat{\wedge} \eta) : s \in Y_k \rangle$ belongs to $\mathcal{F}_{k,g(\eta)}$
- ⊞₃ $\mathcal{G}_{m,n,\varepsilon}^1 := \{g \in \mathcal{G}_{m,n}^1 : g(\langle \rangle) = \varepsilon\}$ for $\varepsilon \leq \zeta$ and $m \leq n < \omega$.

Now the sets $\mathcal{G}_{m,n}^1$ are non-trivial, i.e.

- ⊞₄ if $m \leq n$ and $\varepsilon \leq \zeta$ then $\mathcal{G}_{m,n,\varepsilon}^1 \neq \emptyset$.

[Why? We prove it by induction on n ; first if $n = m$ this is trivial because the unique function g with domain $\{\langle \rangle\}$ and value ε belongs to $\mathcal{G}_{m,n,\varepsilon}^1$. Next, if $m < n$ we choose $f \in \mathcal{F}_{n-1,\varepsilon}$ hence the sequence $\langle \mathcal{G}_{m,n-1,f(s)}^1 : s \in Y_{n-1} \rangle$ is well defined and by the induction hypothesis each set in the sequence is non-empty. As $\text{AC}_{Y_{n-1}}$ holds there is a sequence $\langle g_s : s \in Y_{n-1} \rangle$ such that $s \in Y_{n-1} \Rightarrow g_s \in \mathcal{G}_{m,n-1,f(s)}^1$. Now define g as the function with domain $Y_{m,n}^1$:

$$g(\langle \rangle) = \varepsilon$$

$$g(\nu \hat{\wedge} \langle s \rangle) = g_s(\nu) \text{ for } \nu \in Y_{m,n-1}^1 \text{ and } s \in Y_n.$$

It is easy to check that $g \in \mathcal{G}_{m,n,\varepsilon}^1$ indeed so \boxplus_4 holds.]

- \boxplus_5 if $g, h \in \mathcal{G}_{m,n}^1$ and $g(\langle \rangle) < h(\langle \rangle)$ then there is an (m, n) -witness Z for (h, g) which means (just being an (m, n) -witness means we omit clause (d)):
- (a) $Z \subseteq Y_{m,n}^1$ is closed under initial segments, i.e. if $\eta \in Y_{k,n}^0 \cap Z$ and $m \leq k < \ell \leq n$ then $\eta \upharpoonright [\ell, n) \in Y_{\ell,n}^0 \cap Z$
 - (b) $\langle \rangle \in Z$
 - (c) if $\eta \in Z \cap Y_{k+1,n}^0$, $m \leq k < n$ then $\{s \in Y_k : \langle s \rangle \wedge \eta \in Z\} \in D_k$
 - (d) if $\eta \in Z$ then $g(\eta) < h(\eta)$.

[Why? By induction on n , similarly to the proof of \boxplus_4 .]

- \boxplus_6 (a) we can find $\bar{g} = \langle g_n : n < \omega \rangle$ such that $g_n \in \mathcal{G}_{0,n,\zeta}^1$ for $n < \omega$
- (b) for \bar{g} as above for $n < \omega$, $s \in Y_n$ let $g_{n+1,s} \in \mathcal{G}_{0,n}^1$ be defined by $g_{n+1,s}(\nu) = g_{n+1}(\nu \wedge \langle s \rangle)$ for $\nu \in Y_{0,n}$.

[Why? Clause (a) by \boxplus_4 as AC_{\aleph_0} holds, clause (b) is obvious by the definitions in $\boxplus_2 + \boxplus_3$.]

We fix \bar{g} as in $\boxplus_6(a)$ for the rest of the proof.

- \boxplus_7 There is $\langle \langle Z_{n,s} : s \in Y_n \rangle : n < \omega \rangle$ such that $Z_{n,s}$ witness $(g_n, g_{n+1,s})$ for $n < \omega$, $s \in Y_n$.

[Why? For a given $n < \omega$, $s \in Y_n$ we know that $g_{n+1}(\langle s \rangle) < \zeta = g_n(\langle \rangle)$ hence $Z_{n,s}$ as required exists by \boxplus_5 . By AC_{Y_n} for each n a sequence $\langle Z_{n,s} : s \in Y_n \rangle$ as required exists, and by AC_{\aleph_0} we are done.]

- \boxplus_8 $Z_n := \{\langle \rangle\} \cup \{\nu \wedge \langle s \rangle : s \in Y_{n-1}, \nu \in Z_{n-1,s}\}$ is a $(0, n)$ -witness.

[Why? By our definitions.]

- \boxplus_9 there is \bar{F} such that:
- (a) $\bar{F} = \langle F_{m,n} : m < n < \omega \rangle$
 - (b) $F_{m,n} : Y_{m+1,n+1}^1 \rightarrow \text{dual}(D_m)$
 - (c) $F_{m,n}(\nu)$ is $\{s \in Y_m : \nu \wedge \langle s \rangle \notin Z_{n-1}\}$ when $\nu \in Z_n$ and is \emptyset otherwise.

[Why? As clauses (a),(b),(c) define \bar{F} .]

- \boxplus_{10} \bar{F} witness $\text{IND}(\langle (Y_n, D_n) : n < \omega \rangle)$ fail.

[Why? Clearly $\bar{F} = \langle F_{m,n} : m < n < \omega \rangle$ has the right form.

So toward contradiction assume $\bar{t} = \langle t_n : n < \omega \rangle \in \prod_{n < \omega} Y_n$ is such that

- $(*)_1$ $m < n < \omega \Rightarrow t_m \notin F_{m,n}(\bar{t} \upharpoonright (m, n])$.

Now

- $(*)_2$ $\bar{t} \upharpoonright [m, n) \in Z_n$ for $m \leq n < \omega$.

[Why? For each n , we prove this by downward induction on m . If $m = n$ then $\bar{t} \upharpoonright [m, n) = \langle \rangle$ but $\langle \rangle \in Z_n$ by its definition. If $m < n$ and $\bar{t} \upharpoonright [m+1, n) \in Z_n$ then $t_m \notin F_{m,n-1}(\bar{t} \upharpoonright (m, n])$ by $(*)_1$ so $\bar{t} \upharpoonright [m, n) = \langle t_m \rangle \wedge (\bar{t} \upharpoonright [m+1, n)) \in Z_n$ holds by clause $\boxplus_9(c)$.]

- $(*)_3$ $g_{n+1}(\bar{t} \upharpoonright [m, n]) < g_n(\bar{t} \upharpoonright [m, n])$.

[Why? Note that Z_{n,t_n} is a witness for (g_n, g_{n+1,t_n}) by \boxplus_7 . So by \boxplus_5 (see clause (d) there) we have $\eta \in Z_{n,t_n} \Rightarrow g_{n+1,t_n}(\eta) < g_n(\eta)$. But $m < n \Rightarrow \bar{t} \upharpoonright [m, n] \in Z_{n+1} \Rightarrow \bar{t} \upharpoonright [m, n] \in Z_{n,t_n}$, the first implication by $(*)_2$, the second implication by the definition of Z_{n+1} in \boxplus_8 . Hence by $\boxplus_6(b)$ and the last sentence, and by the sentence before last $g_{n+1}(\bar{t} \upharpoonright [m, n]) = g_{n+1,t_n}(\bar{t} \upharpoonright [m, n]) < g_n(\bar{t} \upharpoonright [m, n])$ as required. So $(*)_3$ holds indeed.]

So for each $m < \omega$ the sequence $\langle g_n(\bar{t} \upharpoonright [m, n]) : n \in [m, \omega) \rangle$ is a decreasing sequence of ordinals, contradiction. Hence there is no \bar{t} as above, so indeed \boxplus_{10} holds. But \boxplus_{10} contradicts an assumption, so we are done. $\square_{3.13}$

Remark 3.16. 1) Note that in the proof of 3.13 there was no use of completeness demands, still natural to assume \aleph_1 -completeness because: if D'_n is the \aleph_1 -completion of D_n then $\text{IND}(\langle D'_n : n < \omega \rangle)$ is equivalent to $\text{IND}(D_n : n < \omega)$.

2) Recall that by [She02, 2.7], iff $\text{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then for every $\lambda > \aleph_\omega$ for infinitely many $n < \omega$ we have $(\forall \mu < \lambda)(\text{cf}(\mu) = \aleph_n \Rightarrow \text{pp}(\mu) \leq \lambda)$.

3) Concerning 3.17 below recall that:

- (A) if Y_n is a regular cardinal, D_n witness Y_n is a measurable cardinal, then clause (a) of 3.17 holds, but [She12, §4] says more
- (B) if $\mu = \mu^{<\mu}$ and \mathbb{P}_μ is the Levy collapse a measurable cardinal $\lambda > \mu$ to be μ^+ with D a normal ultrafilter on λ , then $\Vdash_{\mathbb{P}_\mu}$ “the filter which D generates is as required in (b) with μ in the role of Z_n ”, by Jech-Magidor-Mitchel-Prikry [JMMP80].

So we can force that $n < \omega \Rightarrow Y_n = \aleph_{2n}$.

4) So

- (a) if $\text{pp}(\aleph_\omega) > \aleph_{\omega_1}$ and \aleph_ω divides δ , $\text{cf}(\delta) < \aleph_\omega$ and $\delta < \aleph_\delta$ then $\text{pp}(\aleph_\delta) < \aleph_{|\delta|^+}$
- (b) we can replace \aleph_ω by any singular $\mu < \aleph_\mu$
- (c) if, e.g. $\delta_n < \lambda_n = \aleph_{\delta_n}$, $\delta_n < \delta_{n+1}$ and $\text{cf}(\delta_n) < \aleph_{\delta_0}$ for $n < \omega$, then, except for at most one n , $\text{pp}(\aleph_{\lambda_n}) < \aleph_{\lambda_n^+}$.

5) We had thought that maybe: if μ is singular and $\text{pp}(\mu) \geq \aleph_{\mu^+}$ then some case of IND follows. Why? Because by [She02, 2.8] this holds if $\mu < \aleph_{\mu^+}$ provided that $\mu = \aleph_\delta \wedge |\delta|^{\aleph_0} < \mu$, (even getting $\text{IND}(\langle \text{dual}(J_{\lambda_n}^{\text{bd}}) : n < \omega \rangle)$ for some increasing sequence $\langle \lambda_n : n < \omega \rangle$ of regular cardinals $< \mu$ with limit μ if $\text{cf}(\mu) = \aleph_0$ and $\subseteq \{\lambda^+ : \lambda \in E\}$ for any pre-given club E of μ if $\text{cf}(\mu) > \aleph_0$). If only $\mu = \aleph_\delta \wedge |\delta| < \mu$ then in [She02] we get a weaker version of IND.

Claim 3.17. [DC] For $\mathbf{x} = \langle Y_n, D_n : n < \omega \rangle$ with each D_n being an \aleph_1 -complete filter on Y_n , each of the following is a sufficient condition for $\text{IND}(\mathbf{x})$, letting $Y(< n) := \prod_{m < n} Y_m$ and for $m < n$, let $Z_{m,n} = \{t : t \text{ is a function from } \prod_{\ell=m+1}^{n-1} Y_\ell \text{ into } Y_m\}$ and let $Z_n = \prod_{m < n} Z_{m,n}$

- (a) D_n is a $(\leq Z_n)$ -complete ultrafilter
- (b) • D_n is a $(\leq Z_n)$ -complete filter

- for each n in the following game $\mathfrak{D}_{\mathbf{x},n}$ the non-empty player has a winning strategy. A play last ω -moves. In the k -th move the empty player chooses $A_k \in D_n$ and $\langle X_t^k : t \in Z_n \rangle$, a partition of A_k and the non-empty player chooses $t_k \in Z_n$. In the end the non-empty player wins the play if $\bigcap_{k < \omega} X_{t_k}^k$ is non-empty
- (c) like clause (b) but in the second part the non-empty player instead t_k chooses $S_k \subseteq Z_n$ satisfying $|S_k| \leq_X |S|$ and every $D_{\mathbf{x},n}$ is $(\leq S)$ -complete, S is infinite
- (d) if $m < n < \omega$ then D_m is $(\leq \prod_{k=m+1}^n Y_k)$ -complete⁵

Proof. Straightforward. E.g.

Clause (b):

Let $\langle \mathbf{st}_n : n < \omega \rangle$ be such that \mathbf{st}_n is a winning strategy of the non-empty player in the game $\mathfrak{D}_{\mathbf{x},n}$.

Let $\bar{F} = \langle F_{m,n} : m < n < \omega \rangle \in \text{alg}(\mathbf{x})$ and we should find a member of $\prod_n Y_n$ as required in Definition 3.12(2). We now, by induction on $i < \omega$, choose the following objects satisfying the following condition

- (*)_i (a) for $k < m$ and $j < i$, $G_{j,k,m}$ is a function from $\prod_{\ell=k+1}^m Y_\ell$ into Y_k
- (b)(α) for $m < \omega$, $\langle (\bar{X}_{j,m}, \mathbf{t}_{j,m}) : j < i \rangle$ is an initial segment of a play of the game $\mathfrak{D}_{\mathbf{x},m}$ in which the non-empty player uses the strategy \mathbf{st}_m ;
- (β) we have $\bar{X}_{j,m} = \langle X_{j,m,t} : \mathbf{t} \in Z_m \rangle$ so $X_{j,m,\mathbf{t}} \subseteq Y_m$
- (γ) $\mathbf{t}_{j,m} = \langle t_{j,k,m} : k < m \rangle$ and $t_{j,k,m} \in Z_k$
- (δ) $X_{j,m,\mathbf{t}} = \bigcap_{k < m} X_{j,k,m,t_k}$, see clause (e) when $\mathbf{t} = \langle t_k : k < m \rangle \in Z_m, \bigwedge_k t_k \in Z_{k,m}$
- (c)(α) $Y_{j,m}$ is Y_m if $j = 0$
- (β) $Y_{j,m}$ is $\bigcap \{X_{\ell,m,k,t_{j,k,m}} : \ell < j\} \subseteq Y_m$ if $j \in (0, i)$
- (d)(α) if $j = 0 < i$ then $G_{j,k,m}$ is $F_{k,m}$
- (β) if $j \in (0, i)$ then $G_{j,k,m}$ is defined by: for $\langle y_{k+1}, \dots, y_m \rangle \in \prod_{\ell=k+1}^m Y_\ell$ we have $G_{j,k,m}(\langle y_{k+1}, \dots, y_m \rangle) = G_{j-1,k,m+1}(\langle y_{k+1}, \dots, y_{m+1} \rangle)$ for any $y_{m+1} \in Y_{j,m+1}$ (so the value does not depend on y_{m+1} !)
- (e) for $k < m$ and $t \in Z_{k,m}$ let $X_{j,k,m,t}$ be $\{y \in Y_m : \text{if } \langle y_{k+1}, \dots, y_{m-1} \rangle \in \prod_{\ell=k+1}^{m-1} Y_\ell \text{ then } G_{j,k,m}(y_{k+1}, \dots, y_{m-1}, y) = (y_{k+1}, \dots, y_{m-1})\}$.

⁵So the Y_k 's are not well ordered! But, on the one hand, if $\alpha < \text{hrtg}(Y_n) \Rightarrow D_n$ is $|\alpha|^+$ -complete then $\alpha^{Y_n}/D_n \cong \alpha$. On the other hand, if D_n is \aleph_1 -complete and $\alpha^{Y_n}/D \cong \alpha$ then D projects onto a uniform \aleph_1 -complete filter on some $\mu \leq \alpha$ and those projections cover the ultra-power.

Clearly $(*)_0$ holds emptyly.

For $i \geq 1$, let $j = i - 1$ clearly $\langle Y_{j,m} : m < \omega \rangle$ is well defined by clause (c), hence we can define $\langle X_{j,k,m,t} : t \in Z_{k,m} \rangle$ by clause (e) and let $X_{j,m,\mathbf{t}} = \cap \{X_{j,k,m,t_k} : k < m\}$ when $\mathbf{t} = \langle t_k : k < m \rangle$.

So $\bar{X}_{j,m} = \langle X_{j,m,\mathbf{t}} : \mathbf{t} \in Z_m \rangle$ is a legal j -move of the empty player in the game $\mathfrak{D}_{\mathbf{x},m}$, so we can use \mathbf{st}_m to define $\mathbf{t}_{j,m} = \langle t_{j,k,m} : k < m \rangle$ as the j -th move of the non-empty player.

Lastly, the function $G_{j,k,m}$ is well defined. Having carried the induction, for each m clearly $\langle (\bar{X}_{j,m}, \mathbf{t}_{j,m}) : j < \omega \rangle$ is a play of the game $\mathfrak{D}_{\mathbf{x},m}$ in which the non-empty player uses the strategy \mathbf{st}_m hence win in the play, so $\cap \{X_{j,m,\mathbf{t}_{j,m}} : j < \omega\}$ is non-empty so by AC_{\aleph_0} we can choose $\bar{y} = \langle y_m : m < \omega \rangle$ such that $y_m \in \cap \{X_{j,m,\mathbf{t}_{j,m}} : j < \omega\}$.

It is easy to see that \bar{y} is as required in Definition 3.12(2). □_{3.17}

Conclusion 3.18. [DC] Assume $\langle \kappa_n : n \rangle$ is increasing and κ_n is measurable as witnessed by the ultrafilter D_n or just D_n is a uniform⁶ $\Upsilon(\mathcal{P}(\kappa_{n-1}))$ -complete ultrafilter on κ_n .

Then for every ordinal ζ , for every large enough n we have $\text{o-Depth}_{D_n}^+(\zeta) \leq \zeta$.

Proof. By 3.17 we know that $\text{IND}(\langle D_n : n < \omega \rangle)$ and by 3.13(2) we get the desired conclusion. □_{3.18}

Claim 3.19. (ZFC for simplicity).

If (A) then (B) where

(A) (a) $\lambda_n = \text{cf}(\lambda_n)$ and $(\lambda_n)^{<\lambda_n} < \lambda_{n+1}$ and $\mu = \Sigma\{\lambda_n : n < \omega\}$ and $\lambda = \mu^+$

(b) \mathbb{P}_n is the natural λ_n -complete λ_n^+ -c.c. forcing adding $\langle f_{n,\alpha} : \alpha < \lambda \rangle$ of members of ${}^{\lambda_n}(\lambda_n)$, $<_{J_{\lambda_n}^{\text{bd}}}$ -increasing

(c) \mathbb{P} is the product $\prod_n \mathbb{P}_n$ with full support

(B) in $\mathbf{V}^{\mathbb{P}}$ we have $\text{NIND}(\langle \text{dual}(J_{\lambda_n}^{\text{bd}}) : n < \omega \rangle)$ and a cardinal θ is not collapsed if $\theta \notin (\mu^+, \mu^{\aleph_0}]$.

Proof. So $p \in \mathbb{P}_n$ iff p is a function from some $u \in [\lambda^+]^{<\lambda_n}$ into $\cup \{\zeta(\lambda_n) : \zeta < \lambda_n\}$, ordered by $\mathbb{P}_n \models "p \leq q"$ iff $\alpha \in \text{Dom}(q) \Rightarrow \alpha \in \text{Dom}(p) \wedge p(\alpha) \subseteq q(\alpha)$. Now use 3.13. □_{3.19}

§ 3(C). **Bounds on the Depth.** We continue 3.2. We try to get a bound for singulars of uncountable cofinality say for the depth, recalling that depth, rank and $\text{ps-}T_D$ are closely related.

Hypothesis 3.20. D an \aleph_1 -complete filter on a set Y .

Remark 3.21. Some results do not need the \aleph_1 -completeness.

⁶Recall $\Upsilon(A) = \min\{\theta : \text{there is no one-to-one function from } \theta \text{ into } A\}$.

Claim 3.22. Assume $\bar{\alpha} \in {}^Y \text{Ord}$.

- 1) [DC] (No-hole-Depth) If $\zeta + 1 \leq \text{ps-o-Depth}_D^+(\bar{\alpha})$ then for some $\bar{\beta} \in {}^Y \text{Ord}$, we have $\bar{\beta} \leq \bar{\alpha} \text{ mod } D$ and $\zeta + 1 = \text{ps-o-Depth}^+(\bar{\beta})$.
- 2) In Definition 3.1 we may allow $\mathcal{F}_\varepsilon \subseteq {}^Y \text{Ord}$ such that $g \in \mathcal{F}_\varepsilon \Rightarrow g < f \text{ mod } D$.
- 3) If $\bar{\beta} \in {}^Y \text{Ord}$ and $\bar{\alpha} = \bar{\beta} \text{ mod } D$ then $\text{ps-o-Depth}^+(\bar{\alpha}) = \text{ps-o-Depth}^+(\bar{\beta})$.
- 4) If $\{y \in Y : \alpha_y = 0\} \in D^+$ then $\text{ps-o-Depth}^+(\bar{\alpha}) = 1$.
- 5) Similarly for the other versions of depth from Definition 3.10.

Proof. 1) By DC without loss of generality there is no $\bar{\beta} <_D \bar{\alpha}$ such that $\zeta + 1 \leq \text{ps-o-Depth}^+(\bar{\beta})$. Without loss of generality $\bar{\alpha}$ itself fails the desired conclusion hence $\zeta + 1 < \text{ps-o-Depth}^+(\bar{\alpha})$. By parts (3),(4) without loss of generality $s \in Y \Rightarrow \alpha_s > 0$. As $\zeta + 1 < \text{ps-o-Depth}^+(\bar{\alpha})$ there is a $<_D$ -increasing sequence $\langle \mathcal{F}_\varepsilon : \varepsilon < \zeta + 1 \rangle$ with \mathcal{F}_ε a non-empty subset of $\Pi \bar{\alpha}$. Now any $\bar{\beta} \in \mathcal{F}_\zeta, \zeta + 1 \leq \text{ps-o-Depth}^+(\bar{\beta})$ as witnessed by $\langle \mathcal{F}_\varepsilon : \varepsilon < \zeta \rangle$, recalling part (2); contradicting the extra assumption on $\bar{\alpha}$ (being $<_D$ -minimal such that...).

- 2) Let $\mathcal{F}'_\varepsilon = \{f^{[\bar{\alpha}]} : f \in \mathcal{F}_\varepsilon\}$ where $f^{[\bar{\alpha}]}(s)$ is $f(s)$ if $f(s) < \alpha_s$ and is zero otherwise.
- 3),4) Obvious.
- 5) Similarly. □_{3.22}

Claim 3.23. [DC + AC_Y] If $\bar{\alpha}, \bar{\beta} \in {}^Y \text{Ord}$ and D is a filter on Y and $s \in Y \Rightarrow |\alpha_s| = |\beta_s|$ then $\text{ps-T}_D(\bar{\alpha}) = \text{ps-T}_D(\bar{\beta})$.

Proof. Straightforward. □_{3.23}

Assuming full choice the following is a relative of Galvin-Hajnal theorem.

Theorem 3.24. [DC + AC_Y] Assume $\alpha(1) < \alpha(2) < \lambda^+$, $\text{ps-o-Depth}^+(\lambda) \leq \lambda^{+\alpha(1)}$ and $\xi = \text{hrtg}({}^Y \alpha(2)/D)$. Then $\text{ps-o-Depth}_D^+(\lambda^{+\alpha(2)}) < \lambda^{+(\alpha(1)+\xi)}$.

Proof. Let $\Lambda = \{\mu : \lambda^{+\alpha(1)} < \mu \leq \lambda^{\alpha(1)+\xi}\}$ and for every $\mu \in \Lambda$ let

- (*)₁ $\mathcal{F}_\mu = \mathcal{F}(\mu) = \{f : f \in {}^Y \{\lambda^{+\alpha} : \alpha < \alpha(2)\} \text{ and } \mu = \text{ps-Depth}_D^+(f)\}$
- (*)₂ obviously $\langle \mathcal{F}_\mu : \mu \in \Lambda \rangle$ is a sequence of pairwise disjoint subsets of ${}^Y \alpha(2)$ closed under equality modulo D .

By the no-hole-depth claim 3.22(1) above we have

- (*)₃ if $\mu_1 < \mu_2$ are from Λ and $f_2 \in \mathcal{F}_{\mu_2}$ then for some $f_1 \in \mathcal{F}_{\mu_1}$ we have $f_1 < f_2 \text{ mod } D$
- (*)₄ $\xi > \sup\{\zeta + 1 : \mathcal{F}(\lambda^{+(\alpha+\zeta)}) \neq \emptyset\}$ implies the conclusion.

Lastly, as $\xi = \text{hrtg}({}^Y \alpha(2)/D)$ we are done. □_{3.24}

Remark 3.25. 0) Compare this with conclusion 1.11.

1) We may like to lower ξ to $\text{ps-Depth}_D^+(\alpha(2))$, toward this let

- (*)₁ $\mathcal{F}'_\mu = \{f \in \mathcal{F}_\mu : \text{there is no } g \in \mathcal{F}_\mu \text{ such that } g < f \text{ mod } D\}$.

By DC

- (*)₂ if $f \in \mathcal{F}_\mu$ then there is $g \in \mathcal{F}'_\mu$ such that $g \leq_D f \text{ mod } D$.

2) Still the sequence of those \mathcal{F}'_μ is not $<_D$ -increasing.

Instead of counting cardinals we can count regular cardinals.

Theorem 3.26. [DC+AC_Y] *The number of regular cardinals in the interval $(\lambda^{+\alpha(1)}, \text{ps-depth}_D^+(\lambda^{+\alpha(2)})$ is at most $\text{hrtg}^Y(\alpha(2)/D)$ when:*

- (a) $\alpha(1) < \alpha(2) < \lambda^+$
- (b) $\kappa > \aleph_0$
- (c) D is a κ -complete filter on Y
- (d) $\lambda^{+\alpha(1)} = \text{ps-Depth}_D(\lambda)$.

Proof. Straightforward, using the No-Hole Claim 1.13.

□_{3.26}

REFERENCES

- [JMMP80] Thomas Jech, Menachem Magidor, William Mitchell, and Karel Prikry, *On precipitous ideals*, J. of Symb. Logic **45** (1980), 1–8.
- [She] Saharon Shelah, *PCF without choice*, arXiv: math/0510229.
- [She82] ———, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982. MR 675955
- [She94] ———, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912
- [She97] ———, *Set theory without choice: not everything on cofinality is possible*, Arch. Math. Logic **36** (1997), no. 2, 81–125, arXiv: math/9512227. MR 1462202
- [She00] ———, *The generalized continuum hypothesis revisited*, Israel J. Math. **116** (2000), 285–321, arXiv: math/9809200. MR 1759410
- [She02] ———, *PCF and infinite free subsets in an algebra*, Arch. Math. Logic **41** (2002), no. 4, 321–359, arXiv: math/9807177. MR 1906504
- [She06] ———, *More on the revised GCH and the black box*, Ann. Pure Appl. Logic **140** (2006), no. 1-3, 133–160, arXiv: math/0406482. MR 2224056
- [She12] ———, *PCF arithmetic without and with choice*, Israel J. Math. **191** (2012), no. 1, 1–40, arXiv: 0905.3021. MR 2970861
- [She16] ———, *ZF + DC + AX₄*, Arch. Math. Logic **55** (2016), no. 1-2, 239–294, arXiv: 1411.7164. MR 3453586

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: <http://shelah.logic.at>