PRESERVING OLD $([\omega]^{\aleph_0}, \supseteq^*)$ IS PROPER
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Abstract. We give some sufficient and necessary conditions on a forcing notion $\mathbb{Q}$ for preserving the forcing notion $([\omega]^{\aleph_0}, \supseteq^*)$ being proper. They cover many reasonable forcing notions.

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Anotated Content

§ 0. Introduction

We investigate the question "$\mathrm{Pr}_1^+(\mathbb{Q}, \mathbb{R})$", which means that the proper forcing $\mathbb{Q}$ preserves that the (old) $\mathbb{R}$ is proper for various $\mathbb{R}$'s. In what follows, $B \subseteq^* A$ means $|B \backslash A| < \aleph_0$, and $A \supseteq^* B$ means the same.

Recall:

Definition 0.1. properness:

(a) Assume that $N \prec (\mathscr{H}(\chi), \in), \mathbb{P} \in N$ is a forcing notion and $q \in \mathbb{P}$. We say that q is $(N, \mathbb{P})$-generic iff for every dense $D \subseteq \mathbb{P}$, if $D \in N$ then $D \cap N$ is pre-dense above q.

(b) A forcing notion $\mathbb{P}$ is proper iff for every sufficiently large regular χ and every countable $N \prec (\mathscr{H}(\chi), \in)$, if $p, \mathbb{P} \in N$ then there is a condition $q \in \mathbb{P}, q \geq p$ such that q is $(N, \mathbb{P})$-generic.

Gitman proved that $\mathrm{Pr}_1^+(\mathbb{Q}, \mathbb{P}_{\mathscr{P}(\omega)[\mathbf{V}]})$ (see definition below, where $\mathbb{P}_{\mathscr{P}(\omega)[\mathbf{V}]}$ is the forcing notion $(\{A \in \mathbf{V} : A \subseteq \omega, |A| = \aleph_0\}, \supseteq^*)$, when $\mathbb{Q}$ is adding Cohen reals (or just Cohen subsets even $> 2^{\aleph_0}$ many). But no other examples were known even Sacks forcing. Also for e.g. $\mathbf{V} \models$ "$V = L$", we did not know a forcing making it not proper.

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Let us state the problem and relatives. We are interested mainly in the case $\mathbb{Q}$ is proper.

Definition 0.2. 1) Let $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P})$ means: $\mathbb{Q}, \mathbb{P}$ are forcing notions and $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$, i.e. $\mathbb{P}^{\mathbf{V}}$ is a proper forcing".
1A) Let $\mathrm{Pr}_1^+(\mathbb{Q}, \mathbb{P})$ be defined similarly but adding "$\mathbb{Q}$ is proper".
2) For $\mathscr{A} \subseteq \mathscr{P}(\omega)$ let $\mathbb{P}_{\mathscr{A}}$ be $\mathscr{A} \backslash [\omega]^{<\aleph_0}$ ordered by $\supseteq^*$, inverse almost inclusion.
3) Let $\mathscr{A}_* = \mathscr{A}_*[\mathbf{V}] = ([\omega]^{\aleph_0})^{\mathbf{V}}$.

Observation 0.3. *A necessary condition for* $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P})$ *is:*

$(*)_1$ *if* χ *is regular and large enough,* $N \prec (\mathscr{H}(\chi), \in)$ *is countable,* $\mathbb{Q}, \mathbb{P} \in N, q_1 \in \mathbb{Q}$ *is* $(N, \mathbb{Q})$*-generic and* $r_1 \in N \cap \mathbb{P}$ *then we can find* (q_2, r_2) *such that:*

$\odot$ (a) $q_1 \leq_{\mathbb{Q}} q_2$
(b) $r_1 \leq_{\mathbb{P}} r_2$
(c) $q_2 \Vdash$ "r_2 *is* $(N[\underset{\sim}{G}_{\mathbb{Q}}], \mathbb{P})$*-generic*".

Definition 0.4. 1) We define $\mathrm{Pr}^-(\mathbb{Q}, \mathbb{P}) = \mathrm{Pr}_2(\mathbb{Q}, \mathbb{P})$ as the necessary condition from 0.3.
2) Let $\mathrm{Pr}_3(\mathbb{Q}, \mathbb{P})$ mean that $\mathbb{Q}, \mathbb{P}$ are forcing notions and for some λ and stationary $S \subseteq [\lambda]^{\aleph_0}$ from $\mathbf{V}$ we have $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ is S-proper", and note that S remains stationary of course.
3) $\mathrm{Pr}_4(\mathbb{Q}, \mathbb{P})$ is defined similarly but $S \in \mathbf{V}^{\mathbb{Q}}$, still $S \subseteq ([\lambda]^{\aleph_0})^{\mathbf{V}}$, so S is actually $\underset{\sim}{S}$, a $\mathbb{Q}$-name.
4) $\mathrm{Pr}_5(\mathbb{Q}, \mathbb{P})$ is the statement (A) of 0.5(4) below.
5) Let $\mathrm{Pr}_\ell^+(\mathbb{Q}, \mathbb{P})$ means $\mathrm{Pr}_\ell(\mathbb{Q}, \mathbb{P})$ and $\mathbb{Q}$ is a proper forcing, for $\ell = 2, 3, 4, 5$.

Claim 0.5. *1)* $\mathrm{Pr}_2(\mathbb{Q},\mathbb{P})$ *means that for* λ *large enough, letting* $S = ([\lambda]^{\aleph_0})^{\mathbf{V}}$*, we have* $\Vdash_{\mathbb{Q}}$ *"*$\mathbb{P}$ *is* S*-proper".*
2) $\mathrm{Pr}_1(\mathbb{Q},\mathbb{P}) \Rightarrow \mathrm{Pr}_2(\mathbb{Q},\mathbb{P}) \Rightarrow \mathrm{Pr}_3(\mathbb{Q},\mathbb{P})$*; similarly for* Pr^+*.*
3) Also $\mathrm{Pr}_3(\mathbb{Q},\mathbb{P}) \Rightarrow \mathrm{Pr}_4(\mathbb{Q},\mathbb{P}) \Rightarrow \mathrm{Pr}_5(\mathbb{Q},\mathbb{P})$*; similarly for* Pr^+*.*
4) If $\mathbb{Q},\mathbb{P}$ *are forcing notions,* χ *large enough and regular,* then *(A)* $\Leftrightarrow$ *(B) where*

- (A) *for some countable* $N \prec (\mathscr{H}(\chi),\in)$ *and for some* $q \in \mathbb{Q}, p \in \mathbb{P}$ *we have*
  - (a) q *is* $(N,\mathbb{Q})$*-generic*
  - (b) $q \Vdash_{\mathbb{Q}}$ *"*p *is* $(N[\underset{\sim}{G}_{\mathbb{Q}}],\mathbb{P})$*-generic"*
- (B) *for some* $q_* \in \mathbb{Q}, p_* \in \mathbb{P}$ *we have* $\mathrm{Pr}_4(\mathbb{Q}_{\geq q_*},\mathbb{P}_{\geq p_*})$*.*

Proof. Easy. $\square_{0.5}$

Notation 0.6. $<^*_\chi$ denotes a well ordering of $\mathscr{H}(\chi)$.

Recall (Balcar-Pelant-Simon [BPS80], or see, e.g. Blass [Bla])

Definition 0.7. $\mathfrak{h}$ is the following cardinal invariant, it is the minimal cardinality χ (necessarily regular) such that forcing with $\mathbb{P}_{\mathscr{A}_*}$ adds a new sequence of ordinals of length χ.

Notation 0.8. If $\mathscr{T}$ is a tree, then $\mathrm{suc}_{\mathscr{T}}(p)$ is the set of immediate successors of $p \in \mathscr{T}$ in the tree order.

§ 1. Properness of $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ and CH

Claim 1.1. *Assume* $\mathbf{V}_0 \models$ CH, $\mathbf{V}_1 \supseteq \mathbf{V}_0$, *e.g.* $\mathbf{V}_1 = \mathbf{V}_0^{\mathbb{Q}}$ *and let* $\mathscr{A} = \mathscr{A}_*[\mathbf{V}_0]$.

(a) *If* $\aleph_1^{\mathbf{V}_0}$ *is a countable ordinal in* $\mathbf{V}_1$, *then* $\mathbf{V}_1 \models$ "$\mathbb{P}_{\mathscr{A}}$ *is proper*".

(b) *If* $\aleph_1^{\mathbf{V}_0} = \aleph_1^{\mathbf{V}_1}$ *and* $\mathbf{V}_1 \models$ "$({}^{\omega}2)^{\mathbf{V}_0}$ *is non-meagre*", *then* $\mathbf{V}_1 \models$ "$\mathbb{P}_{\mathscr{A}}$ *is proper*".

In both cases, if $\mathbf{V}_1$ *is a generic extension of* $\mathbf{V}_0$ *by the forcing notion* $\mathbb{Q}$ *then it means that* $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}})$ *holds.*

Proof. Assume that $\mathbf{V}_1 \supseteq \mathbf{V}_0$.

If $\mathbf{V}_1 \models$ "$\aleph_1^{\mathbf{V}_0}$ is countable" then recalling $\mathbf{V}_0 \models$ CH clearly $\mathbf{V}_1 \models$ "$\mathscr{A}$ is countable" so we know that $\mathbb{P}_{\mathscr{A}}$ is proper in $\mathbf{V}_1$, thus proving clause (a). So from now on we assume $\aleph_1^{\mathbf{V}_0}$ is not collapsed.

In $\mathbf{V}_0$ let $\mathscr{T} = {}^{\omega_1 >}(\omega_1)$ and choose a subset $\mathscr{A}' \subseteq \mathscr{A}$ such that $\mathscr{A}'$ is $\subseteq^*$-dense in $\mathscr{A}$ and $(\mathscr{A}', \supseteq^*)$ is tree-isomorphic to $\mathscr{T}$. Let π be the isomorphism between these trees[1]. Notice that all this is done in $\mathbf{V}_0$ (recalling that $\mathbf{V}_0 \models$ CH). In $\mathbf{V}_0$ there is a sequence $\bar{\mathscr{T}} = \langle \mathscr{T}_\alpha : \alpha < \omega_1 \rangle$ which is $\subseteq$-increasing continuous with union $\mathscr{T}$ and each $\mathscr{T}_\alpha$ countable. Also there is $\bar{C} = \langle C_\delta : \delta < \omega_1, \delta \text{ is a limit ordinal} \rangle \in \mathbf{V}_0$ such that $C_\delta \subseteq \delta = \sup(C_\delta)$, $\mathrm{otp}(C_\delta) = \omega$. Let $\mathscr{T}'_\delta = \mathscr{T}_\delta \restriction \{\eta \in \mathscr{T}_\delta : \ell g(\eta) \in C_\delta\}$.

In $\mathbf{V}_1$ choose a sufficiently large regular cardinal χ, and let $N \prec (\mathscr{H}(\chi), \in)$ be countable such that $\mathscr{A}, \pi, \bar{\mathscr{T}} \in N$ and let $\delta = \omega_1 \cap N$, clearly $\mathscr{T} \cap N = \mathscr{T}_\delta$. We have to prove the statement:

$(*)_0$ "for every $p \in \mathbb{P}_{\mathscr{A}} \cap N$ there is $q \in \mathbb{P}_{\mathscr{A}}$ above p which is $(N, \mathbb{P}_{\mathscr{A}})$-generic".

As $\mathbf{V}_0 \models$ CH and the density of $\mathscr{A}'$ in $\mathscr{A}$ and $(\mathscr{A}', \supseteq^*)$ being isomorphic in $\mathbf{V}_0$ by π to $\mathscr{T}$ this is equivalent (in $\mathbf{V}_1$, of course) to:

$(*)_1$ for every $\nu \in \mathscr{T} \cap N = \mathscr{T}_\delta$ there is $\eta \in \mathscr{T}$ which is $(N, \mathscr{T})$-generic and $\nu \leq_{\mathscr{T}} \eta$.

In $\mathbf{V}_0$ we let $\bar{S} = \langle S_\delta : \delta < \omega_1 \text{ a limit ordinal} \rangle$ where $S_\delta = \{\bar{\nu} : \bar{\nu} = \langle \nu_n : n < \omega \rangle$ is $<_{\mathscr{T}}$-increasing, $\nu_n \in \mathscr{T}'_\delta$, moreover $\ell g(\nu_n)$ is the n-th member of $C_\delta\}$.

As $(\forall \nu \in \mathscr{T}_\delta)(\exists \rho)(\nu <_{\mathscr{T}} \rho \in \mathscr{T}'_\delta)$, and [$\bar{\nu} \in S_\delta \Rightarrow$ there is a $<_{\mathscr{T}}$-upper bound $\rho \in \mathscr{T}$ of $\bar{\nu}$, in $\mathbf{V}_0$, of course] recalling $\mathscr{T}_\delta, S_\delta \in \mathbf{V}_0$ clearly $(*)_1$ is equivalent (in $\mathbf{V}_1$, of course) to

$(*)_2$ for every $\nu \in \mathscr{T}'_\delta$ there is $\bar{\nu} \in S_\delta$ such that $\nu \in \mathrm{Rang}(\bar{\nu})$ and $\bar{\nu}$ induce a subset of $\mathscr{T}_\delta$ generic over N (i.e. $(\forall A)[A \in N$ is a dense open subset of $\mathscr{T} \Rightarrow A \cap \{\nu_n : n < \omega\} \neq \emptyset]$.

Now a sufficient condition for $(*)_2$ is

$(*)_3$ S_δ, as a set of ω-branches of the tree $\mathscr{T}'_\delta$, is non-meagre.

But in $\mathbf{V}_0$, $\mathscr{T}'_\delta$ and ${}^{\omega >}\omega$ are isomorphic and S_δ is the set of all ω-branches of $\mathscr{T}'_\delta$, so by an assumption from part (b), $(*)_3$ holds so we are done. $\square_{1.1}$

Discussion 1.2. However, there can be $\mathscr{A} \subseteq \mathscr{P}(\omega)$ such that $(\mathscr{A}, \subseteq^*)$ is a variation of Souslin tree.

[1]this is trivial as $\mathbf{V}_0 \models$ CH, however always there is a dense tree with $\mathfrak{h}$ levels by the celebrated theorem of Balcar-Pelant-Simon

Claim 1.3. *1) We have* $\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ *when:*

(a) $\aleph_1^{\mathbf{V}[\mathbb{Q}]} = \aleph_1$

(b) $\Vdash_{\mathbb{Q}}$ "$|\lambda| = \aleph_1$ *where* $\lambda = (2^{\aleph_0})^{\mathbf{V}}$"

(c) *moreover letting* $\langle \underset{\sim}{u}_i : i < \aleph_1 \rangle$ *be a* $\mathbb{Q}$*-name of a* $\subseteq$*-increasing continuous sequence of countable subsets of* λ *with union* λ*, the* $\mathbb{Q}$*-name* $\underset{\sim}{S} = \{i : u_i \in \mathbf{V}\}$ *is forced to contain a club (of* $\aleph_1$*)*

(d) *forcing with* $\mathbb{Q}$ *preserves* "$({}^{\omega}2)^{\mathbf{V}}$ *is non-meagre".*

2) Assume the forcing notion $\mathbb{Q}$ *satisfies (a) + (d),* $\mathrm{Pr}_4(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ *as witnessed by* S *and* $\mathbb{Q}$ *is proper and* $\underset{\sim}{S}$ *is forced to be stationary.*

Then the forcing notion $\mathbb{Q} * \mathrm{Levy}(\aleph_1, (|\mathbb{Q}|^{\aleph_0})^{\mathbf{V}}) * \mathbb{Q}_{\underset{\sim}{S}}$ *preserves* "$\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ *is proper" where* $\mathbb{Q}_S$ *is the (well known) shooting of a club through the stationary subsets of* ω_1 *(to make clause (c) hold).*

Proof. Like 1.1. $\square_{1.3}$

In what follows we prove that many forcing notions destroy properness. We need a preliminary concept.

Definition 1.4. For $\lambda > \kappa$ we say that a forcing notion $\mathbb{Q}$ is (λ, κ)-newly proper (omitting κ means $\kappa = \aleph_0$ and we define $(\lambda, < \chi)$-newly proper similarly) when: if $\bar{N} = \langle (N_\eta, \nu_\eta) : \eta \in {}^{\omega >}\lambda \rangle$ satisfies $\circledast$ below and $\mathbb{Q} \in N_{<>}$ and $p \in \mathbb{Q} \cap N_{<>}$ then we can find $q, \underset{\sim}{\eta}$ such that $\boxtimes$ below holds where:

$\circledast$ for some cardinal $\chi > \lambda$

(a) $N_\eta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is countable

(b) if $\nu \triangleleft \eta$ then $N_\nu \prec N_\eta$

(c) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$ if $\kappa = \aleph_0$ and $N^\kappa_{\eta_1} \cap N^\kappa_{\eta_2} = N^\kappa_{\eta_1 \cap \eta_2}$ generally where $N^\kappa_\eta := \cup\{v \in N_\eta : |v| \le \kappa\}$

(d) $\nu_\eta \in N_\eta \backslash \cup \{N^\kappa_{\eta \restriction m} : m < \ell g(\eta)\}$ hence $\nu_\eta \notin \cup\{N_\nu : \neg(\eta \trianglelefteq \nu)$ and $\nu \in {}^{\omega >}\lambda\}$

(e) $\nu_\eta \in {}^{\ell g(\eta)}\lambda$ and $\ell < \ell g(\eta) \Rightarrow \nu_{\eta \restriction \ell} \trianglelefteq \nu_\eta$

$\boxtimes$ (a) $p \le_{\mathbb{Q}} q$

(b) $q \Vdash_{\mathbb{Q}}$ "$\cup \{N_{\underset{\sim}{\eta} \restriction n}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}] : n < \omega\} \cap \mathbf{V} = \cup\{N_{\underset{\sim}{\eta} \restriction n} : n < \omega\}$"

(c) $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in {}^{\omega}\lambda$ is new, i.e. $\underset{\sim}{\eta} \notin ({}^{\omega}\lambda)^{\mathbf{V}}$"

(c)$^+$ moreover if $\kappa > \aleph_0$ and $\mathscr{T} \in \mathbf{V}$ is a sub-tree of ${}^{\omega >}\lambda$ of cardinality $\le \kappa$ then $\underset{\sim}{\eta} \notin \lim(\mathscr{T})$, i.e. $\{\underset{\sim}{\eta} \restriction n : n < \omega\} \notin \mathscr{T}$.

Observation 1.5. *If* $\langle N_\eta : \eta \in {}^{\omega >}\lambda \rangle$ *satisfies clauses (a),(b),(c) of* $\circledast$ *of Definition 1.4, then the following conditions are equivalent:*

- $\bullet_1$ *there is* $\langle \nu_\eta : \eta \in {}^{\omega >}\lambda \rangle$ *such that clauses (d),(e) of* $\circledast$ *of Definition 1.4*
- $\bullet_2$ *if* $\eta \in {}^{\omega >}\lambda$*, then* $N_\eta \cap \lambda \nsubseteq \cup\{N_{\eta \restriction \ell} : \ell < \ell g(\eta)\}$.

For a proper forcing notion adding a new real it is quite easy to be $\aleph_1$-newly proper; e.g.

Claim 1.6. *Assuming* $2^{\aleph_0} \ge \lambda = \mathrm{cf}(\lambda) > \aleph_1$*, sufficient conditions for "*$\mathbb{Q}$ *is* λ*-newly proper" are:*

(a) $\mathbb{Q}$ *is c.c.c. and adds a new real*
(b) $\mathbb{Q}$ *is Sacks forcing*
(c) $\mathbb{Q}$ *is a tree-like creature forcing in the sense of Roslanowski-Shelah* [RS99].

Proof. Easy; for clause (a) we use $q = p$ for $\boxplus$ of the definition noting that: if $\eta \in {}^{\omega >}\lambda$ then p is $(N_\eta, \mathbb{Q})$-generic. For clauses (b),(c) we use fusion but in the n-th step use members of $N_\eta \cap \mathbb{Q}$ for $\eta \in {}^n\lambda$, we get as many distinct η's as we can. $\square_{1.6}$

Theorem 1.7. *We have* $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ *is not proper*" *when:*

(a) $\mathbf{V} \models 2^{\aleph_0} \geq \aleph_2$
(b) λ *is regular,* $\aleph_2 \leq \lambda \leq 2^{\aleph_0}$ *and*[2] $\alpha < \lambda \Rightarrow \mathrm{cf}([\alpha]^{\aleph_0}, \subseteq) < \lambda$ *hence (by* [She93]*) there is a stationary* $\mathscr{U}_\alpha \subseteq [\alpha]^{\aleph_0}$ *of cardinality* $< \lambda$
(c) $\mathfrak{h} < \lambda$
(d) *the forcing notion* $\mathbb{Q}$ *adds at least one real and is* λ*-newly proper.*

Proof. Let χ be large enough and for transparency, $x \in \mathscr{H}(\chi)$.

By Rubin-Shelah [RS87], see more [She98, Ch.XI] in $\mathbf{V}$ there is a sequence $\langle N_\eta : \eta \in {}^{\omega >}\lambda\rangle$ such that:

$\boxdot_1$ (a) $N_\eta \prec (\mathscr{H}(\chi), \in)$
(b) $\mathbb{Q}, x \in N_\eta$
(c) N_η is countable
(d) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$.

Now for each $\eta \in {}^\omega\lambda$ let $N_\eta = \cup\{N_{\eta \restriction k} : k < \omega\}$; we can easily add:

(e) there is $\mathscr{W}$ such that:
 (α) $\mathscr{W}$ is a subtree of ${}^{\omega >}\lambda$
 (β) $\langle\rangle \in \mathscr{W}$
 (γ) if $\eta \in \mathscr{W}$ then $(\exists^\lambda \alpha)(\eta\hat{\ }\langle\alpha\rangle \in \mathscr{W})$
 (δ) if $\eta \in \lim(W)$ then $\eta \in {}^\omega\lambda$ is increasing, and $\sup(N_\eta \cap \lambda) = \sup(\mathrm{Rang}(\eta))$
 (ε) we can choose $\nu_\eta \in N_\eta$ for $\nu \in \mathscr{W}$ as in clauses (d),(e) of $\circledast$ of 1.4.

By Balcar-Pelant-Simon [BPS80] there is $\mathscr{T} \subseteq [\omega]^{\aleph_0}$ such that

$\boxdot_2$ (α) $(\mathscr{T}, \supseteq^*)$ is a tree with $\mathfrak{h}$ levels ($\mathfrak{h}$ is the cardinal invariant from 0.7, a regular cardinal $\in [\aleph_1, 2^{\aleph_0}]$), the tree $\mathscr{T}$ has a root and each node has $2^{\aleph_0}$ many immediate successors, i.e. $\mathscr{T}$ has splitting to $2^{\aleph_0}$)
(β) $\mathscr{T}$ is dense in $([\omega]^{\aleph_0}, \supseteq^*)$, i.e. in $\mathbb{P}_{\mathscr{P}(\omega)[\mathbf{V}]} = \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ recalling 0.2(2).

Choose $\bar{h}$ such that

$\boxdot_3$ $\bar{h} = \langle h_p : p \in \mathscr{T}\rangle$ satisfies: h_p is a one-to-one function from $\mathrm{suc}_{\mathscr{T}}(p)$ onto $2^{\aleph_0} \backslash \{h_{p_0}(p_1) : p_0 <_{\mathscr{T}} p_1 <_{\mathscr{T}} p \text{ and } p_1 \in \mathrm{suc}_{\mathscr{T}}(p_0)\}$.

So without loss of generality

[2] If $\lambda = \aleph_2$ the rest of clause (b) follows.

$\boxdot_4$ $\mathscr{T} \in N_{<>}, \mathfrak{h} \in N_{<>}$ and $\bar{h} \in N_{<>}$.

As $\mathbb{Q}$ is λ-newly proper there are $\underset{\sim}{\eta}, q$ as in $\boxtimes$ of Definition 1.4. Let $\mathbf{G} \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$ such that $q \in \mathbf{G}$, let $\eta = \underset{\sim}{\eta}[G]$ and $M_2 := N_{\underset{\sim}{\eta}[G]} := \cup\{N_{\eta \restriction n}[\mathbf{G}] : n < \omega\}$, so $M_2 \prec (\mathscr{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathscr{H}(\chi)^{\mathbf{V}}, \in)$ is countable, pedantically $(|M_2|, \mathscr{H}(\chi)^{\mathbf{V}} \cap |M_2|, \in \restriction |M_2|) \prec (\mathscr{H}(\chi)^{\mathbf{V}[\mathbf{G}]}, \mathscr{H}(\chi)^{\mathbf{V}}, \in \restriction \mathscr{H}(\chi)^{\mathbf{V}[\mathbf{G}]})$.

By $\boxtimes$ of 1.4, i.e. the choice of $\underset{\sim}{\eta}, q$ as $q \in \mathbf{G}$ we have $M_1 = M_2 \cap \mathscr{H}(\chi)^{\mathbf{V}}$ is $\cup\{N_{\eta \restriction n} : n < \omega\}$, and of course $\tilde{M}_1 \prec (\mathscr{H}(\chi), \in)$. Toward contradiction assume $\mathbf{V}[\mathbf{G}] \models$ "$\mathscr{P}_{\mathscr{A}_*[\mathbf{V}]}$ is proper", hence some $p_* \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ is $(M_2, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$-generic. But $\mathscr{T}$ is dense in $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ so without loss of generality $p_* \in \mathscr{T}$ and p_* is $(M_2, \mathscr{T})$-generic.

Since $\mathfrak{h} \in N_{<>}$ and $\mathfrak{h} < \lambda$, without loss of generality $\eta \in {}^{\omega >}\lambda \Rightarrow N_\eta \cap \mathfrak{h} = N_{<>} \cap \mathfrak{h}$. For any $\alpha < \lambda$ let

$$\mathscr{I}_\alpha = \{p \in \mathscr{T} : \text{ for some } p_0 \in \mathscr{T} \text{ we have } p \in \text{ suc}_{\mathscr{T}}(p_0) \text{ and } h_{p_0}(p) = \alpha\}$$

and letting $\mathscr{T}_\alpha$ be the α-th level of $\mathscr{T}$ and let

$$\mathscr{I}^+_\alpha = \{p \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]} : p \text{ is above some member of } \mathscr{T}_\alpha\}.$$

Now clearly (in $\mathbf{V}$ and in $\mathbf{V}[\mathbf{G}]$):

$(*)_1$ (a) $\mathscr{I}_\alpha$ is a pre-dense subset of $\mathscr{T}$ (and of $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$)
(b) $\mathscr{I}^+_\alpha$ is dense open decreasing with α
(c) if $p \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ <u>then</u> for every large enough $\alpha < \lambda, p \notin \mathscr{I}^+_\alpha$
(d) if $p \in \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ and $\alpha < \lambda$ <u>then</u> there is $q \in \mathscr{I}_\alpha$ such that $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]} \models$ "$p \le q$".

Also clearly the sequence $\langle \mathscr{I}_\alpha : \alpha < \lambda \rangle$ belongs to $N_{\langle\rangle}$ hence if $\alpha \in \lambda \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ then $\mathscr{I}_\alpha \in N_{\underset{\sim}{\eta}[\mathbf{G}]}$ and the set $\{p \in \mathscr{T} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]} : p \le_{\mathscr{T}} p_* \text{ and } p \in \mathscr{T}_\alpha\}$ is not empty.

Now

$(*)_2$ in $\mathbf{V}[\mathbf{G}]$ the following functions $h_\bullet, h_*$ are well defined
(a) $\mathrm{Dom}(p_\bullet) = \mathrm{Dom}(h_*) = N_{<>} \cap \mathfrak{h}$
(b) $h_\bullet(\gamma)$ is the unique $p \in N_{\underset{\sim}{\eta}[\mathbf{G}]} \cap \mathscr{T}$ of level γ which is $\le_{\mathscr{T}} p_*$
(c) if $\gamma < \mathfrak{h}$ then $h_*(\gamma) = h_{\gamma+1}(h_\bullet(\gamma + 1))$

$(*)_3$ if $\alpha \in \mathfrak{h} \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ then $h_*(\alpha) \in N_{\underset{\sim}{\eta}[\mathbf{G}]} \cap \mathfrak{h} = N_{<>} \cap \mathfrak{h}$

also by the choice of $\bar{h}$ (and genericity) clearly

$(*)_4$ $\mathrm{Rang}(h_*)$ is equal to $u := (2^{\aleph_0}) \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$.

Lastly,

$(*)_5$ $h_* \in \mathbf{V}$.

[Why? As its domain, $N_{<>} \cap \mathfrak{h}$ belongs to $\mathbf{V}$ and $h_*(\gamma)$ is defined from $\langle \mathscr{T}, \bar{h}, \gamma, p_* \rangle \in \mathbf{V}$ and $\mathscr{T}$ is a tree.]

$(*)_6$ (a) from $u := \lambda \cap N_{\underset{\sim}{\eta}[\mathbf{G}]}$ we can define $\underset{\sim}{\eta}[\mathbf{G}]$
(b) $u = \cup\{N_{\underset{\sim}{\eta} \restriction n[\mathbf{G}]} \cap \lambda : n < \omega\}$.

[Why? By the choice of $\bar{N}$.]

Together we get that $\underset{\sim}{\eta}[\mathbf{G}] \in \mathbf{V}$, contradiction. $\square_{1.7}$

Claim 1.8. *We have* $\neg\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$ *when*

- (a) $2^{\aleph_0} \geq \lambda = \mathrm{cf}(\lambda) > \kappa = \mathfrak{h}$
- (b) $\alpha < \lambda \Rightarrow \mathrm{cf}([\alpha]^{\leq \kappa}, \subseteq) < \lambda$
- (c) $\mathbb{Q}$ *is* (λ, κ)*-newly proper.*

Proof. Similar to 1.7. $\square_{1.8}$

Conclusion 1.9. *If* $\mathfrak{h} < 2^{\aleph_0}$ *and* $\mathbb{Q}$ *is a* $(\mathfrak{h}^+, \mathfrak{h})$*-newly proper* *then* $\neg\mathrm{Pr}_1(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$.

§ 2. General sufficient conditions

Claim 2.1. *Assume* $\mathbf{V} \models$ CH.

If $\mathbb{Q}$ *is c.c.c. then* $\Pr_2(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]})$.

Remark 2.2. 1) This works replacing $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ by any $\aleph_1$-complete $\mathbb{P}$ and strengthening the conclusions to $\Pr_1$, see 2.3.
2) See Definition 0.4(1).

Proof. Let $\mathbb{P} = \mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$. Clearly it suffices to prove:

- $(*)$ if $r \in \mathbb{P}$ and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{I}}$ is a dense open subset of $\mathbb{P}$" then there is r' such that:
  - (a) $r \leq_{\mathbb{P}} r'$
  - (b) $\Vdash_{\mathbb{Q}}$ "$r' \in \underset{\sim}{\mathscr{I}} \subseteq \mathbb{P}$".

Why $(*)$ holds? We try (all in $\mathbf{V}$) to choose (r_α, q_α) by induction on $\alpha < \omega_1$ but choosing q_α together with $r_{\alpha+1}$ such that:

- $\circledast$ (a) $r_0 = r$
  - (b) $r_\alpha \in \mathbb{P}$ is $\leq_{\mathbb{P}}$-increasing
  - (c) $q_\alpha \in \mathbb{Q}$
  - (d) q_α, q_β are incompatible in $\mathbb{Q}$ for $\beta < \alpha$
  - (e) $q_\alpha \Vdash_{\mathbb{Q}}$ "$r_{\alpha+1} \in \underset{\sim}{\mathscr{I}}$".

We cannot succeed in carrying the induction ω_1 many steps because $\mathbb{Q} \models$ c.c.c.

For $\alpha = 0$ no problem as only clause (a) is relevant.

For α limit - easy as $\mathbb{P}$ is $\aleph_1$-complete (and the only relevant clause is (b)).

For $\alpha = \beta + 1$, we first ask:

Question: Is $\langle q_\gamma : \gamma < \beta \rangle$ a maximal antichain of $\mathbb{Q}$?

If yes, then r_β is as required in $(*)$ on r'; why? if $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$ is generic over $\mathbf{V}$ to which r_β belongs, then for some $\gamma < \beta, q_\gamma \in \mathbf{G}_{\mathbb{Q}}$ hence $r_{\gamma+1} \in \underset{\sim}{\mathscr{I}}[\mathbf{G}_Q]$ but $\underset{\sim}{\mathscr{I}}[\mathbf{G}_{\mathbb{Q}}]$ is a dense subset of $\mathbb{P}$ and is open and $r_{\gamma+1} \leq_{\mathbb{P}} r_\beta$ so $r_\beta \in \underset{\sim}{\mathscr{I}}[\mathbf{G}_{\mathbb{Q}}]$.
If no, let $q^\beta \in \mathbb{Q}$ be incompatible with q_γ for every $\gamma < \beta$. Recalling $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{I}}$ is dense and open" the set $X_\beta = \{r \in \mathbb{P}$: for some $q, q^\beta \leq_{\mathbb{Q}} q$ and $q \Vdash$ "$r \in \underset{\sim}{\mathscr{I}}$"$\}$ is a dense subset of $\mathbb{P}$ hence there is a member of X_β above r_β, let r_α be such member. By $r_\alpha \in X_\beta$, there is $q, q^\beta \leq q$ such that $q \Vdash$ "$r_\alpha \in \underset{\sim}{\mathscr{I}}$". So we choose q_β as such q, so we can carry the induction step.

As said above we cannot carry the induction for all $\alpha < \omega_1$ because then $\{q_\alpha : \alpha < \omega_1\}$ contradicts "$\mathbb{Q}$ satisfies the c.c.c." So for some α we cannot continue, α is neither 0 nor limit hence for some $\beta, \alpha = \beta + 1$. So the answer to the question is yes, hence we get the desired conclusion of $(*)$. $\square_{2.1}$

We can weaken the demand on the second forcing (above, it is $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$).

Claim 2.3. *If (A) then (B) where:*

- (A) (a) $\mathbb{P}, \mathbb{Q}$ *are forcing notions*
  - (b) $\mathbb{Q}$ *is c.c.c. moreover* $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ *is c.c.c.*"
  - (c) *forcing with* $\mathbb{P}$ *adds no new* ω*-sequences,*[3] *from* λ

[3] if you assume $\mathbb{P}$ is proper, $\lambda = \aleph_0$ the proof may be easier to read

(d) *$\mathbb{Q}$ has cardinality $\leq \lambda$*

(B) (a) *if $\mathbb{P}$ is proper in* $\mathbf{V}$ *then* $\Pr_2(\mathbb{Q}, \mathbb{P})$

(b) *for every $\mathbb{Q}$-name $\underset{\sim}{\mathscr{I}}$ of a dense open subset of $\mathbb{P}$, the set $\mathscr{J}$ is dense and open in $\mathbb{P}$ where:*

($*$) *$\mathscr{J} = \mathscr{J}_{\underset{\sim}{\mathscr{I}}}$ is the set of $r \in \mathbb{P}$ such that some $\bar{q}$ witnesses it, i.e. witness it belongs to $\mathscr{J}$ which means:*

- *$\bar{q} = \langle q_\alpha : \alpha < \alpha_* \rangle$ is a maximal antichain of $\mathbb{Q}$*
- *for each $\alpha < \alpha_*$, the set $\{r' \in \mathbb{P} : q_\alpha \Vdash \text{“}r' \in \underset{\sim}{\mathscr{I}}\text{”}\}$ is an open subset of $\mathbb{P}$ dense above r.*

Proof. First, we prove clause (b); so fix $\underset{\sim}{\mathscr{I}}$ and $\mathscr{J}$ as there. Let $\langle q_\varepsilon : \varepsilon < \kappa := |\mathbb{Q}| \rangle$ list $\mathbb{Q}$.

For every $r \in \mathbb{P}$ we define a sequence η_r of ordinals $< \kappa \leq \lambda$ as follows:

$\circledast_1$ $\eta_r(\alpha)$ is the minimal ordinal $\varepsilon < \kappa$ such that (so $\ell g(\eta_r) = \alpha$ when there is no such ε):

(*a*) $q_\varepsilon \Vdash \text{“}r \in \underset{\sim}{\mathscr{I}}\text{”}$

(*b*) if $\beta < \alpha$ then $q_\varepsilon, q_{\eta_r(\beta)}$ are incompatible in $\mathbb{Q}$.

Now

$\circledast_2$ (*a*) η_r is well defined

(*b*) $\ell g(\eta_r) < \omega_1$.

[Why? Obviously η_r is a well defined sequence of ordinals, i.e. clause (a) and clause (b) holds because $\mathbb{Q} \models$ c.c.c.]

Note

$\circledast_3$ if $r_1 \leq_{\mathbb{P}} r_2$ then <u>either</u> $\eta_{r_1} \trianglelefteq \eta_{r_2}$ <u>or</u> for some $\alpha < \ell g(\eta_{r_1})$ we have

$$\eta_{r_1} \restriction \alpha = \eta_{r_2} \restriction \alpha$$

$$\eta_{r_1}(\alpha) > \eta_{r_2}(\alpha).$$

[Why? Think about the definition.]

For $s \in \mathbb{P}$ let η'_s be $\cap\{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$, i.e. the longest common initial segment of $\{\eta_{s_1} : s \leq_{\mathbb{P}} s_1\}$; clearly $s_1 \leq_{\mathbb{P}} s_2 \Rightarrow \eta'_{s_1} \trianglelefteq \eta'_{s_2}$. So

$\circledast_4$ $\underset{\sim}{\eta}^* = \cup\{\eta'_s : s \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}\}$ is a $\mathbb{P}$-name of a sequence of ordinals $< \kappa$ such that $\langle q_{\underset{\sim}{\eta}^*(i)} : i < \ell g(\underset{\sim}{\eta}^*) \rangle$ is a sequence of pairwise incompatible members of $\mathbb{Q}$.

But by clause (A)(b) of the claim, forcing with $\mathbb{P}$ preserve “$\mathbb{Q} \models$ c.c.c.”, so $\ell g(\underset{\sim}{\eta}^*)$ is countable in $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$. By clause (A)(c) of the claim, forcing by $\mathbb{P}$ adds no new ω-sequences to $\kappa = |\mathbb{Q}|$ (and $\mathbb{Q}$ is infinite) and $\mathbf{V}[\mathbf{G}_{\mathbb{P}}]$ has the same $\aleph_1$ as $\mathbf{V}$, so

$\circledast_5$ $\underset{\sim}{\eta}^*$ is a sequence of countable length of ordinals $< \kappa$ so is old.

Hence

$\circledast_6$ the following set is dense open in $\mathbb{P}$

$$\mathscr{J} = \{r \in \mathbb{P} : r \text{ forces in } \mathbb{P} \text{ that } \underset{\sim}{\eta}^* = \eta^*_r \text{ for some } \eta^*_r \in \mathbf{V}\}$$

As for clause (a), let χ, N, q_1, r_1 be as in the assumption of $(*)_1$ of 0.3, so $\mathbb{P}, \mathbb{Q} \in N$. We have to find q_2, r_2 as there.

Let $q_2 = q_1$ and let $r_2 \in \mathbb{P}$ be $(N, \mathbb{P})$-generic and above r_1, exists as $\mathbb{P}$ is a proper forcing in $\mathbf{V}$.

We shall show that (r_2, q_2) is as required, i.e. $q_2 \Vdash_{\mathbb{Q}}$ "r_2 is $(N[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}}], \mathbb{P})$-generic". Let $\mathbf{G}_{\mathbb{Q}} \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$ such that $q_2 \in \mathbf{G}_{\mathbb{Q}}$ and we should prove that $\mathbf{V}[\mathbf{G}_{\mathbb{Q}}] \models$ "r_2 is $(N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})$-generic". So let $\mathscr{I} \in N[\mathbf{G}_{\mathbb{Q}}]$ be a dense open subset of $\mathbb{P}$, and we should prove that $\mathbf{V}[\mathbf{G}_{\mathbb{Q}}] \models$ "$\mathscr{I} \cap N[\mathbf{G}_{\mathbb{Q}}]$ is pre-dense above r_2".

It suffices to prove:

$(*)$ if $r_2 \leq_{\mathbb{P}} r_3$ then r_3 is compatible (in $\mathbb{P}$) with some $r \in \mathscr{J} \cap N$.

So fix $r_3 \in \mathbb{P}$; by the definition of $N[\mathbf{G}_{\mathbb{Q}}]$ there is a $\mathbb{Q}$-name $\underset{\sim}{\mathscr{I}}$ such that $\mathscr{I} = \underset{\sim}{\mathscr{I}}[\mathbf{G}_{\mathbb{Q}}]$, for some $\underset{\sim}{\mathscr{I}} \in N$; without loss of generality $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{I}}$ is a dense open subset of $\mathbb{P}$". Let $\mathscr{J} = \mathscr{J}_{\underset{\sim}{\mathscr{I}}} = \{r \in \mathbb{P} : r$ has an $\underset{\sim}{\mathscr{I}}$-witness $\bar{q}_* = \langle q^*_\alpha : \alpha < \alpha_* \rangle\}$, see clause (B)(b) of the claim. Clearly $\mathscr{J} \in N$ hence $\mathscr{J} \cap N$ is pre-dense in $\mathbb{P}$ over r_2 hence also over r_3 hence there are $r_4, r_5 \in \mathbb{P}$ such that $r_3 \leq_{\mathbb{P}} r_5, r_4 \leq_{\mathbb{P}} r_5$ and $r_4 \in N \cap \mathscr{J}$. By the definition of $\mathscr{J}$ there is an $\underset{\sim}{\mathscr{I}}$-witness $\bar{q}_* = \langle q^*_\alpha : \alpha < \alpha_* \rangle$ for $r_4 \in \mathscr{J}$.

But $\underset{\sim}{\mathscr{I}}, r_4 \in N$ hence without loss of generality $\bar{q}_* \in N$ and $\bar{q}_*$ has countable length, so $\{q^*_\alpha : \alpha < \alpha_*\} \subseteq N$. As $\bar{q}_*$ is a witness, necesarily it is a maximal antichain of $\mathbb{Q}$ hence for some $\alpha < \alpha_*$ we have $q^*_\alpha \in \mathbf{G}_{\mathbb{Q}}$, as $\bar{q}_*$ is a witness for $r_4 \in \mathscr{J}_{\underset{\sim}{\mathscr{I}}}$, necessarily $\mathscr{I}_1 = \{r \in \mathbb{P} : q^*_\alpha \Vdash_{\mathbb{Q}}$ "$r \in \underset{\sim}{\mathscr{I}}$"$\}$ is an open subset of $\mathbb{P}$ dense above r_4.

Clearly $\mathscr{I}_1 \in N$ is an open subset of $\mathbb{P}$, dense above r_4 and $r_4 \leq_{\mathbb{P}} r_5$ hence $\mathscr{I}_1 \cap N$ is pre-dense above r_5 hence there are $r_6 \leq_{\mathbb{P}} r_7$ from $\mathbb{P}$ such that $r_6 \in \mathscr{I}_1 \cap N$ and $r_5 \leq_{\mathbb{P}} r_7$.

Clearly $r_6 \in \mathscr{I}[\mathbf{G}_{\mathbb{Q}}] \cap N$ and r_6 is compatible with r_3 in $\mathbb{P}$, so we are done proving r_2 is $(N[\mathbf{G}_{\mathbb{Q}}], \mathbb{P})$-generic.

So we are done. $\square_{2.3}$

Remark 2.4. In 2.1, 2.3 we can replace "c.c.c." by "strongly proper".

<u>But</u> such $\mathbb{Q}$ preserves "$({}^\omega 2)^{\mathbf{V}}$-non-meagre".

Claim 2.5. *1) There is a proper forcing $\mathbb{Q}$ which forces "$\mathbb{P}_{\mathscr{A}_*}[\mathbf{V}]$ as a forcing notion is not proper", (i.e. $\neg \mathrm{Pr}_1(\mathbb{Q}, \mathbb{P})$).*
2) Even (A) of 0.5(3) fails, i.e. $\neg \mathrm{Pr}_5(\mathbb{Q}, \mathbb{P}_{\mathscr{A}_*}[\mathbf{V}])$.

Proof. We use the proof of [She98, Ch.17,Sec.2] and see references there. We repeat in short.

We use a finite iteration so let $\mathbb{P}_0$ be the trivial forcing notion, $\mathbb{P}_{k+1} = \mathbb{P}_k * \underset{\sim}{\mathbb{Q}}_k$ for $k \leq 3$ and the $\mathbb{P}_k$-name $\underset{\sim}{\mathbb{Q}}_k$ is defined below.

<u>Step A</u>: $\mathbb{Q}_0 =$ Levy$(\aleph_1, 2^{\aleph_0})$ so $\Vdash_{\mathbb{Q}_0}$ "CH".

<u>Step B</u>: $\mathbb{Q}_1$ is Cohen forcing.

<u>Step C</u>: In $\mathbf{V}^{\mathbb{P}_2}, \mathbb{Q}_2$ in the Levy collapse of $2^{2^{\aleph_0}}$ to $\aleph_1$, i.e. $\mathbb{Q}_2 =$ Levy$(\aleph_1, \beth_2)^{\mathbf{V}[\mathbb{P}_2]}$.

Step D: Let $\mathscr{T} = ({}^{(\omega_1>)}\omega_1)^{\mathbf{V}[\mathbb{P}_1]} = ({}^{(\omega_1>)}\omega_1)^{\mathbf{V}[\mathbb{P}_0]}$ be a tree, so we know that $\lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_1]} = \lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_2]} = \lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_3]}$ hence has cardinality $\aleph_1$ in $\mathbf{V}^{\mathbb{P}_3}$ and

$(*)_1$ in $\mathbf{V}^{\mathbb{P}_1}, \mathscr{T}$ is isomorphic to a dense subset of $\mathbb{P}_{\mathscr{A}_*[\mathbb{P}_1]} = \mathbb{P}_{\mathscr{A}_*[\mathbb{P}_0]}$.

So in $\mathbf{V}^{\mathbb{P}_3}$ there is a list $\langle \eta^*_\varepsilon : \varepsilon < \omega_1 \rangle$ of $\lim_{\omega_1}(\mathscr{T})^{\mathbf{V}[\mathbb{P}_1]}$ and let $\langle \eta^*_\varepsilon {\restriction} [\gamma_\varepsilon, \omega_1) : \varepsilon < \omega_1 \rangle$ be pairwise disjoint end segments so $\gamma_\varepsilon < \omega_1, \langle \gamma_\varepsilon : \varepsilon < \omega_1 \rangle \in \mathbf{V}^{\mathbb{P}_3}$ and $\varepsilon_1 < \varepsilon_2 < \omega_1 \wedge \beta_1 \in [\gamma_{\varepsilon_1}, \omega_1) \wedge \beta_2 \in [\gamma_{\varepsilon_2}, \omega_1) \Rightarrow \eta^*_{\varepsilon_1} {\restriction} \gamma_1 \neq \eta^*_{\varepsilon_2} {\restriction} \gamma_2$.

Step E: In $\mathbf{V}^{\mathbb{P}_3}$ there is $\mathbb{Q}_3$, a c.c.c. forcing notion specializing $\mathscr{T}$ in the sense of [She78], i.e. there is $h_* \in \mathbf{V}^{\mathbb{P}_4}$ such that $h_* : \mathscr{T} \to \omega, h_*$ is increasing in $\mathscr{T}$ except being constant on each end segment $\underset{\sim}{\eta}^*_\varepsilon {\restriction} [\gamma_\varepsilon, \omega_1)$ for $\varepsilon < \omega_1$, i.e. $\rho <_{\mathscr{T}} \nu \wedge h_*(\rho) = h_*(\nu) \Rightarrow (\exists \varepsilon)[\rho, \nu \in \{\eta^*_\varepsilon {\restriction} \gamma : \gamma \in [\gamma_\varepsilon, \underset{\sim}{\omega_1})\}$.

Now

⊠ after forcing with $\mathbb{P}_4 = \mathbb{Q}_0 * \mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2 * \mathbb{Q}_3$, i.e. in $\mathbf{V}^{\mathbb{P}_4}$ the forcing notion $\mathbb{P}_{\mathscr{A}_*[\mathbf{V}]}$ is not proper, in fact it collapses $\underset{\sim}{\aleph}_1$.

Why? Recall $(*)_1$ and note

$(*)_2$ $\mathscr{I}_n := \{\rho \in \mathscr{T} : (\forall \nu)(\rho \leq_{\mathscr{T}} \nu \to h_*(\nu) \neq n\}$ is dense open in $\mathscr{T}$

and trivially

$(*)_3$ $\bigcap\limits_n \mathscr{I}_n = \emptyset$; in fact if $\mathbf{G} \subseteq \mathscr{T}$ is generic, then:

- (A) $\mathbf{G}$ is a branch of $\mathscr{T}$ of order type $\omega_1^{\mathbf{V}}$ let its name be $\langle \underset{\sim}{\rho}_\gamma : \gamma < \omega_1 \rangle$
- (B) letting $\underset{\sim}{\gamma}_n = \text{Min}\{\gamma < \omega_1 : \underset{\sim}{\rho}_\gamma \in \mathscr{I}_n\}$ we have $\Vdash_{\mathscr{T}}$ "$\{\underset{\sim}{\gamma}_n : n < \omega\}$ is unbounded in ω_1".

$\square_{2.5}$

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