CREATURE FORCING AND LARGE CONTINUUM: THE JOY OF HALVING

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Abstract. For $f, g \in \omega^\omega$ let $c^\forall_{f,g}$ be the minimal number of uniform g-splitting trees needed to cover the uniform f-splitting tree, i.e., for every branch ν of the f-tree, one of the g-trees contains ν. Let $c^\exists_{f,g}$ be the dual notion: For every branch ν, one of the g-trees guesses $\nu(m)$ infinitely often. We show that it is consistent that $c^\exists_{f_\epsilon,g_\epsilon} = c^\forall_{f_\epsilon,g_\epsilon} = \kappa_\epsilon$ for continuum many pairwise different cardinals κ_ϵ and suitable pairs (f_ϵ, g_ϵ). For the proof we introduce a new mixed-limit creature forcing construction.

Introduction

We continue the investigation in [4] of the following cardinals invariants:

Let f, g be functions from ω to ω such that $f(n) > g(n)$ for all n and furthermore $\lim(f(n)/g(n)) = \infty$. An (f, g)-slalom is a sequence $Y = (Y(n))_{n\in\omega}$ such that $Y(n) \subseteq f(n)$ and $|Y(n)| \leq g(n)$ for all $n \in \omega$. A family $\mathcal{Y}$ of (f, g)-slaloms is a $(\forall, f, g)$-cover, if for all $r \in \prod_{n\in\omega} f(n)$ there is an $Y \in \mathcal{Y}$ such that $r(n) \in Y(n)$ for all $n \in \omega$. The cardinal characteristic $c^\forall_{f,g}$ is defined as the minimal size of a $(\forall, f, g)$-cover.

There is also a dual notion: A family $\mathcal{Y}$ of (f, g)-slaloms is an $(\exists, f, g)$-cover, if for all $r \in \prod_{n\in\omega} f(n)$ there is an $Y \in \mathcal{Y}$ such that $r(n) \in Y(n)$ for infinitely many $n \in \omega$. We define $c^\exists_{f,g}$ to be the minimal size of an $(\exists, f, g)$-cover

It is easy to see that $\aleph_0 < c^\exists_{f,g} \leq c^\forall_{f,g} \leq 2^{\aleph_0}$.

Answering a question of Blass related to [1], Goldstern and the second author [2] showed how to force $\aleph_1$ many different values to $c^\forall_{f,g}$. More specifically, assuming CH and given a sequence $(f_\epsilon, g_\epsilon, \kappa_\epsilon)_{\epsilon\in\aleph_1}$ of natural functions f_ϵ, g_ϵ with "sufficiently different growth rate" and cardinals κ_ϵ satisfying $\kappa_\epsilon^{\aleph_0} = \kappa_\epsilon$, there is a cardinality preserving forcing notion that forces $c^\forall_{f_\epsilon,g_\epsilon} = \kappa_\epsilon$ for all $\epsilon \in \aleph_1$. In [4] we additionally forced $c^\exists_{f_\epsilon,g_\epsilon} = c^\forall_{f_\epsilon,g_\epsilon} = \kappa_\epsilon$.

In this paper, we improve[1] this result to *continumm many* characteristics $c^\exists_{f_\epsilon,g_\epsilon} = c^\forall_{f_\epsilon,g_\epsilon}$ in the extension (something which is a lot easier for $c^\forall$ only; this was done in [3]).

So the main theorem is:

Main Theorem. *Assume that CH holds, that $\mu = \mu^{\aleph_0}$, and that $\kappa_\epsilon < \mu$ satisfies $\kappa_\epsilon^{\aleph_0} = \kappa_\epsilon$ for all $\epsilon \in \mu$. Then there is an ω^ω-bounding, cardinality preserving forcing notion P that forces the following: $2^{\aleph_0} = \mu$, and there are functions f_ϵ, g_ϵ for $\epsilon \in \mu$ such that $c^\exists_{f_\epsilon,g_\epsilon} = c^\forall_{f_\epsilon,g_\epsilon} = \kappa_\epsilon$.*

(We can find such μ and $(\kappa_\epsilon)_{\epsilon\in\mu}$ such that the κ_ϵ are pairwise different. Then we get continuum many pairwise different invariants in the extension.)

The construction builds on the theory of creature forcing, which is described in the monograph [5] by Rosłanowski and the second author. However, this paper should (at least formally) be quite self contained concerning creature forcing theory; we do however (in 2.1) cite a result of [4].

This paper has two parts: In the first part, we introduce a new creature forcing construction (to give some "creature keywords": somewhat in between a restricted product and an iteration, with countable support,

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[1]Note that once we have $\aleph_1$ many different cardinals between $\aleph_0$ and the continumm, then the continumm has to be much bigger than $\aleph_1$.

basically a lim-inf construction but allowing for lim-sup conditions as well). Using this construction, we get a much nicer and more general proof of properness compared to the construction in [4].

This construction (actually a simple case, in particular a pure lim-inf case without downwards memory) is used the second part to construct the required forcing. It turn out that we can use very similar proofs to the ones in [4] to show that the forcing notion constructed this way actually does what we want.

1. The creature forcing construction

1.1. The basic definitions.

Definition 1.1. Let I^* be some (index) set, and for each $i \in I^*$ and $n \in \omega$ fix a finite set $\mathrm{POSS}^*_{=n,\{i\}}$. For $u \subseteq I^*$ and $n \in \omega$ we set

$$\mathrm{POSS}_{n,u} = \{\eta : \eta \text{ is a function, } \mathrm{dom}(\eta) = n \times u, \text{ and } \eta(m,i) \in \mathrm{POSS}^*_{=m,\{i\}} \text{ for all } m \in n \text{ and } i \in u\}.$$

The name POSS is chosen because this is the set of possibile trunks of conditions, see below.

We will use the following notation for restrictions of $\eta \in \mathrm{POSS}_{n,u}$: For $0 \le m \le n$ and for $w \subseteq u$ we use $\eta \restriction m \in \mathrm{POSS}_{m,u}$, $\eta \restriction w \in \mathrm{POSS}_{n,w}$ and $\eta \restriction (m \times w) \in \mathrm{POSS}_{m,w}$ (with the obvious meaning). We will sometimes identify an $\eta \in \mathrm{POSS}_{n,\{i\}}$, i.e., a function with domain $n \times \{i\}$, with the according function with domain n.

Definition 1.2. $\mathrm{VAL}_{n,u}$ is the set of functions $\mathbf{f} : \mathrm{POSS}_{n,u} \to \mathrm{POSS}_{n+1,u}$ satisfying $\mathbf{f}(\eta) \restriction n = \eta$ for all $\eta \in \mathrm{POSS}_{n,u}$.

(This is the set of possible elements of the value-set $\mathrm{val}(\mathfrak{c})$ of an n-ml-creature, see below.)

Definition 1.3. Fix $n \in \omega$. An n-ml-creature parameter $\mathfrak{p}_n$ consists of

- $\mathbf{K}(n)$, the set of n-ml-creatures,
- the functions supp, $\mathrm{supp}^{\mathrm{ls}}$, nor, $\mathrm{nor}^{\mathrm{ls}}$, val and $\mathbf{\Sigma}$, all with domain $\mathbf{K}(n)$,

satisfying the following (for $\mathfrak{c} \in \mathbf{K}(n)$):

1. $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{c}) \subseteq \mathrm{supp}(\mathfrak{c})$ are finite[2] subsets of I^*. We call $\mathrm{supp}(\mathfrak{c})$ the support of $\mathfrak{c}$.
2. $\mathrm{nor}(\mathfrak{c})$ (called norm) and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{c})$ are nonnegative reals.[3]
3. $\mathrm{val}(\mathfrak{c})$ is a nonempty subset of $\mathrm{VAL}_{n,\mathrm{supp}(\mathfrak{c})}$.
   For $\eta \in \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}$, we set $\mathfrak{c}[\eta] := \{\mathbf{f}(\eta) : \mathbf{f} \in \mathrm{val}(\mathfrak{c})\}$. So $\mathfrak{c}[\eta]$ is a nonempty subset of $\mathrm{POSS}_{n+1,\mathrm{supp}(\mathfrak{c})}$, and every $\nu \in \mathfrak{c}[\eta]$ extends η.
4. $\mathbf{\Sigma}(\mathfrak{c})$, the set of ml-creatures that are stronger than (or: successors of) $\mathfrak{c}$, is a subset of $\mathbf{K}(n)$ such that for all $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ the following holds:
   (a) if $\mathfrak{d}' \in \mathbf{\Sigma}(\mathfrak{d})$, then $\mathfrak{d}' \in \mathbf{\Sigma}(\mathfrak{c})$ (i.e., $\mathbf{\Sigma}$ is transitive).
   (b) $\mathfrak{c} \in \mathbf{\Sigma}(\mathfrak{c})$ (i.e., $\mathbf{\Sigma}$ is reflexive).
   (c) $\mathrm{supp}(\mathfrak{d}) \supseteq \mathrm{supp}(\mathfrak{c})$ and $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{d}) \cap \mathrm{supp}(\mathfrak{c}) \subseteq \mathrm{supp}^{\mathrm{ls}}(\mathfrak{c})$.
   (d) $\mathfrak{d}[\eta] \restriction \mathrm{supp}(\mathfrak{c}) \subseteq \mathfrak{c}[\eta \restriction \mathrm{supp}(\mathfrak{c})]$ for every $\eta \in \mathrm{POSS}(n, \mathrm{supp}(\mathfrak{d}))$.

Of course, with $\mathfrak{d}[\eta] \restriction \mathrm{supp}(\mathfrak{c})$ we mean $\{\nu \restriction \mathrm{supp}(\mathfrak{c}) : \nu \in \mathfrak{d}[\eta]\}$.

Remarks 1.4.
- "ml" stands for "mixed limit" (the construction mixes lim-sup and lim-inf aspects). "ls" stands for lim sup; $\mathrm{supp}^{\mathrm{ls}}$ and $\mathrm{nor}^{\mathrm{ls}}$ will correspont to the part of the forcing that corresponds to a lim-sup sequence. The objects supp and nor will correspond to the lim-inf part.
- Our application will be a "pure lim-inf" forcing: We can completely ignore $\mathrm{supp}^{\mathrm{ls}}$ and $\mathrm{nor}^{\mathrm{ls}}$, or, more formally, we can set $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{c}) = \mathrm{supp}(\mathfrak{c})$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}) = n$ for all n-ml-creatures $\mathfrak{c}$.
- Usually we will also have: if $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ then $\mathrm{nor}(\mathfrak{d}) \le \mathrm{nor}(\mathfrak{c})$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{d}) \le \mathrm{nor}^{\mathrm{ls}}(\mathfrak{c})$, but this is not required for the following proofs.

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[2] We will later even require: There is a functions maxsupp : $\omega \to \omega$ such that every n-ml-creature $\mathfrak{c}$ satisfies $|\mathrm{supp}(\mathfrak{c})| < \mathrm{maxsupp}(n)$.

[3] More particularly, elements of some countable set containing $\mathbb{Q}$ and closed under the functions we need, such as ln etc. We can even restrict nor and $\mathrm{nor}^{\mathrm{ls}}$ to values in $\mathbb{N}$. However, this sometimes leads to slightly cumbersome and less natural definitions.

- In our application (as well as in other potential applications) we will not really use $\mathrm{val}(\mathfrak{c})$ (i.e., a set of functions $\mathbf{f}$ each mapping every possible trunk η af height n to one of height $n+1$). Instead, we will only need $(\mathfrak{c}[\eta])_{\eta\in\mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}}$ (i.e., the function that assigns to each η the (nonempty, finite) set of possible extensions $\mathfrak{c}[\eta]$).

  We can formalize this simplification in our framework as the following additional requirement:

  Assume that $\mathbf{f}\in\mathrm{VAL}_{n,\mathrm{supp}(\mathfrak{c})}$ is such that for all $\eta\in\mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}$ there is a $\mathbf{g}\in\mathrm{val}(\mathfrak{c})$ such that $\mathbf{f}(\eta)=\mathbf{g}(\eta)$. Then $\mathbf{f}\in\mathrm{val}(\mathfrak{c})$. Or, in other words: $\mathbf{f}\in\mathrm{VAL}_{n,\mathrm{supp}(\mathfrak{c})}$ is in $\mathrm{val}(\mathfrak{c})$ iff $\mathbf{f}(\eta)\in\mathfrak{c}[\eta]$ for all $\eta\in\mathrm{POSS}_{n,u^{\mathfrak{c}}}$.
- We could have required the following, stronger property instead of 1.3.(4d) (however, in the case referred to in the previous item, the two versions are equivalent anyway):

  For all $\mathbf{f}\in\mathrm{val}(\mathfrak{d})$ there is some $\mathbf{g}\in\mathrm{val}(\mathfrak{c})$ such that for each $\eta\in\mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{d})}$
  $$\mathbf{f}(\eta)\restriction\mathrm{supp}(\mathfrak{c})=\mathbf{g}(\eta\restriction\mathrm{supp}(\mathfrak{c})).$$
- Our application will even have the following property: $\mathfrak{c}[\eta]$ is essentially independent of η; there is no "downwards memory", the creature does not look at what is going on below.

  More exactly: We will define $\mathfrak{p}_n$ in a way so that for all η,η' in $\mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}$ and $\nu\in\mathfrak{c}[\eta]$ the possibility $\eta'\cup(\nu\cap(\{n\}\times I))$ is in $\mathfrak{c}[\eta']$.
- So while the application in this paper only uses a simpler setting, we give the proof of properness for the more general setting. The reason is that this properness-proof is not more complicated for the general case, and we hope that the general case can be used for other applications.

Definition 1.5. A forcing parameter $\mathfrak{p}$ is a sequence $(\mathfrak{p}_n)_{n\in\omega}$ such that each $\mathfrak{p}_n$ is an n-ml-creature parameter. Given such a $\mathfrak{p}$, we define the forcing notion $Q_{\mathfrak{p}}$: A condition p consists of $\mathrm{trnklg}(p)\in\omega$, the n-ml-creatures $p(n)$ for $n\geq\mathrm{trnklg}(p)$ (i.e., $p(n)$ is in the $\mathbf{K}(n)$ defined by $\mathfrak{p}_n$), and an object $\mathrm{trunk}(p)$ such that:

- $\mathrm{supp}(p(n))\subseteq\mathrm{supp}(p(n+1))$ for all $n\geq\mathrm{trnklg}(p)$.
- We set $\mathrm{dom}(p):=\bigcup_{n\in\omega}\mathrm{supp}(p(n))$, and for $i\in\mathrm{dom}(p)$ we set $\mathrm{trnklg}(p,i)=\min\{n\geq\mathrm{trnklg}(p):i\in\mathrm{supp}(p(n))\}$.
- $\mathrm{trunk}(p)$ is a function with domain $\{(m,i):i\in\mathrm{dom}(p),m<\mathrm{trnklg}(p,i)\}$ such that $\mathrm{trunk}(p)(m,i)$ is in $\mathrm{POSS}^*_{=m,\{i\}}$. For $i\in\mathrm{dom}(p)$, we set $\mathrm{trunk}(p,i)=\mathrm{trunk}(p)\restriction\{i\}$ (which we identify with a function with domain $\mathrm{trnklg}(p,i)$).
- $\liminf_{n\to\infty}\mathrm{nor}(p(n))=\infty$.
- For each $i\in\mathrm{dom}(p)$ the set $X=\{\mathrm{nor}^{\mathrm{ls}}(p(n)):i\in\mathrm{supp}^{\mathrm{ls}}(p(n))\}$ is unbounded, in other words: $\limsup(X)=\infty$. In particular there are infinitely many n with $i\in\mathrm{supp}^{\mathrm{ls}}(p(n))$.

For better readability, we will write $\mathrm{supp}(p,n)$ instead of $\mathrm{supp}(p(n))$, and the same for nor etc.

Note that $Q_{\mathfrak{p}}$ could be empty (for example, if all norms of ml-creatures are bounded by a universal constant). In the following we will always assume that $Q_{\mathfrak{p}}$ is nonempty.

We still have to define the order on $Q_{\mathfrak{p}}$. Before we can do this, we need another notion: $\mathrm{poss}(p,n)$, the sets of elements of $\mathrm{POSS}_{n,\mathrm{dom}(p)}$ that are "compatible with p":

Definition 1.6. For a condition p (or just an according finite sequence of creatures together with a sufficient part of the trunk), we define $\mathrm{poss}(p,n)$ as a subset of $\mathrm{POSS}_{n,\mathrm{dom}(p)}$ by induction on n. If $n\leq\mathrm{trnklg}(p)$, then $\mathrm{poss}(p,n)$ contains the singleton $\mathrm{trunk}(p)\restriction(n\times\mathrm{dom}(p))$. Otherwise $\mathrm{poss}(p,n)$ consists of those $\nu\in\mathrm{POSS}_{n,\mathrm{dom}(p)}$ such that ν is compatible[4] with $\mathrm{trunk}(p)$ and such that $\nu\restriction n\times\mathrm{supp}(p,n-1)\in p(n-1)[\eta\restriction(n-1)\times\mathrm{supp}(p,n-1)]$ for some $\eta\in\mathrm{poss}(p,n-1)$.

Definition 1.7. For $p,q\in Q_{\mathfrak{p}}$, we set $q\leq p$ if the following holds:

- $\mathrm{trnklg}(q)\geq\mathrm{trnklg}(p)$.
- If $n\geq\mathrm{trnklg}(q)$ then
  - $q(n)\in\mathbf{\Sigma}(p,n)$,
  - $\mathrm{supp}(q,n)\cap\mathrm{dom}(p)=\mathrm{supp}(p,n)$, (This implies: $\mathrm{trnklg}(q,i)$ is the maximum of $\mathrm{trnklg}(p,i)$ and $\mathrm{trnklg}(q)$ for all $i\in\mathrm{dom}(p)$.)
  - $\mathrm{supp}^{\mathrm{ls}}(q,n)\cap\mathrm{dom}(p)\subseteq\mathrm{supp}^{\mathrm{ls}}(p,n)$.

[4] I.e., $\nu(m,i)=\mathrm{trunk}(p)(m,i)$ for all $m<\min(n,\mathrm{trnklg}(p,i))$.

- $\mathrm{trunk}(q)$ extends $\mathrm{trunk}(p)$ (as function), i.e., $\mathrm{trunk}(q)(m,i) = \mathrm{trunk}(p)(m,i)$ whenever $i \in \mathrm{dom}(p)$ and $m < \mathrm{trnklg}(p,i)$.
- $\mathrm{trunk}(q) \restriction (\mathrm{trnklg}(q) \times \mathrm{dom}(p)) \in \mathrm{poss}(p, \mathrm{trnklg}(q))$.

Remark 1.8. Note that our ml-creatures have an "answer" $\mathfrak{c}[\eta]$ to all $\eta \in \mathrm{POSS}_{n,\mathrm{supp}(c)}$; so in particular $p(n)$ has answers to all $\eta \notin \mathrm{poss}(p,n)$. In this respect, our creatures carry a lot of seemingly irrelevant information. This is neccessary, however, to allow simple proofs of properness and rapid reading: this way we can, e.g., start with a condition p, then increase the trunk to some height h, strengthen this new condition to some q, and then "merge" p and q, by setting $r(n) = p(n)$ for $n < h$ and $r(n) = q(n)$ otherwise. This would not be possible if we dropped the information about "impossible" $\eta \in \mathrm{POSS}_{n,\mathrm{supp}(c)}$ from the creatures.

Facts 1.9.
- Assume that p is a Q_{p} condition, $n > \mathrm{trnklg}(p)$, choose u such that $\mathrm{supp}(p, n-1) \subseteq u \subseteq \mathrm{dom}(p)$ and $\eta \in \mathrm{POSS}_{n,u}$. Then we can modify p by enlarging the trunk-length to n and replacing part of the trunk by η. Let us call the resulting creature $p \wedge \eta$. (More formally: $\mathrm{trunk}(p \wedge \eta)(m,i) = \eta(m,i)$ if $m < n$ and $i \in u$, and $\mathrm{trunk}(p)(m,i)$ otherwise.)
- $p \wedge \eta \leq p$ if $\eta \in \mathrm{poss}(p,n)$.
- $\{p \wedge \eta :\ \eta \in \mathrm{poss}(p,n)\}$ is predense below p.
- We set $\underset{\sim}{y}^{\mathrm{gen}}$ to be the name for $\bigcup_{p \in G} \mathrm{trunk}(p)$. So Q_{p} forces that $\underset{\sim}{y}^{\mathrm{gen}}$ is a function with domain $\omega \times J$ for some $J \subseteq I^*$. Note that it is not guaranteed that $J = I^*$. (But p forces that $\mathrm{dom}(p) \subseteq J$ and that $\underset{\sim}{y}^{\mathrm{gen}} \restriction (n \times \mathrm{dom}(p)) \in \mathrm{poss}(p,n)$ for all $n \in \omega$.)
- If $\eta \in \mathrm{poss}(p,n)$, then $p \wedge \eta \Vdash \varphi$ iff $p \Vdash \eta \subset \underset{\sim}{y}^{\mathrm{gen}} \rightarrow \varphi$.

One simple way to guarantee that $J = I^*$ is the following: Given $i \in I^*$ and a creature $\mathfrak{c}$, we can strengthen $\mathfrak{c}$ by increasing the support by (not much more than) $\{i\}$ while not decreasing the norm too much:

Lemma 1.10. *Assume that for all $i \in I^*$ there is an $M \in \omega$ and a $u \in [I^*]^{<\aleph_0}$ containing i sucht that for all $n > M$ and all $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > M$ there is a $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ such that*

- $\mathrm{nor}(\mathfrak{d}) > \mathrm{nor}(\mathfrak{c}) - M$ *and* $\mathit{nor}^{ls}(\mathfrak{d}) > \mathit{nor}^{ls}(\mathfrak{c}) - M$,
- $\mathit{supp}(\mathfrak{d}) = \mathit{supp}(\mathfrak{c}) \cup u$ *and* $\mathit{supp}^{ls}(\mathfrak{d}) = \mathit{supp}^{ls}(\mathfrak{c}) \cup u$.

Then the domain of $\underset{\sim}{y}^{gen}$ is forced to be $\omega \times I^*$.

Proof. Given $p \in Q_{\mathrm{p}}$ and $i \in I^*$ we can find a $q \leq p$ such that $i \in \mathrm{supp}(q)$: For sufficiently large n we get $\mathrm{nor}(p,n) > M$ and $\mathrm{dom}(p) \cap u \subseteq \mathrm{supp}(p,n)$. So we can set set $q(n) = \mathfrak{d} \in \mathbf{\Sigma}(p(n))$ as above. □

1.2. **Properness: Bigness and halving.**

Definition 1.11.
- For $\mathfrak{c}$ in $\mathbf{K}(n)$ and $x > 0$ we write $\mathfrak{d} \in \mathbf{\Sigma}^x_+(\mathfrak{c})$ if $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$, $\mathrm{supp}(\mathfrak{d}) = \mathrm{supp}(\mathfrak{c})$, $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{d}) = \mathrm{supp}^{\mathrm{ls}}(\mathfrak{c})$, $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - x$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{d}) \geq \mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}) - x$.
- The n-ml-creature $\mathfrak{c}$ is (B,x)-big, if for all functions $G : \mathrm{POSS}_{n+1,\mathrm{supp}(\mathfrak{c})} \rightarrow B$ there is a $\mathfrak{d} \in \mathbf{\Sigma}^x_+(\mathfrak{c})$ and a $G' : \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})} \rightarrow B$ such that $G(\eta) = G'(\nu)$ for all $\eta \in \mathfrak{d}[\nu]$. I.e., modulo $\mathfrak{d}$ the value of $G(\eta)$ only depends on $\eta \restriction n$.
- $\mathbf{K}(n)$ is (B,x)-big, if all $\mathfrak{c} \in \mathbf{K}(n)$ with norm bigger than 1 are (B,x)-big. (Note that we do not require that $\mathfrak{c}$ has large $\mathrm{nor}^{\mathrm{ls}}$.)[5]

Definition 1.12.
- A condition p decides a name $\underset{\sim}{\tau}$, if there is an element $x \in V$ such that p forces $\underset{\sim}{\tau} = \check{x}$.
- $\underset{\sim}{\tau}$ is n-decided by p, if $p \wedge \eta$ decides $\underset{\sim}{\tau}$ for each $\eta \in \mathrm{poss}(p,n)$.
- p essentially decides $\underset{\sim}{\tau}$, if $\underset{\sim}{\tau}$ is n-decided by p for some n.

[5]Of course there are some other natural definitions for bigness. We briefly mention two of them, however the reader can safely skip this. In our setting, all these notions are more or less equivalent: Firstly, we will assume that $k := |\mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}|$ is "very small" compared to the bigness B. Secondly, $\mathrm{val}(\mathfrak{c})$ will be determined by the sequence $(\mathfrak{c}[\eta])$.

- The n-ml-creature $\mathfrak{c}$ is weakly-(B,x)-big, if for all $\eta \in \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}$ and all $G : \mathfrak{c}[\eta] \rightarrow B$ there is a $\mathfrak{d} \in \mathbf{\Sigma}^x_+(\mathfrak{c})$ such that $G \restriction \mathfrak{d}[\eta]$ is constant.
- The n-ml-creature $\mathfrak{c}$ is (B,x)-big*, if for all $G : \mathrm{val}(\mathfrak{c}) \rightarrow B$ there is a $\mathfrak{d} \in \mathbf{\Sigma}^x_+(\mathfrak{c})$ such that G restricted to $\mathrm{val}(\mathfrak{d})$ is constant.

We obviously get: (B,x)-big implies weakly-(B,x)-big.
Weakly-$(B,x/k)$-big implies (B,x)-big: We just iterate bigness for all $\eta \in \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}$, i.e., at most k times.
(B^k,x)-big* implies (B,x)-big: Apply big* to the function that maps $\mathbf{f} \in \mathrm{val}(\mathfrak{c})$ to the sequence $(\mathbf{f}(\eta))_{\eta \in \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}}$.

- Let $\underset{\sim}{r} : \omega \to \omega$ be a $Q_{\mathfrak{p}}$-name. p reads $\underset{\sim}{r}$ continuously, if p essentially decides $\underset{\sim}{r}(n)$ for all n.
- p rapidly reads $\underset{\sim}{r}$ (above M), if $\underset{\sim}{r} \restriction n$ is n-decided by p for all n (bigger than M).

Sufficient bigness gets us from continuous to rapid reading:

Lemma 1.13. *Fix $B : \omega \to \omega$. Assume that*

- $\mathbf{K}(n)$ *is* $(\prod_{m<n} B(m), 1)$*-big for all* $n \in \omega$*.*
- p *continuously reads*[6] $\underset{\sim}{r} \in \prod B$*.*
- $M \geq \mathrm{trnklg}(p)$*, and* $\mathrm{nor}(p, m) > 1$ *for all* $m \geq M$*.*

Then there is a $q \leq p$ *such that*

- $\mathrm{trnklg}(q) = \mathrm{trnklg}(p)$, $\mathrm{trunk}(q) = \mathrm{trunk}(p)$*, and* $q(n) = p(n)$ *for* $\mathrm{trnklg}(p) \leq n < M$*,*
- $q(n) \in \mathbf{\Sigma}^1_+(p(n))$ *for* $n \geq M$*,*
- q *rapidly reads* $\underset{\sim}{r}$*. I.e.,* $\underset{\sim}{r} \restriction n$ *is* n*-decided by* q *for all* $n > M$*.*

Proof. For $n \in \omega$, let $h(n) \geq 0$ be maximal such that $\underset{\sim}{r} \restriction h(n)$ is n-decided by p. So $h(n)$ is a weakly increasing, unbounded function. Set

$$x_{n,l} = \underset{\sim}{r} \restriction \min(h(n), l).$$

Note that $x_{n,n}$ is n-decided by p, and that there are at most $\prod_{m<l} B(m)$ many possibilities for $x_{n,l}$.

For all $n \geq M$, we define by downward induction for $l = n, n-1, \dots, M+1, M$ the creatures $\mathfrak{d}_{n,l} \in \mathbf{\Sigma}^1_+(p(l))$ and the function $\psi_{n,l}$ with domain $\mathrm{poss}(p, n)$:

- $\mathfrak{d}_{n,n} = p(n)$, $\psi_{n,n}(\eta)$ is the value of $x_{n,n}$ as forced by $p \wedge \eta$.
- For $l < n$ and $\eta \in \mathrm{poss}(p, l+1)$ we know by induction that $\psi_{n,l+1}(\eta)$ is a potential value for $x_{n,l+1}$. Let $\psi^-_{n,l+1}(\eta)$ be the corresponding value of $x_{n,l}$. Using bigness, we get a $\mathfrak{d}_{n,l} \in \mathbf{\Sigma}^1_+(p(l))$ such that $\psi^-_{n,l+1}(\eta)$ only depends on $\eta \restriction l \in \mathrm{poss}(p, l)$. We set $\psi_{n,l}(\eta \restriction l)$ to be this value $\psi^-_{n,l+1}(\eta)$.

For every $n \in \omega$, set $y_n = (\mathrm{val}(\mathfrak{d}_{n,l}), \psi_{n,l})_{M \leq l \leq n}$. For all l there are only finitely many values for $\mathrm{val}(\mathfrak{d}_{n,l})$ and for $\psi_{n,l}$. So the set of the sequences y_n together with their initial sequences form a finite splitting tree. Using König's Lemma, we get an infinite branch: A sequence $(\mathfrak{d}^*_l, \psi^*_l)_{l \geq M}$ such that $\mathfrak{d}^*_l \in \mathbf{\Sigma}^1_+(p(l))$ and such that for all n the sequence $y^*_n = (\mathrm{val}(\mathfrak{d}^*_l), \psi^*_l)_{M \leq l < n}$ is initial sequence of y_m for some $m > n$.

We define $q \leq p$ by $q(l) = p(l)$ for $n < M$ and $q(l) = \mathfrak{d}^*_l$ otherwise (and, of course, $\mathrm{trunk}(q) = \mathrm{trunk}(p)$).

Fix $n > M$. We claim that $\underset{\sim}{r} \restriction n$ is n-decided by q.

Pick some m such that $h(m) > n$ and some k such that y^*_m is initial sequence of y_k. Recall the inductive construction of $\mathfrak{d}_{k,l}$:

(1.1) Modulo p and $\mathfrak{d}_{k,n}, \mathfrak{d}_{k,n-1}, \dots, \mathfrak{d}_{k,k}$ any $\eta \in \mathrm{poss}(p, n)$ already decides $x_{k,n}$.

Also, $x_{k,n}$ contains $\underset{\sim}{r} \restriction n$ (since $h(k) > n$). In fact even $h(m) > n$, so $\underset{\sim}{r} \restriction n$ is decided by $p \wedge \nu$ for all $\nu \in \mathrm{poss}(p, m)$. Therefore we can improve the previous equation:

(1.2) Modulo p and $\mathfrak{d}_{k,m-1}, \dots, \mathfrak{d}_{k,k}$ any $\eta \in \mathrm{poss}(p, n)$ already decides $x_{k,n}$.

Now recall that $\mathfrak{d}_{k,m-1}, \dots, \mathfrak{d}_{k,k}$ are conditions in q, so $x_{k,n}$ (and therefore $\underset{\sim}{r} \restriction n$) is n-decided by q. □

To get properness, we need another well established creature forcing concept:

Definition 1.14. The n-ml-creature $\mathfrak{c}$ is x-halving, if there is a $\mathrm{half}(\mathfrak{c}) \in \mathbf{\Sigma}^x_+(\mathfrak{c})$ satisfying the following: If $\mathfrak{d} \in \mathbf{\Sigma}(\mathrm{half}(\mathfrak{c}))$ has non-zero norm, then there is a $\mathfrak{d}'$ (called the un-halved version of $\mathfrak{d}$) satisfying:

- $\mathfrak{d}' \in \mathbf{\Sigma}(\mathfrak{c})$,
- $\mathrm{supp}(\mathfrak{d}') = \mathrm{supp}(\mathfrak{d})$, and $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{d}') = \mathrm{supp}^{\mathrm{ls}}(\mathfrak{d})$,
- $\mathrm{nor}(\mathfrak{d}') \geq \mathrm{nor}(\mathfrak{c}) - x$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{d}') \geq \mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}) - x$,
- $\mathfrak{d}'[\eta] \subseteq \mathfrak{d}[\eta]$ for all $\eta \in \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{d})}$.[7]

$\mathbf{K}(n)$ is x-halving, if all $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ are x-halving. (Note that we do not require $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}) > 1$.)

Definition 1.15. A forcing parameter $\mathfrak{p}$ has sufficient bigness and halving, if there is an increasing function $\mathrm{maxposs} : \omega \to \omega$ such that for all $n \in \omega$

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[6] I.e., $\underset{\sim}{r}$ is a name, p forces that $\underset{\sim}{r}(m) < B(m)$ for all $m \in \omega$, and p continuously reads $\underset{\sim}{r}$.

[7] An alternative, stronger definition would be: $\mathrm{val}(\mathfrak{d}') \subseteq \mathrm{val}(\mathfrak{d})$. In the special case mentioned in Remark 1.4 these versions are equivalent.

(1) $|\text{poss}(p,n)| < \text{maxposs}(n)$ for all $p \in Q_{\mathfrak{p}}$.
(2) $\mathbf{K}(n)$ is $(2,1)$-big.
(3) $\mathbf{K}(n)$ is $1/\text{maxposs}(n)$-halving.

Remark 1.16. The natural way to guarantee (1) is the following: There is an increasing function maxsupp : $\omega \to \omega$ such that for every $n \in \omega$

- every n-ml-creature $\mathfrak{c}$ satisfies $|\text{supp}(\mathfrak{c})| < \text{maxsupp}(n)$,
- There is an $M(n) \in \omega$ such that $|\text{POSS}^*_{=m,\{i\}}| < M(n)$ for all $i \in I^*$ and $m < n$, and
- $\text{maxposs}(n) \geq M(n)^{(n\cdot\text{maxsupp}(n-1))}$.

A bit of care will be required to construct such creatures, since on the other hand we will also need

- the norm of a creature does not decrease by, say, more than 1 if we "make the support twice as big" (we need this to prove $\aleph_2$-cc, cf. Definition 1.20), and
- there is an n-ml-creature c with $\text{nor}(\mathfrak{c}) \geq n$ (this guarantees that $Q_{\mathfrak{p}}$ is nonempty).

Lemma 1.17. *Assume that $\mathfrak{p}$ has sufficient bigness and halving, that $\underset{\sim}{\tau}$ is the name for an element of V, that $p_0 \in Q_{\mathfrak{p}}$, that $M_0 \geq \text{trnklg}(p_0)$, $n_0 \geq 1$ and $\text{nor}(p_0,m) \geq n_0+2$ for all $m \geq M_0$. Then there is a $q \leq p_0$ such that*[8]

- *q essentially decides $\underset{\sim}{\tau}$,*
- *$q(m) = p_0(m)$ for $\text{trnklg}(p_0) \leq m < M_0$,*
- *$\text{nor}(q,m) \geq n_0$ for all $m \geq M_0$.*

Then the usual standard argument gives us properness and ω^ω-boundedness, and Lemma 1.13 gives us rapid reading:

Corollary 1.18. *Assume that $\mathfrak{p}$ has sufficient bigness and halfing.*

- *$Q_{\mathfrak{p}}$ is proper and ω^ω-bounding. Moreover, for each condition p_0 and name $\underset{\sim}{r} : \omega \to \omega$ there is a $q \leq p_0$ continuously reading $\underset{\sim}{r}$.*
- *If additionally every $\mathbf{K}(n)$ is $(\prod_{m<n} B(m), 1)$-big, we get rapid reading: If $\underset{\sim}{r}$ is a name for an element of $\prod B$ then for every p there is a $q \leq p$ such that $\underset{\sim}{r} \restriction m$ is m-decided by q for all $m \in \omega$.*

Let us first give a sketch of the (standard) argument of the Corollary:

Proof.
- ω^ω-bounding: Assume that $\underset{\sim}{r}$ is a name for a function from ω to ω and that p_0 is in $Q_{\mathfrak{p}}$. Using the previous lemma, we iteratively construct $M_n \in \omega$ and $p_{n+1} \leq p_n$ such that
  - M_n is big enough to satisfy the following: for some $i \in \text{dom}(p_n,n)$ (chosen by suitable bookkeeping) there is an $m < M_n$ such that $i \in \text{supp}^{\text{ls}}(p_n,m)$ and $\text{nor}^{\text{ls}}(p_n,m) > n$. Also, $M_n > \text{trnklg}(p_0) = \text{trnklg}(p_n)$ and $\text{nor}(p_n,k) > n+2$ for all $k \geq M_n$.
  - $p_{n+1}(m) = p_n(m)$ for all $m < M_n$,
  - $\text{nor}(p_{n+1},m) > n$ for all $m \geq M_n$.
  - p_{n+1} essentially decides $\underset{\sim}{r}(n)$,

  This guarantees that the sequence of the p_n's has a limit q, which essentially decides each $\underset{\sim}{r}(n)$. This in turn implies that (modulo q) there are only finitely many possibilities for each $\underset{\sim}{r}(n)$, which gives us ω^ω-boundedness.
- Properness: Fix $N \prec H(\chi)$ and $p_0 \in N$. We need a $q \leq p$ which is N-generic, i.e., which forces that $\underset{\sim}{\tau}[G] \in N$ for all names for ordinals that are in N. Enumerate all these names as $\{\underset{\sim}{\tau}_0, \underset{\sim}{\tau}_1 \dots\}$. Now do the same as above, but instead of $\underset{\sim}{r}(n)$ use τ_n; and construct each p_n inside of N. (The whole sequence of the p_n's cannot be in N, of course.) Then q leaves only finitely many possibilities for each $\underset{\sim}{\tau}_n$, each possibility being element of N, which gives properness. □

Proof of Lemma 1.17. **(a) Halving, the single step $S^{\text{e}}(p,M,n)$:**

Assume that

- $p \in P$,
- $M \geq \text{trnklg}(p)$,
- $n \geq 1$, $\text{nor}(p,m) > n$ for all $m \geq M$.

[8] note that in contrast to the previous lemma, the supports of $q(n)$ will generally be bigger than those of $p(n)$.

We now define $S^{\mathrm{e}}(p, M, n) \leq p$. Enumerate $\mathrm{poss}(p, M)$ as $\eta^1, \ldots, \eta^l$. So $l \leq \mathrm{maxposs}(M)$. Set $p^0 = p$. For $1 \leq k \leq l$, pick p^k such that

- $\mathrm{trnklg}(p^k) = M$ and $p^k \leq p^{k-1} \wedge \eta_k$. (So in particular, $\mathrm{trunk}(p^k) \restriction \mathrm{dom}(p) = \eta_k$.)
- For all $m \geq M$, $\mathrm{nor}(p^k, m) > n - k/\mathrm{maxposs}(M)$.
- One of the following cases holds:
  dec: p^k essentially decides $\underaccent{\tilde}{\tau}$, or
  half: it is not possible to satisfy case *dec*, then $p^k(m) = \mathrm{half}(p^{k-1}(m))$ for all $m > M$.

So in case **half**, we get $\mathrm{dom}(p^k) = \mathrm{dom}(p^{k-1})$, but in case **dec** the domain will generally increase.

We now define $q = S^{\mathrm{e}}(p, M, n)$ by $q(m) = p(m)$ for $m < M$ and $q(m) = p^l(m)$ otherwise.[9] Note that $\mathrm{nor}(q, m) > n - 1$ for all $m \geq M$.

(b) Iterating the single step:

Given p_0, M_0 and n_0 as in the Lemma, we inductively construct p_k and M_k for $k \geq 1$:

- Choose by some bookkeeping an $\alpha \in \mathrm{dom}(p_{k-1})$.
- Choose

$$M_k > k + M_0 \tag{1.3}$$

  big enough such that
  - there is an $l < M_k$ with $\alpha \in \mathrm{supp}^{\mathrm{ls}}(p_{k-1}, l)$ and $\mathrm{nor}^{\mathrm{ls}}(p_{k-1}, l) > k$,
  - $\mathrm{nor}(p_{k-1}(m)) > k + n_0 + 2$ for all $m > M_k$.
- Let p_k be $S^{\mathrm{e}}(p_{k-1}, M_k, k + n_0 + 2)$.

Assuming adequate bookkeeping, the sequence p_k has a limit $q_0 \leq p_0$, and $\mathrm{nor}(q_0, m) > n_0 + 1$ for all $m \geq M_0$.

(c) Bigness, thinning out q_0

We now thin out q_0, using bigness in a way similar to the proof of Lemma 1.13.

For all $n \in \omega$, we define by downward induction for $l = n, n-1, \ldots, M_0 + 1, M_0$, a subset $\Lambda_{n,l}$ of $\mathrm{poss}(q_0, l)$ and ml-creatures $\mathfrak{d}_{n,l} \in \mathbf{\Sigma}^1_+(q_0(l))$:

- $\mathfrak{d}_{n,n} = q_0(n)$; and $\eta \in \Lambda_{n,n}$ iff $q_0 \wedge \eta$ essentially decides $\underaccent{\tilde}{\tau}$.
- For $l < n$, we use bigness to get $\mathfrak{d}_{n,l} \in \mathbf{\Sigma}^1_+(q_0(l))$ such that for all $\eta \in \mathrm{poss}(q_0, l)$ either $\mathfrak{d}_{n,l}[\eta] \subseteq \Lambda_{n,l+1}$ or $\mathfrak{d}_{n,l}[\eta] \cap \Lambda_{n,l+1} = 0$. We set $\Lambda_{n,l}$ to be the set of those $\eta \in \mathrm{poss}(q_0, l)$ such that $\mathfrak{d}_{n,l}[\eta] \subseteq \Lambda_{n,l+1}$.

So by this construction we get: If $\eta \in \mathrm{poss}(q_0, M_0) \cap \Lambda_{n,M_0}$ then every $\nu \in \mathrm{poss}(q_0, n)$ that extends η and is compatible with $(\mathfrak{d}_{n,l})_{M_0 \leq l < n}$ satisfies $q_0 \wedge \nu$ essentially decides $\underaccent{\tilde}{\tau}$.

If on the other hand

- $\eta \in \mathrm{poss}(q_0, M_0) \setminus \Lambda_{n,M_0}$,
- ν is in $\mathrm{poss}(q_0, M)$ for some $M_0 \leq M \leq n$,
- ν extends η, and
- ν is compatible with $(\mathfrak{d}_{n,l})_{M_0 \leq l < M}$, then

$$q_0 \wedge \nu \text{ does not essentially decide } \underaccent{\tilde}{\tau}. \tag{1.4}$$

We claim that there is some $n_0 \geq M_0$ such that

$$\mathrm{poss}(q_0, M_0) \subseteq \Lambda_{n_0, M_0}. \tag{1.5}$$

Then we define $q \leq q_0$ by $q(m) = \mathfrak{d}_{n_0,m}$ for $M_0 \leq m \leq n_0$ and $q(m) = q_0(m)$ for $m > n_0$. According to the definition of Λ_{n_0,M_0}, we know that $q_0 \wedge \nu$ essentially decides $\underaccent{\tilde}{\tau}$ for all $\nu \in \mathrm{poss}(q, n_0)$, so q essentially decides $\underaccent{\tilde}{\tau}$. This finishes the proof of the Lemma, since q satisfies the other requirements as well.

So it remains to show (1.5). For every $n \in \omega$, we define the finite sequence

$$x_n = (\mathrm{val}(\mathfrak{d}_{n,l}), \Lambda_{n,l})_{M_0 \leq l \leq n}.$$

For each l, there are only finitely many possibilities for $\mathrm{val}(\mathfrak{d}_{n,l})$ and for $\Lambda_{n,l}$, so the set of the sequences x_n together with their initial sequences form a finite splitting tree. Using König's Lemma, we get an infinite branch. So we get a sequence $(\mathfrak{d}^*_l, \Lambda^*_l)_{M_0 \leq l \leq \omega}$ such that $\mathfrak{d}^*_l \in \mathbf{\Sigma}^1_+(q_0(l))$ and for all n there is an $m > n$ such that the sequence

$$x^*_n = (\mathrm{val}(\mathfrak{d}^*_l), \Lambda^*_l)_{M_0 \leq l \leq n}$$

[9] And, of course, we set $\mathrm{trunk}(q, i) = \mathrm{trunk}(p, i)$ if $i \in \mathrm{dom}(p)$ and $\mathrm{trunk}(q, i) = \mathrm{trunk}(p^l, i)$ otherwise.

is an inital sequence of x_m.

We claim

$$\mathrm{poss}(q_0, M_0) \subseteq \Lambda^*_{M_0}. \tag{1.6}$$

Then we get (1.5) by picking any n_0 such that $\Lambda_{n_0,M_0} = \Lambda^*_{M_0}$.

To show (1.6), assume towards a contradiction that there is some $\eta_0 \in \mathrm{poss}(q_0, M_0) \setminus \Lambda^*_{M_0}$. Define $q_1 \leq q_0$ by $q_1(l) = q_0(l)$ if $l < M_0$ and $q_1(l) = \mathfrak{d}^*_l$ otherwise. Find an $s \leq q_1 \wedge \eta_0$ deciding $\underaccent{\tilde}{\tau}$. Without loss of generality, $\mathrm{trnklg}(s) = M_k > M_0$ for some k, where M_k was chosen in (1.3). Also we can assume $\mathrm{nor}(s, m) > 2$ for all $m > \mathrm{trnklg}(s)$. Let $\mathrm{trunk}(s)$ extend some $\nu \in \mathrm{poss}(q_1, M_k) \subseteq \mathrm{poss}(q_0, M_k)$. In particular, ν extends η_0. We claim:

$$q_0 \wedge \nu \text{ does not essentially decide } \underaccent{\tilde}{\tau} \tag{1.7}$$

Pick m such that x_m extends $x^*_{M_k}$. In particular, $\Lambda_{m,M_0} = \Lambda^*_{M_0}$, so $\eta_0 \notin \Lambda_{m,M_0}$. Since $\nu \in \mathrm{poss}(q_1, M_k)$, ν is compatible with the sequence $\mathrm{val}(\mathfrak{d}^*_l)_{M_0 \leq l < M_k}$ and $\mathrm{val}(\mathfrak{d}^*_l) = \mathrm{val}(\mathfrak{d}_{m,l})$. So by (1.4) we get that $q_0 \wedge \nu$ does not essentially decide $\underaccent{\tilde}{\tau}$. This proves (1.7).

By (1.7) we know: when we were dealing with ν in stage k, we were in the **half**-case. In particular, s is stronger than some p^l_{k-1} that resulted from halving p^{l-1}_{k-1}. Let M' be such that $\mathrm{nor}(s, m) > k + n_0 + 2$ for all $m \geq M'$. We can now un-halve $s(m)$ for all $M_k \leq m < M'$ (and leave it unchanged above M'), resulting in a condition s' that is stronger than p^{l-1}_{k-1} and essentially decides $\underaccent{\tilde}{\tau}$, a contradiction to the fact that p^l_{k-1} was constructed using the **half**-case. So we have shown (1.6). □

Remark 1.19. The proof actually shows that it is not required that all n-ml-creatures are $1/\mathrm{maxposs}(n)$-halving. It is enough to have an infinite set $w \subseteq \omega$ such that for all $M \in w$ and $n \geq M$ every n-ml-creature is $1/\mathrm{maxposs}(M)$-halving. (Just choose all the M_k in the proof to be in w.)

1.3. $\aleph_2$-cc.

To preserve all cofinalities, we will use $\aleph_2$-cc in addition to properness. To guarantee that $Q_\mathfrak{p}$ is $\aleph_2$-cc, we need additional properties of $\mathfrak{p}$ and we have to assume CH in the ground model.

We will argue as follows: Assume towards a contradiction that A is an antichain of size $\aleph_2$. By a standard Δ-system argument we can assume that any two conditions in A have (more or less) disjoint domain; we assume that there are only continuum many different conditions "modulo isomorphism of the domain"; and then we have to argue that two identical (modulo domain) conditions with disjoint domain are compatible.

There are many ways to achieve this, one sufficient conditions is the following:

Definition 1.20. Fix $n \in \omega$. The n-creature-parameter $\mathfrak{p}(n)$ has the local Δ-property, if we can assign one of continuum many[10] "local types" to each pair $(\mathfrak{c}, \bar{i})$, where $\mathfrak{c}$ is an n-ml-creatue and $\bar{i} : |\mathrm{supp}(\mathfrak{c})| \to \mathrm{supp}(\mathfrak{c})$ is bijective, such that the following holds:

If

- $(\mathfrak{c}_1, \bar{i}_1)$ and $(\mathfrak{c}_2, \bar{i}_2)$ are as above and have the same local type,
- $\mathrm{nor}(\mathfrak{c}_1) = \mathrm{nor}(\mathfrak{c}_2) > 1$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}_1) = \mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}_2)$,
- the enumerations $\bar{i}_1$ and $\bar{i}_2$ agree on $\mathrm{supp}(\mathfrak{c}_1) \cap \mathrm{supp}(\mathfrak{c}_2)$.
  More formally: if $i \in \mathrm{supp}(\mathfrak{c}_1) \cap \mathrm{supp}(\mathfrak{c}_2)$, then there is an m such that $\bar{i}_1(m) = \bar{i}_2(m) = i$,

then there is a $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c}_1) \cap \mathbf{\Sigma}(\mathfrak{c}_2)$ such that

- $\mathrm{supp}(\mathfrak{d}) = \mathrm{supp}(\mathfrak{c}_1) \cup \mathrm{supp}(\mathfrak{c}_2)$ and $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{d}) = \mathrm{supp}^{\mathrm{ls}}(\mathfrak{c}_1) \cup \mathrm{supp}^{\mathrm{ls}}(\mathfrak{c}_2)$,
- $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}_1) - 1$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{d}) \geq \mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}_1) - 1$.

Lemma 1.21. *Assume CH and that $\mathfrak{p}(n)$ has the local Δ-property for all n. Then $Q_\mathfrak{p}$ is $\aleph_2$-cc.*

Proof. Assume towards a contradiction that A is an antichain of size $\aleph_2$. We can assume that there is a $\Delta \subseteq I^*$ such that $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \Delta$ for all $p \neq q$ in A, and that $|\mathrm{dom}(p)| = M \leq \omega$ for all $p \in A$. Pick for all $p \in A$ a bijection $\bar{i}^p : M \to \mathrm{dom}(p)$.

We can also assume that the following objects and statements do not depend on the choice of $p \in A$ for $i^\Delta \in \Delta, m < M$ and $n \in \omega$:

- The trunk of p "modulo the enumeration of the domain", i.e., $\mathrm{trnklg}(p)$, $\mathrm{trnklg}(p, \bar{i}^p(m))$ and $\mathrm{trunk}(p, \bar{i}^p(m))$.
- The norms, $\mathrm{nor}(p, n)$, $\mathrm{nor}^{\mathrm{ls}}(p, n)$.

[10] In practise, we can get finitely many.

- The local type of $(p(n), \bar{j}^p_n)$, where $\bar{j}^p_n$ is $\bar{\imath}^p$ restricted to $\mathrm{supp}(p, n)$.[11]
- Whether $\bar{\imath}^p(m) \in \mathrm{supp}(p, n)$.
- Whether $\bar{\imath}^p(m) = i^\Delta$.

Now pick $p \neq q$ in A. We show towards a contradiction that p and q are compatible: Pick h such that $\mathrm{nor}(p, n) > 1$ for all $n \geq h$. The local types of $(p(n), \bar{j}^p_n)$ and $(q(n), \bar{j}^q_n)$ are the same. If $i^\Delta \in \mathrm{supp}(p, n) \cap \mathrm{supp}(q, n)$, then $i^\Delta = \bar{\imath}^p(m) = \bar{\imath}^q(m)$ for some $m < M$, and $\bar{\imath}^p(k) \in \mathrm{supp}(p, n)$ iff $\bar{\imath}^q(k) \in \mathrm{supp}(q, n)$ for all $k \leq m$, therefore $i^\Delta = \bar{j}^p_n(l) = \bar{j}^q_n(l)$ for some l. So we can apply the local Δ property and get $\mathfrak{d} \in \mathbf{\Sigma}(p(n)) \cap \mathbf{\Sigma}(q(n))$. The sequence of these creatures, together with the union of the stems of p and q, form a condition $r \leq p, q$. $\square$

2. Continuum many invariants

We now apply this creature forcing construction (actually, only the pure lim-sup case and the simplified setting described in Remark 1.4) to improve the result of *Decisive Creatures* [4]. We have to make sure to define the ml-creatures and the norms in a way to satisfy sufficient bigness and halfing (see Definition 1.15 and the Remark following it). Once we have done this, it turns out that the rest of the proof of the Main Theorem is a rather straightforward modification of the proof in [4].

2.1. **Atomic creatures, decisiveness.** We will build the ml-creatures from simpler creatures, which we call atomic creatures. An atomic parameter is a tuple $a = (A, \mathbf{K}, \mathrm{val}, \mathrm{nor}, \mathbf{\Sigma})$ such that

- A is a finite set.
- $\mathbf{K}$ is a finite set (the set of a-atomic creatures),
- val, nor and $\mathbf{\Sigma}$ are functions with domain $\mathbf{K}$

such that for all a-atomic creatures $w \in \mathbf{K}$ the following holds:

- $\mathrm{nor}(w) \geq 0$,
- $\mathrm{val}(w) \subseteq A$ is nonempty,
- $\mathbf{\Sigma}(w)$ is a subset of $\mathbf{K}$,
- $w \in \mathbf{\Sigma}(w)$; and if $w_2 \in \mathbf{\Sigma}(w_1)$ and $w_3 \in \mathbf{\Sigma}(w_2)$ then $w_3 \in \mathbf{\Sigma}(w_1)$,
- if $v \in \mathbf{\Sigma}(w)$ then $\mathrm{val}(v) \subseteq \mathrm{val}(w)$ and $\mathrm{nor}(v) \leq \mathrm{nor}(w)$,
- if $|\mathrm{val}(w)| = 1$ then $\mathrm{nor}(w) < 1$.

As usual we get notions of bigness and halving, as well as decisiveness as introduced in [4]:

- $v \in \mathbf{\Sigma}^x_+(w)$ means $v \in \mathbf{\Sigma}(w)$ and $\mathrm{nor}(v) > \mathrm{nor}(w) - x$.
- $w \in \mathbf{K}$ is (B, x)-big, if for all $F : \mathrm{val}(w) \to B$ there is a $v \in \mathbf{\Sigma}^x_+(w)$ such that $F \restriction \mathrm{val}(v)$ is constant.
- w is hereditary (B, x)-big, if every $v \in \mathbf{\Sigma}(w)$ with norm at least 1 is (B, x)-big.
- The atomic parameter a is (B, x)-big, if every $w \in \mathbf{K}$ with norm at least 1 is (B, x)-big.
- $w \in \mathbf{K}$ is x-halving, if there is a $\mathrm{half}(w) \in \mathbf{\Sigma}^x_+(w)$ such that for all $v \in \mathbf{\Sigma}(\mathrm{half}(w))$ with norm bigger than 0 there is a $v' \in \mathbf{\Sigma}^x_+(w)$ with $\mathrm{val}(v') \subseteq \mathrm{val}(v)$. We call this v' "unhalved version of v", or we say that we "unhalve v" to get v'.
- The atomic parameter a is x-halving, if every $w \in \mathbf{K}$ with norm bigger than 1 is x-halving.
- $w \in \mathbf{K}$ is (K, m, x)-decisive, if there are $v^-, v^+ \in \mathbf{\Sigma}^x_+(w)$ such that

$$|\mathrm{val}(v^-)| \leq K \quad \text{and} \quad v^+ \text{ is hereditarily } (2^{K^m}, x)\text{-big.} \tag{2.1}$$

  v^- is called a *K-small successor*, and v^+ a *K-big successor* of w.
- w is (m, x)-decisive if w is (K, m, x)-decisive for some K.
- $\mathbf{K}$ is (m, x)-decisive if every $w \in \mathbf{K}$ with $\mathrm{nor}(w) > 1$ is (m, x)-decisive.
- An atomic-parameter is M-nice with maximal norm m, if it is $(2^M, 1/M^2)$-big, $1/M$-halving and $(M, 1/M^2)$-decisive and $m = \max(\mathrm{nor}(w) : w \in \mathbf{K})$.

Facts 2.1. (1) Given $M, m \in \omega$ there is an M-nice atomic-parameter with maximal norm m.
Another way to formulate this:
For all $M, m \in \omega$ there is a $K(M, n) \in \omega$ such that for all $k > K(M, n)$ there is an atomic-parameter $a = (A, \mathbf{K}, \mathrm{val}, \mathrm{nor}, \mathbf{\Sigma})$ which is M-nice with maximal norm m such that $A = k$.

[11] More formally, $\bar{j}^p_n : |\mathrm{supp}(p, n)| \to \mathrm{supp}(p, n)$ is defined by $\bar{j}^p_n(l) = \bar{\imath}^p(k)$ for the minimal k such that $\bar{\imath}^p(k) \in \mathrm{supp}(p, n) \setminus \bar{j}^p_n{}``l$.

(2) Assume that an atomic paramter is M-nice, that $\mathrm{nor}(w_i) > 2$ for all $i \in M$, and that $F : \prod_{i\in M} \mathrm{val}(w_i) \to 2^M$. Then there are $v_i \in \mathbf{\Sigma}_+^{1/M}(w_i)$ such that $F \restriction \prod_{i\in m} \mathrm{val}(v_i)$ is constant.

Proof. This is shown in [4]: (1) is Lemma 6.1, (2) is Corollary 4.4. □

2.2. **The forcing.**

Definition 2.2. We define by induction on $n \in \omega$ the natural numbers $\mathrm{maxposs}(n)$, $\mathrm{maxnor}(n)$, $\mathrm{maxsupp}(n)$, $B^{\min}(n)$, $k^*(n)$, $g^{\min}(n)$ and $f^{\max}(n)$; as well as $f_{n,m}$ and $g_{n,m}$ for $m \in k^*(n)$:

(1) Set $f^{\max}(-1) = \mathrm{maxsupp}(-1) = 1$.
(2) Set $\mathrm{maxposs}(n) = 1 + (f^{\max}(n-1))^{n\,\mathrm{maxsupp}(n-1)}$.
(By induction, we will see that $|\mathrm{poss}(p, n)| < \mathrm{maxposs}(n)$ for every condition p.)
(3) Set $\mathrm{maxnor}(n) = 1 + 2^{n\cdot \mathrm{maxposs}(n)}$.
(This will later be used to guarantee there is an n-ml-creature with norm n, i.e., that $Q_{\mathfrak{p}}$ is nonempty.)
(4) Set $\mathrm{maxsupp}(n) = 1 + 2^{\mathrm{maxnor}(n)}$.
(We will later define the n-ml-creatures so that $|\mathrm{supp}(\mathfrak{c})| \leq \mathrm{maxsupp}(n)$ for all $\mathfrak{c} \in \mathbf{K}(n)$.)
(5) Pick $B^{\min}(n)$ large with respect to $\mathrm{maxsupp}(n)$.
More specifically: larger than $f^{\max}(n-1)^{n f^{\max}(n-1)^{1+(n\,\mathrm{maxsupp}(n))}}$ and larger than $2\,\mathrm{maxsupp}(n)^2$.
(6) Pick $k^*(n)$ large with respect to $B^{\min}(n)$, which means that we can fix a $B^{\min}(n)$-nice atomic paramter $a_{n,*} = (k^*_n, \mathbf{K}_{n,*}, \mathrm{val}_{n,*}, \mathrm{nor}_{n,*}, \mathbf{\Sigma}_{n,*})$ with maximal norm $\mathrm{maxnor}(n)$. (Use 2.1(1).)
(7) Pick $g^{\min}(n) = g_{n,0}$ large with respect to $k^*(n)$.
More specifically, we will need: larger than $f^{\max}(n-1)^{n\,\mathrm{maxsupp}(n)} \cdot \mathrm{maxposs}(n) \cdot k^*(n)^{\mathrm{maxsupp}(n)}$ and than $f^{\max}(n-1)^{n f^{\max}(n-1)}$.
(8) Pick $f_{n,m}$ large with respect to $g_{n,m}$, which means that we can fix an $g_{n,m}$-nice atomic parameter $a_{n,m} = (f_{n,m}, \mathbf{K}_{n,m}, \mathrm{val}_{n,m}, \mathrm{nor}_{n,m}, \mathbf{\Sigma}_{n,m})$ with maximal norm $\mathrm{maxnor}(n)$. (Again, use 2.1(1).)
(9) Pick $g_{n,m+1}$ large with respect to $f_{n,m}$.
More specifically, we need: larger than $(f_{n,m})^{f_{n,m}{}^{k^*(n)}}$.
(10) Set $f^{\max}(n) = f_{n,k^*(n)-1}$.

We choose an index set I^* containing μ and sets I_ϵ for all $\epsilon \in \mu$:

- For every ϵ in μ, pick some I_ϵ of size κ_ϵ such that μ and all the I_ϵ are pairwise disjoint. Set $I^* = \mu \cup \bigcup_{\epsilon\in\mu} I_\epsilon$.
- We define $\varepsilon : I^* \setminus \mu \to I^*$ by $\varepsilon(\alpha) = \epsilon$ for $\alpha \in I_\epsilon$. A subset u of I^* is ε-closed, if for all $\varepsilon(\alpha) \in u$ for all $\alpha \in u \setminus \mu$.

For $\epsilon \in \mu$ we set $\mathrm{POSS}_{=m,\{\epsilon\}}$ to be $k^*(m)$, and for $\alpha \in I^* \setminus \mu$ we set $\mathrm{POSS}_{=m,\{\alpha\}}$ to be $f^{\max}(m)$.

Definition 2.3. We define the ml-parameter $\mathfrak{p}(n)$: An n-ml-creature $\mathfrak{c}$ is a triple $(u^{\mathfrak{c}}, \bar{w}^{\mathfrak{c}}, d^{\mathfrak{c}})$ satisfying the following:

- $u^{\mathfrak{c}} \subset I^*$ is nonempty, ϵ-closed, and of size at most $\mathrm{maxsupp}(n)$.
- $\bar{w}^{\mathfrak{c}}$ consists of the sequences $(w^{\mathfrak{c}}_\epsilon)_{\epsilon\in u^{\mathfrak{c}}\cap\mu}$ and $(w^{\mathfrak{c}}_{\alpha,k})_{\alpha\in u^{\mathfrak{c}}\cap I_\epsilon, k\in \mathrm{val}(w^{\mathfrak{c}}_\epsilon)}$ such that $w^{\mathfrak{c}}_\epsilon$ is an $a_{n,*}$-atomic-creature and $w^{\mathfrak{c}}_{\alpha,k}$ is an $a_{n,k}$-atomic-creature. We will write $A^{\mathfrak{c}}_\epsilon$ (or $A^{\mathfrak{c}}_{\alpha,k}$) for $\mathrm{val}(w^{\mathfrak{c}}_\epsilon)$ (or $\mathrm{val}(w^{\mathfrak{c}}_{\alpha,k})$, respectively).
- $d^{\mathfrak{c}} \in \mathbb{R}_{\geq 0}$.[12]

Given such an ml-creature $\mathfrak{c}$, we define the creature-properties of $\mathfrak{c}$ as follows:

- $\mathrm{supp}(\mathfrak{c}) := u^{\mathfrak{c}}$.
- $\mathrm{val}(\mathfrak{c})$ is the set of those $\mathbf{f} \in \mathrm{VAL}_{n,u^{\mathfrak{c}}}$ that satisfy the following for all $\eta \in \mathrm{POSS}_{n,u^{\mathfrak{c}}}$: If $\epsilon \in u^{\mathfrak{c}} \cap \mu$, then $\mathbf{f}(\eta)(n, \epsilon) \in A^{\mathfrak{c}}_\epsilon$, and if $\alpha \in u^{\mathfrak{c}} \cap I_\epsilon$ and $\mathbf{f}(\eta)(n, \epsilon) = k$ then $\mathbf{f}(\eta)(n, \alpha) \in A^{\mathfrak{c}}_{\alpha,k}$.
- $\mathrm{nor}(\mathfrak{c}) := (1/\,\mathrm{maxposs}(n)) \cdot \log_2 [\mathrm{minnor}(\mathfrak{c}) - \log_2(|\mathrm{supp}(\mathfrak{c})|) - d]$, where we set minnor to be the minimum of the norms of all atomic creatures used, i.e.,

$$\mathrm{minnor}(\mathfrak{c}) := \min\Big(\{\mathrm{nor}_{n,*}(w^{\mathfrak{c}}_\epsilon) : \epsilon \in u \cap \mu\} \cup \{\mathrm{nor}_{n,k}(w^{\mathfrak{c}}_{\alpha,k}) : \alpha \in u \cap I_\epsilon, k \in A^{\mathfrak{c}}_\epsilon\}\Big). \tag{2.2}$$

(If $\mathrm{nor}(c)$ would be negative or undefined when calculated this way, we set it 0.)
- $\mathrm{supp}^{\mathrm{ls}}(\mathfrak{c}) := \mathrm{supp}(\mathfrak{c})$ and $\mathrm{nor}^{\mathrm{ls}}(\mathfrak{c}) := n$ (so here we have the pure lim-inf case).

[12] We could restrict this to a countable set; moreover given $\bar{w}^{\mathfrak{c}}$ we can even restrict $d^{\mathfrak{c}}$ to a finite set.

So our ml-creatures have rather "restricted memory", they do not "look down" at all, and horizontally only "look from α to $\epsilon(\alpha)$". More exactly:

Fact. $\eta \in \text{poss}(p, n)$ iff

- η is compatible with $\text{trunk}(p)$,
- for all m with $\text{trnklg}(p) \leq m < n$, $\mathfrak{c} := p(m)$, and $\alpha \in I_\epsilon \cap \text{supp}(\mathfrak{c})$ we have: $\eta(m, \epsilon) \in A^{\mathfrak{c}}_\epsilon$ and $\eta(m, \alpha) \in A^{\mathfrak{c}}_{\alpha,\eta(m,\epsilon}$.

Lemma 2.4.

- $\mathbf{K}(n)$ *is* $(f^{max}(n-1)^{nf^{max}(n-1)}, 1)$*-big.*
- $\mathbf{K}(n)$ *is* $1/\text{maxposs}(n)$*-halving.*
- $\mathfrak{p}$ *satisfies the local* Δ*-property.*
- *The generic element lives on all of* I^* *(i.e., the domain of the generic sequence is* $\omega \times I^*$*).*

So we can use Lemma 1.21 and Corollary 1.18 (since $\text{maxposs}(n)$ witnesses that $\mathfrak{p}$ has sufficient bigness and halving, as defined in 1.15), and get:

Corollary 2.5. $Q_\mathfrak{p}$ *is proper,* ω^ω*-bounding and* $\aleph_2$*-cc. If* $p \in Q_\mathfrak{p}$ *forces that* $r(n) < f^{max}(n)^{f^{max}(n)}$ *for all n, then there is a* $q \leq p$ *that n-decides* $r \restriction n$ *for all n.*

Proof of Lemma 2.4. First note a few obvious facts: For all n-ml-creatures $\mathfrak{c}$, we have

$$\left|\text{POSS}_{n,\text{supp}(\mathfrak{c})}\right| \leq f^{\max}(n-1)^{n\,\text{maxsupp}(n)} \tag{2.3}$$

and for a condition p we get, according to 2.2(2),

$$\left|\text{poss}(p, n)\right| \leq f^{\max}(n-1)^{n\,\text{maxsupp}(n-1)} < \text{maxposs}(n), \tag{2.4}$$

According to 2.2(4), we get: If $|\text{supp}(\mathfrak{c})| \geq \text{maxsupp}(n)/2$, then

$$\text{nor}(\mathfrak{c}) \leq 1/\text{maxposs}(n)\ \log_2\left(\text{maxnor}(n) - \log_2(\text{maxsupp}(n)) + 1\right) = 0. \tag{2.5}$$

The local Δ property: We only have to check that "taking the union of identical creatures with disjoint domains" decreases the norm by at most one, the rest is just notation:

Given an n-ml creature $(u^\mathfrak{c}, \bar{w}^\mathfrak{c}, d^\mathfrak{c})$ and an enumeration $\bar{i} : |u^\mathfrak{c}| \to u^\mathfrak{c}$, we define the local type to contain the following information for $m, m' < |u^\mathfrak{c}|$: $d^\mathfrak{c}$, $|u^\mathfrak{c}|$, whether $\bar{i}(m) \in \mu$, whether $\varepsilon(\bar{i}(m)) = \bar{i}(m')$, and the sequence of the atomic creatures (enumerated by $\bar{i}$).[13] Take $\mathfrak{c}_1$ and $\mathfrak{c}_2$ as in the Definition 1.20 of the local Δ property. Since $\text{nor}(\mathfrak{c}_1) > 1$, we know by (2.5) that $|\text{supp}(\mathfrak{c})| < \text{maxsupp}(n)/2$. So we can define the n-ml-creature $\mathfrak{d}$ by $d^\mathfrak{d} = d^{\mathfrak{c}_1} = d^{\mathfrak{c}_2}$; $u^\mathfrak{d} = u^{\mathfrak{c}_1} \cup u^{\mathfrak{c}_2}$; and for $\epsilon \in \mu$ we set $w^\mathfrak{d}_\epsilon$ to be $w^{\mathfrak{c}_1}_\epsilon$ or $w^{\mathfrak{c}_2}_\epsilon$, whichever is defined (if both are defined, they have to be equal, since the type is the same); and in the same way we define $w^\mathfrak{d}_{\alpha,k}$ for $\alpha \in I_\epsilon$ and $k \in A^\mathfrak{d}_\epsilon$.

As already mentioned, the only thing we have to check is that $\text{nor}(\mathfrak{d}) \geq \text{nor}(\mathfrak{c}) - 1$ (for $\mathfrak{c} = \mathfrak{c}_1$ or $\mathfrak{c} = \mathfrak{c}_2$, which does not make any difference). Since $\mathfrak{d}$ consists of the same atomic creatures as $\mathfrak{c}$, we get $\text{minnor}(\mathfrak{d}) = \text{minnor}(\mathfrak{c})$, and therefore

$$\begin{aligned}
\text{nor}(\mathfrak{d}) &\geq 1/\text{maxposs}(n)\ \log_2\left(\text{minnor}(\mathfrak{d}) - \log_2(2|\text{supp}(\mathfrak{c})|) - d\right) \\
&\geq 1/\text{maxposs}(n)\ \log_2\left(\left(\text{minnor}(\mathfrak{c}) - \log_2(|\text{supp}(\mathfrak{c})|) - d\right)/2\right) \\
&= \text{nor}(\mathfrak{c}) - 1/\text{maxposs}(n).
\end{aligned}$$

The domain of the generic: Given $\alpha \in I^*$, we can just enlarge any n-ml-creature creature $\mathfrak{c} = (u^\mathfrak{c}, \bar{w}^\mathfrak{c}, d^\mathfrak{c})$ in the following way: Increase the domain by α and (if $\alpha \notin \mu$) additionally by $\varepsilon(\alpha)$, and pick for the new positions atomic creatures with norm $\text{maxnor}(n)$. The same argument as for the local Δ-property shows that the norm of the new creature decreases by at most $1/\text{maxposs}(n)$. So we can modify any condition to a stronger condition with a domain containing α (as in Lemma 1.10).

Halving: Halving follows directly from the definition of the norm: Given $\mathfrak{c} = (u^\mathfrak{c}, \bar{w}^\mathfrak{c}, d^\mathfrak{c})$, set $\text{half}(\mathfrak{c}) = (u^\mathfrak{c}, \bar{w}^\mathfrak{c}, d')$ with

$$d' = d^\mathfrak{c} + 1/2\left[\text{minnor}(\mathfrak{c}) - \log_2(\text{supp}(\mathfrak{c})) - d^\mathfrak{c}\right].$$

Fix $\mathfrak{d} = (u^\mathfrak{d}, \bar{w}^\mathfrak{d}, d^\mathfrak{d}) \in \mathbf{\Sigma}(\text{half}(\mathfrak{c}))$ (so in particular, $d^\mathfrak{d} \geq d'$). We can unhalve $\mathfrak{d}$ to $\tilde{\mathfrak{d}} = (u^\mathfrak{d}, \bar{w}^\mathfrak{d}, d^\mathfrak{c})$. Straightforward calculations show that the halving properties are satisfied. In particular: If $\text{nor}(\mathfrak{d}) > 0$, then

$$\text{minnor}(\mathfrak{d}) - \log_2(\text{supp}(\mathfrak{d})) - d^\mathfrak{d} > 1.$$

[13] More formally: the sequences $(w^\mathfrak{c}_{\bar{i}(m)})_{m<|u|,\bar{i}(m)\in\mu}$ and $(w^\mathfrak{c}_{\bar{i}(m),k})_{m<|u|,\bar{i}(m)\notin\mu,k\in A^\mathfrak{c}_{\varepsilon(\bar{i}(m))}}$.

To calculate $\mathrm{nor}(\tilde{\mathfrak{d}})$, we use

$$\mathrm{minnor}(\mathfrak{d}) - \log_2(\mathrm{supp}(\mathfrak{d})) - d^{\mathfrak{c}} > 1 + d^{\mathfrak{d}} - d^{\mathfrak{c}} \geq 1 + d' - d^{\mathfrak{c}} > \\ > 1/2\left[\mathrm{minnor}(\mathfrak{c}) - \log_2(\mathrm{supp}(\mathfrak{c})) - d^{\mathfrak{c}}\right].$$

So $\mathrm{nor}(\tilde{\mathfrak{d}}) \geq \mathrm{nor}(\mathfrak{c}) - 1/\mathrm{maxposs}(n)$.

Bigness: Let $\mathfrak{c}$ be an n-ml-creature. Set $B := f^{\max}(n-1)^{n f^{\max}(n-1)}$. To show $(B,1)$-bigness, we pick some $G : \mathrm{POSS}_{n+1,\mathrm{supp}(\mathfrak{c})} \to B$, and we have to find a $\mathfrak{d} \in \Sigma^1_+(\mathfrak{c})$ such that G only depends on $\eta \restriction n$. (More formally: there is a $G' : \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})} \to B$ such that $G(\eta) = G'_0(\nu)$ for all $\eta \in \mathfrak{d}[\nu]$.)

Set $S = \mathrm{POSS}_{n,\mathrm{supp}(\mathfrak{c})}$ and $M = \prod_{\epsilon \in \mathrm{supp}(\mathfrak{c}) \cap \mu} A^{\mathfrak{c}}(\epsilon)$. ($S$ and M stand for "small" and "medium", respectively.) Note that according to (2.3) and 2.2(7),

$$|S \times M| \leq f^{\max}(n-1)^{n\,\mathrm{maxsupp}(n)} \cdot k^*(n)^{\mathrm{maxsupp}(n)} < g^{\min}(n). \tag{2.6}$$

If we fix $\eta \in S$ and $x \in M$, then G can be written as a function from $\prod_{\alpha \in \mathrm{supp}(\mathfrak{c}) \setminus \mu} A^{\mathfrak{c}}_{\alpha, x(\varepsilon(\alpha))}$ to B.

We get:

- All the atomic creatures involved are $g^{\min}(n)$-nice.
- $|\mathrm{supp}(\mathfrak{c}) \setminus \mu| < \mathrm{maxsupp}(n) < g^{\min}(n)$.
- $B < 2^{g^{\min}(n)}$.

So we can apply Fact 2.1(2) and get successors $\nu_\alpha \in \Sigma^{1/g^{\min}(n)}_+(w^{\mathfrak{c}}_{\alpha,x(\varepsilon(\alpha))})$ such that G is constant (with respect to the new creatures).

We can iterate this for all $(\eta, x) \in S \times M$, each time decreasing the norm of some of the atomic creatures on $\mathrm{supp}(\mathfrak{c}) \setminus \mu$ by at most $1/g^{\min}(n)$. By (2.6), in the end we get $\nu_{\alpha,k} \in \Sigma^1_+(w^{\mathfrak{c}}_{\alpha,k})$ for all $\alpha \in u^{\mathfrak{c}} \setminus \mu$ and $k \in A^{\mathfrak{c}}_{\varepsilon(\alpha)}$ such that (modulo these new creatures) G only depends on $(\eta, x) \in S \times M$; or, in other words, G can be written as function fomr M to B^S.

It remains to get rid of the dependence on M. For this, just note that all the atomic creatures $w^{\mathfrak{c}}_\epsilon$ (for $\epsilon \in u^{\mathfrak{c}} \cap \mu$) are $B^{\min}(n)$-nice, $\mathrm{maxsupp}(n) < B^{\min}(n)$ and $B^{\min}(n) > B^S$, so we can find successors on which G is constant. □

2.3. **Proof of the main theorem.**

Definition 2.6.
- $\nu_i := \underset{\sim}{y}^{\mathrm{gen}} \restriction \{i\}$ for all $i \in I^*$. (We interpret ν_i as a function from ω to ω.)
- $f_\epsilon(n) := f_{n,\nu_\epsilon(n)}$ for $\epsilon \in \mu$, and analogously for g_ϵ.
- $c^\forall_\epsilon := c^\forall_{f_\epsilon, g_\epsilon}$ for $\epsilon \in \mu$, and analogously for $c^\exists_\epsilon$.

So $Q_{\mathfrak{p}}$ forces that $\nu_\epsilon(n) < k^*(n)$ for all $n \in \omega$, and that $\nu_\alpha(n) < f_\epsilon(n)$ for all but finitely many n. (There might be finitely many exceptions, since the initial trunk at α might not fit to the initial trunk at $\varepsilon(\alpha)$.)

To prove the main theorem, it is enough to show the following:

$Q_{\mathfrak{p}}$ forces $2^{\aleph_0} = \mu$ and $c^\exists_\epsilon = c^\forall_\epsilon = \kappa_\epsilon$ for all $\epsilon \in \mu$.

This will be done in Lemmas 2.7, 2.3 and 2.12.

Lemma 2.7. *$Q_{\mathfrak{p}}$ forces $2^{\aleph_0} = \mu$.*

Proof. First note that trivially all ν_i are different: Fix $p \in Q_{\mathfrak{p}}$ and $i \neq j$ in I^*. We already know that $Q_{\mathfrak{p}}$ forces that the domain of the generic is $\omega \times I^*$, in particular we can assume that $i, j \in \mathrm{dom}(p)$. Choose n so that $\mathrm{nor}(p, n) > 1$. In particular, all the atomic creatures involved have norm bigger than 1 and therefore more than one possible value. So we can choose an $\eta \in \mathrm{poss}(p, n+1)$ such that $\eta(n, i) \neq \eta(n, j)$. Then $p \wedge \eta$ forces $\nu_i \neq \nu_j$.

This shows that the continuum has size at least μ in the extension.

Due to continuous reading of names, every real r in the extension corresponds to a condition p in $Q_{\mathfrak{p}}$ together with a continuous way to read r off p.

More formally: For each $n \in \omega$ there are $h(n) \in \omega$ and a function $\mathrm{eval}(n) : \mathrm{poss}(p, h(n)) \to \omega$ such that $p \wedge \eta$ forces $\underset{\sim}{r}(n) = \mathrm{eval}(n)(\eta)$ for all $\eta \in \mathrm{poss}(p, h(n))$.

Since there are only $\mu^{\aleph_0} = \mu$ many such pairs of conditions and continuous readings, there can be at most μ many reals in the extension. □

We now mention a simple but useful property of the atomic creatures:

Lemma 2.8. *Assume w_0 and w_1 are two atomic creatures that appear in some n-ml-creature $\mathfrak{c}$. Then there are $v_i \in \Sigma_+^{2/B^{\min}(n)}(w_i)$ (for $i \in \{0,1\}$) such that* $\mathrm{val}(v_0) \cap \mathrm{val}(v_1) = \emptyset$.

Proof. Apply decisiveness to get successors w^s of w_0 and w^b of w_1 (or the other way round) such that the norms decrease by at most $1/B^{\min}(n)$ and $|\mathrm{val}(w^s)| < K$ and w^b is hereditarily $K+1$-big for some $K \in \omega$.

[In more detail: Since w_0 is decisive, there is a natural number K such that there is a K-small successor w_0^s as well as a K-big successor w_0^b of w_0. On the other hand, again using decisiveness, w_1 has a successor w_1' that is either K-small (then we set $w^s = w_1'$ and $w^b = w_0^b$) or K-big (then we set $w^b = w_1'$ and $w^s = w_0^s$).]

Enumerate $\mathrm{val}(w^s)$ as $\{x_0, \dots, x_{K-1}\}$, and define G from $\mathrm{val}(w^b)$ to $K+1$ as follows: If $l \in \mathrm{val}(w^b)$ is equal to x_k (for some $k \in K$), then set $G(l) = k+1$. Otherwise, set $G(l) = 0$.

Using $K+1$-bigness, we get a G-homogeneous successor v of w^b, i.e., $G \restriction \mathrm{val}(v)$ is constant m for some $m \in K+1$. Of course m has to be 0. (Otherwise $\mathrm{val}(v) = \{x_{m+1}\}$ is a singleton and therefore $\mathrm{nor}(v) = 0$.) Therefore $\mathrm{val}(v) \cap \mathrm{val}(w^s) = \emptyset$, so v and w^s are the required successors w_0 and w_1. □

A simple application of this Lemma gives us "separated support":

Lemma 2.9. *For $p \in Q_{\mathrm{p}}$ there is a $q \leq p$ such that $q(n) \in \Sigma_+^1(p(n))$ for all $n \geq \mathrm{trnklg}(q)$ and $A_{\epsilon_0}^{q(n)} \cap A_{\epsilon_1}^{q(n)} = \emptyset$ for all n and $\epsilon_0 \neq \epsilon_1$ in $supp(q,n) \cap \mu$.*

Proof. Fix n and a pair $\epsilon_0 \neq \epsilon_1$ in $\mathrm{supp}(p,n) \cap \mu$. According to Lemma 2.8, we can find $v_{\epsilon_i} \in \Sigma_+^{2/B(n)}(w_{\epsilon_i}^{p(n)})$ for $i \in \{0,1\}$ with disjoint values. Iterate this for all pairs in $\mathrm{supp}(p,n) \cap \mu$ (note that there are less than $\mathrm{maxsupp}(n)^2 < B^{\min}(n)/2$ many, according to 2.2(5)). □

Lemma 2.10. *Fix $\epsilon_0 \in \mu$. Then Q_{p} forces that $c_{\epsilon_0}^{\forall} \leq \kappa_{\epsilon_0}$.*

Proof. Set $I' = \{\epsilon_0\} \cup I_{\epsilon_0}$. We will show that in the Q_{p} extension of V the family of those $(f_{\epsilon_0}, g_{\epsilon_0})$-slaloms that can (in V) be read continuously from I' alone form a $\forall$-cover. This proves the Lemma, since there are only $\kappa_{\epsilon_0}^{\aleph_0} = \kappa_{\epsilon_0}$ many continuous readings on I'.

Assume that r is a name for an element of $\prod f_{\epsilon_0}$. Fix $p \in Q_{\mathrm{p}}$. Using Corollary 2.5, without loss of generality we can assume that p rapidly reads r (i.e., $r \restriction n$ is n-decided by p) and that it satisfies separated support as in the previous Lemma.

We will construct a $q \leq p$ and a name for an $(f_{\epsilon_0}, g_{\epsilon_0})$-slalom Y that can be continuously read from $q \restriction I'$ such that q forces $r(n) \in Y(n)$ for all but finitely many $n \in \omega$. (This proves the Lemma.)

Fix n_0 such that $\mathrm{nor}(p,n) > 2$ for all $n \geq n_0$ and set $q(n) = p(n)$ for $n < n_0$. We construct $Y(n)$ and $q(n)$ by induction on $n \geq n_0$. We set $\mathrm{supp}(q,n) := \mathrm{supp}(p,n)$ and $\mathrm{trunk}(q) := \mathrm{trunk}(p)$. I.e., the supports and trunks do not change at all. So by induction $\mathrm{poss}(q,n) \subseteq \mathrm{poss}(p,n)$.

Let us denote the n-ml-creature $p(n)$ by $\mathfrak{c}$. We have to define the n-ml-creature $q(n)$ (let us call it $\mathfrak{d}$) with $u^{\mathfrak{d}} = u^{\mathfrak{c}}$ (call it u). We set $d^{\mathfrak{d}} := d^{\mathfrak{c}}$. On μ, we do not change anything: For $\epsilon \in u \cap \mu$ we set $w_\epsilon^{\mathfrak{d}} := w_\epsilon^{\mathfrak{c}}$ (call it w_ϵ, and set $A_\epsilon := \mathrm{val}(w_\epsilon) = A_\epsilon^{\mathfrak{c}} = A_\epsilon^{\mathfrak{d}}$). It remains to define $w_{\alpha,k}^{\mathfrak{d}} \in \Sigma_+^1(w_{\alpha,k}^{\mathfrak{c}})$ for $\alpha \in u \cap I_\epsilon$ and $k \in w_\epsilon$. Then, since the norms of all the atomic creatures only decrease by 1, we know that $\mathrm{nor}(\mathfrak{d})$ will definitely be bigger than $\mathrm{nor}(\mathfrak{c}) - 1$, as required.

Let T (for "trunk") be the set of pairs (η, x) such that $\eta \in \mathrm{poss}(q,n)$ and $x \in \prod_{\epsilon \in u \cap \mu} A_\epsilon$.

$$|T| \leq g^{\min}(n). \tag{2.7}$$

We now partition $\mathrm{supp}(\mathfrak{c}) \setminus \mu$ into sets called S, M, L (small, medium, large): Set $M = \mathrm{supp}(\mathfrak{c}) \cap I_{\epsilon_0}$. Using separated support, we know that every $\epsilon \neq \epsilon_0$ in $u \cap \mu$ satisfies either $x(\epsilon) < x(\epsilon_0)$ (then we put all elements of $I_\epsilon \cap u$ into S) or $x(\epsilon) > x(\epsilon_0)$ (then we put them into L).

Rapid reading implies that (modulo the pair (η, x)) the natural number $r(n)$ can be interpreted as a function

$$r(n) : \prod_S \times \prod_M \times \prod_L \;\rightarrow\; f_{n,x(\epsilon_0)}.$$

where we set (for $X \in \{S, M, L\}$)

$$\prod_X := \prod_{\alpha \in X} A_{\alpha, x(\varepsilon(\alpha))}.$$

Our goal is to get a name $Y(n)$ for a small subset of $f_{n,x(\epsilon_0)}$ that only depends on M and and contains $r(n)$.

First note that we can rewrite $r(n)$ as a function

$$r(n) : \prod_L \to f^{\Pi_S \times \Pi_M}_{n,x(\epsilon_0)}.$$

Using the fact that the atomic creatures in L are nice enough,[14] we can find successors of these creatures that evaluate $r(n)$ to a constant value, and such that the norms decrease by less than $1/g^{\min}(n)$. We define $w'_{\alpha,x(\epsilon)}$ to be these successors for $\epsilon \in L$; and leave the other atomic creatures unchanged. Now for every $y \in \Pi_M$ there are only $|\Pi_S|$ many possible values for $r(n)$, call this sets of possible values $Y(\eta, x, y)$.

Iterate this procedure for all pairs $(\eta, x) \in T$. The same atomic creature may be decreased more than once, but at most $g^{\min}(n)$ many times, according to (2.7). So in the end, the norms of the resulting atomic creatures decrease by less than 1. This finishes the definition of $q(n)$.

We still have to define $Y(n)$ as a function from the possible values (k_0, y_0) on $\{\epsilon_0\} \cup I_{\epsilon_0}$, i.e., as a function with domain $\{(k_0, y_0) : k_0 \in A_{\epsilon_0}, y_0 \in \prod_{\alpha \in I_{\epsilon_0}} A^{\mathfrak{d}}_{\alpha,k_0}\}$. We set $Y(n)$ to be $\bigcup_{(\eta,x)\in T,\ x(\epsilon_0)=k_0} \tilde{Y}(\eta, x, y_0)$. This set has size less than g_{n,k_0}, as required.[15] $\square$

Lemma 2.11. *Fix $|J| \leq \text{maxsupp}(n)$ and for each $i \in J$ an atomic creature w_i that is $(\text{maxsupp}(n), 1/g^{min}(n))$-decisive. Then there are $w'_i \in \mathbf{\Sigma}^{1/k^*(n)}_+(w_i)$ for all $i \in J$ and a linear order $\leq_J$ on J such that each w'_i is hereditarily $\prod_{j<_J i} |\text{val}(w_i)|$ big.*

Proof. For any $i \in J$, apply decisiveness to the atomic creature w_i. This gives some K_i and a K_i-big as well as a K_i-small successor of w_i. Pick the i with a minimal K_i, let this i be the first element of the $<_J$-order, set w'_i to be the K_i-small successor, and pick for all other j the K_j-big successor. Repeat this construction for $J \setminus \{i\}$.

So in the end we order the whole set J, decreasing each creature at most maxsupp(n) many times by at most $1/g^{\min}(n)$. $\square$

It remains to be shown:

Lemma 2.12. *$Q_{\mathfrak{p}}$ forces that $c^{\exists}_{\epsilon_0} \geq \kappa_{\epsilon_0}$.*

Proof. Note that it is forced that $f_{\epsilon_0}(n)/g_{\epsilon_0}(n)$ converges to infinity, therefore (by the usual diagonalization) it is forced that $c^{\exists}_{\epsilon_0} > \aleph_0$. So if $\kappa_{\epsilon_0} = \aleph_1$ there is nothing to do.

So assume that $\aleph_1 \leq \lambda < \kappa_{\epsilon_0}$ and assume towards a contradiction that some p_0 forces $\{Y_\zeta : \zeta \in \lambda\}$ is an $\exists$-cover.

For each $\zeta \in \lambda$ we can find a maximal antichain A_ζ below p_0 such that every condition in A_ζ rapidly reads Y_ζ. Let D be the union of the domains of all elements of any of the A_ζ for $\zeta \in \lambda$. Due to $\aleph_2$-cc, D has size $|\aleph_0 \times \aleph_1 \times \lambda| = \lambda$ which is less than κ_{ϵ_0}. So we can pick a $\beta \in I_{\epsilon_0} \setminus D$ and a $p_1 \leq p_0$ deciding the Y_ζ that $\exists$-covers ν_β. From now an, we will call Y_ζ just Y. Pick some $p \leq p_1$ that is stronger than some element of A_ζ. To summarize:

p restricted to $\text{dom}(p) \setminus \{\beta\}$ rapidly reads Y. (I.e., Y does not depend on the values at β.)

p forces that $Y(n)$ is a subset of $f_{\epsilon_0}(n)$ of size less than $g_{\epsilon_0}(n)$ for all n,

p forces that there are infinitely many n such that $\nu_\beta(n) \in Y(n)$

We will now derive the desired contradiction: We will find an $n_0 \in \omega$ and a $q \leq p$ forcing that $\nu_\beta(n) \notin Y(n)$ for all $n \geq n_0$.

Pick n_0 such that $\text{nor}(p, n) > 2$ for all $n \geq n_0$. We will construct $q(n) =: \mathfrak{d}$ by induction on $n \geq n_0$. Denote $p(n)$ by $\mathfrak{c}$. We set $\text{trunk}(q) := \text{trunk}(p)$ and $u^{\mathfrak{d}} := u^{\mathfrak{c}}$ (call it u), so the supports and the trunks do not change at all, and by induction $\text{poss}(q, n) \subseteq \text{poss}(p, n)$. We also set $d^{\mathfrak{d}} := d^{\mathfrak{c}}$. On μ, nothing changes: For $\epsilon \in u \cap \mu$ set $w^{\mathfrak{d}}_\epsilon := w^{\mathfrak{c}}_\epsilon$ (call it w_ϵ, and set $A_\epsilon = \text{val}(w_\epsilon) = A^{\mathfrak{c}}_\epsilon = A^{\mathfrak{d}}_\epsilon$).

It remains to construct $w^{\mathfrak{d}}_{\alpha,k} \in \mathbf{\Sigma}^1_+(w^{\mathfrak{c}}_{\alpha,k})$ for $\alpha \in u \cap I_\epsilon$ and $k \in A_\epsilon$.

Let T (for "trunk") consist of all pairs (η, x) such that $\eta \in \text{poss}(q, n)$ and $\eta \in \prod_{\epsilon \in u \cap \mu} A_\epsilon$. Note that $|T|$ is smaller than $g^{\min}(n)$, as already stated in (2.7).

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[14] they all satisfy $g_{n,x(\epsilon_0)+1}$ niceness, and in 2.2(9) we assumed that $g_{n,x(\epsilon_0)+1}$ is bigger than $f^{\Pi_S \times \Pi_M}_{n,x(\epsilon_0)}$, since $\prod S \times \prod M$ has size less than $f^{k^*(n)}_{n,x(\epsilon_0)}$. Now use Fact 2.1(2). So the norms decrease at most by $1/g_{n,x(\epsilon_0)+1} < 1/g^{\min}(n)$.

[15] $|Y(\eta, x, y)| \leq |\Pi_S|$, so $|Y(n)| \leq |T \times \Pi_S| \leq \text{maxposs}(n) \cdot k^*(n)^{\text{maxsupp}(n)} \cdot f^{\text{maxsupp}(n)}_{n,k_0-1}$, which is smaller than g_{n,k_0} according to 2.2(9).

Given (η, x) in T, we apply the previous Lemma to $J := u \setminus \mu$ and the sequence $(w^{c}_{\alpha,x(\varepsilon(\alpha))})_{\alpha\in J}$. This gives us successor creatures $(w'_\alpha)_{\alpha\in J}$ as well as an order $<_J$ of J. Partition J into $S = \{i <_J \beta\}$, $\{\beta\}$, and $L = \{i >_J \beta\}$.

So (given η and x), we can write $Y(n)$ (which does not depend on β) as a function from $\prod_{\alpha\in L} \mathrm{val}(w'_\alpha) \times \prod_{\alpha\in S} \mathrm{val}(w'_\alpha)$ to the family of subsets of $f_{n,x(\epsilon_0)}$ of size less than $g_{n,x(\epsilon_0)}$. Therefore we can use bigness to once more strenghen the atomic creatures indexed by L and thus remove the dependence of $Y(n)$ from L. We now take the union $\tilde{Y}$ of all the remaining possibilities for $Y(n)$ and get a set of size less than $g_{n,x(\epsilon_0)} \cdot |\prod_{\alpha\in S} \mathrm{val}(w'_\alpha)|$, which is smaller than the bigness of w'_β. So (just as in the proof of Lemma 2.8) we can strengthen this creature w'_β to be disjoint to $\tilde{Y}$.

As usual, we now iterate this construction for all pairs $(\eta, x) \in T$. The resulting n-ml-creature $q(n)$ guarantees that $\nu_\beta(n)$ is not in $Y(n)$, as required. $\square$

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