A DEPENDENT THEORY WITH FEW INDISCERNIBLES

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Abstract. We give a full solution to the question of existence of indiscernibles in dependent theories by proving the following theorem: for every θ there is a dependent theory T of size θ such that for all κ and δ, $\kappa \to (\delta)_{T,1}$ iff $\kappa \to (\delta)_{\theta}^{<\omega}$. This means that unless there are good set theoretical reasons, there are large sets with no indiscernible sequences.

1. Introduction

Indiscernible sequences play a very important role in model theory. Let us recall the definition.

Definition 1.1. Suppose M is some structure, $A \subseteq M$, $(I,<)$ is some linearly ordered set, and α some ordinal. A sequence $\bar{a} = \langle a_i \,|\, i \in I \rangle \in (M^\alpha)^I$ is called *indiscernible over* A if for all $n < \omega$, every increasing n-tuple from $\bar{a}$ realizes the same type over A. When A is omitted, it is understood that $A = \emptyset$.

A very important fact about indiscernible sequences is that they exist in the following sense:

Fact 1.2. [TZ12, Lemma 5.1.3] *Let $(I,<_I)$, $(J,<_J)$ be infinite linearly ordered sets, α some ordinal, M a structure and $A \subseteq M$. Suppose $\bar{b} = \langle b_j \,|\, j \in J \rangle$ is some sequence of tuples from M^α. Then there exists an indiscernible sequence $\bar{a} = \langle a_i \,|\, i \in I \rangle$ of tuples of length α in some elementary extension N of M such that:*

- $\star$ *For any $n < \omega$ and formula φ, if $M \models \varphi\left(b_{j_0}, \ldots, b_{j_{n-1}}\right)$ for every $j_0 <_J \cdots <_J j_{n-1}$ from J then $N \models \varphi\left(a_{i_0}, \ldots, a_{i_{n-1}}\right)$ for every $i_0 <_I \cdots <_I i_{n-1}$ from I.*

This is proved using Ramsey and compactness.

Sometimes, however, we want a stronger result. For instance we may want that given any set of elements, there is an indiscernible sequence in it. This gives rise to the following definition:

Definition 1.3. Let T be a complete first order theory, and let $\mathfrak{C}$ be a monster model of T (i.e., a very big saturated model). For a cardinal κ, $n \leq \omega$ and an ordinal δ, the notation $\kappa \to (\delta)_{T,n}$ means:

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- $\star$ For every set $A \subseteq \mathfrak{C}^n$ of size κ, there is a non-constant sequence of elements of A of length δ which is indiscernible.

This definition was suggested by Grossberg and Shelah in [She86, pg. 208, Definition 3.1(2)] with a slightly different form[1].

As we remarked above, the mere existence of indiscernibles as in Fact 1.2 follows from Ramsey. It is therefore no surprise that if a cardinal λ enjoys a Ramsey-like property, then for any countable theory T we would have $\lambda \to (\omega)_{T,n}$.

For a cardinal κ, denote by $[\kappa]^{<\omega}$ the set of all increasing finite sequences of ordinals below κ.

Definition 1.4. For cardinals κ, θ and an ordinal δ, the notation $\kappa \to (\delta)_\theta^{<\omega}$ means:

- $\star$ For every function $f : [\kappa]^{<\omega} \to \theta$ there is a homogeneous sub-sequence of order-type δ (i.e., there exists an increasing sequence $\langle \alpha_i \,|\, i < \delta \rangle \in {}^\delta\kappa$ and $\langle c_n \,|\, n < \omega \rangle \in {}^\omega\theta$ such that $f\left(\alpha_{i_0}, \ldots, \alpha_{i_{n-1}}\right) = c_n$ for every $i_0 < \cdots < i_{n-1} < \delta$).

Proposition 1.5. *Let κ, θ be cardinals and $\delta \geq \omega$ a limit ordinal. If $\kappa \to (\delta)_\theta^{<\omega}$ then for every $n \leq \omega$ and every theory T of cardinality $|T| \leq \theta$, $\kappa \to (\delta)_{T,n}$.*

This will be proved below, see Proposition 5.1.

Definition 1.6. For an ordinal α, the Erdös cardinal $\kappa(\alpha)$ is the least non-zero cardinal λ such that $\lambda \to (\alpha)_2^{<\omega}$.

The cardinal $\kappa(\alpha)$ may not always exist, indeed, it depends on the model of ZFC we are in.

Fact 1.7. [Kan09, Proposition 7.15] *Suppose $\alpha \geq \omega$ is a limit ordinal, then:*

(1) For any $\gamma < \kappa(\alpha)$, $\kappa(\alpha) \to (\alpha)_\gamma^{<\omega}$.
(2) $\kappa(\alpha)$ is an uncountable strongly inaccessible cardinal.

In [She86, pg. 209] it is proved that there is a countable simple unstable theory such that for a limit ordinal $\delta \geq \omega$, if $\kappa \to (\delta)_{T,1}$ then $\kappa \to (\delta)_2^{<\omega}$. It is also very easy to find such a theory with the property that if $\kappa \to (\delta)_{T,1}$ then $\kappa \to (\delta)_\omega^{<\omega}$ (T would be the model completion of the empty theory in the language $\{R_{n,m} \,|\, n, m < \omega\}$ where $R_{n,m}$ is an n-ary relation).

There it is conjectured that in dependent (NIP) theories (see Definition 2.1 below), such phenomenon cannot happen:

Conjecture 1.8. [She86, pg. 209, Conjecture 3.3] *If T is dependent, then for every cardinal μ there is some cardinal λ such that $\lambda \to (\mu)_{T,1}$.*

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[1] The definition there is: $\kappa \to (\delta)_{T,n}$ if and only if for each sequence of length κ (of n-tuples), there is an indiscernible sub-sequence of length δ. For us there is no difference because we are dealing with examples where $\kappa \not\to (\mu)_{T,n}$. It is also not hard to see that when δ is an infinite cardinal these two definitions are equivalent.

By Proposition 1.5, if $\kappa(\mu)$ exists then Conjecture 1.8 holds for μ and every theory T (regardless of NIP) with $|T| < \kappa(\mu)$.

In stable theories, Conjecture 1.8 holds in any model of ZFC:

Fact 1.9. *For any λ satisfying $\lambda = \lambda^{|T|}$, $\lambda^+ \to (\lambda^+)_{T,n}$.*

This was proved by Shelah (See [She90]), and follows from local character of non-forking.

Conjecture 1.8 also holds in strongly dependent[2] theories:

Fact 1.10. [She12] *If T is strongly dependent, then for all $\lambda \geq |T|$, $\beth_{|T|^+}(\lambda) \to (\lambda^+)_{T,n}$ for all $n < \omega$.*

Conjecture 1.8 is connected to a result by Shelah and Cohen: in [CS09], they proved that a theory is stable iff it can be presented in some sense in a free algebra with a fixed vocabulary, allowing function symbols with infinite arity. If this result could be extended to saying that a theory is dependent iff it can be represented as an algebra with ordering, then this could be used to prove Conjecture 1.8.

In the previous paper [KS12, Theorem 2.11], we have shown that:

Theorem 1.11. *There exists a countable dependent theory T such that:*

For any two cardinals $\mu \leq \kappa$ with no uncountable strongly inaccessible cardinals in $[\mu, \kappa]$, $\kappa \not\to (\mu)_{T,1}$.

Thus, if V is a model of ZFC without strongly inaccessible cardinals, then Conjecture 1.8 fails in V (so this conjecture is false in general). Still, one might hope that this is the only restriction. However, we show that in fact one needs Erdös cardinals to exist. Namely, we show that there is a dependent theory, of any given cardinality, such that the only reason for which Conjecture 1.8 could hold for it is Proposition 1.5, thus getting the best possible result.

Main Theorem A. *For every θ there is a dependent theory T of size θ such that for all cardinals κ and limit ordinals $\delta \geq \omega$, $\kappa \to (\delta)_{T,1}$ iff $\kappa \to (\delta)_{\theta}^{<\omega}$.*

Note that by Fact 1.7, Main Theorem A is a generalization of Theorem 1.11.

We also should note that a related result can be found in an unpublished paper in Russian by Kudajbergenov that states that for every ordinal α there exists a dependent theory (but it may be even strongly dependent) T_α such that $|T_\alpha| = |\alpha| + \aleph_0$ and $\beth_\alpha(|T_\alpha|) \not\to (\aleph_0)_{T_\alpha,1}$ and thus seem to indicate that the bound in Fact 1.10 is tight.

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[2]For more on strongly dependent theories, see Section 6.

1.1. **The idea of the proof.** The theory T is a "tree of trees" with functions between the trees. More precisely, for all η in the base tree $\mathbb{S} = 2^{<\omega}$ we have a unary predicate P_η and an ordering $<_\eta$ such that $(P_\eta, <_\eta)$ is a discrete tree. In addition we will have functions $G_{\eta,\eta\hat{}\{i\}} : P_\eta \to P_{\eta\hat{}\langle i\rangle}$ for $i = 0, 1$. The idea is to prove that if $\kappa \not\to (\delta)^{<\omega}_\theta$ then $\kappa \not\to (\delta)_{T,1}$ by induction on κ, i.e., to prove that we can find a subset of $P_{\langle\rangle}$ of size κ without an indiscernible sequence in it. For κ regular but not strongly inaccessible or κ singular the proof is similar to the one in [KS12]: we just push our previous examples into deeper levels.

The main case is when κ is strongly inaccessible.

We have a function $\mathbf{c}$ that witnesses that $\kappa \not\to (\delta)^{<\omega}_\theta$ and we build a model $M_{\mathbf{c}}$. In this model, the base tree will be ω and not $2^{<\omega}$, i.e., for each $n < \omega$ we have a predicate P_n with tree-ordering $<_n$ and functions $G_n : P_n \to P_{n+1}$. In addition, $P_0 \subseteq \kappa$. On P_n we will define an equivalence relation E_n refining the neighboring relation (x, y are neighbors if they succeed the same element) so that every class of neighbors (neighborhood) is a disjoint union of less than κ many classes of E_n. We will prove that if there are indiscernibles in P_0, then there is some $n < \omega$ such that in P_n we get an indiscernible sequence $\langle t_i \,|\, i < \delta\rangle$ that looks like a fan, i.e., there is some u such that $t_i \wedge t_j = u$ and t_i is the successor of u, and in addition t_i and t_j are not E_n equivalent for $i \neq j$.

Now embed $M_{\mathbf{c}}$ into a model of our theory (i.e., now the base tree is again $2^{<\omega}$), and in each neighborhood we send every E_n class to an element from the model we get from the induction hypothesis (as there are less than κ many classes, this is possible).

By induction, we get there is no indiscernible sequence in P_0 and finish.

1.2. **Description of the paper.** In Section 2 we give some preliminaries on dependent and strongly dependent theories and trees. In Section 3 we describe the theory and prove quantifier elimination and dependence. In Section 4 we deal with the main technical obstacle, namely the inaccessible case.

In Section 5 we prove the main theorem. In Section 6 we give a parallel result for ω-tuples in strongly dependent theories.

2. Preliminaries

Notation. We use standard notation. a, b, c are elements, and $\bar{a}, \bar{b}, \bar{c}$ are finite or infinite tuples of elements.

$\mathfrak{C}$ will be the monster model of the theory (i.e., a very big, saturated model).

For a set $A \subseteq \mathfrak{C}$, $S_n(A)$ is the set of complete n-types over A, and $S^{\mathrm{qf}}_n(A)$ is the set of all quantifier free complete n-types over A. For a finite set of formulas with a partition of variables, $\Delta(\bar{x};\bar{y})$, $S_{\Delta(\bar{x};\bar{y})}(A)$ is the set of all Δ-types over A, i.e., maximal consistent subsets of $\left\{\varphi(\bar{x},\bar{a}), \neg\varphi(\bar{x},\bar{a}) \,\middle|\, \varphi(\bar{x},\bar{y}) \in \Delta \,\&\, \bar{a} \in A^{\lg(\bar{y})}\right\}$. Similarly we define $\mathrm{tp}_{\Delta(\bar{x};\bar{y})}\left(\bar{b}/A\right)$ as the set of formulas $\varphi(\bar{x},\bar{a})$ such that $\varphi(\bar{x},\bar{y}) \in \Delta$ and $\mathfrak{C} \models \varphi\left(\bar{b},\bar{a}\right)$.

When α and β are ordinals, we use left exponentiation ${}^{\beta}\alpha$ to denote the set of functions from β to α, as to not to confuse with ordinal (or cardinal) exponentiation. If there is no room for confusion, and A and B are some sets we use A^B instead. The set $\alpha^{<\beta}$ is the set of sequences (functions) $\bigcup\{{}^{\gamma}\alpha \mid \gamma<\beta\}$. Similarly, for a set A, $A^{<\beta}=\bigcup\{A^{\gamma} \mid \gamma<\beta\}$.

For a sequence $\bar{s}$ (finite or infinite), we denote by $\lg(\bar{s})$ its length. If f is a function from some ordinal α, then $\lg(f)=\alpha$.

Dependent theories. For completeness, we give here the definitions and basic facts we need on dependent theories.

Definition 2.1. A first order theory T is *dependent* (sometimes also NIP) if it does not have the independence property: there is no formula $\varphi(\bar{x},\bar{y})$ and tuples $\left\langle \bar{a}_i,\bar{b}_s \mid i<\omega, s\subseteq\omega\right\rangle$ from $\mathfrak{C}$ such that $\mathfrak{C}\models\varphi\left(\bar{a}_i,\bar{b}_s\right)$ iff $i\in s$.

We recall the following fact, which is a consequence of both the so-called Sauer-Shelah lemma (apparently first proved by Vapnik and Chervonenkis, then rediscovered by Sauer and again by Shelah in the model theoretic setting, more or less at the same time) and the fact that if a theory has the independence property then there is a formula $\varphi(x,\bar{y})$ with $\lg(x)=1$ that witnesses this:

Fact 2.2. [She90, II, 4] *Let T be any theory. Then for all $n<\omega$, T is dependent if and only if $\square_n$ if and only if $\square_1$ where for all $n<\omega$,*

> $\square_n$ *For every finite set of formulas $\Delta(\bar{x},\bar{y})$ with $n=\lg(\bar{x})$, there is a polynomial f over $\mathbb{N}$ such that for every finite set $A\subseteq M\models T$, $|S_{\Delta}(A)|\leq f(|A|)$.*

Strongly dependent theories. In [She12, She09], the author asks what is a possible solution to the equation dependent / $x=$ stable / superstable. There, he discusses several possible strengthening of NIP, namely strongly^l dependent theories for $l=1,2,3,4$. These are subclasses of dependent theories and each one refines the previous one. Strongly^1 dependent theories are usually just called strongly dependent, and strongly^2 theories are sometimes called strongly$^+$ theories. These two classes and related notions (such as dp-rank) were studied much more than the other two, so we will not mention strongly^3 or strongly^4 dependent theories. For instance, strongly^2 dependent groups are discussed in [KS11]. The theories of the reals and the p-adics are both strongly dependent, but neither is strongly^2 dependent.

Here are the definitions:

Definition 2.3. A theory T is said to be <u>not</u> *strongly dependent* if there exists a sequence of formulas $\langle\varphi_i(\bar{x},\bar{y}_i)\rangle$ (where $\bar{x},\bar{y}_i$ are tuples of variables), an array $\langle\bar{a}_{i,j} \mid i,j<\omega\rangle$ in $\mathfrak{C}$ (where $\lg(\bar{a}_{i,j})=\lg(\bar{y}_i)$) and tuples $\left\langle\bar{b}_{\eta} \mid \eta:\omega\to\omega\right\rangle$ ($\lg\left(\bar{b}_{\eta}\right)=\lg(\bar{x})$) in $\mathfrak{C}$ such that $\models\varphi_i\left(\bar{b}_{\eta},\bar{a}_{i,j}\right)\Leftrightarrow\eta(i)=j$.

Definition 2.4. A theory T is said to be <u>not</u> *strongly*^{2} *dependent* if there exists a sequence of formulas $\langle \varphi_i(\bar{x}, \bar{y}_i, \bar{y}_{i-1}, \ldots, \bar{y}_0) \mid i < \omega \rangle$, an array $\langle \bar{a}_{i,j} \mid i, j < \omega \rangle$ in $\mathfrak{C}$ (where $\lg(\bar{a}_{i,j}) = \lg(\bar{y}_i)$) and tuples $\langle \bar{b}_\eta \mid \eta : \omega \to \omega \rangle$ ($\lg(\bar{b}_\eta) = \lg(\bar{x})$) in $\mathfrak{C}$ such that $\models \varphi_i(\bar{b}_\eta, \bar{a}_{i,j}, \bar{a}_{i-1,\eta(i-1)}, \ldots, \bar{a}_{0,\eta(0)}) \Leftrightarrow \eta(i) = j$.

See [She12, Claim 2.9] for more details.

We will use the following criterion:

Lemma 2.5. *Suppose T is a theory such that for every number $n < \omega$ there exists some number $N_n < \omega$ such that for every finite set of formulas $\Delta(\bar{x}, \bar{y})$ with $n = \lg(\bar{x})$, there is a polynomial f over $\mathbb{N}$ of degree $\leq N_n$ such that for every finite set $A \subseteq M \models T$, $|S_\Delta(A)| \leq f(|A|)$. Then T is strongly^2 dependent.*

Proof. Suppose not, then by Definition 2.4, we have a sequence of formulas $\langle \varphi_i(\bar{x}, \bar{y}_i, \bar{y}_{i-1}, \ldots, \bar{y}_0) \mid i < \omega \rangle$ and an array $\langle \bar{a}_{i,j} \mid i, j < \omega \rangle$. Suppose $N = N_{\lg(\bar{x})} < K < \omega$. Let l be a bound on $\lg(\bar{a}_{i,j})$ for $i < K$, and for $j < \omega$ let $A_j = \bigcup \{ \bigcup \bar{a}_{i,j'} \mid i < K, j' < j \}$. Let $\Delta(\bar{x}; \bar{y}) = \{ \varphi_i(\bar{x}, \bar{y}_i, \ldots, \bar{y}_0) \mid i < K \}$. So the number of Δ-types over A_j is at least j^K (as the number of functions $\eta : K \to j$). By assumption, $|S_\Delta(A_j)| \leq c \cdot |A_j|^N \leq c \cdot (l \cdot j \cdot K)^N$ for some $c \in \mathbb{N}$. But for big enough j, $c \cdot (l \cdot j \cdot K)^N < j^K$ — contradiction. □

Trees. Let us remind the reader of the basic definitions and properties of trees.

Definition 2.6. A *tree* is a partially ordered set $(A, <)$ such that for all $a \in A$, the set $A_{<a} = \{ x \mid x < a \}$ is linearly ordered.

Definition 2.7. We say that a tree A is *well ordered* if $A_{<a}$ is well ordered for every $a \in A$. Assume now that A is well ordered.

- For every $a \in A$, denote $\operatorname{lev}(a) = \operatorname{otp}(A_{<a})$ — the *level* of a is the order type of $A_{<a}$.
- The *height* of A is $\sup \{ \operatorname{lev}(a) \mid a \in A \}$.
- $a \in A$ is a *root* if it is minimal.
- A is *normal* when for all limit ordinals δ, and for all $a, b \in A$, if 1) $\operatorname{lev}(a) = \operatorname{lev}(b) = \delta$, and 2) $A_{<a} = A_{<b}$, then $a = b$.
- If $a < b$ then we denote by $\operatorname{suc}(a, b)$ the *successor* of a in the direction of b, i.e., $\min \{ c \leq b \mid a < c \}$.
- We write $a <_{\text{suc}} b$ if $b = \operatorname{suc}(a, b)$.
- We call A *standard* if it is well ordered, normal, and has a root.

For a standard tree $(A, <)$, define $a \wedge b = \max \{ c \mid c \leq a \,\&\, c \leq b \}$.

3. Construction of the theory

In this section we shall introduce the theory T_S, attached to a standard tree S. Then, for $\mathbb{S} = 2^{<\omega}$, this theory (or a variant of it, given by adding constants) will be the theory that will exemplify Main Theorem A.

In the first part, we construct the theory $T_S^{\forall}$ which is universal (i.e., all its axioms are of the form $\forall x \varphi$ where φ is quantifier free). As we said in the introduction, the idea is that for every $\eta \in S$, we have a predicate P_η, and whenever $\eta_1 <_{\text{suc}} \eta_2$ there is a function from P_{η_2} to P_{η_1}. Then we would like to take a model completion T_S of this theory (see below). If we put no further restriction on the theory $T_S^{\forall}$, this is easily done (using AP and JEP, see below), and the model completion will be the theory of dense trees with functions (if S is a finite tree, then it is even ω-categorical). This is what we did in [KS12, Theorem 2.11], but this does not seem to suffice to deal with inaccessible cardinals. For that reason we further complicate the theory by making the trees discrete, adding successors and predecessors. This require some constraint on the functions involved — "regressiveness" — which is needed for quantifier elimination.

Recall that for a given (first order) theory T in a language L, a *model companion* of T is another theory T' in L such that every model of T can be embedded in a model of T' and vice versa and in addition T' is *model complete*, i.e., if M_1 is a substructure of M_2 and $M_1, M_2 \models T'$ then M_1 is an elementary substructure of M_2. A model companion of a theory is unique if it exists. A model companion T' is called a *model completion* when for every model M of T, $T' \cup \mathrm{Diag}_{\mathrm{qf}}(M)$ is complete ($\mathrm{Diag}_{\mathrm{qf}}(M)$ is the theory in the language $L \cup \{c_a \mid a \in M\}$ that contains all atomic formulas that hold in M). If T' is a model completion of T and T is universal, then T' eliminates quantifiers for non-sentences. If in addition T has JEP (see below) then T' is complete. For more, see e.g., [Hod93].

A theory T has the *joint embedding property (JEP)* if given any two models A, B of T, there is a model C and embeddings $f : A \to C$, $g : B \to C$.

A theory T has the *amalgamation property (AP)* if given any three models A, B and C of T, and embeddings $f : A \to B$, $g : A \to C$, there is a model D and embeddings $h : B \to D$, $i : C \to D$ such that $h \circ f = i \circ g$.

By [Hod93, Theorem 7.4.1], if a universal theory T in a finite language is uniformly locally finite (i.e., there is a function $f : \omega \to \omega$ such that for all $M \models T$ and finite $A \subseteq M$, the size of the structure generated by A is $f(|A|)$) and has AP and JEP, then it has a model completion T' which is also ω-categorical (this is related to Fraïssé limits). In [KS12, Theorem 2.11] we used exactly this criterion to construct the model completion. Here, however, substructures are not finite (since we have the successor function), so we cannot apply this theorem.

Instead, we show that the the class of existentially closed models of $T_S^{\forall}$ is elementary (recall that a model M of a theory T is an *existentially closed model of* T if for any extension $N \supseteq M$

such that $N \models T$, every quantifier free formula $\varphi(x)$ over M that has a realization in N has one in M). In fact we show that every two existentially closed models of $T_S^\forall$ are elementary equivalent (this uses the fact that $T_S^\forall$ has JEP). We call their theory T_S. In the process we show that T_S also eliminates quantifiers. Thus, this is the model completion of $T_S^\forall$.

In the second part, we show that T_S is dependent, and that if S is finite then it is strongly[2] dependent (using Lemma 2.5 and quantifier elimination).

Finally we add constants to the language so that its cardinality will be θ, and call the resulting theory T_S^θ.

The first order theory. The language:

Let S be a standard tree, and let L_S be the language:

$$\left\{P_\eta, <_\eta, \wedge_\eta, G_{\eta_1,\eta_2}, \mathrm{suc}_\eta, \mathrm{pre}_\eta, \lim_\eta \mid \eta, \eta_1, \eta_2 \in S, \eta_1 <_{\mathrm{suc}} \eta_2\right\}.$$

Where:

P_η is a unary predicate, $<_\eta$ is a binary relation symbol, $\wedge_\eta$ and suc_η are binary function symbols, G_{η_1,η_2}, pre_η and $\lim_\eta$ are unary function symbols.

Definition 3.1. Let $L'_S = L_S \setminus \left\{\mathrm{pre}_\eta, \mathrm{suc}_\eta \mid \eta \in S\right\}$.

The theory:

Definition 3.2. The theory $T_S^\forall$ says:

- $(P_\eta, <_\eta)$ is a tree.
- $\eta_1 \neq \eta_2 \Rightarrow P_{\eta_1} \cap P_{\eta_2} = \emptyset$.
- $\wedge_\eta$ is the meet function: $x \wedge_\eta y = \max\{z \in P_\eta \mid z \leq_\eta x \,\&\, z \leq_\eta y\}$ for $x, y \in P_\eta$ (so its existence is part of the theory).
- suc_η is the successor function — for $x, y \in P_\eta$ with $x <_\eta y$, $\mathrm{suc}_\eta(x, y)$ is the successor of x in the direction of y. The axioms are:
  - $\forall x <_\eta y \, (x <_\eta \mathrm{suc}_\eta(x, y) \leq_\eta y)$, and
  - $\forall x \leq_\eta z \leq_\eta \mathrm{suc}_\eta(x, y) \, [z = x \vee z = \mathrm{suc}_\eta(x, y)]$.
- $\lim_\eta(x)$ is the greatest limit element below x. Formally,
  - $\lim_\eta : P_\eta \to P_\eta$, $\forall x \lim_\eta(x) \leq_\eta x$, $\forall x <_\eta y \, (\lim_\eta(x) \leq_\eta \lim_\eta(y))$,
  - $\forall x <_\eta y \, (\lim_\eta(\mathrm{suc}_\eta(x, y)) = \lim_\eta(x))$, $\forall x \lim_\eta(\lim_\eta(x)) = \lim_\eta(x)$.
- Let the successor elements be those x's such that $\lim_\eta(x) <_\eta x$, and denote

$$\mathrm{Suc}(P_\eta) = \{x \in P_\eta \mid \lim_\eta(x) <_\eta x\}.$$

- pre_η is the immediate predecessor function from $\mathrm{Suc}(P_\eta)$ to P_η —
  $\forall x \neq \lim_\eta(x) \left(\mathrm{pre}_\eta(x) < x \wedge \mathrm{suc}_\eta\left(\mathrm{pre}_\eta(x), x\right) = x\right)$.

- (**regressiveness**) If $\eta_1 <_{\mathrm{suc}} \eta_2$ then G_{η_1,η_2} satisfies: $G_{\eta_1,\eta_2} : \mathrm{Suc}\left(P_{\eta_1}\right) \to P_{\eta_2}$ and if $x <_{\eta_1} y$, both x and y are successors, and $\lim_\eta (x) = \lim_\eta (y)$, then $G_{\eta_1,\eta_2}(x) = G_{\eta_1,\eta_2}(y)$.
- In all the axioms above, for elements or pairs outside of the domain of any of the functions $\wedge_\eta, \lim_\eta, G_{\eta_1,\eta_2}, \mathrm{suc}_\eta$ or pre_η, these functions are the identity on the leftmost coordinate, so for example if $(x,y) \notin P_\eta^2$, then $x \wedge_\eta y = x$.

Remark 3.3. We need the regressiveness axiom so that $T_S^\forall$ would have a model completion. Indeed, suppose $S = \{0,1\}$ and we remove this axiom, and suppose that T is a model completion of $T_S^\forall$. Then every model of T is an existentially closed model of $T_S^\forall$. Suppose $M \models T$ and $a <_0^M b \in \mathrm{Suc}\left(P_0^M\right)$. Then if b is greater than $\mathrm{suc}_0^M\left(\cdots\left(\mathrm{suc}_0^M(a,b)\right)\right)$ for every finite number of compositions then there is some $a <_0^M c <_0^M b$ in M such that $G_{0,1}^M(c) \neq G_{0,1}^M(b)$ (because there is an extension of M to a model of $T_S^\forall$ where such a c exists). So by compactness there is some n such that for every model $M \models T$ and every $a <_0^M b \in \mathrm{Suc}\left(P_0^M\right)$, if b is greater than n successors of a, then there is some c with $a <_0^M c <_0^M b$ and $G_{0,1}^M(c) \neq G_{0,1}^M(b)$. But there is a model M' of $T_S^\forall$ with some $a <_0^{M'} b$ such that b is the $(n+1)$'th successor of a and $G_{0,1}^{M'}$ is constant on the interval $(a,b]$. Since every model of $T_S^\forall$ can be extended to a model of T this is a contradiction.

Model completion. Here we will prove the existence of the model completion T_S of $T_S^\forall$.

Notation 3.4. If S_1, S_2 are standard trees, we shall treat them as structures in the language $\{<_{\mathrm{suc}}, <\}$, so when we write $S_1 \subseteq S_2$, we mean that S_1 is a substructure of S_2 in this language (which means that if b is the successor of a in S_1, it remains such in S_2).

When M is a model of T_S, we may write $<$, suc, etc. instead of suc_η, $<_\eta$ etc. or suc_η^M, $<_\eta^M$ etc. where M and η are clear from the context.

Remark 3.5. Let S be a standard tree. The following is not hard to see:

(1) $T_S^\forall$ is a universal theory.
(2) $T_S^\forall$ has the joint embedding property (JEP).
(3) If $S_1 \subseteq S_2$ then $T_{S_1}^\forall \subseteq T_{S_2}^\forall$ and moreover, if $M \models T_{S_2}^\forall$ is existentially closed, $M \restriction L_{S_1}$ is an existentially closed model of $T_{S_1}^\forall$.

We will need some technical closure operators.

Definition 3.6. Assume S is a <u>finite</u> standard tree.

(1) Suppose Σ is a finite set of <u>terms</u> from L_S. We define the following closure operators on terms:
   (a) $\mathrm{cl}_\wedge^S(\Sigma) = \Sigma \cup \bigcup\left\{\wedge_\eta\left(\Sigma^2\right) \mid \eta \in S\right\} = \Sigma \cup \{t_1 \wedge_\eta t_2 \mid t_1, t_2 \in \Sigma, \eta \in S\}$.
   (b) $\mathrm{cl}_G^S(\Sigma) = \Sigma \cup \bigcup\{G_{\eta_1,\eta_2}(\Sigma) \mid \eta_1 <_{\mathrm{suc}} \eta_2 \in S\}$.
   (c) $\mathrm{cl}_{\lim}^S(\Sigma) = \Sigma \cup \bigcup\{\lim_\eta(\Sigma) \mid \eta \in S\}$.

(d) $\mathrm{cl}^{0,S}(\Sigma)=\mathrm{cl}_G^S\left(\mathrm{cl}_{\lim}^S\left(\mathrm{cl}_\wedge^S\left(\cdots\left(\mathrm{cl}_G^S\left(\mathrm{cl}_{\lim}^S\left(\mathrm{cl}_\wedge^S(\Sigma)\right)\right)\right)\right)\right)\right)$ where the number of compositions is the length of the longest branch in S.

(e) $\mathrm{cl}_{\mathrm{suc}}^S(\Sigma)=\bigcup\left\{\mathrm{suc}_\eta\left(\Sigma^2\right)\cup\mathrm{pre}_\eta(\Sigma)\,|\,\eta\in S\right\}\cup\Sigma$.

(f) $\mathrm{cl}^S(\Sigma)=\mathrm{cl}^{0,S}\left(\mathrm{cl}_{\mathrm{suc}}^S(\Sigma)\right)$.

(2) Denote $\mathrm{cl}^{(0),S}=\mathrm{cl}^{0,S}$ and for a number $0<k<\omega$, $\mathrm{cl}^{(k),S}(\Sigma)=\mathrm{cl}^S\left(\mathrm{cl}^{(k-1),S}(S)\right)$.

(3) If $\bar{t}=\langle t_i\,|\,i<n\rangle$ is an n-tuple of terms then $\mathrm{cl}^S(\bar{t})$ is $\mathrm{cl}^S(\{t_i\,|\,i<n\})$, and similarly define the other closure operators for tuples of terms.

(4) For a model $M\models T_S^\forall$, and $\bar{a}\in M^{<\omega}$, define $\mathrm{cl}^{0,S}(\bar{a})=\left(\mathrm{cl}^{0,S}(\bar{x})\right)^M(\bar{a})$ where $\bar{x}$ is a sequence of variables in the length of $\bar{a}$. Similarly define $\mathrm{cl}_\wedge^S(\bar{a}),\mathrm{cl}_{\lim}^S(\bar{a}),\mathrm{cl}_G^S(\bar{a}),\mathrm{cl}_{\mathrm{suc}}^S(\bar{a})$ and $\mathrm{cl}^{(k),S}(\bar{a})$. For a set $A\subseteq M$, define $\mathrm{cl}^{0,S}(A)=\mathrm{cl}^{0,S}(\bar{a})$ where $\bar{a}$ is an enumeration of A, and similarly for the other closure operators.

We will usually omit the superscript S when it is clear from the context.

Claim 3.7. Assume S is a finite standard tree. For $A\subseteq M\models T_S^\forall$, $\mathrm{cl}^0(A)$ is closed under $\wedge_\eta$, $\lim_\eta$ and G_{η_1,η_2} for all η and $\eta_1<_{\mathrm{suc}}\eta_2$ in S. So it is the substructure generated by A in the language L'_S (recall that $L'_S=L_S\backslash\left\{\mathrm{pre}_\eta,\mathrm{suc}_\eta\,|\,\eta\in S\right\}$).

Proof (sketch). Note that $\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(A))$ is closed under $\lim_\eta$ and $\wedge_\eta$ for all $\eta\in S$.

For $n<\omega$, let $\mathrm{cl}^{0,(n)}(A)=\mathrm{cl}_G(\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(\cdots(\mathrm{cl}_G(\mathrm{cl}_{\lim}(\mathrm{cl}_\wedge(A)))))))$ where there are n compositions. For $\eta\in S$, let $r(\eta)=|\{\nu\in S\,|\,\nu\leq\eta\}|$, so $\mathrm{cl}^0=\mathrm{cl}^{0,(\max\{r(\eta)|\,\eta\in S\})}$.

Let $B\supseteq A$ be the closure of A in M under $\wedge_\eta$, $\lim_\eta$ and G_{η_1,η_2} for all η and $\eta_1<_{\mathrm{suc}}\eta_2$ in S. Then B is in fact $\mathrm{cl}^{0,(\omega)}(A)=\bigcup\left\{\mathrm{cl}^{0,(n)}(A)\,|\,n<\omega\right\}$. Now, by induction on $r(\eta)$ it is easy to see that $B\cap P_\eta=\mathrm{cl}^{0,(r(\eta))}(A)\cap P_\eta$. Hence $B=\mathrm{cl}^0(A)$. □

Claim 3.8. Assume S is a finite standard tree. For every $k<\omega$, there is a polynomial f_k^S such that for every finite subset A of a model M of $T_S^\forall$, $\left|\mathrm{cl}^{(k)}(A)\right|\leq f_k^S(|A|)$. Moreover, we can choose f_k^S so that it is linear (i.e., of degree 1).

Proof. The fact that f_k^S exists is trivial. For the moreover part, letting $U=\{\wedge,G,\lim,\mathrm{suc}\}$, it is enough to show that there are $\{d_\square\in\mathbb{N}\,|\,\square\in U\}$ such that for every finite A, $\square\in U$, $|\mathrm{cl}_\square(A)|\leq d_\square\cdot|A|$.

We can choose $d_{\lim}=2$ and $d_G=2^{|S|^2}$.

For $\square=\wedge$, note that for all $a\in M$, $\mathrm{cl}_\wedge(A\cup\{a\})=\mathrm{cl}_\wedge(A)\cup\{a,\max\{a\wedge_\eta b\,|\,b\in A\}\}$ where $a\in P_\eta$ (this follows from the fact that if $a\wedge b<b\wedge b'$ then $a\wedge b'=a\wedge b$). So by induction on $|A|$, $|\mathrm{cl}_\wedge(A)|\leq 2|A|$.

For $\square=\mathrm{suc}$, note that for $a\in M$ such that for no $b\in A$, $b\geq a$, $\mathrm{cl}_{\mathrm{suc}}(A\cup\{a\})\subseteq\mathrm{cl}_{\mathrm{suc}}(A)\cup\left\{a,\mathrm{pre}_\eta(a),\mathrm{suc}_\eta(a',a)\right\}$ where $a\in P_\eta$ and $a'=\max\{b\in A\,|\,b<_\eta a\}$ (it may be that this set is empty or that a is a limit element, so the closure may be smaller). Hence by induction on $|A|$, $|\mathrm{cl}_{\mathrm{suc}}(A)| \leq 3|A|$. □

Remark 3.9. Note that although the degree of f_k^S in Claim 3.8 is 1, the coefficients do depend on k and S.

Definition 3.10. Assume S is a finite standard tree.

(1) For a term t of L_S, we define its *successor rank* as follows: if suc and pre do not appear in t, then $r_{\mathrm{suc}}(t) = 0$. For two terms t_1, t_2: $r_{\mathrm{suc}}(\mathrm{suc}_\eta(t_1, t_2)) = \max\{r_{\mathrm{suc}}(t_1), r_{\mathrm{suc}}(t_2)\} + 1$, $r_{\mathrm{suc}}(\mathrm{pre}_\eta(t_1)) = r_{\mathrm{suc}}(t_1) + 1$, $r_{\mathrm{suc}}(t_1 \wedge t_2) = \max\{r_{\mathrm{suc}}(t_1), r_{\mathrm{suc}}(t_2)\}$, $r_{\mathrm{suc}}(G_{\eta_1,\eta_2}(t_1)) = r_{\mathrm{suc}}(t_1)$ and $r_{\mathrm{suc}}(\lim_\eta(t_1)) = r_{\mathrm{suc}}(t_1)$.

(2) For a quantifier free formula φ in L_S, let $r_{\mathrm{suc}}(\varphi)$ be the maximal rank of a term appearing in φ.

(3) For $k < \omega$ and an n-tuple of variables $\bar{x}$, denote by $\Delta_k^{\bar{x},S}$ the set of all atomic formulas $\varphi(\bar{x})$ in L_S such that for every term t in φ, $t \in \mathrm{cl}^{(k)}(\bar{x})$. Note that since $\mathrm{cl}^{(k)}(\bar{x})$ is a finite set, so is $\Delta_k^{\bar{x},S}$.

Claim 3.11. Suppose S is a finite standard tree. Assume that $M \models T_S^\forall$, $n < \omega$, $\bar{a} \in M^n$ and $\bar{x}$ a tuple of n variables. Then $\mathrm{cl}^{(k)}(\bar{a}) = \{t^M(\bar{a}) \,|\, r_{\mathrm{suc}}(t(\bar{x})) \leq k\}$.

Proof. The inclusion $\subseteq$ is clear. The other direction follows by induction on k and t.

For instance, suppose $r_{\mathrm{suc}}(t(\bar{x})) = k$ and $t = G_{\eta_1,\eta_2}(t_1)$, then by induction there is some $t_2 \in \mathrm{cl}^{(k)}(\bar{x})$ such that $t_1^M(\bar{a}) = t_2^M(\bar{a})$. If $t_2(\bar{a}) \notin \mathrm{Suc}(P_{\eta_1}^M)$, then $t_2^M(\bar{a})$ is not in the domain of G_{η_1,η_2}^M and so $t^M(\bar{a}) = t_2^M(\bar{a})$. If $t_2^M(\bar{a}) \in \mathrm{Suc}(P_{\eta_1}^M)$, then by the proof of Claim 3.7, there is some $t_3(\bar{x}) \in \mathrm{cl}^{0,(r(\eta_1))}\left(\mathrm{cl}_{\mathrm{suc}}\left(\mathrm{cl}^{(k-1)}(\bar{x})\right)\right)$ such that $t_2^M(\bar{a}) = t_3^M(\bar{a})$ (if $k = 0$, then $t_3(\bar{x}) \in \mathrm{cl}^{0,(r(\eta_1))}(\bar{x})$). So $t_4 = G_{\eta_1,\eta_2}(t_3) \in \mathrm{cl}^{(k)}(\bar{x})$ and $t^M(\bar{a}) = t_4^M(\bar{a})$. If $t = s_1 \wedge_\eta s_2$, then by induction there are $s_3, s_4 \in \mathrm{cl}^{(k)}(\bar{x})$ such that $t^M(\bar{a}) = s_3^M(\bar{a}) \wedge_\eta s_4^M(\bar{a})$. Since $\mathrm{cl}^{(k)}(\bar{a})$ is closed under $\wedge$ (by Claim 3.7), there is some $s_5 \in \mathrm{cl}^{(k)}(\bar{x})$ such that $t^M(\bar{a}) = s_5^M(\bar{a})$. □

Definition 3.12. Suppose S is a finite standard tree and $k < \omega$. Let $M_1, M_2 \models T_S^\forall$.

(1) Suppose $n < \omega$ and $\bar{a} \in M_1^n$, $\bar{b} \in M_2^n$. We say that $\bar{a} \equiv_k^S \bar{b}$ if there is an isomorphism of L'_S-structures from $\mathrm{cl}^{(k)}(\bar{a})$ to $\mathrm{cl}^{(k)}(\bar{b})$ taking $\bar{a}$ to $\bar{b}$ (recall that $L'_S = L_S \setminus \{\mathrm{pre}_\eta, \mathrm{suc}_\eta \,|\, \eta \in S\}$). In this notation we assume that M_1, M_2 are clear from the context.

(2) If $A \subseteq M_1$, $B \subseteq M_2$ are two finite subsets of M_1 and M_2, we write $A \xrightarrow[k]{S,f} B$ when f extends some L'_S-isomorphism $f' : \mathrm{cl}^{(k)}(A) \to \mathrm{cl}^{(k)}(B)$ such that $f'(A) = B$. So this is equivalent to saying that $B = \{f(a) \,|\, a \in A\}$, $\langle a \,|\, a \in A\rangle \equiv_k^S \langle f(a) \,|\, a \in A\rangle$ and $f \restriction \mathrm{cl}^{(k)}(A)$ witnesses this.

Recall (from the notation section in the beginning of Section 2), that for a finite set of formulas Δ, by writing $\Delta(\bar{x};\bar{y})$ we mean that we assign to it a partition of the free variables appearing in it. In that case, for $\bar{b}$ of the same length as $\bar{x}$, $\mathrm{tp}_{\Delta(\bar{x};\bar{y})}\left(\bar{b}/A\right)$ is the set of formulas $\varphi\left(\bar{x},\bar{a}\right)$ such that $\varphi\left(\bar{x},\bar{y}\right)\in\Delta$, $\bar{a}\in A^{\lg(\bar{y})}$ and $\mathfrak{C}\models\varphi\left(\bar{b},\bar{a}\right)$. If the partition $(\bar{x};\bar{y})$ is clear, then we omit it from the notation.

Recall also that $\Delta_k^{\bar{x},S}$ is the set of all atomic formulas $\varphi\left(\bar{x}\right)$ in L_S such that for every term t in φ, $t\in\mathrm{cl}^{(k)}\left(\bar{x}\right)$.

Definition 3.13. Suppose S is a finite standard tree. For $M\models T_S^{\forall}$, $\bar{a}\in M^{<\omega}$, $A\subseteq M$ a finite set, and $k<\omega$, let $\mathrm{tp}_k^S\left(\bar{a}/A\right)=\mathrm{tp}_{\Delta_k^{\bar{x}\bar{y},S}}\left(\bar{a}/A\right)$ where $\lg\left(\bar{x}\right)=\lg\left(\bar{a}\right)$ and $\bar{y}$ is of length $|A|$. This is the *k-type of $\bar{a}$ over A*.

In Definitions 3.10, 3.12 and 3.13, we omit S from the superscript when it is clear from the context.

Claim 3.14. Suppose S is a finite standard tree. Assume $M_1,M_2\models T_S^{\forall}$. Assume that $\bar{a}\in M_1^n,\bar{b}\in M_2^n$ for some $n<\omega$, $\bar{x}$ a tuple of n variables and assume $k<\omega$. Then the following are equivalent:

(1) $\bar{a}\equiv_k\bar{b}$.
(2) $\mathrm{tp}_k\left(\bar{a}\right)=\mathrm{tp}_k\left(\bar{b}\right)$.
(3) For every quantifier free formula $\varphi\left(\bar{x}\right)$ in L_S with $r_{\mathrm{suc}}\left(\varphi\right)\leq k$, $M_1\models\varphi\left(\bar{a}\right)\Leftrightarrow M_2\models\varphi\left(\bar{b}\right)$.
(4) The tuples $\left\langle t\left(\bar{a}\right)\middle|t\in\mathrm{cl}^{(k)}\left(\bar{x}\right)\right\rangle$ and $\left\langle t\left(\bar{b}\right)\middle|t\in\mathrm{cl}^{(k)}\left(\bar{x}\right)\right\rangle$ have the same quantifier free type in L'_S.

Proof. (1) implies (2): assume $\bar{a}\equiv_k\bar{b}$ and $f:\mathrm{cl}^{(k)}\left(\bar{a}\right)\to\mathrm{cl}^{(k)}\left(\bar{b}\right)$ is an L'_S-isomorphism taking $\bar{a}$ to $\bar{b}$. It is easy to see by induction on t and k that for every term $t\in\mathrm{cl}^{(k)}\left(\bar{x}\right)$, $f\left(t\left(\bar{a}\right)\right)=t\left(\bar{b}\right)$, and so $\mathrm{tp}_k\left(\bar{a}\right)=\mathrm{tp}_k\left(\bar{b}\right)$.

(2) implies (3): this follows from Claim 3.11 — for every term $t\left(\bar{x}\right)$ with rank $r_{\mathrm{suc}}\left(t\right)\leq k$ there is a term $t'\in\mathrm{cl}^{(k)}\left(\bar{x}\right)$ such that $M_1\models t'\left(\bar{a}\right)=t\left(\bar{a}\right)$. By induction on k and t, one can show that since $\mathrm{tp}_k\left(\bar{a}\right)=\mathrm{tp}_k\left(\bar{b}\right)$, $M_2\models t'\left(\bar{b}\right)=t\left(\bar{b}\right)$ and this suffices. For instance, suppose $t=s_1\wedge_\eta s_2$. By induction, there are $s_3,s_4\in\mathrm{cl}^{(k)}\left(\bar{x}\right)$ such that $s_1^{M_1}\left(\bar{a}\right)=s_3^{M_1}\left(\bar{a}\right)$ and $s_2^{M_1}\left(\bar{a}\right)=s_4^{M_1}\left(\bar{a}\right)$ and the same equations hold with M_2 instead of M_1 and $\bar{b}$ instead of $\bar{a}$. Since $\mathrm{cl}^{(k)}\left(\bar{a}\right)$ is closed under $\wedge$, there is some $s_5\in\mathrm{cl}^{(k)}\left(\bar{x}\right)$ such that $M_1\models s_3\left(\bar{a}\right)\wedge_\eta s_4\left(\bar{a}\right)=s_5\left(\bar{a}\right)$, so

$$s_5^{M_1}\left(\bar{a}\right)=\max\left\{s^{M_1}\left(\bar{a}\right)\middle|s\in\mathrm{cl}^{(k)}\left(\bar{x}\right),M_1\models s\left(\bar{a}\right)\leq_\eta s_3\left(\bar{a}\right),s_4\left(\bar{a}\right)\right\}.$$

By (2), the same equation holds if we replace M_1 with M_2 and $\bar{a}$ with $\bar{b}$. Since $\mathrm{cl}^{(k)}\left(\bar{b}\right)$ is closed under $\wedge$, it follows that $M_2\models s_3\left(\bar{b}\right)\wedge_\eta s_4\left(\bar{b}\right)=s_5\left(\bar{b}\right)$.

(3) implies (4): since formulas in L'_S do not increase the successor rank, this is clear.

(4) implies (1): the map taking $t\left(\bar{a}\right)$ to $t\left(\bar{b}\right)$ for every term $t\in\mathrm{cl}^{(k)}\left(\bar{x}\right)$ is a well defined isomorphism of L'_S structures. $\square$

Similarly, we have:

Claim 3.15. Suppose S is a finite standard tree. Let $M \models T_S^\forall$, $n < \omega$, $\bar{a}, \bar{b} \in M^n$, $\bar{x}$ a tuple of n variables and $k, k_1, k_2 < \omega$.

(1) if $\bar{a} \equiv_k \bar{b}$ then there is a unique isomorphism that shows it. Namely, for each $t \in \mathrm{cl}^{(k)}(\bar{x})$, the isomorphism f must satisfy $f(t(\bar{a})) = t(\bar{b})$.
(2) Assume $k_2 \geq k_1$. Then $\bar{a} \equiv_{k_2} \bar{b}$ implies $\bar{a} \equiv_{k_1} \bar{b}$.
(3) If $\bar{a}\bar{a}' \equiv_k \bar{b}\bar{b}'$ then $\bar{a} \equiv_k \bar{b}$.
(4) If $\bar{a} \equiv_{k+1} \bar{b}$, witnessed by f, then $\mathrm{cl}(\bar{a}) \xrightarrow[k]{S,f} \mathrm{cl}(\bar{b})$.
(5) If $S' \subseteq S$, and $\bar{a} \equiv_k^S \bar{b}$ then $\bar{a} \equiv_k^{S'} \bar{b}$ (when $\bar{a}$ and $\bar{b}$ are considered as tuples in $M_1 \restriction L_{S'}$ and $M_2 \restriction L_{S'}$).

Before proceeding to prove the main quantifier elimination lemma, let us give two more important definitions:

Definition 3.16. Suppose S is a standard tree. Suppose $M \models T_S^\forall$, $\eta \in S$ and $a, b \in P_\eta^M$. We say that the *distance* between a and b is n if $a <_\eta b$ and b is the n-th successor of a or vice-versa. We say the distance is infinite if for no $n < \omega$ the distance is n. Denote this by $d(a,b) = n$.

For a set $A \subseteq M \models T_S^\forall$, we denote by $\mathrm{Suc}(A)$ the set of all successors in A.

Definition 3.17. Suppose S is a standard tree, $\eta \in S$, $M \models T_S^\forall$ and $A \subseteq M$. Let $R_\eta^A \subseteq \mathrm{Suc}(A)^2$ be the following relation: $(x,y) \in R_\eta^A$ iff $\lim(x) = \lim(y)$ and x and y are comparable ($x <_\eta y$ or $y \leq_\eta x$). Let $\sim_\eta^A$ be the the transitive closure of R_η^A (so it is an equivalence relation on $\mathrm{Suc}(A)$).

So the equivalence relation $\sim_\eta$ determines the function $G_{\eta,\eta'}$ for $\eta <_{\mathrm{suc}} \eta'$ from S: if $a, b \in P_\eta^M$ for $M \models T_S^\forall$ and $a \sim_\eta^M b$ then $G_{\eta,\eta'}(a) = G_{\eta,\eta'}(b)$.

Lemma 3.18. *(Quantifier elimination lemma) For every finite standard tree S, and $m_1, n, k < \omega$, there is $m_2 = m_2(m_1, k, S) < \omega$ such that if:*

- *$M_1, M_2 \models T_S^\forall$ are existentially closed.*
- *$\bar{a} \in M_1^n$ and $\bar{b} \in M_2^n$.*
- *$\bar{a} \equiv_{m_2} \bar{b}$.*

Then for all $\bar{c} \in M_1^k$ there is some $\bar{d} \in M_2^k$ such that $\bar{c}\bar{a} \equiv_{m_1} \bar{d}\bar{b}$.

(Note that m_2 does not depend on n.)

Proof. The proof is by induction on $|S|$. Given S, we will show that the lemma holds for all m_1 and k. Without loss of generality $k = 1$: by induction one can choose $m_2(m_1, k+1, S) = m_2(m_2(m_1, k, S), 1, S)$. We may also assume that $m_1 > 0$.

We may assume that $m_2(m_1, k, S') > \max\{m_1, k, |S'|\}$ for all $S' \subsetneq S$ (by enlarging m_2 if necessary).

For $|S| = 0$ the claim is trivial because $T_S^{\forall}$ is just the theory of a set with no structure.

Assume $0 < |S|$. Let η_0 be the root of S, $S_0 = \{\eta_0\}$ and partition S as $S = \bigcup \{S_i \mid i < m\}$ where for $i \geq 1$, the S_i's are the connected components of S above η_0 (note that $S_i \subseteq S$, see Notation 3.4). Let

$$m_2 = m_2(m_1, 1, S) = \max\{2m_2(m_1, K, S_i) \mid 1 \leq i < m\} + 2m_1 + 1$$

where $K = 3$.

Suppose M_1, M_2, $\bar{a}$ and $\bar{b}$ are as in the lemma and let $c \in M_1$.

By assumption there is a unique L'_S-isomorphism $f : \mathrm{cl}^{(m_2)}(\bar{a}) \to \mathrm{cl}^{(m_2)}(\bar{b})$.

For $i \leq m$, let $P_{S_i} = \bigvee\{P_\eta \mid \eta \in S_i\}$, $A_i = \mathrm{cl}^{(m_1)}(\bar{a}) \cap P_{S_i}^{M_1}$ and $B_i = \mathrm{cl}^{(m_1)}(\bar{b}) \cap P_{S_i}^{M_2}$.

Since $\bar{a} \equiv_{m_2} \bar{b}$, it follows that $\mathrm{cl}^{(m_2(m_1,K,S_i))}(\bar{a}) \xrightarrow[m_2(m_1,K,S_i)]{f} \mathrm{cl}^{(m_2(m_1,K,S_i))}(\bar{b})$ and in particular $A_i \xrightarrow[m_2(m_1,K,S_i)]{S_i,f} B_i$ (see Claim 3.15 (4) and (5)).

We divide into cases:

Case 1. $c \notin P_\eta^{M_1}$ for every $\eta \in S$.

Here finding d is easy due to the fact that M_1 and M_2 are existentially closed.

Case 2. $c \in P_{S_i}^{M_1}$ for some $1 \leq i \leq m$.

$A_i \xrightarrow[m_2(m_1,K,S_i)]{S_i,f} B_i$ (as subsets of $M_1 \restriction L_{S_i}$ and $M_2 \restriction L_{S_i}$), so by the induction hypothesis (and by Remark 3.5 (3)) we can find $d \in M_2$ and extend $f \restriction \mathrm{cl}^{(m_1)}(A_i)$ to an L'_{S_i}-isomorphism $f' : \mathrm{cl}^{(m_1)}(\{c\} \cup A_i) \to \mathrm{cl}^{(m_1)}(\{d\} \cup B_i)$ taking c to d. Note that f' is also an L'_S-isomorphism. It follows that

$$f \restriction \mathrm{cl}^{(m_1)}(\bar{a}) \cup f' \restriction \mathrm{cl}^{(m_1)}(c\bar{a})$$

is an L'_S-isomorphism from $\mathrm{cl}^{(m_1)}(c\bar{a})$ to $\mathrm{cl}^{(m_1)}(d\bar{b})$ that shows that $c\bar{a} \equiv_{m_1} d\bar{b}$ (note that $P_{S_j}^{M_1} \cap \mathrm{cl}^{(m_1)}(\bar{a}c) = A_j$ for $j \neq i$ and that if $x \in \mathrm{cl}^{(m_1)}(\bar{a}c) \cap P_{S_i}^{M_1}$ then $x \in \mathrm{cl}^{(m_1)}(\{c\} \cup A_i)$, and so the domain is indeed $\mathrm{cl}^{(m_1)}(c\bar{a})$).

Case 3. $c \in P_{\eta_0}$.

For notational simplicity, let $<$ be $<_{\eta_0}$, lim be $\lim_{\eta_0}$, $\sim$ be $\sim_{\eta_0}$ and $\wedge$ be $\wedge_{\eta_0}$.

Let $A'_0 = \mathrm{cl}^{(0)}(\bar{a}) \cap P_{\eta_0}^{M_1}$ (so this is the closure of $\bar{a}$ inside P_{η_0} under $\wedge$ and lim), $B'_0 = \mathrm{cl}^{(0)}(\bar{b}) \cap P_{\eta_0}^{M_2}$, $F = \mathrm{cl}^{(m_1)}(A'_0 \cup \{c\}) \cap P_{\eta_0}^{M_1}$ and $\eta_i = \min(S_i)$ for $1 \leq i \leq m$.

Note that F is really just $\mathrm{cl}_{\mathrm{suc}}^{(m_1)}\left(\mathrm{cl}^{(0)}(A'_0 \cup \{c\})\right)$.

Say that an element of F is *new* if it is a successor and is not $\sim^F$-equivalent to any element from A_0 (note: A_0 and <u>not</u> A'_0). We will prove the following claim:

Claim I. (1) There are at most K many $\sim^F$-equivalence classes of new elements in F. For each one choose a representative. Enumerate them as $\langle c_l \mid l < K' \rangle$ for $K' \leq K$.

(2) There is a model M_3' of $T_{S_0}^{\forall}$, an L_{S_0}'-isomorphism f' and $d' \in M_3'$ such that $M_3' \supseteq P_{\eta_0}^{M_2}$, $f' \restriction A_0 = f$, $A_0' \cup \{c\} \xrightarrow[m_1]{S_0, f'} B_0' \cup \{d'\}$ and $f'(c) = d'$ (so the domain of f' is F).

(3) Moreover, for $l < K'$, $f'(c_l)$ are pairwise non-$\sim^{M_3}$-equivalent and they are not $\sim^{M_3}$-equivalent to any element from $\mathrm{Suc}\left(P_{\eta_0}^{M_2}\right)$.

Suppose first that Claim I holds.

For $1 \leq i \leq m$ let $c_l^i = G_{\eta_0, \eta_i}(c_l)$.

Fix $1 \leq i \leq m$. By assumption, $A_i \xrightarrow[m_2(m_1, K, S_i)]{S_i, f} B_i$, so by the induction hypothesis there are $d_l^i \in M_2$ for $l < K'$ and an L_{S_i}'-isomorphism g_i extending $f \restriction \mathrm{cl}^{(m_1)}(A_i)$ such that $g_i\left(c_l^i\right) = d_l^i$ and $A_i \cup \left\{c_l^i \,|\, l < K'\right\} \xrightarrow[m_1]{S_i, g_i} B_i \cup \left\{d_l^i \,|\, l < K'\right\}$.

Claim II. There exists a model $M_3 \models T_S^{\forall}$ satisfying $P_{\eta_0}^{M_3} = P_{\eta_0}^{M_3'}$, $M_3 \supseteq M_2$ and $G_{\eta_0, \eta_i}^{M_3}(f'(c_l)) = d_l^i$ for $l < K'$ and $1 \leq i \leq m$.

Proof. (of Claim II) Since $M_3' \models T_{S_0}^{\forall}$, $M_2 \models T_S^{\forall}$ and $P_{\eta_0}^{M_3'} \supseteq P_{\eta_0}^{M_2}$ the only thing we must show is that G_{η_0, η_i} defined in Claim II is well defined and can be extended to a regressive function. This follows directly from Claim I (3). □

Define

$$g = f \restriction \mathrm{cl}^{(m_1)}(\bar{a}) \cup f' \restriction \mathrm{cl}^{(m_1)}(\bar{a}c) \cup \bigcup \left\{g_i \restriction \mathrm{cl}^{(m_1)}(\bar{a}c) \,|\, 1 \leq i < m\right\}.$$

We claim that g is an L_S'-isomorphism extending $f \restriction \mathrm{cl}^{(m_1)}(\bar{a})$ from $\mathrm{cl}^{(m_1)}(\bar{a}c)$ to $\mathrm{cl}^{(m_1)}(\bar{a}d')$ sending c to d. It is easy to see that g is well defined as a function. To see that it is an L_S'-isomorphism we only need to show that if $e \in \mathrm{cl}^{(m_1)}(\bar{a}c)$ is a successor and $1 \leq i \leq m$ then $G_{\eta_0, \eta_i}^{M_3}(f'(e)) = g_i\left(G_{\eta_0, \eta_i}^{M_1}(e)\right)$. Suppose $e \sim^F b$ where $b \in A_0$, then $f'(e) \sim^{M_3} f'(b)$, $G_{\eta_0, \eta_i}^{M_1}(e) = G_{\eta_0, \eta_i}^{M_1}(b)$ and $G_{\eta_0, \eta_i}^{M_3}(f'(e)) = G_{\eta_0, \eta_i}^{M_3}(f'(b))$. Now we are done since:

$$G_{\eta_0, \eta_i}^{M_3}(f'(b)) = G_{\eta_0, \eta_i}^{M_2}(f(b)) = f\left(G_{\eta_0, \eta_i}^{M_1}(b)\right) = g_i\left(G_{\eta_0, \eta_i}^{M_1}(b)\right).$$

Suppose e is new. Then $e \sim^F c_l$ for some $l < K'$. But then $G_{\eta_0, \eta_i}^{M_1}(e) = G_{\eta_0, \eta_i}^{M_1}(c_l) = c_l^i$, and $g_i\left(c_l^i\right) = d_l^i$, while $f'(e) \sim^{M_3} f'(c_l)$, so $G_{\eta_0, \eta_i}^{M_3}(f'(e)) = G_{\eta_0, \eta_i}^{M_3}(f'(c_l)) = d_l^i$ by Claim II.

So $c\bar{a} \equiv_{m_1} d'\bar{b}$, i.e., $\mathrm{tp}_{m_1}(c\bar{a}) = \mathrm{tp}_{m_1}\left(d'\bar{b}\right)$, and if Ψ is the conjunction of all formulas appearing in $\mathrm{tp}_{m_1}(c\bar{a})$ then $M_3 \models \exists x \Psi\left(x\bar{b}\right)$. As M_2 is existentially closed there is some $d \in M_2$ such that $\Psi\left(d\bar{b}\right)$, i.e., $c\bar{a} \equiv_{m_1} d\bar{b}$.

We will be done once we prove Claim I.

Proof. (of Claim I) Again we need to divide into cases:

Case i. $c \in A_0'$: there is nothing to do.

Case ii. c is in a branch of A_0', i.e., there is $c < y \in A_0'$ and assume y is minimal in this sense (it exists since A_0' is closed under $\wedge$). We again divide into cases:

Case a. There is no $x \in A_0'$ below c. This means that $c < x$ for all $x \in A_0'$, and even for all $x \in A_0$ (since for all $x \in A_0$, there is $x' \in A_0'$ such that $\lim(x) = \lim(x')$) and that $y = \lim(y)$. There is exactly one $\sim^F$-class of new elements, which is $[\mathrm{suc}(c, y)]_{\sim^F}$. In this case (2) and (3) are easy: just let d' be a new element below $P_{\eta_0}^{M_2}$ with the same distance from its limit as $d(c, \lim(c))$ (which can be infinite, and if $d(c, \lim(c)) > 2m_1$, we can choose $d(d', \lim(d')) = 2m_1 + 1$).

Case b. There is some $x \in A_0'$ such that $x < c$. Assume x is maximal in this sense.

If $\lim(x) < \lim(y)$ then necessarily $\lim(x) \le x < c < \lim(y) = y$. If $\lim(x) < \lim(c)$, then there is one $\sim^F$-class of new elements — $[\mathrm{suc}(c, y)]_{\sim^F}$. Again (2) and (3) are easy: let $\lim(d')$ be a new limit element below $f(y)$ and above all elements from M_2 below $f(y)$ and let d' be with the right distance from $\lim(d')$. If $\lim(x) = \lim(c)$, then there are no new $\sim^F$-classes. Moreover, we can choose $M_3' = M_2 \restriction L_{S_0}$ and $d' \in M_2$.

If $\lim(x) = \lim(y)$ (so also $= \lim(c)$), then again there are no new $\sim^F$-classes. For (2) and (3), we must make sure that the distance between $f(x)$ and $f(y)$ is big enough, so that we can place d' in the right spot between them. In $F \backslash A_0$ we may add m_1 successors to c in the direction of y and m_1 predecessors. This is why we chose $m_2 \ge 2m_1 + 1$.

Case iii. c starts a new branch in A_0', i.e., there is no $y \in A_0'$ such that $c < y$. In this case, let $c' = \{\max(c \wedge b) \mid b \in A_0'\}$. Note that if there is an element in $\mathrm{cl}_\wedge(A_0' \cup \{c\}) \backslash A_0 \cup \{c\}$, it must be c'. Adding c' falls under Case ii above (if it is indeed new), so the $\sim^F$-classes of new elements will be those which come from c' as before, and perhaps more. Namely, it can be that $\lim(c) < c'$ (so $\lim(c) = \lim(c')$) in which case that is all, or we should add $[\mathrm{suc}(\lim(c), c)]_{\sim^F}$ and $[\mathrm{suc}(c', c)]_{\sim^F}$.

By the previous case, we can first find $M_3'' \supseteq P_{\eta_0}^{M_2}$, an L_{S_0}'-isomorphism f'' and $d'' \in M_3''$ such that $f'' \restriction A_0 = f$, $A_0' \cup \{c'\} \xrightarrow[m_1]{S_0, f'} B_0' \cup \{d''\}$ and $f''(c') = d''$. Then we can just add a new branch starting at d'' to construct M_3'.

□

Claim 3.19. Let S be a finite standard tree. For every formula $\varphi(\bar{x})$ (with free variables) there is a quantifier free formula $\psi(\bar{x})$ such that for every existentially closed model $M \models T_S^{\forall}$, we have $M \models \psi \equiv \varphi$.

Proof. It is enough to check formulas of the form $\exists y \varphi(y, \bar{x})$ where φ is quantifier free and $\lg(\bar{x}) = n > 0$. Let $k = r_{\text{suc}}(\varphi)$. Let $m = m_2(k, 1, S)$ from Lemma 3.18. By Claim 3.14, if $M_1, M_2 \models T_S^{\forall}$ are existentially closed and $\bar{a} \in M_1, \bar{b} \in M_2$ are of length n and $\bar{a} \equiv_m \bar{b}$, then $M_1 \models \exists y \varphi(y, \bar{a})$ iff $M_2 \models \exists y \varphi(y, \bar{b})$.

Assume $|\Delta_m^{\bar{x}}| = N$ and let $\{\varphi_i \,|\, i < N\}$ be an enumeration of $\Delta_m^{\bar{x}}$. For every $\eta : N \to 2$, let $\varphi_\eta^m(\bar{x}) = \bigwedge_{i<N} \varphi_i^{\eta(i)}(\bar{x})$ (where $\varphi^0 = \neg\varphi$ and $\varphi^1 = \varphi$).

Let

$$R = \left\{\eta : N \to 2 \,\middle|\, \exists \text{ e.c. } M \models T_S^{\forall} \,\&\, \exists \bar{c} \in M \left(M \models \varphi_\eta^m(\bar{c}) \wedge \exists y \varphi(y, \bar{c})\right)\right\}.$$

Let $\psi(\bar{x}) = \bigvee_{\eta \in R} \varphi_\eta^m(\bar{x})$. By Claim 3.14 it follows that ψ is the desired formula. □

Corollary 3.20. *If M_1 and M_2 are two existentially closed models of $T_S^{\forall}$ then $M_1 \equiv M_2$ and their theory eliminates quantifiers.*

Proof. Assume first that $M_1 \subseteq M_2$, then $M_1 \prec M_2$: for formulas with free variables it follows directly from the previous claim, and for a sentence φ we consider the formula $\varphi \wedge (x = x)$.

Now the corollary follows from the fact that the theory is universal (so every model can be extended to an existentially closed one) and has JEP. □

Definition 3.21. Let S be a finite standard tree. Let T_S be the theory of all existentially closed models of $T_S^{\forall}$.

From Corollary 3.20 and the definition of model completion, we deduce:

Corollary 3.22. *Let S be a finite standard tree. Then T_S is the model completion of $T_S^{\forall}$. The theory T_S eliminates quantifiers. Thus $T_S^{\forall}$ has AP.*

NIP. In this section we will show that T_S is dependent. The idea is to count the number of Δ-types for finite Δ over a finite set of parameters A, and to show that this number is polynomial in $|A|$. Thus, from Fact 2.2 it follows that T_S is dependent. In fact, we will show that we can find such polynomials f_Δ such that their degree does not depend on Δ, but only on the number of free variables and on S. From this, by Lemma 2.5 we will conclude that T_S is not just dependent but even strongly[2] dependent.

Definition 3.23. Suppose S is a finite standard tree. Assume $A \subseteq M \models T_S$ is a finite set and $k < \omega$.

(1) We say that $a, b \in M$ are k-isomorphic over A, denoted by $a \equiv^S_{A,k} b$ iff for some (any) enumeration $\bar{a}$ of A, $a\bar{a} \equiv^S_k b\bar{a}$.
(2) Similarly for tuples from $M^{<\omega}$.

Claim 3.24. Suppose S is a finite standard tree. Assume $M \models T^{\forall}_S$, $k < \omega$, $A \subseteq M$ is finite and $\bar{a}, \bar{b} \in M^{<\omega}$. Then $\bar{a} \equiv_{A,k} \bar{b}$ iff $\mathrm{tp}_k(\bar{a}/A) = \mathrm{tp}_k(\bar{b}/A)$ iff for every quantifier free formula $\varphi(\bar{x})$ over A such that $r_{\mathrm{suc}}(\varphi) \leq k$, $M \models \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$.

Proof. Follows from the definitions and from Claim 3.14. □

Proposition 3.25. *Assume $|S| = 1$ and $k < \omega$. Then there is a polynomial p_k over $\mathbb{N}$ such that for every model $M \models T^{\forall}_S$ and for every finite set $A \subseteq M$, $|\{M/\equiv_{A,k}\}| \leq p_k(|A|)$. Moreover, we can choose $\langle p_k \mid k < \omega \rangle$ so that p_k is linear for all k.*

Proof. As $|S| = 1$, we can forget the index η and write $<, \lim$, etc. instead of $<_\eta, \lim_\eta$, etc.

Suppose $M \models T^{\forall}_S$. Given $a < b \in M$, the k-distance between them is defined by

$$d_k(a,b) = \min\{d(a,b), 2k+1\}.$$

Assume $a \in M$ and $A \subseteq M$ is finite.

Let $B = \mathrm{cl}^{(0)}(A)$ and $l = |B|$. Recall that $l \leq f^S_0(|A|)$ where f^S_0 is a linear function (see Claim 3.8). We will divide the possible k-isomorphism type of a over A into finitely many cases, and in each case the number of possible types will be linear in l (so linear in $|A|$).

Case 1. $a \notin P$. Here there is no structure, so the number of types is $|A| + 1$.

Case 2. $a \in P$, and there is some $b \in B$ such that $a \leq b$. We further divide into sub-cases:

- *Case* i. $a \in B$. In that case there are at most l types.
- *Case* ii. There is no $b \in B$ such that $b < a$. In that case, since B is closed under $\wedge$, a is smaller than b for all $b \in B$. In this case it is enough to know the k-distance between a and $\lim(a)$. So there are $2k+1$ types.
- *Case* iii. There is some $b \in B$ such that $b < a$. Choose $b_0, b_1 \in B$ such that b_1 is minimal with the property that $a < b_1$ and b_0 is maximal such that $b_0 < a$. Since B can also be viewed as a finite graph-theoretic tree and as such has $l - 1$ edges, we have at most $l - 1$ such pairs.
  - *Case a.* $\lim(b_0) < \lim(b_1)$. Note that it follows that $\lim(b_1) = b_1$.
    - *Case* 1. $\lim(b_0) < \lim(a)$. Then the type is determined by the k-distance between a and $\lim(a)$, so there are at most $2k+1$ types here.

Case 2. $\lim(b_0) = \lim(a)$. The type is determined by the k-distance between a and b_0, so again there are at most $2k+1$ types.

Case b. $\lim(b_0) = \lim(b_1)$. In this case $\lim(b_1) = \lim(a)$. The type is determined by the k-distance between a and b_0 and the k distance between a and b_1. So totally there are at most $4k+2$ types.

So in this case (Case iii) there are at most $(l-1)\cdot(4k+2)$ many types.

Case 3. $a \in P$, and there is no $b \in B$ such that $a \leq b$. Let $a' = \max\{a \wedge b \,|\, b \in B\}$. Since there is some $b \in B$ such that $a' \leq b$, the number of possible k-isomorphism types of a' over A is bounded by $h(l)$ where h is a linear map. Fix $\mathrm{tp}_k(a'/A)$.

Case i. $\lim(a) = \lim(a')$. Here the type is determined by the k-distance between a and a', so there are at most $2k+1$ types.

Case ii. $\lim(a) > \lim(a')$. Here the type is determined by the k-distance between a and $\lim(a)$, so there are at most $2k+1$ types.

So in this case (Case 3) there are at most $h(l)\cdot(4k+2)$ types.

□

Definition 3.26. Let S be a finite standard tree, and $n < \omega$. Say that S is *n-nice* if there is a number $N < \omega$ and a sequence of polynomials $\langle p_k^S \,|\, k < \omega \rangle$ over $\mathbb{N}$, <u>whose degrees are bounded by N</u> such that for every model $M \models T_S^{\forall}$ and finite $A \subseteq M$, $|\{M^n / \equiv_{A,k}\}| \leq p_k^S(|A|)$. Say that S is nice if it is *n-nice* for all $n < \omega$.

From Proposition 3.25 we get:

Corollary 3.27. *If* $|S| = 1$, *then* S *is* 1*-nice.*

Lemma 3.28. *Suppose* S *is a* 1*-nice finite standard tree. Then it is nice.*

Proof. We may restrict our attention to models of T_S (i.e., existentially closed models of $T_S^{\forall}$), since every model of $T_S^{\forall}$ extends to a model of T_S, and the number of k-isomorphism types can only increase.

The proof is by induction on n. For $n = 1$ this is the assumption, so assume it holds for every $l \leq n$. Fix some polynomials $\langle p_{k,l} \,|\, k < \omega, 0 < l \leq n \rangle$ that witness l-niceness for all $l \leq n$. We will show that the polynomials defined by $p_{k,n+1}^S(X) = p_{k',n}(X) \cdot p_{k,1}(X+1)$ with $k' = m_2(k,n,S)$ (see Lemma 3.18) bound the number of k-isomorphism types. By induction, their degree is bounded by a constant number, regardless of k.

We use Claim 3.24, namely that we can identify the number of k-isomorphism types and the number of k-types (see Definition 3.13).

Suppose A is a finite subset of a model $M \models T_S$. For every $k, m < \omega$ let $\Delta_k^m = \Delta_k^{\bar{x}\bar{y}}$ where $\lg(\bar{x}) = m$ and $\lg(\bar{y}) = |A|$. Let $Q = S_{\Delta_k^{n+1}}(A)$. For each type $r \in Q$, choose a realization $(\bar{a}_r, b_r)$ where $\lg(\bar{a}_r) = n$. Let E be the equivalence relation on Q defined by $r \; E \; r'$ iff $b_r \equiv_{A,k'} b_{r'}$. Without loss of generality, for all $r, r' \in Q$, if $r \; E \; r'$ then $b_r = b_{r'}$: choose representatives $\langle r_i \,|\, i < l \rangle$ for all the E-classes. Fix some $i < l$ and $r \; E \; r_i$. Enumerate A as $\bar{a}$. Since $b_r\bar{a} \equiv_{k'} b_{r_i}\bar{a}$, by Lemma 3.18 there is some $\bar{a}'_r \in M^n$ such that $\bar{a}_r b_r \bar{a} \equiv_k \bar{a}'_r b_{r_i} \bar{a}$, i.e., $\bar{a}_r b_r \equiv_{A,k} \bar{a}'_r b_{r_i}$, so we can replace $(\bar{a}_r, b_r)$ by $(\bar{a}'_r, b_{r_i})$. Now for each E-equivalence class $C \subseteq Q$, the map $r \mapsto \mathrm{tp}_k^S(\bar{a}_r / A \cup \{b_r\})$ from C to $S_{\Delta_k^n}(A \cup \{b_r\})$ is injective, so $|C| \leq p_{k,n}^S(|A| + 1)$. The number of E-classes is bounded by $p_{k',1}^S(|A|)$, so we are done. □

Theorem 3.29. *Suppose S is a finite standard tree. Then it is nice.*

Proof. The proof is by induction on $|S|$. For $|S| = 1$ it follows from Proposition 3.25 and Lemma 3.28 (and for $|S| = 0$ it is obvious).

Assume $1 < |S|$. By Lemma 3.28, it is enough to show that S is 1-nice.

Let η_0 be the root of S, $S_0 = \{\eta_0\}$ and let $S = \bigcup \{S_i \,|\, i < m\}$ where for $1 \leq i < m$ the S_i's are the connected components of S above η_0. For $i \leq m$, let $P_{S_i} = \bigvee \{P_\eta \,|\, \eta \in S_i\}$. For $i < m$, let $\eta_i = \min(S_i)$. Suppose $\left\langle p_{k,n}^i \,|\, k, n < \omega, i < m \right\rangle$ witness that S_i are nice. Suppose the degree of $p_{k,n}^i$ is bounded by N_n for all $k, n < \omega$ and $i < m$. We may assume that $p_{k,n}^i \leq p_{k,n+1}^i$ and $N_n \leq N_{n+1}$ for all $k, n < \omega$ and $i < m$.

Assume $A \subseteq M \models T_S^{\forall}$ is finite and $a \in M$. We will divide the possible k-isomorphism types of a over A into finitely many cases. In each case we will have a polynomial bound (in terms of $|A|$) on the number of types. This polynomial will have degree at most $m \cdot N_K$ where $K = 3$. Since M, A and a were arbitrary this will show that S is 1-nice.

Let $A_i = \mathrm{cl}^{(k)}(A) \cap P_{S_i}^M$.

Case 1. $a \notin P_\eta^M$ for all $\eta \in S$. In that case there are at most $|A| + 1$ types.

Case 2. $a \in P_{\eta_i}^M$ for some $1 \leq i < m$. It is enough to determine $\mathrm{tp}_k^{S_i}(a/A_i)$. If $\mathrm{tp}_k^{S_i}(a/A_i) = \mathrm{tp}_k^{S_i}(b/A_i)$, then $a \equiv_{A_i,k} b$ (by Claim 3.24), so there is an L'_{S_i} isomorphism f' : $\mathrm{cl}^{(k)}(A_i a) \to \mathrm{cl}^{(k)}(A_i b)$ taking a to b and fixing A_i. Define $f : \mathrm{cl}^{(k)}(Aa) \to \mathrm{cl}^{(k)}(Ab)$ by

$$\left(f' \restriction \mathrm{cl}^{(k)}(A \cup \{a\}) \cap P_{S_i}^M\right) \cup \left(\mathrm{id} \restriction \mathrm{cl}^{(k)}(A)\right).$$

This is an isomorphism. Now, note that $|A_i| \leq f_k^{S_i}(|A|)$ which is linear in $|A|$ (see Claim 3.8), and the number of types over A_i is bounded by $p_{k,1}^i(|A_i|) \leq p_{k,1}^i\left(f_k^{S_i}(|A|)\right)$.

Case 3. $a \in P_{\eta_0}$. Let $B = A \cap P_{\eta_0}^M$. First we determine $\mathrm{tp}_k^{S_0}(a/B)$, for this we have at most $p_{k,1}^0(|A|)$ many possibilities. Fix one such type.

Suppose $a \equiv_{B,k}^{S_0} b$. Let f' be an L'_{S_0}-isomorphism such that $B \cup \{a\} \xrightarrow[k]{S_0, f'} B \cup \{b\}$, f' fixes B and takes a to b. Let $F = \mathrm{cl}^{(k)}(A \cup \{a\}) \cap P_{\eta_0}^M$ and $F' = f'(F)$, so that

f is an L'_{S_0} isomorphism between F and F'. By Claim I (1) in the proof of Lemma 3.18, there are at most K (i.e., 3) $\sim^F_{\eta_0}$-classes in F that are not already in $\mathrm{cl}^{(k)}(A)$, suppose there are $K' \leq K$ such classes. Let $\bar{b}$ be an enumeration of B, and $\bar{y}$ a tuple of variables of the same length. If $\langle t_i(x,\bar{y}) \,|\, i < K' \rangle$ are terms from $\mathrm{cl}^{(k),S_0}(x\bar{y})$ such that the new classes are exactly $\left\{ \left[t_i\left(a,\bar{b}\right)\right]_{\sim^F_{\eta_0}} \,|\, i < K' \right\}$, then the new classes in F' are $\left\{ \left[t_i\left(b,\bar{b}\right)\right]_{\sim^{F'}_{\eta_0}} \,|\, i < K' \right\}$. This means that we can fix such terms depending only on $\mathrm{tp}^{S_0}_k(a/B)$. Now it is enough to determine $\mathrm{tp}^{S_i}_k\left(\left\langle G_{\eta_0,\eta_i}\left(t_l\left(a,\bar{b}\right)\right) \,|\, l < K' \right\rangle / A_i\right)$ for each $1 \leq i < m$.

Indeed, suppose that a, b and f' are as above and moreover for each $1 \leq i < m$, $\left\langle G_{\eta_0,\eta_i}\left(t_l\left(a,\bar{b}\right)\right) \,|\, l < K' \right\rangle \equiv_{k,A_i} \left\langle G_{\eta_0,\eta_i}\left(t_l\left(b,\bar{b}\right)\right) \,|\, l < K' \right\rangle$. Let g_i be an L'_{S_i}-isomorphism fixing A_i witnessing this. Then

$$\mathrm{id} \restriction \mathrm{cl}^{(k)}(A) \cup f' \cup \bigcup_{1 \leq i < m} \left(g_i \restriction \mathrm{cl}^{(k)}(A \cup \{a\}) \cap P^M_{S_i} \right)$$

is an L'_S-isomorphism showing that $a \equiv^k_A b$. This follows from the fact that if $e \sim^F_{\eta_0} e'$ then $G_{\eta_0,\eta_i}(e) = G_{\eta_0,\eta_i}(e')$.

In this case there are at most $p^0_{k,1}(|A|) \cdot \prod_{1 \leq i < m} p^i_{k,K}\left(f^{S_i}_k(|A|)\right)$ types (here we used the assumption that $p^i_{k,K'} \leq p^i_{k,K}$).

□

Corollary 3.30. *Suppose S is a finite standard tree. Then T_S is strongly^2-dependent.*

Proof. We will apply Lemma 2.5.

Let $\Delta(\bar{x};\bar{y})$ be a finite set of formulas. By quantifier elimination, we may assume that Δ is quantifier free. Let $k = \max\{r_{\mathrm{suc}}(\varphi) \,|\, \varphi \in \Delta\}$ and $m = \left|S_{\Delta(\bar{x};\bar{y})}(A)\right|$. Let $\{\bar{c}_i \,|\, i < m\}$ be a set of tuples satisfying all the different types in $S_{\Delta(\bar{x};\bar{y})}(A)$ in some model M of T_S. If $i \neq j$ then $\mathrm{tp}_k(\bar{c}_i/A) \neq \mathrm{tp}_k(\bar{c}_j/A)$ (by Claim 3.24), so $m \leq \left|\left\{M^{\lg(\bar{x})}/\equiv_{A,k}\right\}\right|$, and hence we are done by Theorem 3.29. □

So far we mostly assumed that S is finite. Now we will let S be any standard tree.

Corollary 3.31. *Suppose S is a standard tree. If $M \models T^{\forall}_S$ then since*

$$Th(M) = \bigcup \{Th(M \restriction L_{S_0}) \,|\, S_0 \subseteq S \,\&\, |S_0| < \aleph_0\},$$

by Remark 3.5, Corollary 3.20 is true in the case where S is infinite. So T_S is well defined in this case as well and it is in fact $\bigcup\{T_{S_0} \,|\, S_0 \subseteq S \,\&\, |S_0| < \aleph_0\}$. It eliminates quantifiers and is dependent.

Adding Constants. We want to find an example of every cardinality, and so we add constants to the language. For a cardinal θ, the theory T_S^θ will be T_S augmented with the quantifier free diagram of a model of $T_S^\forall$ of cardinality θ. The simplest thing to do is to add θ-many constants that do not belong to any P_η. The problem with this approach is that the induction would not work in the proof of the main theorem. So instead we put a tree of constants in every P_η. Formally:

Definition 3.32. Let S be a standard tree. For a cardinal θ, let $L_S^\theta = L_S \cup \{e_{\eta,i} \,|\, i < \theta, \eta \in S\}$ where $\{e_{\eta,i} \,|\, i < \theta, \eta \in S\}$ are new constants. Let $T_S^{\forall,\theta}$ be the theory $T_S^\forall$ with the axioms stating that for all $\eta, \eta_1, \eta_2 \in S$ and $i, j, i', j' < \theta$ such that $\eta_1 <_{\text{suc}} \eta_2$,

- $e_{\eta,i} \in P_\eta$,
- $i \neq j \Rightarrow e_{\eta,i} \neq e_{\eta,j}$,
- $i \neq j, i' \neq j' \Rightarrow e_{\eta,i} \wedge_\eta e_{\eta,j} = e_{\eta,i'} \wedge_\eta e_{\eta,j'}$,
- $\eta_1 <_{\text{suc}} \eta_2 \Rightarrow G_{\eta_1,\eta_2}(e_{\eta_1,i}) = e_{\eta_2,i}$,
- $\lim_\eta (e_{\eta,i} \wedge e_{\eta,j}) = e_{\eta,i} \wedge e_{\eta,j}$ and
- $\text{suc}_\eta (e_{\eta,i} \wedge e_{\eta,j}, e_{\eta,i}) = e_{\eta,i}$.

Corollary 3.33. *Suppose S is a standard tree.*

(1) $T_S^{\forall,\theta}$ has JEP and AP.

(2) $T_S^{\forall,\theta}$ has a model completion — T_S^θ — that is complete, dependent and has quantifier elimination.

(3) Given any model $M \models T_S^\forall$, there is a model $M' \models T_S^{\forall,\theta}$ satisfying $M' \restriction L_S \supseteq M$.

(4) If S is finite then T_S^θ is strongly[2] dependent.

Proof. (1) This follows from Corollary 3.22 (noting that JEP for $T_S^{\forall,\theta}$ follows from AP for $T_S^\forall$).

(2) Since T_S is the model completion of $T_S^\forall$ and $T_S^{\forall,\theta}$ is the quantifier free diagram of a model of $T_S^\forall$, $T_S^\theta = T_S \cup T_S^{\forall,\theta}$ is a complete theory. Since we only added constants, T_S^θ is dependent and has quantifier elimination.

(3) This follows from JEP for $T_S^\forall$.

(4) This follows from Corollary 3.30. $\square$

4. The inaccessible case

In this section we will deal with the main technical obstacle in proving Main Theorem A. The proof, which will be described in Section 5, is by induction in the following sense: for $\mathbb{S} = 2^{<\omega}$, cardinals κ, θ and a limit ordinal $\delta \geq \omega$ such that $\kappa \not\to (\delta)_\theta^{<\omega}$, we will find a model $M \models T_{\mathbb{S}}^{\forall,\theta}$ and a set $A \subseteq P_{\langle\rangle}^M$ of size $|A| \geq \kappa$ with no non-constant indiscernible sequence in A^δ. We are allowed to use induction since $\lambda \not\to (\delta)_\theta^{<\omega}$ for all $\lambda < \kappa$. We divide into cases, namely $\kappa \leq \theta$, κ singular and κ regular but not strongly inaccessible. The main problem is in the remaining case, i.e., when κ is strongly inaccessible. In all other cases, the proof will follow by induction without using explicitly the fact that $\kappa \not\to (\delta)^{<\omega}_{\theta}$.

Assumption 4.1. *Assume for this section that $\theta < \kappa$ are cardinals, $\delta \geq \omega$ is a limit ordinal and that κ is strongly inaccessible such that $\kappa \not\to (\delta)^{<\omega}_{\theta}$.*

This section is divided into two subsections.

In the first subsection we define a class $\mathcal{T}$ of models of $T^{\forall,\theta}_{\omega}$ (here $S = \omega$, with the tree structure being the usual order on ω). We will analyze sequences of elements in models in $\mathcal{T}$ that are close to being indiscernible. There are two main results here, the first (Proposition 4.13) says that sequences (of singletons) that are closed to being indiscernible can have two forms: "almost increasing" and "fan". "Almost increasing" means that $s_i \wedge s_{i+1} < s_{i+1} \wedge s_{i+2}$, and "fan" means that $s_i \wedge s_j$ is constant. The second result (Corollary 4.16) deals with applying a specific definable map on sequences. Given an almost increasing sequence $\bar{s}$, let $H(\bar{s}) = \bar{t}$ where $t_i = G(\mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1}))$ (where G is some $G_{n,n+1}$, recall that here $S = \omega$). We will show that if applying H again and again we always get an almost increasing sequence, then this almost increasing sequence will satisfy $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$.

In the second subsection we will construct a model in $\mathcal{T}$ that uses explicitly a witness of $\kappa \not\to (\delta)^{<\omega}_{\theta}$. For this model, $P_0 = \kappa$. We will show, applying the analysis, that if we have an indiscernible sequence in P_0 such that applying H to it again and again results in almost increasing sequences, then there is a homogeneous sub-sequence of κ of length δ, contradicting the assumption. So after applying H finitely many times we must get a fan. This model will come equipped with equivalence relations on the trees P_n, which refines the neighboring relation (x, y are neighbors if they succeed the same element). The point is that the number of classes inside a given neighborhood will be less than κ. This will enable us to use the induction hypothesis in the proof of Main Theorem A.

The models in $\mathcal{T}$ will be standard in the following sense:

Definition 4.2. Suppose S is a standard tree. Call a model of $T^{\forall}_{S}$ *standard* if for every $\eta \in S$, $(P_\eta, <_\eta)$ is a standard tree, and $\wedge_\eta, \lim_\eta, \mathrm{suc}_\eta$ are all interpreted in the natural way (so $\lim_\eta(a)$ is the greatest element $\leq a$ of a limit level).

Let us fix some notation:

Notation 4.3. Suppose S is the standard tree ω with the usual ordering. Assume $M \models T^{\forall}_{S}$ and $x, y \in P^M_\eta$.

(1) When we say indiscernible, we shall always mean indiscernible for quantifier free formulas.
(2) We say that $x \equiv 0 \pmod{\omega}$ when $x = \lim(x)$. For $n < \omega$, we say that $x \equiv n+1 \pmod{\omega}$ where $x \neq \lim_\eta(x)$ and $\mathrm{pre}_\eta(x) \equiv n \pmod{\omega}$. Note that for a fixed n, the set

$\{x \mid x \equiv n \pmod{\omega}\}$ is quantifier free definable. In addition, if M is standard, then for every x there is some $n < \omega$ such that $x \equiv n \pmod{\omega}$ (where n is the unique number satisfying $\operatorname{lev}(x) = \alpha + n$ for a limit ordinal α).

(3) Say that $x \equiv y \pmod{\omega}$ if there is $n < \omega$ such that $x \equiv n \pmod{\omega}$ and $y \equiv n \pmod{\omega}$.

(4) Instead of $G_{n,n+1}$ we write G_n.

Analysis of indiscernibles in $\mathcal{T}$.

Definition 4.4. Let $\mathcal{T}$ be the class of models $M \models T_\omega^\forall$ that satisfy:

(1) M is standard (see Definition 4.2).
(2) For $t \in P_n$, $\operatorname{lev}(G_n(t)) \leq \operatorname{lev}(t)$.
(3) $G_n : \operatorname{Suc}(P_n) \to \operatorname{Suc}(P_n)$ (i.e., we demand that the image is also a successor).
(4) If $\langle s_i \mid i < \delta \rangle$ is an increasing sequence in $\operatorname{Suc}(P_n)$ such that $s_i \equiv s_j \pmod{\omega}$ for all $i < j < \delta$ then $i < j \Rightarrow G_n(s_i) \neq G_n(s_j)$.

Notation 4.5. For $M \in \mathcal{T}$ and $n < \omega$,

(1) We say that $s, t \in P_n^M$ are neighbors, denoted by $t \mathrel{E^{\mathrm{nb}}} s$ when $\{x \mid x < t\} = \{x \mid x < s\}$. This is an equivalence relation. As P_n is a normal tree, for t of a limit level its E^{nb}-class is $\{t\}$.
(2) Let $\operatorname{Suc}(M) = \bigcup \left\{\operatorname{Suc}\left(P_n^M\right) \mid n < \omega\right\}$.
(3) $\bar{s}$, $\bar{t}$ and $\bar{r}$ will denote δ-sequences, e.g., $\bar{s} = \langle s_i \mid i < \delta \rangle$.
(4) If $\bar{s}$ is contained in some P_n^M and n is clear from the context or insignificant, then we write $<$ instead of $<_n$ etc.

Definition 4.6. Recall that given $\delta^* \geq \omega$ and an indiscernible sequence $\bar{s} = \langle s_i \mid i < \delta^* \rangle$, its quantifier free Ehrenfeucht-Mostowski type (or in short quantifier free EM-type) is defined as $\left\langle \operatorname{tp}_{\mathrm{qf}}(s_0, \ldots, s_{n-1}) \mid n < \omega \right\rangle$. In general, a quantifier free EM-type is a sequence $\bar{p} = \langle p_n \mid n < \omega \rangle$ such that $p_n \in S_n^{\mathrm{qf}}(\emptyset)$.

We need the following generalization of indiscernible sequences for $\mathcal{T}$:

Definition 4.7. A sequence $\bar{s} = \langle s_i \mid i < \delta \rangle$ is called *nearly indiscernible* (in short *NI*) if:

(1) There is $n < \omega$ and an EM-type $\bar{p} = \left\langle p_k \in S_k^{\mathrm{qf}}(\emptyset) \mid k < \omega \right\rangle$ such that if $i_0 < \cdots < i_{k-1} < \delta$ and $i_j + n \leq i_{j+1}$ for all $j < k$, then $\left(s_{i_0}, \ldots, s_{i_{k-1}}\right) \models p_k$. (So for $\delta^* \leq \delta$ every sub-sequence $\left\langle s_{i_j} \mid j < \delta^* \right\rangle$ with $i_j + n \leq i_{j+1} < \delta$ is indiscernible and its quantifier free EM-type is $\bar{p}$.) We call this property *sparseness*.
(2) For $i, j < \delta$ and $k < \omega$, $\operatorname{tp}_{\mathrm{qf}}(s_i, \ldots, s_{i+k}) = \operatorname{tp}_{\mathrm{qf}}(s_j, \ldots, s_{j+k})$. We call this property *sequential homogeneity*.

Definition 4.8. A sequence $\bar{s} = \langle s_i \,|\, i < \delta \rangle$ is called *hereditarily nearly Indiscernible* (in short *HNI*) if:

For every term $\sigma(x_0, \ldots, x_{n-1})$, the sequence $\bar{t} = \langle t_i \,|\, i < \delta \rangle$ defined by $t_i = \sigma(s_i, \ldots, s_{i+n-1})$ is NI.

Remark 4.9. If $\bar{s}$ is HNI then it is NI, and for every term $\sigma(x_0, \ldots, x_{n-1})$, the sequence $\bar{t} = \langle t_i \,|\, i < \delta \rangle$ defined by $t_i = \sigma(s_i, \ldots, s_{i+n-1})$ is HNI. Indeed, for any term $\tau(x_0, \ldots, x_{k-1})$, let

$$\tau'(x_0, \ldots, x_{n+k-2}) = \tau(\sigma(x_0, \ldots, x_{n-1}), \ldots, \sigma(x_{k-1}, \ldots, x_{n+k-2})),$$

then the sequence $\bar{r} = \langle r_i \,|\, i < \delta \rangle$ defined by $r_i = \tau(t_i, \ldots, t_{i+k-1})$ is equal to $\tau'(s_i, \ldots, s_{i+n+k-2})$ thus it is NI.

Example 4.10. If $\bar{s} = \langle s_i \,|\, i < \delta \rangle$ is indiscernible, then it is HNI.

Proof. Suppose $\sigma(x_0, \ldots, x_{n-1})$ is a term. If $t_i = \sigma(s_i, \ldots, s_{i+n-1})$, then any sub-sequence of $\bar{t} = \langle t_i \,|\, i < \delta \rangle$ where the distance between two consecutive elements is at least n is an indiscernible sequence with a constant quantifier free EM-type. This shows sparseness.

For sequential homogeneity, note that for a quantifier free formula φ,

$$\varphi(t_i, \ldots, t_{i+k}) = \varphi(\sigma(s_i, \ldots, s_{i+n-1}), \ldots, \sigma(s_{i+k}, \ldots, s_{i+n+k-1})).$$

Let $i, j < \delta$. As $\mathrm{tp}_{\mathrm{qf}}(s_i, \ldots, s_{i+n+k-1}) = \mathrm{tp}_{\mathrm{qf}}(s_j, \ldots, s_{j+n+k-1})$, it follows that

$$\mathrm{tp}_{\mathrm{qf}}(t_i, \ldots, t_{i+k}) = \mathrm{tp}_{\mathrm{qf}}(t_j, \ldots, t_{j+k}).$$

□

Definition 4.11. Assume $M \in \mathcal{T}$.

(1) $\mathrm{ind}(M)$ is the set of all non-constant indiscernible sequences $\bar{s} \in \mathrm{Suc}(M)^{\delta}$.
(2) $\mathrm{HNind}(M)$ is the set of all non-constant HNI sequences $\bar{s} \in \mathrm{Suc}(M)^{\delta}$.
(3) $\mathrm{ai}(M)$ is the set of sequences $\bar{s}$ such that for some $n < \omega$, $\bar{s} \in \left(P_n^M\right)^{\delta}$ and $s_i \wedge s_{i+1} < s_{i+1} \wedge s_{i+2}$ (ai stands for "almost increasing", note that if $\bar{s}$ is increasing then it is here).
(4) $\mathrm{ind}_f(M)$ is the set of all sequences $\bar{s} \in \mathrm{ind}(M)$ such that $s_i \wedge s_j$ is constant for all $i < j < \delta$ (f stands for "fan").
(5) $\mathrm{ind}_i(M)$ is the set of all increasing sequences $\bar{s} \in \mathrm{ind}(M)$.
(6) $\mathrm{ind}_{\mathrm{ai}}(M) = \mathrm{ind}(M) \cap \mathrm{ai}(M)$.
(7) Define $\mathrm{HNind}_f(M)$, $\mathrm{HNind}_i(M)$ and $\mathrm{HNind}_{\mathrm{ai}}(M)$ similarly, but we demand that the sequences are HNI.

From now on, assume $M \in \mathcal{T}$.

Remark 4.12. If $\bar{s} \in \mathrm{ai}(M)$, then $s_i \wedge s_{i+n} = s_i \wedge s_{i+1}$ for all $2 \leq n < \omega$ and $i < \delta$ (prove by induction on n, using the fact that if $a \wedge b < b \wedge b'$ then $a \wedge b' = a \wedge b$).

Proposition 4.13. $\mathrm{HNind}(M) = \mathrm{HNind}_{\mathrm{ai}}(M) \cup \mathrm{HNind}_f(M)$.

Proof. Assume that $\bar{s} \in \mathrm{HNind}(M)$. Since $\bar{s}$ is NI, there is some $n < \omega$ that witnesses sparseness. As for $i < j < k$, $s_i \wedge s_j$ is comparable with $s_j \wedge s_k$, by Ramsey there is an infinite subset $A \subseteq \omega$ that satisfies one of the following possibilities:

(1) For all $i < j < k \in A$, $s_i \wedge s_j = s_j \wedge s_k$, or
(2) For all $i < j < k \in A$, $s_i \wedge s_j < s_j \wedge s_k$.

(note that it cannot be that $s_j \wedge s_k < s_i \wedge s_j$ because the trees are well ordered).

Assume (1) is true.

It follows that if $i < j < k < l \in A$ then $s_i \wedge s_j = s_j \wedge s_k = s_k \wedge s_l$. If $n \leq j - i, k - j, l - k$, then by the choice of n, the same is true for all $i < j < k < l < \delta$ where the distances are at least n. Moreover, given $i < j, k < l$ such that $n \leq j - i$ and $n \leq l - k$, then $s_i \wedge s_j = s_{\max\{j,l\}+n} \wedge s_{\max\{j,l\}+2n}$, and the same is true for $s_k \wedge s_l$. It follows that $s_i \wedge s_j = s_k \wedge s_l$.

Choose some $0 < i < n$.

Assume for contradiction that $s_0 \wedge s_i < s_i \wedge s_{2i}$, then by sequential homogeneity $\langle s_{i\alpha} \mid \alpha < \delta \rangle \in \mathrm{ai}(M)$. In this case, by Remark 4.12, $s_0 \wedge s_i < s_i \wedge s_{2i} = s_i \wedge s_{ni+i}$. But $s_0 \wedge s_{ni+i} = s_i \wedge s_{ni+i}$, and so on the one hand $s_0 \wedge s_i < s_0 \wedge s_{ni+i}$, and on the other hand $s_0 \wedge s_{ni+i} \leq s_i$ — together it's a contradiction.

It cannot be that $s_0 \wedge s_i > s_i \wedge s_{2i}$ since the trees are well ordered.

So (again by the sequential homogeneity) it must be that $s_0 \wedge s_i = s_i \wedge s_{2i} = \cdots = s_{ni} \wedge s_{ni+i}$. So necessarily $s_0 \wedge s_i \leq s_0 \wedge s_{ni}$, but in addition $s_0 \wedge s_{ni} = s_0 \wedge s_{ni+i}$ (since the distance is at least n) and so $s_0 \wedge s_i = s_{ni} \wedge s_{ni+i} \geq s_0 \wedge s_{ni}$, and hence $s_0 \wedge s_i = s_0 \wedge s_{ni} = s_0 \wedge s_n$.

It follows that $s_{i_0} \wedge s_{i_0+i} = s_{i_0} \wedge s_{i_0+n} = s_0 \wedge s_n$ for every $i_0 < \delta$. This is true for all i such that $i_0 + i < \delta$ and so $s_i \wedge s_j = s_0 \wedge s_n$ for all $i < j < \delta$. So in this case $\bar{s} \in \mathrm{HNind}_f(M)$.

Assume (2) is true. Assume that $i < j < k \in A$ and the distances are at least n. Then, as $s_i \wedge s_j < s_j \wedge s_k$, it follows from sparseness that $\langle s_{n\alpha} \mid \alpha < \delta \rangle \in \mathrm{ai}(M)$ and that $\langle s_0, s_{n+1}, s_{3n}, s_{4n}, \ldots \rangle \in \mathrm{ai}(M)$. In particular, by Remark 4.12, $s_0 \wedge s_n = s_0 \wedge s_{3n} = s_0 \wedge s_{n+1}$.

If $s_0 \wedge s_1 < s_1 \wedge s_2$, then $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$ by sequential homogeneity and we are done, so assume this is not the case.

It cannot be that $s_0 \wedge s_1 > s_1 \wedge s_2$ (because the trees are well ordered).

Assume for contradiction that $s_0 \wedge s_1 = s_1 \wedge s_2$. By sequential homogeneity it follows that $s_0 \wedge s_1 = s_n \wedge s_{n+1}$. We also know that $s_0 \wedge s_n = s_0 \wedge s_{n+1}$, and together we have $s_0 \wedge s_1 = s_0 \wedge s_{n+1}$, and again by sequential homogeneity, $s_n \wedge s_{2n+1} = s_n \wedge s_{n+1}$, and so $s_n \wedge s_{2n+1} = s_0 \wedge s_n$ — a contradiction (because the distances are at least n). $\square$

Definition 4.14. Define the function $H : \mathrm{HNind}_{\mathrm{ai}}(M) \to \mathrm{HNind}(M)$ as follows: given $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$, let $H(\bar{s}) = \bar{t}$ where $t_i = G(\mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1}))$. (Recall that $G = G_n$ where the sequence $\bar{s}$ is contained in P_n^M.)

Remark 4.15. H is well defined: if $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$ then $H(\bar{s})$ is in $\mathrm{HNind}(M)$. This is because $\bar{t} = H(\bar{s})$ is not constant — by Clause (4) of Definition 4.4 (it is applicable: the sequence $\langle s_i \wedge s_{i+1} \mid i < \delta \rangle$ is NI and increasing, so there is some $n < \omega$ such that $s_i \wedge s_{i+1} \equiv n \pmod{\omega}$ for all $i < \delta$, and hence $\langle \lim(s_i \wedge s_{i+1}) \mid i < \delta \rangle$ is increasing).

As usual, we denote $H^{(0)}(\bar{s}) = \bar{s}$ and $H^{(n)}(\bar{s}) = H\left(H^{(n-1)}(\bar{s})\right)$ for $n > 0$.

Corollary 4.16. *Let $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M)$. If for no $n < \omega$, $H^{(n)}(\bar{s}) \in \mathrm{HNind}_f(M)$, then for all $n < \omega$, $H^{(n)}(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M)$. Moreover, in this case there exists some $K < \omega$ such that for all $n \geq K$, if $\bar{t} = H^{(n)}(\bar{s})$ then $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$.*

Proof. By Proposition 4.13, it follows by induction on $n < \omega$ that $H^{(n)}(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M)$ and so $H^{(n+1)}(\bar{s})$ is well defined.

For $n < \omega$, let $\bar{s}_n = H^{(n)}(\bar{s})$, and let us enumerate this sequence as $\bar{s}_n = \langle s_{n,i} \mid i < \delta \rangle$.

$\mathrm{lev}(\lim(s_{n,0} \wedge s_{n,1})) < \mathrm{lev}(s_{n,0})$ because $\mathrm{lev}(s_{n,0})$ is a successor ordinal (by Clause (3) of Definition 4.4) while $\mathrm{lev}(\lim(x))$ is a limit ordinal for all $x \in M$.

So $\mathrm{lev}(\mathrm{suc}(\lim(s_{n,0} \wedge s_{n,1})), s_{n,1}) \leq \mathrm{lev}(s_{n,0})$, and so by Clause (2) of Definition 4.4,

$$\langle \mathrm{lev}(s_{n,0}) \mid n < \omega \rangle$$

is a $\leq$-decreasing sequence.

Hence there is some $K < \omega$ and some α such that $\mathrm{lev}(s_{n,0}) = \alpha$ for all $K \leq n$. Assume without loss of generality that $K = 0$.

Let $n < \omega$. We know that

$$\begin{aligned} \mathrm{lev}(s_{n+1,0}) \leq \mathrm{lev}(\mathrm{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,1})) &= \\ &= \mathrm{lev}(\mathrm{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,0})) \\ &\leq \mathrm{lev}(s_{n,0}) \end{aligned}$$

But the left hand side and the right hand side are equal and $\mathrm{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,0}) \leq s_{n,0}$, so

$$\mathrm{suc}(\lim(s_{n,0} \wedge s_{n,1}), s_{n,0}) = s_{n,0}.$$

By sequential homogeneity, $\mathrm{suc}(\lim(s_{n,i} \wedge s_{n,i+1}), s_{n,i}) = s_{n,i}$ for all $i < \delta$ as desired. □

Constructing a model in $\mathcal{T}$. By Assumption 4.1, we have a function $\mathbf{c} : [\kappa]^{<\omega} \to \theta$ that witnesses the fact that $\kappa \not\to (\delta)_{\theta}^{<\omega}$ (the letter $\mathbf{c}$ stands for "coloring"). Fix $\mathbf{c}$, and also a pairing function (a bijection) $\mathrm{pr} : \theta \times \theta \to \theta$ and projections $\pi_1, \pi_2 : \theta \to \theta$ (defined so that $\pi_1 (\mathrm{pr} (\alpha, \beta)) = \alpha$ and $\pi_2 (\mathrm{pr} (\alpha, \beta)) = \beta$). For us, 0 is considered to be a limit ordinal. For an ordinal α, let $\mathrm{Lim} (\alpha) = \{\beta < \alpha \,|\, \beta \text{ is a limit}\}$.

Definition 4.17. $\mathbf{F} = \mathbf{F}_{\theta,\kappa}$ is the set of triples $\mathbf{f} = (d, M, E) = (d_{\mathbf{f}}, M_{\mathbf{f}}, E_{\mathbf{f}})$ such that:

(1) M is a standard model of $T^{\forall}_{\{\emptyset\}}$ and $M = P_{\emptyset}^{M}$ (i.e., M is just a standard tree). Some notation:
  (a) We write $<_{\mathbf{f}}$ instead of $<_{\emptyset}^{M_{\mathbf{f}}}$ etc., or omit $\mathbf{f}$ when it is clear from the context.
  (b) Let $\mathrm{Suc}_{\mathrm{lim}} (M)$ be the set of all $t \in \mathrm{Suc} (M)$ such that $\mathrm{lev} (t) - 1$ is a limit.

(2) E is an equivalence relation refining E^{nb} (see Notation 4.5). Moreover, for levels that are not $\alpha + 1$ for limit α it equals E^{nb}. By normality E is equality on limit elements, so it is interesting only on $\mathrm{Suc}_{\mathrm{lim}} (M)$.

(3) For every E^{nb} equivalence class C, $|C/E| < \kappa$.

(4) d is a function from $\left\{\eta \in \mathrm{Suc}_{\mathrm{lim}} (M)^{<\omega} \,|\, \eta (0) < \cdots < \eta (\lg (\eta) - 1)\right\}$ to θ.

(5) We say that $\mathbf{f}$ is *hard* if there is no increasing sequence of elements $\bar{s}$ of length δ from $\mathrm{Suc}_{\mathrm{lim}} (M)$ such that:

  For all $n < \omega$ there is $c_n < \theta$ such that for every $i_0 < \cdots < i_{n-1} < \delta$, $d \left(s_{i_0}, \ldots, s_{i_{n-1}}\right) = c_n$.

Example 4.18. Consider $(\kappa, <)$ as a standard tree. Let $\mathbf{f}_{\mathbf{c}} = (\mathbf{c} \restriction \mathrm{Suc}_{\mathrm{lim}} (\kappa), \kappa, =) \in \mathbf{F}$. Then $\mathbf{f}_{\mathbf{c}}$ is hard.

Definition 4.19. Let $\mathbf{f} = (d_{\mathbf{f}}, M_{\mathbf{f}}, E_{\mathbf{f}}) \in \mathbf{F}$, let x be a variable and $A \subseteq \mathrm{Suc}_{\mathrm{lim}} (M_{\mathbf{f}})$ be a linearly ordered set.

(1) Say that p is a *d-type* over A if p is a consistent set of equations of the form
  $d (a_0, \ldots, a_{n-1}, x) = \varepsilon$ where $n < \omega$, $\varepsilon < \theta$ and $a_0 < \cdots < a_{n-1} \in A$.

(2) Consistency here means that p does not contain a subset of the form

$$\{d (a_0, \ldots, a_{n-1}, x) = \varepsilon, d (a_0, \ldots, a_{n-1}, x) = \varepsilon'\}$$

for $\varepsilon \neq \varepsilon'$.

(3) Say that p is complete if for every increasing sequence $\langle a_0, \ldots, a_{n-1}\rangle$ from A there is such an equation in p.

(4) If $B \subseteq A$ then for a d-type p over A, let

$$p \restriction B = \{d (a_0, \ldots, a_{n-1}, x) = \varepsilon \in p \,|\, a_0, \ldots, a_{n-1} \in B\}.$$

(5) For $t \in \mathrm{Suc}_{\lim}(M_{\mathbf{f}})$,

$$\begin{aligned} \mathrm{dtp}(t/A) &= \{d(a_0,\ldots,a_{n-1},x)=\varepsilon| \\ &\quad a_0<\cdots<a_{n-1}\in A, a_{n-1}<t, d_{\mathbf{f}}(a_0,\ldots,a_{n-1},t)=\varepsilon\}. \end{aligned}$$

For an element $t \in \mathrm{Suc}_{\lim}(M)$, $t \models p$ means that t satisfies all the equations in p when we replace d by d_p.

(6) Let $S_d(A)$ be the set of all complete d-types over A.

Now we define the function $\mathbf{g}$ from $\mathbf{F}$ to $\mathbf{F}$.

Definition 4.20. For $\mathbf{f}=(M_{\mathbf{f}},d_{\mathbf{f}},E_{\mathbf{f}})\in\mathbf{F}$, define $\mathbf{g}=\mathbf{g}(\mathbf{f})=(M_{\mathbf{g}},d_{\mathbf{g}},E_{\mathbf{g}})\in\mathbf{F}$ by:

- $M_{\mathbf{g}}$ is the set of pairs $a=(\Gamma,\eta)=(\Gamma_a,\eta_a)$ such that:
  (1) There is $\alpha<\kappa$ such that $\eta:\alpha\to\mathrm{Suc}_{\lim}(M_{\mathbf{f}})$ and $\Gamma:\mathrm{Lim}(\alpha)\to S_d(M_{\mathbf{f}})$. Denote $\lg(\Gamma,\eta)=\lg(\eta)=\alpha$. If α is a successor ordinal, let $l_{(\Gamma,\eta)}=\eta(\alpha-1)\in M_{\mathbf{f}}$.
  (2) For $\beta<\alpha$ limit, $\Gamma(\beta)\in S_d(\{\eta(\beta')\,|\,\beta'\le\beta\})$.
  (3) If $0<\alpha$ then $\eta(0)\models\Gamma(0)\restriction\emptyset$.
  (4) For $\beta'<\beta<\alpha$, $\eta(\beta')<_{\mathbf{f}}\eta(\beta)$ (η is increasing in $M_{\mathbf{f}}$).
  (5) If $\beta'<\beta<\alpha$ are limit ordinals then $\Gamma(\beta')\subseteq\Gamma(\beta)$.
  (6) If $\beta'<\beta<\alpha$ and β' is a limit ordinal then $\eta(\beta)\models\Gamma(\beta')$.
  (7) For $\beta<\alpha$, there is no $t<_{\mathbf{f}}\eta(\beta)$ that satisfies
    (a) $t\in\mathrm{Suc}_{\lim}(M_{\mathbf{f}})$,
    (b) $\eta(\beta')<_{\mathbf{f}}t$ for all $\beta'<\beta$,
    (c) $t\models\Gamma(0)\restriction\emptyset$, and
    (d) $t\models\Gamma(\beta')$ for all limit $\beta'<\beta$.
  (8) The order on $M_{\mathbf{g}}$ is $(\Gamma,\eta)<_{\mathbf{g}}(\Gamma',\eta')$ iff $\Gamma\triangleleft\Gamma'$ and $\eta\triangleleft\eta'$ (where $\triangleleft$ means first segment). This defines a standard tree structure on $M_{\mathbf{g}}$.
  It follows that for $a=(\Gamma,\eta)$, $\mathrm{lev}(a)=\lg(a)$.
- $d_{\mathbf{g}}$ is defined as follows: suppose $a_0<_{\mathbf{g}}\cdots<_{\mathbf{g}}a_{n-1}\in\mathrm{Suc}_{\lim}(M_{\mathbf{g}})$ and $a_i=(\Gamma_i,\eta_i)$.
  Let $t_i=l_{a_i}=\eta_i(\lg(a_i)-1)$ and $p=\Gamma_{n-1}(\lg(a_{n-1})-1)$. Let $\varepsilon\in\theta$ be the unique color such that $d(t_0,\ldots,t_{n-1},x)=\varepsilon\in p$. Then

$$d_{\mathbf{g}}(a_0,\ldots,a_{n-1})=\mathrm{pr}(\varepsilon,\mathbf{c}(\mathrm{lev}(a_0),\ldots,\mathrm{lev}(a_{n-1}))).$$

- $E_{\mathbf{g}}$ is defined as follows: $(\Gamma_1,\eta_1)\,E_{\mathbf{g}}\,(\Gamma_2,\eta_2)$ iff
  - $\lg(\eta_1)=\lg(\eta_2)$, so equals to some $\alpha<\kappa$,
  - $\eta_1\restriction\beta=\eta_2\restriction\beta,\Gamma_1\restriction\beta=\Gamma_2\restriction\beta$ for all $\beta<\alpha$ (so they are E^{nb}-equivalent),
  - $\Gamma_1(0)\restriction\emptyset=\Gamma_2(0)\restriction\emptyset$, and

– If $\alpha = \beta + n$ for $\beta \in \mathrm{Lim}(\alpha)$ and $n < \omega$ then for all $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < \beta$,

$$d(\eta_1(\alpha_0), \ldots, \eta_1(\alpha_{k-1}), \eta_1(\beta), x) = \varepsilon \in \Gamma_1(\beta) \quad \Leftrightarrow$$

$$d(\eta_2(\alpha_0), \ldots, \eta_2(\alpha_{k-1}), \eta_2(\beta), x) = \varepsilon \in \Gamma_2(\beta)$$

Note that it follows that if $1 < n$, and $(\Gamma_1, \eta_1)\, E^{\mathrm{nb}}\, (\Gamma_2, \eta_2)$, then $\Gamma_1(\beta) = \Gamma_2(\beta)$ and $\eta_1(\beta) = \eta_2(\beta)$, so they are E-equivalent.

In the next claims we assume that $\mathbf{f} \in \mathbf{F}$ and $\mathbf{g} = \mathbf{g}(\mathbf{f})$.

Remark 4.21. $\mathrm{lev}(a) = \mathrm{lg}(a)$ for $a \in M_{\mathbf{g}}$ and $a\, E^{\mathrm{nb}}\, b$ iff $\mathrm{lev}(a) = \mathrm{lev}(b)$ and $a \restriction \alpha = b \restriction \alpha$ for all $\alpha < \mathrm{lev}(a)$.

Claim 4.22. $\mathbf{g} \in \mathbf{F}_{\theta,\kappa}$ and moreover it is hard.

Proof. The fact that $M_{\mathbf{g}}$ is a standard tree is trivial. Also, E refines E^{nb} by definition.

We must show that the number of E-classes inside a given E^{nb}-class is bounded.

Given a (partial) d-type p over $M_{\mathbf{f}}$ and $t \in M_{\mathbf{f}}$, let p^t be the set of equations we get by replacing all appearances of t by a special letter $*$.

Assume that A is an E^{nb}-class contained in $\mathrm{Suc}_{\mathrm{lim}}(M_{\mathbf{g}})$, and that for every $a \in A$, $\mathrm{lev}(a) = \alpha+1$ where α is limit. Assume $a \in A$ and let $B = \{*\} \cup \mathrm{im}(\eta_a) \setminus \{l_a\}$ (since A is an E^{nb}-class, this set does not depend on the choice of a). Consider the map ε defined by $a \mapsto \Gamma_a(\alpha)^{l_a}$. Then, $a, b \in A$ are E equivalent iff $\varepsilon(a) = \varepsilon(b)$. Therefore this map induces an injective map from A/E to this set of types. The size of this set is at most $2^{|B|+\theta+\aleph_0}$. But $|B| = |\alpha| < \kappa$, and $\theta < \kappa$ by assumption, so $|A/E| < \kappa$ (as κ is a strong limit).

$\mathbf{g}$ is hard: if $\bar{s} = \langle s_i \,|\, i < \delta \rangle$ is a counterexample then $\langle \mathrm{lev}(s_i) \,|\, i < \delta \rangle$ would be a homogeneous sub-sequence, contradicting the choice of $\mathbf{c}$. $\square$

Proposition 4.23.

(1) For all $a \in \mathrm{Suc}(M_{\mathbf{g}})$, $\mathrm{lev}_{M_{\mathbf{g}}}(a) \leq \mathrm{lev}_{M_{\mathbf{f}}}(l_a)$.

(2) Assume $t \in \mathrm{Suc}_{\mathrm{lim}}(M_{\mathbf{f}})$. Then there is some $a = (\Gamma, \eta) \in \mathrm{Suc}(M_{\mathbf{g}})$ such that $l_a = t$.

Proof. (1) Let $\mathrm{lev}_{M_{\mathbf{g}}}(a) = \alpha$. Then $\langle \mathrm{pre}_{\mathbf{f}}(\eta_a(\beta)) \,|\, \beta < \alpha \rangle$ is an increasing sequence below l_a, hence $\alpha \leq \mathrm{lev}_{M_{\mathbf{f}}}(l_a)$.

(2) Let Γ be the set of ordinals γ for which there is a sequence $\langle (\Gamma_\alpha, \eta_\alpha) \,|\, \alpha < \gamma \rangle$ such that for every $\alpha < \gamma$:

⋆ $(\Gamma_\alpha, \eta_\alpha) \in M_{\mathbf{g}}$; $\mathrm{lg}(\eta_\alpha) = \alpha$; it is an increasing sequence in $<_{\mathbf{g}}$; $\eta_\alpha(\beta) < t$ for $\beta < \alpha$ and if β is a limit then $\Gamma_\alpha(\beta) = \mathrm{dtp}(t / \{\eta_\alpha(\beta') \,|\, \beta' \leq \beta\})$.

We try to construct such a sequence $\langle (\Gamma_\alpha, \eta_\alpha) \,|\, \alpha < \gamma \rangle$ as long as we can. By (1), $\mathrm{lev}_{M_{\mathbf{f}}}(t)+1 \notin \Gamma$, so $\gamma < \kappa$ and γ must be a successor ordinal. Let $\beta = \gamma - 1$.

Define $\eta = \eta_\beta \cup \{(\beta, t)\}$, $\Gamma = \Gamma_\beta$ unless β is a limit, in which case let $\Gamma(\beta)$ be any complete type in x over $\{\eta(\beta') \mid \beta' \leq \beta\}$ containing $\bigcup \{\Gamma_\beta(\beta') \mid \beta' \in \mathrm{Lim}(\beta)\} \cup \{d(x) = d_{\mathbf{f}}(t)\}$.

By construction, $(\Gamma, \eta) \in M_{\mathbf{g}}$. $\square$

Now we build a model in $\mathcal{T}$ using $\mathbf{F}$:

Definition 4.24.

(1) Define $\mathbf{f}_0 = \mathbf{f}_{\mathbf{c}}$ (see Example 4.18), and for $n < \omega$, let $\mathbf{f}_{n+1} = \mathbf{g}(\mathbf{f}_n)$.

(2) Define $P_n = M_{\mathbf{f}_n}$, $d_n = d_{\mathbf{f}_n}$ and $E_n = E_{\mathbf{f}_n}$.

(3) Let $M_{\mathbf{c}} = \bigcup_{n<\omega} P_n$ (we assume that the P_n's are mutually disjoint). So $P_n^{M_{\mathbf{c}}} = P_n$.

(4) $M_{\mathbf{c}} \models T_\omega^\forall$ when we interpret the relations in the language as they are induced from each P_n and in addition:

(5) Define $G_n^{M_c} : \mathrm{Suc}(P_n) \to \mathrm{Suc}(P_{n+1})$ as follows: let $a \in \mathrm{Suc}(P_n)$ and $a' = \mathrm{suc}(\lim(a), a)$. By Proposition 4.23, there is an element $(\Gamma, \eta)_a \in \mathrm{Suc}(P_{n+1})$ such that $l_{(\Gamma,\eta)_a} = a'$. Choose such an element for each a, and define $G_n^{M_{\mathbf{c}}}(a) = (\Gamma, \eta)_a$.

Corollary 4.25. $M_{\mathbf{c}} \in \mathcal{T}$.

Proof. All the demands of Definition 4.4 are easy. For instance, Clause (2) follows from Proposition 4.23. Clause (4) follows from the fact that if $\langle s_i \mid i < \delta \rangle$ is an increasing sequence in P_n such that $s_i \equiv s_j \pmod{\omega}$ then $\langle \mathrm{suc}(\lim(s_i), s_i) \mid i < \delta \rangle$ is increasing, so $l_{G_n(s_i)} \neq l_{G_n(s_j)}$ for $i \neq j$. $\square$

Notation 4.26. Again, we do not write the index $\mathbf{f}_n$ when it is clear from the context (for instance we write $d(s_0, \ldots, s_k)$ instead of $d_{\mathbf{f}_n}(s_0, \ldots, s_k)$).

The following lemma and corollary will show that starting with any HNI sequence in $M_{\mathbf{c}}$, by applying H to it many times, we must get a fan.

Lemma 4.27. *Assume that $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_{\mathbf{c}})$ and $\bar{t} = H(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M_{\mathbf{c}})$ (see Definition 4.14) satisfy that for all $i < \delta$:*

- $\mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_i) = s_i$, *and*
- $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$.

Then, letting $u_i = \mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1})$ and $v_i = \mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_{i+1})$ for $i < \delta$:

(1) $\langle d(u_i) \mid 1 \leq i < \delta \rangle$ is constant.

(2) $d(u_{i_0}, \ldots, u_{i_n}) = \pi_1(d(v_{i_0}, \ldots, v_{i_{n-1}}))$ for $1 \leq i_0 < \cdots < i_n < \delta$ (recall that π_1 is defined by $\pi_1(\mathrm{pr}(i, j)) = i$).

Proof. (1) By definition, $t_i = G(u_i)$. Denote $t_i = (\Gamma_i, \eta_i)$. As $\langle t_i \wedge t_{i+1} \mid i < \delta \rangle$ is an increasing sequence (because $\bar{t} \in \mathrm{HNind}_{\mathrm{ai}}(M_{\mathbf{c}})$), $0 < \mathrm{lev}(t_1 \wedge t_2)$. Let $p = \Gamma_{t_1 \wedge t_2}(0) \restriction \emptyset$. Then $p = \Gamma_i(0) \restriction \emptyset$ for all $1 \leq i$ (it may be that $t_1 \wedge t_0 = \emptyset$ and in this case we have no information on t_0). Assume that $p = \{d(x) = \varepsilon\}$ for some $\varepsilon < \theta$. Then, by Definition 4.20, Clauses (3) and (6), $d(\eta_i(\beta)) = \varepsilon$ for all $1 \leq i < \delta$ and $\beta < \lg(\eta_i)$. As $u_i = l_{t_i}$ we are done.

(2) Denote $v_i = (\Gamma'_i, \eta'_i)$. By our assumptions on $\bar{t}$, $t_i\ E^{\mathrm{nb}}\ v_i$ hence if $\bar{t}$ is increasing then $\bar{v} = \bar{t}$. Assume that it is not increasing. Then $t_i \wedge t_{i+1} < t_i$ so $\lim(t_i \wedge t_{i+1}) = t_i \wedge t_{i+1}$. Let $\alpha_i = \beta_i + 1 = \mathrm{lev}(t_i) = \lg(\eta'_i)$, then β_i is a limit ordinal and $t_i \restriction \beta_i = v_i \restriction \beta_i$. So for $1 \leq i$, $\Gamma'_i(0) \restriction \emptyset = \Gamma_i(0) \restriction \emptyset = p$ and $\Gamma'_i \restriction \beta_i = \Gamma_i \restriction \beta_i$.

Note that for $1 \leq i$, l_{t_i} and l_{v_i} are both below $u_{i+1} = l_{t_{i+1}}$ (as $v_i \leq t_{i+1}$ and $l_{t_i} = u_i < u_{i+1}$), that they both satisfy p and that they both satisfy the equations in $\Gamma(\beta)$ for each limit $\beta < \beta_i$, so if for instance $l_{t_i} < l_{v_i}$, we will have a contradiction to Definition 4.20, Clause (7).

So, in any case (whether or not $\bar{t}$ is increasing), we have $l_{v_i} = l_{t_i} = u_i$.

By choice of $\bar{v}$ and the assumptions on $\bar{t}$, $\bar{v}$ is increasing so d is defined on finite subsets of it.

Assume $1 \leq i_0 < \cdots < i_n < \delta$. Then for every $\sigma < \theta$, by the choice of d in Definition 4.20:

- $\boxtimes$ $\pi_1\left(d\left(v_{i_0}, \ldots, v_{i_{n-1}}\right)\right) = \sigma$ iff
- $\boxtimes$ $d\left(l_{v_{i_0}}, \ldots, l_{v_{i_{n-1}}}, x\right) = \sigma \in \Gamma'_{i_{n-1}}\left(\beta_{i_{n-1}}\right)$ iff
- $\boxtimes$ $d\left(l_{v_{i_0}}, \ldots, l_{v_{i_{n-1}}}, x\right) = \sigma \in \Gamma'_{i_n}\left(\beta_{i_{n-1}}\right)$ (because $\Gamma'_{i_n} \restriction \alpha_{i_{n-1}} = \Gamma'_{i_{n-1}} \restriction \alpha_{i_{n-1}}$) iff
- $\boxtimes$ $d\left(l_{v_{i_0}}, \ldots, l_{v_{i_{n-1}}}, l_{v_{i_n}}\right) = \sigma$ (this follows from Clause (6) of Definition 4.20) iff
- $\boxtimes$ $d(u_{i_0}, \ldots, u_{i_n}) = \sigma$ (because $l_{v_i} = u_i$).

$\square$

Corollary 4.28. *If $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_{\mathbf{c}})$ then there must be some $n < \omega$ such that $H^{(n)}(\bar{s}) \in \mathrm{HNind}_f(M_{\mathbf{c}})$ (see Definition 4.14).*

Proof. If not, by Corollary 4.16, for all $n < \omega$, $H^{(n)}(\bar{s}) \in \mathrm{HNind}_{\mathrm{ai}}(M_{\mathbf{c}})$. Moreover, there exists some $K < \omega$ such that for all $K \leq n$, if $\bar{t} = H^{(n)}(\bar{s})$ then $\mathrm{suc}(\lim(t_i \wedge t_{i+1}), t_i) = t_i$. Without loss, $K = 0$ (i.e., this is true also for $\bar{s}$).

Claim. If $\bar{s}$ is such a sequence then for all $n < \omega$, $d\left(u_{i_0}, \ldots, u_{i_{n-1}}\right)$ is constant for all $1 \leq i_0 < \cdots < i_{n-1} < \delta$ where $u_i = \mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1})$ for $i < \delta$.

Proof. (of claim) Prove by induction on n using Lemma 4.27. $\square$

But this claim contradicts the fact that for all $k < \omega$, $\mathbf{f}_k$ is hard. $\square$

The next lemma and corollaries are the main conclusion of this section:

Lemma 4.29. *If $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}(M_{\mathbf{c}})$ and $\bar{t} = H(\bar{s}) \in \mathrm{HNind}_f(M_{\mathbf{c}})$ then $\neg(v_i\ E\ v_j)$ for $i < j < \delta$ where $v_i = \mathrm{suc}(\lim(t_{i+1} \wedge t_i), t_i)$.*

Proof. Let $t = t_0 \wedge t_1$, so $t = t_i \wedge t_j$ for all $i < j < \delta$. Let $u_i = \mathrm{suc}(t, t_i)$. As $t_i \neq t_j$ for $i < j < \delta$, $u_i \neq u_j$. In addition

$$l_{u_i} \leq l_{t_i} = \mathrm{suc}(\lim(s_i \wedge s_{i+1}), s_{i+1}) \leq s_{i+1} \wedge s_{i+2}$$

and $\langle s_i \wedge s_{i+1} \,|\, i < \delta \rangle$ is increasing so l_{u_i} and l_{u_j} are comparable.

First assume that $\alpha = \mathrm{lev}\,(t) > 0$. Then $\Gamma_t\,(0) = \Gamma_{t_i}\,(0)$ for $i < \delta$. For all $i < j < \delta$, $l_{u_i} \models \Gamma_{u_j}\,(0) \restriction \emptyset$, l_{u_i} is greater than $\eta_{u_j}\,(\beta) = \eta_t\,(\beta)$ for all $\beta < \alpha$ and $l_{u_i} \models \Gamma_{u_j}\,(\beta) = \Gamma_t\,(\beta)$ for all limit $\beta < \alpha$. So by Definition 4.20, Clause (7), $l_{u_i} = l_{u_j}$, so $\eta_{u_i} = \eta_{u_j}$ for all $i < j < \delta$.

But since $u_i \neq u_j$, it necessarily follows that $\Gamma_{u_i} \neq \Gamma_{u_j}$. If $\alpha = \beta + 1$ for some β, then by definition of the function $\mathbf{g}$, $\Gamma_{u_i} = \Gamma_{u_i} \restriction \alpha = \Gamma_t$ (because Γ was defined only for limit ordinals). So necessarily α is a limit, and it follows that $\lim\,(t) = t$ so $v_i = u_i$. Now it is clear that $\Gamma_{v_i}\,(\alpha) \neq \Gamma_{v_j}\,(\alpha)$ and by definition of E, $\neg\,(v_i \mathrel{E} v_j)$ for all $i < j < \delta$.

If $\alpha = 0$, then as before $v_i = u_i$ (because $\lim\,(t) = t$). We cannot use the same argument (because $\Gamma_t\,(0)$ is not defined), so we take care of each pair $i < j < \delta$ separately. If $\Gamma_{v_i}\,(0) \restriction \emptyset = \Gamma_{v_j}\,(0) \restriction \emptyset$ then the argument above will work and $\neg\,(v_i \mathrel{E} v_j)$. If $\Gamma_{v_i}\,(0) \restriction \emptyset \neq \Gamma_{v_j}\,(0) \restriction \emptyset$, then $\neg\,(v_i \mathrel{E} v_j)$ follows directly from the definition. $\square$

Finally we have

Corollary 4.30. *If $\bar{s} \in \mathrm{HNind}_{\mathrm{ai}}\,(M_{\mathbf{c}})$, then there is some $\bar{v} \in \mathrm{HNind}_f\,(M_{\mathbf{c}})$ such that $v_i = \mathrm{suc}\,(\lim\,(v_i)\,, v_i)$, $v_i \mathrel{E^{\mathrm{nb}}} v_j$ but $\neg\,(v_i \mathrel{E} v_j)$ for $i < j < \delta$.*

Proof. By Corollary 4.28, there is some minimal $n < \omega$ such that $\bar{t} = H^{(n+1)}\,(\bar{s}) \in \mathrm{HNind}_f\,(M_{\mathbf{c}})$. Let $v_i = \mathrm{suc}\,(\lim\,(t_{i+1} \wedge t_i)\,, t_i)$ for $i < \delta$. By Lemma 4.29, we have that $v_i \mathrel{E^{\mathrm{nb}}} v_j$ but $\neg\,(v_i \mathrel{E} v_j)$ for $i < j < \delta$ (in particular $v_i \neq v_j$). So necessarily $t = t_i \wedge t_j$ is a limit and $v_i = \mathrm{suc}\,(t, v_i)$. $\square$

Corollary 4.31. *If there is some $\bar{s} \in \mathrm{ind}\,(M_{\mathbf{c}})$ such that $s_i \in P_0^{M_{\mathbf{c}}}$ for all $i < \delta$, then there is some $\bar{v} \in \mathrm{ind}_f\,(M_{\mathbf{c}})$ such that $v_i \in \mathrm{Suc}_{\lim}\,(M_{\mathbf{c}})$, $v_i \mathrel{E^{\mathrm{nb}}} v_j$ but $\neg\,(v_i \mathrel{E} v_j)$ for $i < j < \delta$.*

Proof. Since $P_0 = \kappa$, any sequence $\bar{s}$ in $\mathrm{ind}\,(M_{\mathbf{c}})$ in P_0 must be increasing. So by the last corollary there is some $\bar{v} \in \mathrm{HNind}_f\,(M_{\mathbf{c}})$ like there. But then by sparseness (see Definition 4.7) there is some $n < \omega$ such that $\langle v_{ni} \,|\, i < \delta \rangle$ is indiscernible. $\square$

Remark 4.32. In this section it becomes clear why we needed to use discrete trees and not dense ones (as in [KS12]). In Corollary 4.31, we started with an increasing sequence in $P_0 = \kappa$, and then applied a definable map on it, to get a new HNI sequence $\bar{s}$, but this sequence might be almost increasing and not increasing (i.e., in $\mathrm{ind}_{\mathrm{ai}}$). Since we wanted the coloring function d to be defined on increasing sequences, we needed again to get an increasing sequence, so this is done by taking $s_i \wedge s_{i+1}$. This sequence is increasing, but in order for the coloring d to affect the coloring of the original sequence (as in Lemma 4.27), we need this definable map to give us a successor of $s_i \wedge s_{i+1}$. Trial and error has shown that adding the function "successor to the meet" instead of just successor results in losing AP, so we needed the successor function. The predecessor function is not necessary (in existentially closed models, if $x > \lim_\eta\,(x)$, x has a predecessor), but there is no price to adding it, and it simplifies the theory a bit.

5. Proof of the main theorem

In this section we prove Main Theorem A.

We start with the easy direction.

Proposition 5.1. *Let κ, θ be cardinals and $\delta \geq \omega$ a limit ordinal. If $\kappa \to (\delta)_\theta^{<\omega}$ then for every $n \leq \omega$ and every theory T of cardinality $|T| \leq \theta$, $\kappa \to (\delta)_{T,n}$.*

Proof. For convenience, let $\bar{x}_i$ for $i < \omega$ be disjoint n-tuples of variables and let $L(T)$ be the set of formulas in T in $\{\bar{x}_i \,|\, i < \omega\}$.

Let $\langle \bar{a}_i \,|\, i < \kappa \rangle$ be a sequence of n-tuples in a model $M \models T$. Define $c : [\kappa]^{<\omega} \to L(T) \cup \{0\}$ as follows:

Given an increasing sequence $\eta \in \kappa^{<\omega}$, if $\lg(\eta)$ is odd, then $c(\eta) = 0$. If not, assume it is $2k$ and that $\eta = \langle \alpha_i \,|\, i < 2k \rangle$. If $\bar{a}_{\alpha_0} \cdots \bar{a}_{\alpha_{k-1}} \equiv \bar{a}_{\alpha_k} \cdots \bar{a}_{\alpha_{2k-1}}$ then $c(\eta) = 0$. If not there is a formula $\varphi(\bar{x}_0, \ldots, \bar{x}_{k-1})$ such that $M \models \varphi(\bar{a}_{\alpha_0}, \ldots, \bar{a}_{\alpha_{k-1}}) \wedge \neg\varphi(\bar{a}_{\alpha_k}, \ldots, \bar{a}_{\alpha_{2k-1}})$, so choose such a φ and define $c(\eta) = \varphi$. By assumption there is a sub-sequence $\langle \bar{a}_{\alpha_i} \,|\, i < \delta \rangle$ on which c is homogeneous. Without loss, assume that $\alpha_i = i$ for $i < \delta$.

It follows that $\langle \bar{a}_i \,|\, i < \delta \rangle$ is an indiscernible sequence:

Suppose there are some $i_0 < i_1 < \cdots < i_{2k-1} < \delta$ such that $\bar{a}_{i_0} \cdots \bar{a}_{i_{k-1}} \not\equiv \bar{a}_{i_k} \cdots \bar{a}_{i_{2k-1}}$. Since δ is limit there are some ordinals $i_{2k}, \ldots, i_{3k-1}$ such that $i_{2k-1} < i_{2k} < \cdots < i_{3k-1} < \delta$.

Since c is homogeneous, there is a formula φ such that $c(\langle i_k, \ldots, i_{3k-1} \rangle) = c(\langle i_0, \ldots, i_{2k-1} \rangle) = \varphi$, meaning that

$$M \models \varphi(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{k-1}}) \wedge \neg\varphi(\bar{a}_{i_k}, \ldots, \bar{a}_{i_{2k-1}})$$

and

$$M \models \varphi(\bar{a}_{i_k}, \ldots, \bar{a}_{i_{2k-1}}) \wedge \neg\varphi(\bar{a}_{i_{2k}}, \ldots, \bar{a}_{i_{3k-1}})$$

— a contradiction.

Now let $i_0 < \cdots < i_{k-1} < \delta$ be any increasing sequence. Let $j < \delta$ be greater than i_{k-1}. Then $\bar{a}_{i_0} \cdots \bar{a}_{i_{k-1}} \equiv \bar{a}_j \cdots \bar{a}_{j+k-1} \equiv \bar{a}_0 \cdots \bar{a}_{k-1}$ and we are done. $\square$

From now on let $\mathbb{S} = 2^{<\omega}$.

As in Notation 4.3, when we say indiscernible, we mean indiscernible for quantifier free formulas.

The proof uses the following construction:

Construction A. Assume $S' \subseteq \mathbb{S}$ is such that $\nu \in S' \Rightarrow \nu \upharpoonright k \in S'$ for every $k \leq \lg(\nu)$. Assume $N \models T_{S'}^{\forall,\theta}$ and that for every $\nu \in S'$, if $\nu\hat{\ }\langle\varepsilon\rangle \notin S'$ for $\varepsilon \in \{0,1\}$, we have a model $M_\nu^\varepsilon \models T_{\mathbb{S}}^{\forall,\theta}$. We may assume all models are disjoint. We build a model $M \models T_{\mathbb{S}}^{\forall,\theta}$ such that $M \upharpoonright L_{S'}^\theta \supseteq N$

and: for every $\nu \in S'$ and $\varepsilon \in \{0,1\}$ such that $\nu\hat{\ }\langle\varepsilon\rangle \notin S'$ and for every $\eta \in \mathbb{S}$, $P^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta} = P^{M^\varepsilon_\nu}_\eta$. In general, for every symbol R_η from $L^\theta_{\mathbb{S}}$, let $R^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta} = R^{M^\varepsilon_\nu}_\eta$. For instance, $e^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta,i} = e^{M^\varepsilon_\nu}_{\eta,i}$ for $i < \theta$ and $G^M_{\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta_1,\nu\hat{\ }\langle\varepsilon\rangle\hat{\ }\eta_2} = G^{M^\varepsilon_\nu}_{\eta_1,\eta_2}$ for $\eta_1 <_{\mathrm{suc}} \eta_2$.

The last thing that remains to be defined is $G^M_{\nu,\nu\hat{\ }\langle\varepsilon\rangle}$. After we have defined it, M is a model. Moreover, for every tuple $\bar{a} \in M^\varepsilon_\nu$ and for every quantifier free formula φ from $L^\theta_{\mathbb{S}}$, there is a formula φ' generated by concatenating $\nu\hat{\ }\langle\varepsilon\rangle$ to every symbol appearing in φ such that $M^\varepsilon_\nu \models \varphi(\bar{a})$ iff $M \models \varphi'(\bar{a})$. In particular, if $I \subseteq M^\varepsilon_\nu$ is an indiscernible sequence in M, it is also such in M^ε_ν.

Main Theorem A follows immediately from Proposition 5.1 and:

Theorem 5.2. *Let $\mathbb{S} = 2^{<\omega}$. For any cardinals θ, κ and a limit ordinal $\delta \geq \omega$, $\kappa \to (\delta)_{T^\theta_{\mathbb{S}},1}$ iff $\kappa \to (\delta)^{<\omega}_\theta$.*

Proof. We shall prove the following: for every cardinal κ and limit ordinal $\delta \geq \omega$ such that $\kappa \not\to (\delta)^{<\omega}_\theta$, there is a model $M \models T^{\forall,\theta}_{\mathbb{S}}$ and a set $A \subseteq P^M_{\langle\rangle}$ of size $|A| \geq \kappa$ with no non-constant indiscernible sequence in A^δ. That will suffice (because M can be extended to a model of $T^\theta_{\mathbb{S}}$).

The proof is by induction on κ. Note that if $\kappa \not\to (\delta)^{<\omega}_\theta$ then also $\lambda \not\to (\delta)^{<\omega}_\theta$ for $\lambda < \kappa$. The case analysis for some of the cases is very similar to the one done in [KS12], but we repeat it for completeness.

Case 1. $\kappa \leq \theta$. Let $M \models T^{\forall,\theta}_S$ be any model and $A = \left\{e^M_{\langle\rangle,i} \,|\, i < \theta\right\}$.

Case 2. κ is singular. Assume that $\kappa = \bigcup\{\lambda_i \,|\, i < \sigma\}$ where $\sigma < \kappa$ and $\lambda_i < \kappa$ for all $i < \sigma$.

Assume that N_0, A_0 are the model and set given by the induction hypothesis for σ. For all $i < \sigma$, let M_i, B_i the models and sets guaranteed by the induction hypothesis for λ_i. Let N_1 be a model of $T^{\forall,\theta}_S$ containing M_i as substructures for all $i < \sigma$ (it exists by JEP) and $A_1 = \bigcup\{B_i \,|\, i < \sigma\}$.

Assume that $\{a_i \,|\, i < \sigma\} \subseteq A_0$ and that $\{b_j \,|\, \bigcup\{\lambda_l \,|\, l < i\} \leq j < \lambda_i\} \subseteq B_i$ are enumerations witnessing that $|A_0| \geq \sigma$, $|B_i| \geq \lambda_i \backslash \bigcup\{\lambda_l \,|\, l < i\}$.

Let $M' \models T^{\forall}_{\{\langle\rangle\}}$ be the standard model (see Definition 4.2) with $P^{M'}_{\langle\rangle} = \kappa$ and $<^{M'}_{\langle\rangle} = \in$.

Let $N \models T^{\forall,\theta}_{\{\langle\rangle\}}$ be a model such that $N \restriction L_{\{\langle\rangle\}} \supseteq M'$. Use Construction A to build a model $M \models T^{\forall,\theta}_{\mathbb{S}}$ with $M^0_{\langle\rangle} = N_0$ and $M^1_{\langle\rangle} = N_1$, and define the functions $G^M_{\langle\rangle,\langle 0\rangle}$ and $G^M_{\langle\rangle,\langle 1\rangle}$ as follows: for a limit $\alpha < \kappa$ and $0 < n < \omega$, define $G^M_{\langle\rangle,\langle 0\rangle}(\alpha + n) = a_{\min\{j<\sigma \,|\, \alpha<\lambda_j\}}$ and $G^M_{\langle\rangle,\langle 1\rangle}(\alpha + n) = b_\alpha$.

Let $A = \kappa = P^{M'}_{\langle 0\rangle}$. Assume that $\bar{s} = \langle s_i \,|\, i < \delta\rangle$ is an indiscernible sequence contained in A.

Obviously it cannot be that $s_1 < s_0$. Assume that $s_0 < s_1$. There are limit ordinals α_i and natural number n_i such that $s_i = \alpha_i + n_i$, i.e., $s_i \equiv n_i \pmod{\omega}$. By indiscernibility, n_i is constant, and denote it by n. So $\langle \mathrm{suc}\,(s_{2i}, s_{2i+1}) = s_{2i} + 1 \,|\, i < \delta \rangle$ is an indiscernible sequence of successor ordinals.

$\left\langle G_{\langle\rangle,\langle 0\rangle}\,(s_{2i}+1) \,|\, i < \delta \right\rangle$ must be constant by the choice of A_0, and assume it is a_{i_0} for $i_0 < \sigma$. It follows that $\alpha_{2i} \in \lambda_{i_0} \backslash \bigcup \{\lambda_l \,|\, l < i_0\}$. This means that $G_{\langle\rangle,\langle 1\rangle}\,(s_{2i}+1) = b_{\alpha_{2i}} \subseteq B_{i_0}$ for all $i < \sigma$, and so α_{2i} must be constant. This means that $\langle s_{2i} \,|\, i < \delta \rangle$ is constant so also $\bar{s}$.

Case 3. κ is regular but not strongly inaccessible. Then there is some $\lambda < \kappa$ such that $2^\lambda \geq \kappa$.

Let $M_0 \models T_S^{\forall,\theta}$ and $A_0 \subseteq P_{\langle\rangle}^{M_0}$ satisfy the induction hypothesis for λ. Assume that $A_0 \supseteq \{a_i \,|\, i \leq \lambda\}$ where $a_i \neq a_j$ for $i \neq j$.

Let $M' \models T_{\{\langle\rangle\}}^{\forall}$ be a standard model such that $P_{\langle\rangle}^{M'} = 2^{\leq \lambda}$ ordered by first segment.

Let $N \models T_{\{\langle\rangle\}}^{\forall,\theta}$ be any model such that $N \restriction L_{\{\langle\rangle\}} \supseteq M'$. We use Construction A to build a model $M \models T_S^{\forall}$ using N and $M_{\langle\rangle}^0 = M_{\langle\rangle}^1 = M_0$. We need to define the functions $G_{\langle\rangle,\langle 0\rangle}$ and $G_{\langle\rangle,\langle 1\rangle}$:

For $f \in P_{\langle\rangle}^{M'}$ such that $\lg(f) = \alpha + n$ for some limit α and $n < \omega$, define $G_{\langle\rangle,\langle 0\rangle}^{M'}(f) = a_\alpha$. There are no further limitations on the functions $G_{\langle\rangle,\langle 0\rangle}^{M'}$ and $G_{\langle\rangle,\langle 1\rangle}^{M'}$ as long as they are regressive.

Let $A = 2^{\leq\lambda} = P_{\langle\rangle}^{M'}$. Assume for contradiction that $\langle s_i \,|\, i < \delta \rangle$ is a non-constant indiscernible sequence contained in A.

It cannot be that $s_1 < s_0$, because by indiscernibility, we would have an infinite decreasing sequence.

It cannot be that $s_0 < s_1$:

In that case, $\langle s_i \,|\, i < \delta \rangle$ is increasing. For all $i < \delta$, let $t_i = \mathrm{suc}\,(s_{2i}, s_{2i+1})$. The sequence $\langle t_i \,|\, i < \delta \rangle$ is an indiscernible sequence contained in $\mathrm{Suc}\left(P_{\langle\rangle}^M\right)$ and so $t_i \equiv n \pmod{\omega}$ for some constant $n < \omega$. Hence $\langle \lg(t_i) - n \,|\, i < \delta \rangle$ is increasing and $\left\langle G_{\langle\rangle,\langle 0\rangle}\,(t_i) = a_{\lg(t_i)-n} \,|\, i < \delta \right\rangle$ is a non-constant indiscernible sequence contained in A_0 — a contradiction.

Denote $r_i = s_0 \wedge s_{i+1}$ for $i < \delta$. This is an indiscernible sequence, and by the same arguments, it cannot decrease or increase. But since $r_i < s_0$, it follows that r_i is constant.

Assume that $s_0 \wedge s_1 < s_1 \wedge s_2$, then $s_1 \wedge s_2 < s_2 \wedge s_3$ and so $s_{2i} \wedge s_{2i+1} < s_{2(i+1)} \wedge s_{2(i+1)+1}$ for all $i < \delta$, and again — $\langle s_{2i} \wedge s_{2i+1} \rangle$ is an increasing indiscernible sequence — we reach a contradiction.

Similarly, it cannot be that $s_0 \wedge s_1 > s_1 \wedge s_2$. As both sides are smaller or equal to s_1, it must be that

$$s_0 \wedge s_2 = s_0 \wedge s_1 = s_1 \wedge s_2.$$

But this is a contradiction (because if $\alpha = \lg(s_0 \wedge s_1)$ then $|\{s_0(\alpha), s_1(\alpha), s_2(\alpha)\}| = 3$, but the range of these functions is $\{0,1\}$).

Case 4. κ is strongly inaccessible.

Assume that M_λ, A_λ are the models and sets given by the induction hypothesis for $\lambda < \kappa$. We may assume they are disjoint. Let N be a model of $T_{\mathbb{S}}^{\forall,\theta}$ containing M_λ for $\lambda < \kappa$ (N exists by JEP), and let $A = \bigcup\{A_\lambda \mid \lambda < \kappa\} \subseteq N$. Recall that we have a function $\mathbf{c} : [\kappa]^{<\omega} \to \theta$ that witnesses the fact that $\kappa \not\to (\delta)_\theta^{<\omega}$, and that in Definition 4.24 we defined a model $M_{\mathbf{c}}$ of $T_\omega^\forall$. Let $N_{\mathbf{c}} \models T_\omega^{\forall,\theta}$ be a model such that $N_{\mathbf{c}} \restriction L_\omega \supseteq M_{\mathbf{c}}$. Let $S' = 1^{<\omega}$ (finite sequences of zeros). We may think of $N_{\mathbf{c}}$ as a model of $T_{S'}^{\forall,\theta}$. Denote $0_n = \langle 0, \dots, 0 \rangle$ where $\lg(0_n) = n$.

We use Construction A and S' to build a model M of $T_{\mathbb{S}}^{\forall,\theta}$:

- For all $n < \omega$, let $M_{0_n}^1 = N$.
- Define $G_{0_n, 0_n\char`\^\langle 1 \rangle}^M$ as follows:
  - Recall that $P_{0_n}^M \supseteq P_n^{M_{\mathbf{c}}} = M_{\mathbf{f}_n}$. Assume that $B \subseteq \mathrm{Suc}_{\lim}\left(P_n^{M_{\mathbf{c}}}\right)$ is an E^{nb} class. By definition, $|B/E_{\mathbf{f}_n}| < \kappa$.
  - Choose some enumeration of the classes $\{c_i \mid i < |B/E_{\mathbf{f}_n}|\}$, and an enumeration $A_{|B/E_{\mathbf{f}_n}|} \supseteq \{a_i \mid i < |B/E_{\mathbf{f}_n}|\}$ of pairwise distinct elements. Now, $G_{0_n, 0_n\char`\^\langle 1 \rangle}^M$ maps every class c_i (i.e., every element in c_i) to a_i. It is easy to see that if a E^{nb} b are distinct in $\mathrm{Suc}_{\lim}\left(P_n^{M_{\mathbf{c}}}\right)$, then a and b are not $\sim^{M_{\mathbf{c}}}$-equivalent (see Definition 3.17). This means that $G_{0_n, 0_n\char`\^\langle 1 \rangle}^M$ is well defined. Outside of $P_n^{M_c}$, define $G_{0_n, 0_n\char`\^\langle 1 \rangle}^M$ arbitrarily as long as it is regressive.

Let $A = \mathrm{Suc}_{\lim}\left(P_{\langle\rangle}^{M_{\mathbf{c}}}\right)$, i.e., $A = \mathrm{Suc}_{\lim}(\kappa)$. Assume for contradiction that A contains a δ-indiscernible sequence.

By Corollary 4.31, there is $n < \omega$ and an indiscernible sequence $\bar{v}$ in $\mathrm{Suc}_{\lim}\left(P_{0_n}^M\right)$ such that for $i < j < \delta$, v_i E^{nb} v_j but $\neg(v_i \mathrel{E} v_j)$. So $\left\langle G_{0_n, 0_n\char`\^\langle 1 \rangle}^M(v_i) \,\middle|\, i < \delta \right\rangle$ is a non-constant indiscernible sequence in $A_{|[v_0]_{E^{\mathrm{nb}}}/E_{\mathbf{f}_n}|}$ — a contradiction.

$\square$

Remark 5.3. Why in the definition of $\mathbf{g}$ (Definition 4.20) we demanded that the image of η is in $\mathrm{Suc}_{\lim}$ and that Γ is defined only in limit levels? Had we given Γ the freedom to give values in every ordinal, then the "fan" (i.e., the sequence in ind_f) which we got in Lemma 4.29 might not have been in a successor to a limit level, so we would have no freedom in applying G on it. As Γ

is relevant only for limit levels, the coloring was defined only on sequence in $\mathrm{Suc}_{\lim}$, so we needed η to give elements from there.

6. Strongly dependent theories

As we said in the introduction, in [She12] it is proved that $\beth_{|T|^+}(\lambda) \to (\lambda^+)_{T,n}$ for strongly dependent T and $n < \omega$.

In [KS12] we show that in RCF there is a similar phenomenon to what we have here, but for ω-tuples: there are sets from all cardinalities with no indiscernible sequence of ω-tuples up to the first strongly inaccessible cardinal. This explains why the theorem mentioned was only proved for $n < \omega$.

The example we described here is not strongly dependent, but it can be modified a bit so that it will be, and then give a similar theorem for strongly dependent theories (or even strongly^2 dependent), but for ω-tuples.

Theorem 6.1. *For every θ there is a strongly^2 dependent theory T of size θ such that for all κ and δ, $\kappa \to (\delta)_{T,\omega}$* iff *$\kappa \to (\delta)_\theta^{<\omega}$.*

Proof. Right to left follows from Proposition 5.1.

For $n < \omega$ let $S_n = 2^{\leq n}$ and let T_n^θ be the theory $T_{S_n}^\theta$ (see Corollary 3.33). Let T be the theory $\sum_{n<\omega} T_n^\theta$: the language is $\{Q_n \,|\, n < \omega\} \cup \left\{R^n \,\middle|\, R \in L_{S_n}^\theta\right\}$ where Q_n are unary predicates, and the theory says that they are mutually disjoint and that each Q_n is a model of T_n^θ. It is easy to see that this theory is complete and has quantifier elimination. Denote $\mathbb{S} = 2^{<\omega}$ as before. If M is a model of $T_{\mathbb{S}}^\theta$, then M naturally induces a model N of T (where $Q_n^N = (M \times \{n\}) \restriction L_{S_n}^\theta$). For all $a \in M$, let $f_a \in \prod_{n<\omega} Q_n^N$ be defined by $f_a(n) = (a, n)$ for $n < \omega$. Now, if $A \subseteq P_{\langle\rangle}^M$ is any set with no δ-indiscernible sequence then the set $\{f_a \,|\, a \in A\}$ is a sequence of ω-tuples with no indiscernible sequence of length δ.

By Corollary 3.33, it follows that each T_n is strongly^2 dependent, and so also T (this can be seen directly by Definition 2.4, or use an equivalent definition using mutually indiscernible sequences [She12, Definition 2.3] and [She12, Claim 2.8 (3)]). $\square$

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