CHAIN CONDITIONS IN DEPENDENT GROUPS

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Abstract. In this note we prove and disprove some chain conditions in type definable and definable groups in dependent, strongly dependent and strongly^2 dependent theories.

1. Introduction

This note is about chain conditions in dependent, strongly dependent and strongly^2 dependent theories.

Throughout, all formulas will be first order, T will denote a complete first order theory, and $\mathfrak{C}$ will be the monster model of T — a very big saturated model that contains all small models. We do not differentiate between finite tuples and singletons unless we state it explicitly.

Definition 1.1. A formula $\varphi(x, y)$ has the independence property in some model if for every $n < \omega$ there are $\langle a_i, b_s \,| i < n, s \subseteq n \rangle$ such that $\varphi(a_i, b_s)$ holds iff $i \in s$.

A (first order) theory T is dependent (sometimes also NIP) if it does not have the independence property: there is no formula $\varphi(x, y)$ that has the independence property in any model of T. A model M is dependent if $Th(M)$ is.

A good introduction to dependent theories appears in [Adl08], but we shall give an exact reference to any fact we use, so no prior knowledge is assumed.

What do we mean by a chain condition? Rather than giving an exact definition, we give an example of such a condition — the first one. It is the Baldwin-Saxl Lemma, which we shall present with the (very easy and short) proof.

Definition 1.2. Suppose $\varphi(x, y)$ is a formula. Then if G is a definable group in some model, and for all $c \in C$, $\varphi(x, c)$ defines a subgroup, then $\{\varphi(\mathfrak{C}, c) \,|\, c \in C\}$ is a family of *uniformly definable subgroups.*

Lemma 1.3. [BS76] *Let G be a group definable in a dependent theory. Suppose $\varphi(x, y)$ is a formula and that $\{\varphi(x, c) \,|\, c \in C\}$ defines a family of subgroups of G. Then there is a number $n < \omega$ such that any finite intersection of groups from this family is already an intersection of n of them.*

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Proof. Suppose not, then for every $n < \omega$ there are $c_0, \ldots, c_{n-1} \in C$ and $g_0, \ldots, g_{n-1} \in G$ (in some model) such that $\varphi(g_i, c_j)$ holds iff $i \neq j$. For $s \subseteq n$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter), then $\varphi(g_s, c_j)$ iff $j \notin s$ — this is a contradiction. $\square$

In stable theories (which we shall not define here), the Baldwin-Saxl lemma is even stronger: every intersection of such a family is really a finite one (see [Poi01, Proposition 1.4]).

The focus of this note is type definable groups in dependent theories, where such a proof does not work.

Definition 1.4. A *type definable group* for a theory T is a type — a collection $\Sigma(x)$ of formulas (maybe over parameters), and a formula $\nu(x, y, z)$, such that in the monster model $\mathfrak{C}$ of T, $\langle \Sigma(\mathfrak{C}), \nu \rangle$ is a group with ν defining the group operation (without loss of generality, $T \models \forall xy \exists^{\leq 1} z (\nu(x, y, z))$). We shall denote this operation by $\cdot$.

In stable theories, their analysis becomes easier as each type definable group is an intersection of definable ones (see [Poi01]).

Remark 1.5. In this note we assume that G is a finitary type definable group, i.e. x above is a finite tuple.

Definition 1.6. Suppose $G \geq H$ are two type definable groups (H is a subgroup of G). We say that the index $[G : H]$ is *unbounded*, or ∞, if for any cardinality κ, there exists a model $M \models T$, such that $\left[G^M : H^M\right] \geq \kappa$. Equivalently (by the Erdős-Rado coloring theorem), this means that there exists (in $\mathfrak{C}$) a sequence of indiscernibles $\langle a_i \,|\, i < \omega \rangle$ (over the parameters defining G and H) such that $a_i \in G$ for all i, and $i < j \Rightarrow a_i \cdot a_j^{-1} \notin H$. In $\mathfrak{C}$, this means that $\left[G^{\mathfrak{C}} : H^{\mathfrak{C}}\right] = |\mathfrak{C}|$. When G and H are definable, then by compactness this is equivalent to the index $[G : H]$ being infinite.

So $[G : H]$ is *bounded* if it is not unbounded.

This leads to the following definition

Definition 1.7. Let G be a type definable group.

(1) For a set A, G_A^{00} is the minimal A-type definable subgroup of G of bounded index.
(2) We say that G^{00} exists if $G_A^{00} = G_\emptyset^{00}$ for all A.

Shelah proved:

Theorem 1.8. [She08] If G is a type definable group in a dependent theory, then G^{00} exists.

Even though fields are not the main concern of this note, the following question is in the basis of its motivation. Recall

Theorem 1.9. [Lan02, Theorem VI.6.4] *(Artin-Schreier) Let* k *be a field of characteristic* p*. Let* ρ *be the polynomial* $X^p - X$*.*

(1) *Given* $a \in k$*, either the polynomial* $\rho - a$ *has a root in* k*, in which case all its root are in* k*, or it is irreducible. In the latter case, if* α *is a root then* $k(\alpha)$ *is cyclic of degree* p *over* k*.*
(2) *Conversely, let* K *be a cyclic extension of* k *of degree* p*. Then there exists* $\alpha \in K$ *such that* $K = k(\alpha)$ *and for some* $a \in k$*,* $\rho(\alpha) = a$*.*

Such extensions are called Artin-Schreier extensions.

The first author, in a joint paper with Thomas Scanlon and Frank Wagner, proved:

Theorem 1.10. [KSW11] *Let* K *be an infinite dependent field of characteristic* $p > 0$*. Then* K *is Artin-Schreier closed — i.e.* ρ *is onto.*

What about the type definable case? What if K is an infinite type definable field?

In simple theories (which we shall not define), we have:

Theorem 1.11. [KSW11] *Let* K *be a type definable field in a simple theory. Then* K *has boundedly many AS extensions.*

But for the dependent case we only proved:

Theorem 1.12. [KSW11] *For an infinite type definable field* K *in a dependent theory there are either unboundedly many Artin-Schreier extensions, or none.*

From these two we conclude:

Corollary 1.13. *If* T *is stable (so it is both simple and dependent), then type definable fields are AS closed.*

The following, then, is still open:

Question 1.14. *What about the dependent case? In other words, is it true that infinite type definable fields in dependent theories are AS-closed?*

Observing the proof of Theorem 1.10, we see that it is enough to find a number n, and $n+1$ algebraically independent elements, $\langle a_i \mid i \leq n \rangle$ in $k := K^{p^\infty}$, such that $\bigcap_{i<n} a_i \rho(K) = \bigcap_{i \leq n} a_i \rho(K)$. So the Baldwin-Saxl applies in the case where the field K is definable. If K is type definable, we may want something similar. But what can we prove?

A conjecture of Frank Wagner is the main motivation question

Conjecture 1.15. *Suppose* T *is dependent, then the following holds*

☺ *Suppose* G *is a type definable group. Suppose* $p(x,y)$ *is a type and* $\langle a_i \mid i<\omega \rangle$ *is an indiscernible sequence such that* $G_i = p(x,a_i) \leq G$*. Then there is some* n*, such that for all finite sets,* $v \subseteq \omega$*, the intersection* $\bigcap_{i\in v} G_i$ *is equal to a sub-intersection of size* n*.*

Let us refer to ☺ as *Property A* (of a theory T) for the rest of the paper. So we have

Fact 1.16. *If Property A is true for a theory* T*, then type definable fields are Artin-Schreier closed.*

In Section 2, we deal with strongly^2 dependent theories (this is a much stronger condition than merely dependence), and among other things, prove that Property A is true for them.

In Section 3, we give some generalizations and variants of Baldwin-Saxl for type definable groups in dependent and strongly dependent theories (which we define below). One of them is joint work with Frank Wagner. We prove that Property A holds for theories with bounded dp-rank.

In Section 4, we provide a counterexample that shows that property A does not hold in stable theories, so Conjecture 1.15 as it is stated is false.

Question 1.17. *Does Property A hold for strongly dependent theories?*

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2. Strongly^2 dependent theories

Notation 2.1. We call an array of elements (or tuples) $\langle a_{i,j} \mid i,j<\omega \rangle$ an *indiscernible array* over A if for $i_0<\omega$, the i_0-row $\langle a_{i_0,j} \mid j<\omega \rangle$ is indiscernible over the rest of the sequence ($\{a_{i,j} \mid i \neq i_0, i,j<\omega\}$) and A, i.e. when the rows are mutually indiscernible.

Definition 2.2. A theory T is said to be <u>not</u> *strongly^2 dependent* if there exists a sequence of formulas $\langle \varphi_i(x,y_i,z_i) \mid i<\omega \rangle$, an array $\langle a_{i,j} \mid i,j<\omega \rangle$ and $b_k \in \{a_{i,j} \mid i<k, j<\omega\}$ such that

- The array $\langle a_{i,j} \mid i,j<\omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x,a_{i,0},b_i) \wedge \neg\varphi_i(x,a_{i,1},b_i) \mid i<\omega\}$ is consistent.

So T is *strongly^2 dependent* when this configuration does not exist.

Note that the roles of i and j are not symmetric.

(In the definition above, x, z_i, y_i can be tuples, the length of z_i and y_i may depend on i).

This definition was introduced and discussed in [She12] and [She09].

Remark 2.3. By [She12, Claim 2.8], we may assume in the definition above that x is a singleton.

Fact 2.4. [She12, Claim 2.9] *An equivalent definition is* T *is not strongly^2 dependent if there exists an array* $\langle a_{i,j} \mid i,j<\omega \rangle$*, a set* A *and some finite tuple* c *such that*

- *The array* $\langle a_{i,j} \mid i,j<\omega \rangle$ *is an indiscernible array over* A*.*
- *For* $i_0<\omega$*, the row* $\bar{a}_{i_0} := \langle a_{i_0,j} \mid j<\omega \rangle$ *is not indiscernible over* $\bigcup_{i<i_0} \bar{a}_i \cup c$*.*

Proposition 2.5. *Suppose T is strongly[2] dependent, then it is impossible to have a sequence of type definable groups $\langle G_i \,|\, i < \omega \rangle$ such that $G_{i+1} \leq G_i$ and $[G_i : G_{i+1}] = \infty$ (see Definition 1.6).*

Proof. Without loss of generality, we shall assume that all groups are type definable over $\emptyset$. Suppose there is such a sequence $\langle G_i \,|\, i < \omega \rangle$. Let $\langle a_{i,j} \,|\, i, j < \omega \rangle$ be an indiscernible array such that for each $i < \omega$, the sequence $\langle a_{i,j} \,|\, j < \omega \rangle$ is a sequence from G_i (in $\mathfrak{C}$) such that $a_{i,j'}^{-1} \cdot a_{i,j} \notin G_{i+1}$ for all $j < j' < \omega$. We can find such an array because of our assumption and Ramsey (for more details, see the proof of Corollary 2.9 below).

For each $i < \omega$, let $\psi_i(x)$ be in the type defining G_{i+1} such that $\neg\psi_i\left(a_{i,j'}^{-1} \cdot a_{i,j}\right)$. By compactness, there is a formula $\xi_i(x)$ in the type defining G_{i+1} such that for all $a, b \in \mathfrak{C}$, if $\xi_i(a) \wedge \xi_i(b)$ then $\psi_i\left(a \cdot b^{-1}\right)$ holds. Let $\varphi_i(x, y, z) = \xi_i\left(y^{-1} \cdot z^{-1} \cdot x\right)$. For $i < \omega$, let $b_i = a_{0,0} \cdot \ldots \cdot a_{i-1,0}$ (so $b_0 = 1$).

Let us check that the set $\{\varphi_i(x, a_{i,0}, b_i) \wedge \neg\varphi_i(x, a_{i,1}, b_i) \,|\, i < \omega\}$ is consistent. Let $i_0 < \omega$, and let $c = b_{i_0}$. Then for $i < i_0$, $\varphi_i(c, a_{i,0}, b_i)$ holds iff $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ holds, but the product $a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}$ is an element of G_{i+1} and ξ_i is in the type defining G_{i+1}, so $\varphi_i(c, a_{i,0}, b_i)$ holds. Now, $\varphi_i(c, a_{i,1}, b_i)$ holds iff $\xi_i\left(a_{i,1}^{-1} a_{i,0} \cdot \ldots \cdot a_{i_0-1,0}\right)$ holds. So if $\varphi_i(c, a_{i,1}, b_i)$ holds then, since $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ holds, by the choice of ξ_i we get

$$\psi_i\left(\left[a_{i,1}^{-1} a_{i,0} \cdot \ldots \cdot a_{i_0-1,0}\right] \cdot [a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}]^{-1}\right),$$

i.e. $\psi_i\left(a_{i,1}^{-1} \cdot a_{i,0}\right)$ holds — a contradiction. □

Remark 2.6. It is well known (see [Poi01]) that in superstable theories the same proposition hold.

The next corollary already appeared in [She12, Claim 0.1] with definable groups instead of type definable (with proof already in [She09, Claim 3.10]).

Corollary 2.7. *Assume T is strongly[2] dependent. If G is a type definable group and h is a definable homomorphism $h : G \to G$ with finite kernel then h is almost onto G, i.e., the index $[G : h(G)]$ is bounded (i.e. $< \infty$). If G is definable, then the index must be finite.*

Proof. Consider the sequence of groups $\left\langle h^{(i)}(G) \,|\, i < \omega \right\rangle$ (i.e. G, $h(G)$, $h(h(G))$, etc.). By Proposition 2.5, for some $i < \omega$, $\left[h^{(i)}(G) : h^{(i+1)}(G)\right] < \infty$. Now the Corollary easily follows from:

Claim. If G is a group, $h : G \to G$ a homomorphism with finite kernel, then $[G : h(G)] + \aleph_0 = [h(G) : h(h(G))] + \aleph_0$.

Proof. (of claim) Let $H = h(G)$. Easily, one has $[H : h(H)] \leq [G : H]$.

We may assume that $[G : H]$ is infinite. Let $\ker(h) = \{g_0, \ldots, g_{k-1}\}$. Suppose that $[G : H] = \kappa$ but $[H : h(H)] < \kappa$. So let $\{a_i \,|\, i < \kappa\} \subseteq G$ are such that $a_i^{-1} \cdot a_j \notin H$ for $i \neq j$. So there must be some coset $a \cdot h(H)$ in H such that for infinitely many $i < \kappa$, $h(a_i) \in a \cdot h(H)$. Let us enumerate them as $\langle a_i \,|\, i < \omega \rangle$. So for $i < j < \omega$, let $C(a_i, a_j)$ be the least number $l < k$ such that there is some $y \in h(G)$ with $y^{-1} a_i^{-1} a_j = g_l$. By Ramsey, we may assume that $C(a_i, a_j)$ is constant. Now pick $i_1 < i_2 < j < \omega$. So we have $y^{-1} a_{i_1}^{-1} a_j = (y')^{-1} a_{i_2}^{-1} a_j$ for some $y, y' \in H$, so $y^{-1} a_{i_1}^{-1} = (y')^{-1} a_{i_2}^{-1}$ and hence $a_{i_1}^{-1} a_{i_2} = y (y')^{-1} \in H$ — a contradiction. □

Corollary 2.8. *If* K *is a strongly*^{2} *dependent field, (or even a type definable field in a strongly*^{2} *dependent theory) then for all* $n < \omega$, $[K^\times : (K^\times)^n] < \infty$.

Corollary 2.9. *Let* G *be a type definable group in a strongly*^{2} *dependent theory* T.

(1) Given a family of uniformly type definable subgroups $\{p(x, a_i) \,|\, i < \omega\}$ such that $\langle a_i \,|\, i < \omega \rangle$ is an indiscernible sequence, there is some $n < \omega$ such that $\bigcap_{j<\omega} p(\mathfrak{C}, a_j) = \bigcap_{j<n} p(\mathfrak{C}, a_j)$. In particular, T has Property A.

(2) Given a family of uniformly definable subgroups $\{\varphi(x, c) \,|\, c \in C\}$, the intersection

$$\bigcap_{c \in C} \varphi(\mathfrak{C}, c)$$

is already a finite one.

Proof. (1) Assume without loss of generality that G is defined over $\emptyset$. Let $G_i = p(\mathfrak{C}, a_i)$, and let $H_i = \bigcap_{j<i} G_i$. By Proposition 2.5, for some $i_0 < \omega$, $[H_{i_0} : H_{i_0+1}] < \infty$. For $r \geq i_0$, let $H_{i_0,r} = \bigcap_{j<i_0} G_j \cap G_r$ (so $H_{i_0+1} = H_{i_0,i_0}$). By indiscerniblity, $[H_{i_0} : H_{i_0,r}] < \infty$. This means (by definition of $H_{i_0}^{00}$) that $H_{i_0}^{00} \leq H_{i_0,r}$ for all $r > i_0$. However, if $H_{i_0,i_0} \neq H_{i_0,r}$ for some $i_0 < r < \omega$, then by indiscerniblity $H_{i_0,r} \neq H_{i_0,r'}$ for all $i_0 \leq r < r'$, and by compactness and indiscerniblity we may increase the length ω of the sequence to any cardinality κ, so that the size of $H_{i_0}/H_{i_0}^{00}$ is unbounded — a contradiction. This means that $H_{i_0+1} \subseteq G_r$ for all $r > i_0$, and so $\bigcap_{i<\omega} G_i = \bigcap_{i<i_0+1} G_i$.

(2) Assume not. Then we can find a sequence $\langle c_i \,|\, i < \omega \rangle$ of elements of C such that

$$\bigcap_{j<i} \varphi(\mathfrak{C}, c_j) \neq \bigcap_{j<i+1} \varphi(\mathfrak{C}, c_j).$$

By Ramsey and compactness (see e.g. [TZ12, Lemma 5.1.3]), there is an indiscernible sequence $\langle a_i \,|\, i < \omega \rangle$ such that for any n, and any formula $\psi(x_0, \ldots, x_{n-1})$, if $\psi(a_0, \ldots, a_{n-1})$ holds then there are $i_0 < \ldots < i_{n-1}$ such that $\psi(c_{i_0}, \ldots, c_{i_{n-1}})$ holds. In particular, $\varphi(\mathfrak{C}, a_i)$ defines a subgroup of G and $\bigcap_{j<i} \varphi(\mathfrak{C}, a_j) \neq \bigcap_{j<i+1} \varphi(\mathfrak{C}, a_j)$. But this contradicts (1). □

As further applications, we show that some theories are not strongly^2 dependent.

Example 2.10. Suppose $\langle G, +, < \rangle$ is an ordered abelian group. Then its theory $\mathrm{Th}(G, +, 0, <)$ is not strongly^2 dependent.

Proof. We work in the monster model $\mathfrak{C}$. Let $G_d = \{x \in \mathfrak{C} \,|\, \forall n < \omega\, (n \mid x)\}$, so it is a type definable divisible ordered subgroup of G. Note that since G is ordered, it is torsion free, so G_d is a $\mathbb{Q}$-vector space. We shall define a descending sequence of infinite type definable groups $G_d^i \leq G_d$ for $i < \omega$ such that $\left[G_d^i : G_d^{i+1}\right] = \infty$, which contradicts Proposition 2.5. Let $G_d^0 = G_d$, and suppose we have chosen G_d^i. Let $a_i \in G_d^i$ be positive. Let $G_d^{i+1} = G_d^i \cap \bigcap_{n<\omega} (-a_i/n, a_i/n)$. This is a type definable subgroup of G_d^i. The sequence $\langle k \cdot a_i \,|\, k < \omega \rangle$ satisfies $(k - l) \cdot a_i \notin (-a_i/2, a_i/2)$ for any $k \neq l$, and by Ramsey (as in the proof of Corollary 2.9 (2)) we get $\left[G_d^i : G_d^{i+1}\right] = \infty$. $\square$

Example 2.11. The theory $Th(\mathbb{R}, +, \cdot, 0, 1)$ is strongly dependent (it is even o-minimal, so dp-minimal — see Definitions 3.8 and 3.5 below). However it is not strongly^2 dependent.

Example 2.12. The theory $Th(\mathbb{Q}_p, +, \cdot, 0, 1)$ of the p-adics is strongly dependent (it is also dp-minimal), but not strongly^2 dependent: The valuation group $(\mathbb{Z}, +, 0, <)$ is interpretable.

Adding some structure to an algebraically closed field, we can easily get a strongly^2 dependent theory which is not stable.

Example 2.13. Let $L = L_{\text{rings}} \cup \{P, <\}$ where L_{rings} is the language of rings $\{+, \cdot, 0, 1\}$, P is a unary predicate and $<$ is a binary relation symbol. Let K be $\mathbb{C}$ (so it is an algebraically closed field), and let $P \subseteq K$ be a countable set of algebraically independent elements, enumerated as $\{a_i \,|\, i \in \mathbb{Q}\}$. Let $M = \langle K, P, < \rangle$ where $a <^M b$ iff $a, b \in P$ and $a = a_i$, $b = a_j$ where $i < j$. Let $T = Th(M)$.

Claim 2.14. T is strongly^2 dependent.

Proof. Note that T is axiomatizable by saying that the universe is an algebraically closed field, P is a subset of algebraically independent elements and $<$ is a dense linear order on P (to see this, take two saturated models of the same size and show that they are isomorphic).

Let us fix some terminology:

- When we write acl, we mean the algebraic closure in the field sense. When we say basis, we mean a transcendental basis.
- When we say that a set is independent / dependent over A for some set A, we mean that it is dependent / independent in the pregeometry induced by $\text{cl}(X) = \text{acl}(AX)$.
- $\text{dcl}(X)$ stands for the definable closure of X.

We work in a saturated model $\mathfrak{C}$ of T.

Suppose X is some set. Let X_0 be some basis for X over P, and let $\text{dcl}^P(X)$ be the set of $p \in P$ such that there exists some minimal finite $P_0 \subseteq P$ with $p \in P_0$ and some $x \in X$ such that $x \in \text{acl}(P_0 X_0)$. Note that this set is contained in $\text{dcl}(X)$ (since P is linearly ordered) and that it does not depend on the choice of X_0.

For a set (or a tuple) A, let $A^P = \text{dcl}^P(A)$.

Subclaim. Suppose $M_1 = (K_1, P_1, <_1)$ and $M_2 = (K_2, P_2, <_2)$ are two saturated models of T and $A \subseteq K_1$ is a small set. Suppose that $K_1 = K_2$ and $\left(A^{P_1}, <^{P_1}\right) = \left(A^{P_2}, <^{P_2}\right)$. Then there is an isomorphism $f : M_1 \to M_2$ fixing $A \cup A^{P_1}$.

Proof. Let $\tau : P_1 \to P_2$ be any isomorphism fixing A^{P_1}. Since both $P_1 \backslash A^{P_1}$ and $P_2 \backslash A^{P_1}$ are algebraically independent over A, $\tau \cup (\mathrm{id} \restriction A)$ is an elementary map in the field language. This map can be extended to an automorphism f of K_1, which is the desired isomorphism. $\square$

Let $\mathrm{tp}_K(a/A)$ be the type of $a \frown (Aa)^P$ (considered as a tuple, ordered by $<^{\mathfrak{C}}$) over $A \cup A^P$ in the field language, and $\mathrm{tp}_P(a/A)$ the type of the tuple $(Aa)^P$ over A^P in the order language.

Subclaim. For finite tuples a, b and a set A, $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ iff $\mathrm{tp}_P(a/A) = \mathrm{tp}_P(b/A)$ and $\mathrm{tp}_K(a/A) = \mathrm{tp}_K(b/A)$.

Proof. Denote by K the field structure of $\mathfrak{C}$. There is an automorphism σ of K that maps $a \frown (Aa)^P$ to $b \frown (Ab)^P$ and fixes $A \cup A^P$ pointwise. Since $\mathrm{tp}_P(a/A) = \mathrm{tp}_P(b/A)$, the restriction $\sigma \restriction A^P \cup (Aa)^P$ is order preserving. Let $\mathfrak{C}' = (K, \sigma(P), \sigma(<))$. By the first subclaim, there is an isomorphism $\tau : \mathfrak{C}' \to \mathfrak{C}$ fixing $Ab \cup (Ab)^P$. Now, $\tau \circ \sigma$ is an automorphism of $\mathfrak{C}$ that takes a to b and fixes A. $\square$

Suppose that $\langle a_{i,j} \mid i, j < \omega \rangle$ is an indiscernible array over a parameter set A as in Definition 2.2 and that c is a singleton such that:

- The sequence $I_0 := \langle a_{0,j} \mid j < \omega \rangle$ is not indiscernible over Ac, and moreover $\mathrm{tp}(a_{0,0}/Ac) \neq \mathrm{tp}(a_{0,1}/Ac)$.
- For $i > 0$, the sequence $I_i := \langle a_{i,j} \mid j < \omega \rangle$ is not indiscernible over $c \cup \bigcup_{k<i} I_k \cup A$.

Suppose that $c \notin \mathrm{acl}(APa_{0,0}a_{0,1})$. Then, by the second subclaim, $\mathrm{tp}(ca_{0,0}/A) = \mathrm{tp}(ca_{0,1}/A)$ — a contradiction. So $c \in \mathrm{acl}(APa_{0,0}a_{0,1})$. Increase the parameter set A by adding the first row $\langle a_{0,j} \mid j < \omega \rangle$. So we may assume that $c \in \mathrm{acl}(AP)$. Since $c \in \mathrm{acl}\left(A(Ac)^P\right)$, we may replace c by a finite tuple contained in $(Ac)^P$ and assume that c is a finite tuple of elements in P (here we use the fact that in general, if I is indiscernible over C then it is also indiscernible over $\mathrm{acl}(C)$).

Expand all the sequences to order type $\omega^* + \omega + \omega$. Let $B = \bigcup \{a_{i,j} \mid i < \omega, j < 0 \vee \omega \leq j\} \cup A$. For each $i < \omega$ and $0 \leq j < \omega$, let $a^P_{i,j}$ be $\mathrm{dcl}^P(a_{i,j}B)$ considered as a tuple ordered by $<^{\mathfrak{C}}$, and let $B^P = \mathrm{dcl}^P(B)$. Then $\left\langle a^P_{i,j} \mid 0 \leq i, j < \omega \right\rangle$ is an indiscernible array over B^P and $\left\langle a_{i,j} \frown a^P_{i,j} \mid 0 \leq i, j < \omega \right\rangle$ is an indiscernible array over $B \cup B^P$.

As both the theories of dense linear orders and algebraically closed fields are strongly[2] dependent (this is easy to check), by Fact 2.4 there is some i_0 such that $\left\langle a^P_{i_0,j} \mid 0 \leq j < \omega \right\rangle$ is indiscernible over $cB^P \cup \left\{a^P_{i,j} \mid i < i_0, 0 \leq j < \omega\right\}$ in the order language and $\left\langle a_{i_0,j} \frown a^P_{i_0,j} \mid 0 \leq j < \omega \right\rangle$ is indiscernible over $cB \cup B^P \cup \left\{a_{i,j} \frown a^P_{i,j} \mid i < i_0, 0 \leq j < \omega\right\}$ in the field language.

Let $C = \bigcup \{a_{i,j} \,|\, i < i_0, 0 \leq j < \omega\}$. We must check that $\langle a_{i_0,j} \,|\, 0 \leq j < \omega \rangle$ is indiscernible over BCc. Let us show, for instance, that $\operatorname{tp}(a_{i_0,0}/BCc) = \operatorname{tp}(a_{i_0,1}/BCc)$. For this we apply the second subclaim. For each $0 \leq i,j < \omega$, let $a'_{i,j}$ be a basis for $a_{i,j}$ over BP. Then, by indiscernibility, $\left\{a'_{i,j} \,|\, i < i_0, 0 \leq j < \omega\right\}$ is a basis for C over BP (this is why we expanded the sequences). Now it follows that $\operatorname{dcl}^P(BCc) = \bigcup\left\{a^P_{i,j} \,|\, i < i_0, 0 \leq j < \omega\right\} \cup B^P \cup c$. Similarly, for $j \geq 0$, $\operatorname{dcl}^P(a_{i_0,j}BCc) = a^P_{i_0,j} \cup \operatorname{dcl}^P(BC) \cup c$. By the second subclaim above, we are done. $\square$

Remark 2.15. With the same proof, one can show that if T is strongly minimal, and $P = \{a_i \,|\, i < \omega\}$ is an infinite indiscernible set in $M \models T$ of cardinality $\aleph_1$, the theory of the structure $\langle M, P, < \rangle$ where $<$ is some dense linear order with no end points on P, is strongly^2 dependent.

We finish this section with the following conjecture:

Conjecture 2.16. *All strongly^2 dependent groups are stable, i.e. if G is a group such that $\operatorname{Th}(G,\cdot)$ is strongly^2 dependent, then it is stable.*

Example 2.10 and Corollary 2.9 show that this might be reasonable. This is related to the conjecture of Shelah in [She12] that all strongly^2 dependent infinite fields are algebraically closed.

3. Baldwin-Saxl type lemmas

The next lemma is the type definable version of the Baldwin-Saxl Lemma (see Lemma 1.3). But first,

Notation 3.1. If $p(x,y)$ is a partial type, then $|p|$ is the size of the set of formulas $\varphi(x, z_1, \ldots, z_n)$ (where z_i is a singleton) such that for some finite tuple $y_1, \ldots, y_n \in y$, $\varphi(x, y_1, \ldots, y_n) \in p$. In this sense, the size of any partial type over $\emptyset$ is bounded by $|T|$.

Lemma 3.2. *Suppose G is a type definable group in a dependent theory T.*

1. *If $p_i(x, y_i)$ is a type for $i < \kappa$ (y_i may be an infinite tuple), $|\bigcup p_i| < \kappa$, and $\langle c_i \,|\, i < \kappa \rangle$ is a sequence of tuples such that $p_i(\mathfrak{C}, c_i)$ is a subgroup of G, then for some $i_0 < \kappa$, $\bigcap_{i<\kappa} p_i(\mathfrak{C}, c_i) = \bigcap_{i<\kappa, i \neq i_0} p_i(\mathfrak{C}, c_i)$.*
2. *In particular, given a family of uniformly type definable subgroups, defined by $p(x,y)$, and C of size $|p|^+$, there is some $c_0 \in C$ such that $\bigcap_{c \neq c_0} p(\mathfrak{C}, c) = \bigcap_{c \in C} p(\mathfrak{C}, c)$.*
3. *In particular, if $\left\{G_i \,\middle|\, i < |T|^+\right\}$ is a family of type definable subgroups (defined with parameters), then there is some $i_0 < |T|^+$ such that $\bigcap G_i = \bigcap_{i \neq i_0} G_i$.*

Proof. (1) Without loss of generality $p_i(x, y_i)$ are closed under finite conjunctions. Let $H_i = p_i(\mathfrak{C}, c_i)$. Suppose not, i.e. for all $i < \kappa$, there is some g_i such that $g_i \in H_j$ iff $i \neq j$. If $d_1, d_2 \in H_i$ then $d_1 \cdot g_i \cdot d_2 \notin H_i$. Hence by compactness there is some formula $\varphi_i(x, c_i) \in p_i(x, c_i)$ such that for all such $d_1, d_2 \in H_i$, $\neg\varphi_i(d_1 g_i d_2, c_i)$ holds. Since $|\bigcup p_i| < \kappa$, we may assume that for $i < \omega$,

φ_i is constant and equals $\varphi(x, y)$. Now for any finite subset $s \subseteq \omega$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter). So we have $\varphi(g_s, c_i)$ iff $i \notin s$ — a contradiction.

(2) and (3) now follow easily from (1). □

In (2) of Lemma 3.2, if C is an indiscernible sequence, then the situation is simpler:

Corollary 3.3. *Suppose G is a type definable group in a dependent theory T. Given a family of uniformly type definable subgroups, defined by $p(x, y)$, and an indiscernible sequence $C = \langle a_i \mid i \in \mathbb{Z} \rangle$, $\bigcap_{i \neq 0} p(\mathfrak{C}, a_i) = \bigcap_{i \in \mathbb{Z}} p(\mathfrak{C}, a_i)$.*

Proof. Assume not. By indiscernibility, we get that for all $i \in \mathbb{Z}$, $\bigcap_{j \neq i} p(\mathfrak{C}, a_j) \nsubseteq p(\mathfrak{C}, a_i)$. Let I be an indiscernible sequence which extends C to length $|p|^+$. Then by indiscernibility and compactness the same is true for this sequence. This contradicts Lemma 3.2. □

Remark 3.4. The above corollary is in the kernel of the proof that G^{00} exists in dependent theories.

If T is strongly dependent, and C is indiscernible, we can even assume that the order type is ω. Let us recall,

Definition 3.5. A theory T is said to be <u>not</u> *strongly dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i) \mid i < \omega \rangle$ and an array $\langle a_{i,j} \mid i, j < \omega \rangle$ such that

- The array $\langle a_{i,j} \mid i, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \mid i < \omega\}$ is consistent.

So T is *strongly dependent* when this configuration does not exist.

Remark 3.6. This definition is not exactly the original definition given in [She12, Definition 1.2], but it is equivalent to it by [She12, Definition 1.2]

Lemma 3.7. *Suppose G is a type definable group in a strongly dependent theory T. Given a family of type definable subgroups $\{p_i(x, a_i) \mid i < \omega\}$ such that $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence and $p_{2i} = p_{2i+1}$ for all $i < \omega$, there is some $i < \omega$ such that $\bigcap_{j \neq i} p_j(\mathfrak{C}, a_j) = \bigcap_{j < \omega} p_j(\mathfrak{C}, a_j)$.*

In particular, this is true when p is constant.

Proof. Without loss of generality $p_i(x, y_i)$ are closed under finite conjunctions. Let $H_i = p_i(\mathfrak{C}, a_i)$. Assume not, i.e. for all $i < \omega$, there exists some $g_i \in G$ such that $g_i \in H_j$ iff $i \neq j$. For each even $i < \omega$ we find a formula $\varphi_i(x, y) \in p_i(x, y)$ such that for all $d_1, d_2 \in H_i$, $\neg\varphi_i(d_1 g_i d_2, a_i)$. Let $n < \omega$, and consider the product $g_n = \prod_{i < n,\, 2|i} g_i$ (the order does not matter). Then for odd $i < n$, $\varphi_{i-1}(g_n, a_i)$ holds (because $\varphi_{i-1} \in p_{i-1} = p_i$ by assumption), and for even $i < n$, $\neg\varphi_i(g_n, a_i)$ holds. By compactness, we can find $g \in G$ such that $\varphi_{i-1}(g, a_i)$ holds for all odd $i < \omega$ and $\neg\varphi_i(g, a_i)$ for all even $i < \omega$. Now expand the sequence by adding a sequence $\langle b_{i,j} \mid j < \omega \rangle$ after each pair a_{2i}, a_{2i+1}. Then the array defined by $a_{i,0} = a_{2i}$, $a_{i,1} = a_{2i+1}$ and $a_{i,j} = b_{i,j-2}$ for $j \geq 2$ will show that the theory is not strongly dependent. □

If the theory is of bounded dp-rank, then we can say even more.

Definition 3.8. A theory T is said to have *bounded dp-rank*, if there is some $n < \omega$ such that the following configuration does not exist: a sequence of formulas $\langle \varphi_i(x, y_i) \,|\, i < n \rangle$ where x is a singleton and an array $\langle a_{i,j} \,|\, i < n, j < \omega \rangle$ such that

- The array $\langle a_{i,j} \,|\, i < n, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \,|\, i < n\}$ is consistent.

T is *dp-minimal* if $n = 2$.

Note that if T has bounded dp-rank, then it is strongly dependent.

Remark 3.9. All dp-minimal theories are of bounded dp-rank. This includes all o-minimal theories and the p-adics.

The name is justified by the following fact:

Fact 3.10. [UOK11] *If T has bounded dp-rank, then for any $m < \omega$, there is some $n_m < \omega$ such that a configuration as in Definition 3.8 with n_m replacing n is impossible for a tuple x of length m (in fact $n_m \le m \cdot n_1$).*

Lemma 3.11. *Let G be type definable group in a bounded dp-rank theory T.*

Given a family of type definable subgroups $\{p_i(x, a_i) \,|\, i < \omega\}$ such that $\langle a_i \,|\, i < \omega \rangle$ is an indiscernible sequence and $p_{2i} = p_{2i+1}$ for all $i < \omega$, there is some $n < \omega$ and $i < n$ such that $\bigcap_{j \neq i, j < n} p_j(\mathfrak{C}, a_j) = \bigcap_{j < n} p_j(\mathfrak{C}, a_j)$.

In particular, if p_i is constant (say p) and $\langle a_i \,|\, i < \omega \rangle$ is an indiscernible set, then $\bigcap_{i<\omega} p(\mathfrak{C}, a_i) = \bigcap_{i<n} p(\mathfrak{C}, a_i)$.

In particular, T has Property A.

Proof. The proof is exactly the same as the proof of Lemma 3.7, but we only need to construct g_n for n large enough. $\square$

Another similar proposition:

Proposition 3.12. *Assume T is strongly dependent, G a type definable group and $G_i \le G$ are type definable normal subgroups for $i < \omega$. Then there is some i_0 such that $\left[\bigcap_{i \neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.*

Proof. Assume not. Then, for each $i < \omega$, we have an indiscernible sequence $\langle a_{i,j} \,|\, j < \omega \rangle$ (over the parameters defining all the groups) such that $a_{i,j} \in \bigcap_{k \neq i} G_k$ and for $j_1 < j_2 < \omega$, $a_{i,j_1}^{-1} \cdot a_{i,j_2} \notin G_i$. Note that if $d_1, d_2, d_3 \in G_i$, then $d_1 \cdot a_{i,j_1}^{-1} \cdot d_2 \cdot a_{i,j_2} \cdot d_3 \notin G_i$, since G_i is normal. By compactness there is a formula $\psi_i(x)$ in the type defining G_i such that for all $d_1, d_2, d_3 \in G_i$, $\neg\psi_i\left(d_1 \cdot a_{i,j_1}^{-1} \cdot d_2 \cdot a_{i,j_2} \cdot d_3\right)$ holds (by indiscernibility it is the same for all $j_1 < j_2$). We may assume, applying Ramsey, that the array $\langle a_{i,j} \mid i,j<\omega\rangle$ is indiscernible (i.e. the sequences are mutually indiscernible). Let $\varphi_i(x,y)=\psi_i\left(x^{-1}\cdot y\right)$.

Now we check that the set $\{\varphi_i(x,a_{i,0})\wedge\neg\varphi_i(x,a_{i,1}) \mid i<n\}$ is consistent for each $n<\omega$. Let $c=a_{0,0}\cdot\ldots\cdot a_{n-1,0}$ (the order does not really matter, but for the proof it is easier to fix one). So $\varphi_i(c,a_{i,0})$ holds iff $\psi_i\left(a_{n-1,0}^{-1}\cdot\ldots\cdot a_{i,0}^{-1}\cdot\ldots\cdot a_{0,0}^{-1}\cdot a_{i,0}\right)$ holds. But since G_i is normal, $a_{i,0}^{-1}\cdot\ldots\cdot a_{0,0}^{-1}\cdot a_{i,0}\in G_i$, so the entire product is in G_i, so $\varphi_i(c,a_{i,0})$ holds. On the other hand, $\psi_i\left(a_{n-1,0}^{-1}\cdot\ldots\cdot a_{i,0}^{-1}\cdot\ldots\cdot a_{0,0}^{-1}\cdot a_{i,1}\right)$ does not hold by the choice of ψ_i. □

The following Corollary is a weaker version of Corollary 2.8:

Corollary 3.13. *If G is an abelian definable group in a strongly dependent theory and $S\subseteq\omega$ is an infinite set of pairwise co-prime numbers, then for almost all (i.e. for all but finitely many) $n\in S$, $[G:G^n]<\infty$. In particular, if K is a definable field in a strongly dependent theory, then for almost all primes p, $\left[K^\times:(K^\times)^p\right]<\infty$.*

Proof. Let $K\subseteq S$ be the set of $n\in S$ such that $[G:G^n]<\infty$. If $S\backslash K$ is infinite, replace S with $S\backslash K$.

For $i\in S$, let $G_i=G^i$ (so it is definable). By Proposition 3.12, there is some $n\in S$ such that $\left[\bigcap_{i\neq n}G_i:\bigcap_{i\in S}G_i\right]<\infty$. If $[G:G_n]=\infty$, then there is an indiscernible sequence $\langle a_i \mid i<\omega\rangle$ of elements of G, such that $a_i^{-1}\cdot a_j\notin G_n$. Suppose $S_0\subseteq S\backslash\{n\}$ is a finite subset and let $r=\prod S_0$. Then $\langle a_i^r \mid i<\omega\rangle$ is an indiscernible sequence in $G^r\subseteq\bigcap_{i\in S_0}G_i$ such that $a_i^{-r}\cdot a_j^r\notin G_n$. So by compactness, we can find such a sequence in $\bigcap_{i\neq n}G_i$ — a contradiction. □

Remark 3.14. The above Proposition and Corollary can be generalized (with almost the same proofs) to the case where the theory is only *strong*. For the definition, see [Adl].

Remark 3.15. This Corollary generalizes in some sense [KP11, Proposition 2.1] (as they only assumed finite weight of the generic type). And so, as in [KP11, Corollary 2.2], we can conclude that if K is a field definable in a strongly stable theory (i.e. the theory is strongly dependent and stable), then $K^p=K$ for almost all primes p.

Problem 3.16. Is Proposition 3.12 is still true without the assumption that the groups are normal?

Note that in strongly dependent[2] theories, this assumption is not needed: Let $H_i=\bigcap_{j<i}G_i$. Then $[H_i:H_{i+1}]<\infty$ for all i big enough by Proposition 2.5. But this implies $\left[\bigcap_{j\neq i}G_j:\bigcap_j G_j\right]<\infty$.

κ-intersection.

This part is joint work with Frank Wagner.

Definition 3.17. For a cardinal κ and a family $\mathfrak{F}$ of subgroups of a group G, the κ-intersection $\bigcap_\kappa \mathfrak{F}$ is $\{g \in G \,|\, |\{F \in \mathfrak{F} \,|\, g \notin F\}| < \kappa\}$.

The following proposition shows that in some sense, the intersection of a family of uniformly type definable subgroups can be understood via its κ-intersection and a small intersection.

Proposition 3.18. *Let G be a type definable group in a dependent theory. Suppose*

- *$\mathfrak{F}$ is a family of uniformly type definable subgroups defined by $p(x, y)$.*

Then for any infinite regular cardinal $\kappa > |p|$ (in the sense of Notation 3.1), and any subfamily $\mathfrak{G} \subseteq \mathfrak{F}$, there is some $\mathfrak{G}' \subseteq \mathfrak{G}$ such that

$\star$ *$|\mathfrak{G}'| < \kappa$ and $\bigcap \mathfrak{G}$ is $\bigcap \mathfrak{G}' \cap \bigcap_\kappa \mathfrak{G}$.*

Remark 3.19. In the context of the proposition, this means that $\mathfrak{G}'$ has the property that for every subset $\mathfrak{G}'' \subseteq \mathfrak{G}$ such that $|\mathfrak{G} \backslash \mathfrak{G}''| < \kappa$, $\bigcap \mathfrak{G} = \bigcap \mathfrak{G}' \cap \bigcap \mathfrak{G}''$.

Proof. (of proposition) Let κ be such a cardinal. Assume that there is some family $\mathfrak{G} = \{H_i \,|\, i < \varkappa\}$, which is a counterexample of the proposition. For $g \in G$, let $J_g = \{i < \varkappa \,|\, g \in H_i\}$. So $g \in \bigcap_\kappa \mathfrak{G}$ iff $|\varkappa \backslash J_g| < \kappa$.

For $i < \kappa$ we define by induction $g_i \in \bigcap_\kappa \mathfrak{G}$, $I_i \subseteq \varkappa$, $R_i \subseteq \varkappa$ and $\alpha_i < \varkappa$ such that:

(1) $R_0 = [0, \alpha_0)$ and for $0 < i$, $R_i = \bigcup_{j<i} R_j \cup \left[[\sup_{j<i} \alpha_j, \alpha_i) \cap \bigcap_{j<i} I_j\right]$ (so $R_i \subseteq \alpha_i$).
(2) $\bigcap_{j \leq i} J_{g_j} \subseteq R_i \cup I_i$ (so by the definition of $\bigcap_\kappa$, and by the regularity of κ, $|\varkappa \backslash (R_i \cup I_i)| < \kappa$).
(3) $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\alpha \in R_i} H_\alpha$.
(4) $I_i \cap [0, \alpha_i] = \emptyset$.
(5) I_i is $\subseteq$-decreasing.
(6) α_i is $<$-increasing.
(7) $I_i \subseteq J_{g_i}$.
(8) For $j < i$, $g_i \in H_{\alpha_j}$, $g_j \in H_{\alpha_i}$ and $g_i \notin H_{\alpha_i}$.

Let $\alpha_0 < \varkappa$ be minimal such that there is some $g_0 \in \bigcap_\kappa \mathfrak{G} \backslash H_{\alpha_0}$ (it must exist, otherwise $\bigcap_\kappa \mathfrak{G} = \bigcap \mathfrak{G}$). Let $I_0 = \{j > \alpha_0 \,|\, g_{\alpha_0} \in H_j\}$.

For α_0, (2), (3), (4), (7) and (8) are true, by the definition of $\bigcap_\kappa$ and the choice of α_0.

Suppose we have chosen g_j, I_j and α_j (so R_j is already defined by (1)) for $j < i$.

Let $J = \bigcap_{j<i} I_j$. Choose $g_i \in \left(\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j}\right) \backslash H_{\alpha_i}$ where $\alpha_i \in J$ is the smallest possible such that this set is nonempty. Suppose for contradiction that we cannot find such α_i, then $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\alpha \in J} H_\alpha$, so

$$\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \varkappa \backslash J} H_j = \bigcap \mathfrak{G}.$$

Let $J' = J \cup \bigcup_{j<i} R_j$, then by (3), $\bigcap \mathfrak{G}$ equals

$$\bigcap_{\kappa} \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \varkappa \backslash J'} H_j.$$

Note that $\bigcap_{j<i} (R_j \cup I_j) \subseteq J'$, so by the regularity of κ, and by (2), $|\varkappa \backslash J'| < \kappa$, so we get a contradiction.

Let $I_i = \{\alpha_i < j \in J \,|\, g_i \in H_j\}$, and let us check the conditions above.

Conditions (4) – (7) are easy.

Condition (2): By induction we have

$$\bigcap_{j \leq i} J_{g_j} = \bigcap_{j<i} J_{g_j} \cap J_{g_i} \subseteq J' \cap J_{g_i} \subseteq R_i \cup (J \cap J_{g_i}).$$

But by (4) and the definition of R_i, letting $\alpha = \sup_{j<i} \alpha_j$, we have

$$J \cap J_{g_i} \subseteq \left[[\alpha, \alpha_i) \cap \bigcap_{j<i} I_j \right] \cup I_i \subseteq R_i \cup I_i.$$

Condition (3) is true by the minimality of α_i: $\bigcap_{\kappa} \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\beta \in J \cap [\alpha, \alpha_i)} H_\beta$, so by the induction hypothesis, we are done.

Condition (8): We show that $g_j \in H_{\alpha_i}$ for $j < i$. We have that $\alpha_i \in J$ so also in I_j which, by (7), is a subset of J_{g_j}, so $g_j \in H_{\alpha_i}$.

Finally, we have that for each $i, j < \kappa$, $g_i \in H_{\alpha_j}$ iff $i \neq j$. But by Lemma 3.2, there is some $i_0 < |p|^+$ such that $\bigcap_{i \neq i_0} H_{\alpha_i} = \bigcap_{i < |p|^+} H_{\alpha_i}$ — a contradiction. □

Remark 3.20. So far we have not found applications for this proposition, but it seems like a very nice proposition in its own right, and it might turn out to be useful.

4. A counterexample

In this section we shall present an example that shows that Property A does not hold in general dependent (or even stable) theories.

Let $S = \{u \subseteq \omega \,|\, |u| < \omega\}$, and $V = \{f : S \to 2 \,|\, |\mathrm{supp}(f)| < \infty\}$ where $\mathrm{supp}(f) = \{x \in S \,|\, f(x) \neq 0\}$. This has a natural group structure as a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

For $n, m < \omega$, define the following groups:

- $G_n = \{f \in V \,|\, u \in \mathrm{supp}(f) \Rightarrow |u| = n\}$
- $G_\omega = \prod_n G_n$
- $G_{n,m} = \{f \in V \,|\, u \in \mathrm{supp}(f) \Rightarrow |u| = n \,\&\, m \in u\}$ (so $G_{0,m} = 0$)
- $H_{n,m} = \{\eta \in G_\omega \,|\, \eta(n) \in G_{n,m}\}$

Now we construct the model:

Let L be the language (vocabulary) $\{P, Q\} \cup \{R_n \,|\, n < \omega\} \cup L_{AG}$ where L_{AG} is the language of abelian groups, $\{0, +\}$; P and Q are unary predicates; and R_n is binary. Let M be the following

L-structure: its universe is $G_\omega \coprod \omega$, $P^M = G_\omega$ (with the group structure), $Q^M = \omega$ and $R_n = \{(\eta, m) \mid \eta \in H_{n,m}\}$. Let $T = Th(M)$.

Let $p(x, y)$ be the type $\{R_n(x, y) \mid n < \omega\}$. Note that since $H_{n,m}$ is a subgroup of G_ω, for each $m < \omega$, $p(M, m)$ is a subgroup of G_ω (and this remains true in elementary extensions).

Claim 4.1. Let $N \models T$ be $\aleph_1$-saturated. For any $m < \omega$, and any distinct $\alpha_0, \ldots, \alpha_m \in Q^N$, $\bigcap_{i \leq m} p(N, \alpha_i)$ is different than any sub-intersection of size m.

Proof. We show that $\bigcap_{i \leq m} p(N, \alpha_i) \subsetneq \bigcap_{i<m} p(N, \alpha_i)$ (the general case is similar). More specifically, we show that

$$\bigcap_{i<m} p(N, \alpha_i) \setminus \bigcap_{i \leq m} R_m(N, \alpha_i) \neq \emptyset.$$

By saturation, it is enough to show that this is the case in M, so we assume $M = N$. Note that if $\eta \in \bigcap_{i \leq m} R_m(M, \alpha_i)$, then $\eta \in H_{m,\alpha_i}$ for all $i \leq m$. So for all $i \leq m$, $u \in \text{supp}(\eta(m)) \Rightarrow |u| = m \,\&\, \alpha_i \in u$. This implies that $\text{supp}(\eta(m)) = \emptyset$, i.e. $\eta(m) = 0$. But we can find $\eta \in \bigcap_{i<m} p(M, \alpha_i)$ such that $\eta(m) \neq 0$. For instance let $\eta(n) = 0$ for all $n \neq m$ while $|\text{supp}(\eta(m))| = 1$ and $\eta(m)(\{\alpha_0, \ldots, \alpha_{m-1}\}) = 1$. $\square$

Next we shall show that T is stable. For this we will use κ-resplendent models. This is a very useful (though not a very well known) tool for proving that theories are stable, and we take the opportunity to promote it.

Definition 4.2. Let κ be a cardinal. A model M is called κ-resplendent if whenever

- $M \prec N$; N' is an expansion of N by less than κ many symbols; $\bar{c}$ is a tuple of elements from M and $\lg(\bar{c}) < \kappa$

There exists an expansion M' of M to the language of N' such that $\langle M', \bar{c} \rangle \equiv \langle N', \bar{c} \rangle$.

The following remarks are not crucial for the rest of the proof.

Remark 4.3. [She]

(1) If κ is regular and $\kappa > |T|$, and $\lambda = \lambda^{<\kappa}$, then T has a κ-resplendent model of size λ.
(2) A κ-resplendent model is also κ-saturated.
(3) If M is κ-resplendent then M^{eq} is also such.

The following is a useful observation:

Claim 4.4. If M is κ-resplendent for some κ, and $A \subseteq M$ is definable and infinite, then $|A| = |M|$.

Proof. Enrich the language with a function symbol f. Let $T' = T \cup \{f : M \to A \text{ is injective}\}$. Then T' is consistent with an elementary extension of M (for example, take an extension N of M where $|A^N| = |M|$, and then take an elementary substructure $N' \prec N$ of size $|M|$ containing M and A^N). Hence we can expand M to a model of T'. $\square$

The main fact is

Theorem 4.5. [She, Main Lemma 1.9] *Assume κ is regular and $\lambda = \lambda^{\kappa} + 2^{|T|}$. Then, if T is unstable then T has $> \lambda$ pairwise nonisomorphic κ-resplendent models of size λ*[1]*. On the other hand, if T is stable and $\kappa \geq \kappa(T) + \aleph_1$ then every κ-resplendent model is saturated.*

Proposition 4.6. *T is stable.*

Proof. We may restrict T to a finite sub-language, $L_n = \{P, Q,\} \cup \{R_i \mid i < n\} \cup L_{AG}$.

Our strategy is to prove that our theory has a unique model in size λ which is κ-resplendent where $\kappa = \aleph_0$, $\lambda = 2^{\aleph_0}$. Let N_0, N_1 be two κ-resplendent models of size λ.

By Claim 4.4, $\left|Q^{N_0}\right| = \left|Q^{N_1}\right| = \lambda$ and we may assume that $Q^{N_0} = Q^{N_1} = \lambda$.

Let $G_0 = P^{N_0}$ and $G_1 = P^{N_1}$ with the group structure. For $i < n$, $j < 2$ and $\alpha < \lambda$, let $H^j_{i,\alpha} = \left\{x \in G_j \,\middle|\, R_i^{N_j}(x, \alpha)\right\}$. This is a definable subgroup of G_j. For $k \leq n$, let $G^k_j = \bigcap_{\alpha<\lambda,\, i \neq k,\, i<n} H^j_{i,\alpha}$. In our original model M, this group is $\{\eta \in G_\omega \mid \forall i \neq k,\, i < n\, (\eta(i) = 0)\}$. Note that $G_j = \sum_{k<n} G^k_j$, and that $G^{k_0}_j \cap \sum_{k<n, k\neq k_0} G^k_j = G^n_j$ (this is true in our original model M, so it is part of the theory). We give each G^k_j the induced L-structure $N^k_j = \left\langle G^k_j, \lambda\right\rangle$, i.e. we interpret $R_i^{N^k_j} = R_i \cap \left(G^k_j \times \lambda\right)$.

Since these groups are definable and infinite, their cardinality is λ, and hence their dimension (over $\mathbb{F}_2$) is λ. In particular there is a group isomorphism $f_n : G^n_0 \to G^n_1$. Note that f_n is an isomorphism of the induced structure on $N^n_j = \left\langle G^n_j, \lambda\right\rangle$ (because it is trivial).

Subclaim. For $k < n$, there is an isomorphism $f_k : G^k_0 \to G^k_1$ which is an isomorphism of the induced structure $N^k_j = \left\langle G^k_j, \lambda\right\rangle$ and extends f_n.

Assuming this claim, we shall finish the proof. Define $f : G_0 \to G_1$ by: given $x \in G_0$, write it as a sum $\sum_{k<n} x_k$ where $x_k \in G^k_0$, and define $f(x) = \sum_{k<n} f_k(x_k)$. This is well defined because if $\sum_{k<n} x_k = \sum_{k<n} x'_k$ then $\sum_{k<n}(x_k - x'_k) = 0$ so for all $k < n$, $x_k - x'_k \in G^n_0$ and

$$\begin{aligned}\sum_{k<n}\left(f_k(x_k) - f_k(x'_k)\right) &= \sum_{k<n}\left(f_k(x_k - x'_k)\right) = \sum_{k<n}\left(f_n(x_k - x'_k)\right) = \\ &= f_n\left(\sum_{k<n} x_k - x'_k\right) = f_n(0) = 0.\end{aligned}$$

It follows similarly that f is a group isomorphism. Also, f is an L_n-isomorphism because if $R_i^{N_0}(a, \alpha)$ holds for some $i < n$, $\alpha < \lambda$ and $a \in G_0$, then write $a = \sum_{k<n} a_k$ where $a_k \in G^k_0$. Since $R_i^{N_0}(a, \alpha)$ holds and $R_i^{N_0}(a_k, \alpha)$ holds for all $k \neq i$, it follows that $R_i^{N_0}(a_i, \alpha)$ holds, so $R_i^{N_1}(f_k(a_k), \alpha)$ holds for all $k < n$, and so $R_i^{N_1}(f(a), \alpha)$ holds. The other direction is similar.

[1] In fact, by [She, Claim 3.1], if T is unstable there are 2^{λ} such models.

Proof. (of subclaim) For a finite set $\mathfrak{b}$ of elements of λ, let $L^j_{\mathfrak{b}} = G^k_j \cap \bigcap_{\alpha\in\mathfrak{b}} H^j_{k,\alpha}$. For $m \le k+1$, let $K^j_m = \sum_{|\mathfrak{b}|=m} L^j_{\mathfrak{b}}$ (as a subspace of G^k_j), so K^j_m is not necessarily definable (however $K^j_0 = G^k_j$ and $K^j_{k+1} = G^n_j$ are). This is a decreasing sequence of subgroups (so subspaces), $G^k_j = K^j_0 \ge \ldots \ge K^j_{k+1} = G^n_j$. Now it is enough to show that:

Subclaim. For $m \le k+1$, there is an isomorphism $f_m : K^0_m \to K^1_m$ which is an isomorphism of the induced structure $\left\langle K^j_m, \lambda \right\rangle$ which extends f_n.

Proof. (of subclaim) The proof is by reverse induction. For $m = k+1$ we already have this. Suppose we have f_{m+1} and we want to construct f_m. Let $\mathfrak{b} \subseteq \lambda$ be of size m. If $m = k$, then it is easy to see that $\left| L^j_{\mathfrak{b}} / \left(K^j_{m+1} \cap L^j_{\mathfrak{b}} \right) \right| = 2$ (this is true in M), so there is an isomorphism $g_{\mathfrak{b}} : L^0_{\mathfrak{b}} / \left(K^0_{m+1} \cap L^0_{\mathfrak{b}} \right) \to L^1_{\mathfrak{b}} / \left(K^1_{m+1} \cap L^1_{\mathfrak{b}} \right)$.

Assume $|\mathfrak{b}| < k$. In our original model M, $L_{\mathfrak{b}} \subseteq K_k$, but here one can find infinitely many pairwise distinct cosets in $L^j_{\mathfrak{b}} / \left(K^j_{m+1} \cap L^j_{\mathfrak{b}} \right)$. Indeed, we can find a type in λ infinitely many variables $\{x_i \mid i < \lambda\}$ over $\mathfrak{b}$ saying that $x_i \in L_{\mathfrak{b}}$ and $x_i - x_j \notin K_{m+1}$ for $i \neq j$ — for all $r < \omega$, it will contain a formula of the form

$$\forall (z_0, \ldots, z_{r-1}) \forall_{t<r} (\bar{y}_t) \left(\left[\bigwedge_{t<r} (z_t \in L_{\bar{y}_t} \wedge |\bar{y}_t| = m+1) \right] \to x_i - x_j \neq \sum_{t=0}^{r-1} z_t \right).$$

To show that this type is consistent, we may assume that $\mathfrak{b} \subseteq Q^M$ so we work in our original model M. For such r and $\mathfrak{b}$, choose distinct $\eta_0, \ldots \eta_{l-1} \in G_\omega$ such that for $s, s' < l$

- $\eta_s(i) = 0$ for $i \neq k$.
- $|\mathrm{supp}(\eta_s(k))| = r+1$.
- $u_1 \in \mathrm{supp}(\eta_s(k))$ & $u_2 \in \mathrm{supp}(\eta_{s'}(k)) \Rightarrow u_1 \cap u_2 = \mathfrak{b}$ (s might be equal to s' but $u_1 \neq u_2$).

Then $\{\eta_s \mid s < l\}$ is such that η_{s_1}, η_{s_2} satisfies the formula above for all $s_1 \neq s_2 < l$: if not, there are $z_0 \in L_{c_0}, \ldots, z_{r-1} \in L_{c_r}$ where $|c_t| = m+1$ such that $\sum_{t<r} z_t = \eta_{s_1} - \eta_{s_2}$. We may assume that

$$\bigcup_{t<r} \mathrm{supp}(z_t(k)) = \mathrm{supp}(\eta_{s_1}(k) - \eta_{s_2}(k)) = \mathrm{supp}(\eta_{s_1}(k)) \cup \mathrm{supp}(\eta_{s_2}(k)),$$

but then for $t < r$, $|\mathrm{supp}(z_t(k))| \le 1$ by our choice of η_s and this is a contradiction.

Now, let N'_j be an elementary extension of N_j with realizations $D = \{c_i \mid i < \lambda\}$ of this type, and we may assume $|N'_j| = \lambda$. Then, add a predicate for the set D, and an injective function from N'_j to D. Finally, by resplendence of N_j, $\left| L^j_{\mathfrak{b}} / \left(K^j_{m+1} \cap L^j_{\mathfrak{b}} \right) \right| = \lambda$.

Hence it has a basis of size λ, and let $g_{\mathfrak{b}} : L^0_{\mathfrak{b}} / \left(K^0_{m+1} \cap L^0_{\mathfrak{b}} \right) \to L^1_{\mathfrak{b}} / \left(K^1_{m+1} \cap L^1_{\mathfrak{b}} \right)$ be an isomorphism of $\mathbb{F}_2$-vector spaces.

Note that $f_{m+1} \restriction K^0_{m+1} \cap L^0_{\mathfrak{b}}$ is onto $K^1_{m+1} \cap L^1_{\mathfrak{b}}$ (this is because f_{m+1} is an isomorphism of the induced structure). We can write $L^j_{\mathfrak{b}} = \left(K^j_{m+1} \cap L^j_{\mathfrak{b}} \right) \oplus W^j$ where $W^j \cong L^j_{\mathfrak{b}} / \left(K^j_{m+1} \cap L^j_{\mathfrak{b}} \right)$, so $g_{\mathfrak{b}}$

induces an isomorphism from W^0 to W^1. Now extend $f_{m+1} \restriction K^0_{m+1} \cap L^0_b$ to $f^b_m : L^0_b \to L^1_b$ using g_b.

Next, note that $\left\{L^j_b \,|\, b \subseteq \lambda, |b| = m\right\}$ is independent over K^j_{m+1}, i.e. for distinct $b_0, \ldots, b_r$, $L^j_{b_r} \cap \sum_{t<r} L^j_{b_t} \subseteq K^j_{m+1}$. Indeed, in our original model M, the intersection $L_{b_r} \cap \sum_{t<r} L_{b_t}$ is equal to $\sum_{t<r} L_{b_r \cup b_t}$, so this is true also in N_j (in fact, this is true for every choice of finite sets b_t — regardless of their size).

Define f_m as follows: given $a \in K^0_m$, we can write $a = \sum_{b \in B} a_b$ where $a_b \in L_b$ for a finite $B \subseteq \{b \subseteq \lambda \,|\, |b| = m\}$, and define $f_m(a) = \sum f^b_m(a_b)$. It is well defined: if $\sum_{b \in B} x_b = \sum_{b' \in B'} y_{b'}$, then for $b_1 \in B \cap B'$, $b_2 \in B \backslash B'$ and $b_3 \in B' \backslash B$, $(x_{b_1} - y_{b_1}), x_{b_2}, y_{b_3} \in K^0_{m+1}$, so

$$\sum_{b \in B} f^b_m(x_b) - \sum_{b' \in B'} f^{b'}_m(y_{b'}) =$$

$$\sum_{b \in B \cap B'} f_{m+1}(x_b - y_b) + \sum_{b \in B \backslash B'} f_{m+1}(x_b) - \sum_{b \in B' \backslash B} f_{m+1}(y_b) = 0.$$

It follows similarly that f_m is a group isomorphism.

We check that f_m is an isomorphism of the induced structure. So suppose $a \in K^0_m$, $\alpha < \lambda$ and $i < \omega$. If $i \neq k$, then since $K^j_m \subseteq G^k_j$ for $j < 2$, both $R^{N_0}_i(a, \alpha)$ and $R^{N_1}_i(f(a), \alpha)$ hold. Suppose $R^{N_0}_k(a, \alpha)$ holds. Write $a = \sum_{b \in B} a_b$ as above. Then, as $a \in L_{\{\alpha\}} \cap \sum_{b \in B} L_b = \sum_{b \in B} L_{b \cup \{\alpha\}}$, we may assume that $b \in B \Rightarrow \alpha \in b$. So by definition of f_m, $R^{N_1}_k(f_m(a), \alpha)$ holds. The other direction holds similarly and we are done. □

Note 4.7. This example is not strongly dependent, because the sequence of formulas $R_n(x, y)$ is a witness of that the theory is not strongly dependent. So as we said in the introduction, it is still open whether Property A holds for strongly dependent theories.

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