CONSISTENCY OF "THE IDEAL OF NULL RESTRICTED TO SOME A IS κ–COMPLETE NOT κ^+–COMPLETE, κ WEAKLY INACCESSIBLE AND $\mathbf{cov}(\text{meagre}) = \aleph_1$"

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In this note we give an answer to the following question of Grinblat (Moti Gitik asked about it in the Oberwolfach meeting:

Grinblat's Question 1. Is it consistent that

(∗∗) for some set X, $\mathbf{cov}(\text{Null} \restriction X) = \lambda$ is a weakly inaccessible cardinal (so X not null of course) while $\mathbf{cov}(\text{Meager})$ is small, say it is $\aleph_1$.

A. The Forcing:

Starting with a universe $\mathbf{V}$ and a cardinal λ of cofinality $> \aleph_0$, regular for simplicity (otherwise the only difference is that J consists of "bounded subsets"), in fact weakly inaccessible for Grinblat's question.

Let $\mathbb{P} = \mathbb{P}_\lambda$ be the result of FS iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \lambda \rangle$ with $\underset{\sim}{\mathbb{Q}}_{2i}$ being the random real forcing, and $\underset{\sim}{\mathbb{Q}}_{2i+1}$ being the Cohen forcing notion. Let $\underset{\sim}{\mathbb{R}}$ be a $\mathbb{P}$–name for the forcing notion adding $\aleph_1$ random reals (i.e., forcing with the measure algebra of Borel subsets of ${}^{\omega_1}2$ of positive Lebesgue measure).

We claim that $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P} * \underset{\sim}{\mathbb{R}}}$ is as required.

Let $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}}$.

As the whole forcing satisfies the ccc, no cardinal is collapsed etc

B. Why $\mathbf{cov}(\text{Meager}) = \aleph_1$?

As forcing by $\underset{\sim}{\mathbb{R}}$ does it (well known).

C.

Let $\underset{\sim}{\eta}_i$ be the $\underset{\sim}{\mathbb{Q}}_i$–generic real for $i < \lambda$. Clearly they are pairwise distinct. Let

$$\underset{\sim}{X} \stackrel{\text{def}}{=} \{\underset{\sim}{\eta}_{2i} : i < \lambda\}.$$

This is a set of cardinality λ. Let J be the ideal of subsets of X of cardinality $< \lambda$ (it is a λ–complete ideal on X).

It is enough to prove

(∗) J is equal to the ideal of null subsets of X.

C1.

Now, for every $\alpha < \lambda$ the set $\{\underset{\sim}{\eta}_{2i} : i < \alpha\}$ is null in $\mathbf{V}_2$. Why? Because $\underset{\sim}{\eta}_{2\alpha+1}$ is Cohen over $\mathbf{V}^{\mathbb{P}_{2\alpha+1}}$ the universe to which the above set belongs and is an inner model of $\mathbf{V}_2$.

This is enough to show that every member of J is null.

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C2.

For the other direction, let $\underset{\sim}{Y}$ be a $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ name of an unbounded subset of λ. We shall prove that

$$\{\underset{\sim}{\eta}_{2i} : i \in \underset{\sim}{Y}\}$$

is forced to be non-null (this clearly suffices).

Let p be a condition in $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ forcing the inverse, so for some $\mathbb{P} * \underset{\sim}{\mathbb{R}}$–name $\underset{\sim}{Z}$ of a null Borel subset of ${}^{\omega}2$, we have

$$p \Vdash \text{“ } \{\underset{\sim}{\eta}_{2i} : i \in \underset{\sim}{Y}\} \subseteq \underset{\sim}{Z} \text{ ”}.$$

We can find $\alpha < \lambda$ such that, in $\mathbf{V}^{\mathbb{P}_\alpha}$, $\underset{\sim}{Z}$ becomes an $\mathbb{R}^{\mathbf{V}^{\mathbb{P}_\alpha}}$–name and p is a member of $\mathbb{R}^{\mathbf{V}^{\mathbb{P}_\alpha}}$.

Now for every i, if $\alpha < 2i < \lambda$ then $\underset{\sim}{\eta}_{2i}$ is random over $\mathbf{V}^{\mathbb{P}_\alpha}$. Hence, by the Fubini theorem (i.e., random reals commute), it is also random over $(\mathbf{V}^{\mathbb{P}_\alpha})^{\mathbb{R}^{\mathbf{V}^{\mathbb{P}_\alpha}}}$. Consequently it does not belongs to $\underset{\sim}{Z}$, so we are done.

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