COMBINATORIAL BACKGROUND FOR NON-STRUCTURE
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Abstract. This was supposed to be an appendix to the book *Non-structure*, and probably will be if it materializes.

It presents relevant material sometimes new, which used in works which were supposed to be part of that book.

In §1 we deal with partition theorems on trees with ω levels; it is self contained. In §2 we deal with linear orders which are countable union of scattered ones with unary predicated, it is self contained. In §3 we deal mainly with pcf theory but just quote. In §4, on normal ideals, we repeat [She86]. This is used in [Shea].

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In reference like [Shea, 1.16=L7.7], the 1.16 is the number of claim (or definition) and L7.7 is its label; so intended just to help the author to correct it if the number will be changed. The author thanks Alice Leonhardt for the beautiful typing.

§ 1. Partitions on trees

See [RS87], [She83, 2.4,2.5], [She82b], [She98, XI3.5,XI3.5A,XI3.7,XI5.3,XV2.6,XV2.6A, XV2.6B,XV2.6C] on those theorems.

See Rubin, Shelah [RS87] pp 47-48 on the history of such theorems, more in [She83].

Definition 1.1. 1) $\mathbf{I}$ is an ideal on S when it is a family of subsets of S including the singleton, closed under union of two and $S \notin S$.
2) An ideal $\mathbf{I}$ is λ-complete if any union of less than λ members of $\mathbf{I}$ is still a member of $\mathbf{I}$.

In [Shec, 1.1=L1.1], [Shec, 1.2=L1.2] we use

Definition 1.2. 1) A tagged tree is a pair $(\mathscr{T}, \bar{\mathbf{I}})$ such that:

- (a) $\mathscr{T}$ is a ω-tree, which in this section means a non-empty set of finite sequences of ordinals such that if $\eta \in \mathscr{T}$ then any initial segment of η belongs to $\mathscr{T}$. We understand that $\mathscr{T}$ is ordered by initial segments, i.e., $\eta \leq_{\mathscr{T}} \nu$ means η is an initial segment of ν that is $\eta \trianglelefteq \nu$
- (b) $\bar{\mathbf{I}}$ is a function but only $\bar{\mathbf{I}} {\restriction} (\mathrm{Dom}(\mathbf{I}) \cap \mathscr{T})$ matters, such that for every $\eta \in \mathscr{T}$: if $\bar{\mathbf{I}}(\eta) = \mathbf{I}_\eta$ is defined then $\bar{\mathbf{I}}(\eta)$ is an ideal of subsets of some set called the domain of $\mathbf{I}_\eta, \mathrm{Dom}(\mathbf{I}_\eta)$ and $\mathrm{Dom}(\mathbf{I}_\eta) \notin \mathbf{I}_\eta$, and

  $\mathrm{Succ}_{\mathscr{T}}(\eta) := \{\nu : \nu \text{ is an immediate successor of } \eta \text{ in } \mathscr{T}\} \subseteq \mathrm{Dom}(\mathbf{I}_\eta)$.

  The interesting case is when $\mathrm{Succ}_{\mathscr{T}}(\eta) \notin \mathbf{I}_\eta$ and usually $\mathbf{I}_\eta$ is $\aleph_2$-complete
- (c) For every $\eta \in \mathscr{T}$ we have $\mathrm{Succ}_{\mathscr{T}}(\eta) \neq \emptyset$.

2) We call $(\mathscr{T}, \bar{\mathbf{I}})$ normal when for every $\eta \in \mathrm{Dom}(\mathbf{I}_\eta)$ we have: $\mathrm{Dom}(\mathbf{I}_\eta) = \mathrm{Succ}_{\mathscr{T}}(\eta)$.

Convention 1.3. 1) For any tagged tree $(\mathscr{T}, \bar{\mathbf{I}})$ we can define the function $\bar{\mathbf{I}}^\dagger$, by:

$$\mathrm{Dom}(\bar{\mathbf{I}}^\dagger) = \{\eta : \eta \in \mathrm{Dom}(\bar{\mathbf{I}}) \text{ and } \mathrm{Succ}_{\mathscr{T}}(\eta) \subseteq \mathrm{Dom}(\mathbf{I}_\eta), \text{ and } \mathrm{Succ}_{\mathscr{T}}(\eta) \notin \mathbf{I}_\eta\}$$

$$\mathbf{I}^\dagger_\eta = \{\{\alpha : \eta\hat{\ }\langle\alpha\rangle \in A\} : A \in \mathbf{I}_\eta\}.$$

2) We sometimes, in an abuse of notation, do not distinguish between $\bar{\mathbf{I}}$ and $\bar{\mathbf{I}}^\dagger$. Also if $\mathbf{I}^\dagger_\eta$ is constantly $\mathbf{I}^*$, we may write $\mathbf{I}^*$ instead of $\bar{\mathbf{I}}$.
3) We use $\mathscr{T}$ only to denote ω–trees.

Definition 1.4. 1) We say that η is a splitting point of $(\mathscr{T}, \bar{\mathbf{I}})$ when $\eta \in \mathscr{T}, \mathbf{I}_\eta$ is defined and $\mathrm{Succ}_{\mathscr{T}}(\eta) \notin \mathbf{I}_\eta$. Let $\mathrm{split}(\mathscr{T}, \bar{\mathbf{I}})$ be the set of splitting points of $(\mathscr{T}, \bar{\mathbf{I}})$. Usually, we will be interested only in trees where each branch meets $\mathrm{split}(\mathscr{T}, \bar{\mathbf{I}})$ infinitely often.
2) For $\eta \in \mathscr{T}$, let $\mathscr{T}^{[\eta]} := \{\nu \in \mathscr{T} : \nu = \eta \text{ or } \nu \triangleleft \eta \text{ or } \eta \triangleleft \nu\}$.

Definition 1.5. We now define several orders between tagged trees:
1) $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \leq (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ if $\mathscr{T}_2 \subseteq \mathscr{T}_1$, and $\mathrm{split}(\mathscr{T}_2, \bar{\mathbf{I}}_2) \subseteq \mathrm{split}(\mathscr{T}_1, \bar{\mathbf{I}}_1)$, and for every $\eta \in \mathrm{split}(\mathscr{T}_2, \bar{\mathbf{I}}_2)$ we have $\bar{\mathbf{I}}_2(\eta) {\restriction} \mathrm{Succ}_{\mathscr{T}_2}(\eta) = \bar{\mathbf{I}}_1(\eta) {\restriction} \mathrm{Succ}_{\mathscr{T}_2}(\eta)$ (where $\mathbf{I} {\restriction} A = \{B : B \subseteq A \text{ and } B \in \mathbf{I}\}$). (So every splitting point of $\mathscr{T}_2$ is a splitting point of $(\mathscr{T}_1, \bar{\mathbf{I}}_1)$, and $\bar{\mathbf{I}}_2 {\restriction} \mathrm{split}(\mathscr{T}_2, \bar{\mathbf{I}}_2)$ is completely determined by $\bar{\mathbf{I}}_1$ and $\mathrm{split}(\mathscr{T}_2, \bar{\mathbf{I}}_2)$ provided that $\bar{\mathbf{I}}_2$ is normal, see 1.2(2).)

2) $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \le^* (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ <u>when</u> $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \le (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ and $\mathrm{split}(\mathscr{T}_2, \bar{\mathbf{I}}_2) = \mathrm{split}(\mathscr{T}_1, \bar{\mathbf{I}}_1) \cap \mathscr{T}_2$.
3) $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \le^{\otimes} (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ <u>if</u> $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \le^* (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ and $\eta \in \mathscr{T}_2 \setminus \mathrm{split}(\mathscr{T}_1, \bar{\mathbf{I}}_1) \Rightarrow \mathrm{Succ}_{\mathscr{T}_2}(\eta) = \mathrm{Succ}_{\mathscr{T}_1}(\eta)$.
4) $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \le^{\otimes}_{\mu} (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ <u>if</u> $(\mathscr{T}_1, \bar{\mathbf{I}}_1) \le^* (\mathscr{T}_2, \bar{\mathbf{I}}_2)$ and $\eta \in \mathscr{T}_2$ and $|\mathrm{Succ}_{\mathscr{T}_1}(\eta)| < \mu \Rightarrow \mathrm{Succ}_{\mathscr{T}_2}(\eta) = \mathrm{Succ}_{\mathscr{T}_1}(\eta)$.

Definition 1.6. 1) For a set $\mathbb{I}$ of ideals, a tagged tree $(\mathscr{T}, \bar{\mathbf{I}})$ is an $\mathbb{I}$-tree <u>if</u> for every splitting point $\eta \in \mathscr{T}$ we have $\mathbf{I}_\eta \in \mathbb{I}$ (up to an isomorphism) or just $\mathbf{I}_\eta$ is isomorphic to $\mathbf{I}{\restriction}A$ for some $\mathbf{I} \in \mathbb{I}, A \subseteq \mathrm{dom}(\mathbf{I}), A \notin \mathbf{I}$; but we usually use restriction-closed $\mathbb{I}$, see Definition 1.9(2).
2) For a set $\mathbf{S}$ of regular cardinals, an $\mathbf{S}$-tree $\mathscr{T}$ is a tree such that for any point $\eta \in \mathscr{T}$ we have: $|\mathrm{Succ}_{\mathscr{T}}(\eta)| \in \mathbf{S}$ or $|\mathrm{Succ}_{\mathscr{T}}(\eta)| = 1$.
3) We may omit $\bar{\mathbf{I}}$ and denote a tagged tree $(\mathscr{T}, \bar{\mathbf{I}})$ by $\mathscr{T}$ whenever $\mathscr{T} \subseteq \mathrm{Dom}(\bar{\mathbf{I}})$ and $\mathbf{I}_\eta = \{A \subseteq \mathrm{Succ}_T(\eta) : |A| < |\mathrm{Succ}_{\mathscr{T}}(\eta)|\}$ and $|\mathrm{Succ}_{\mathscr{T}}(\eta)| \in \mathrm{IRCar} \cup \{1\}$ for every $\eta \in \mathscr{T}$, recalling IRCar is the class of infinite regular cardinals.
4) For a tree $\mathscr{T}, \lim(\mathscr{T})$ is the set of branches of $\mathscr{T}$, i.e. all ω-sequences of ordinals, such that every finite initial segment of them is a member of $\mathscr{T}$, that is $\lim(\mathscr{T}) = \{\eta \in {}^{\omega}\mathrm{Ord} : (\forall n)\, \eta{\restriction}n \in \mathscr{T}\}$.
5) A subset J of a tree $\mathscr{T}$ is a front if: $\eta \neq \nu \in J$ implies none of them is an initial segment of the other, and every $\eta \in \lim(\mathscr{T})$ has an initial segment which is a member of J.
6) $(\mathscr{T}, \bar{\mathbf{I}})$ is standard <u>if</u> for every non-splitting point $\eta \in \mathscr{T}$ we have $|\mathrm{Succ}_{\mathscr{T}}(\eta)| = 1$.
7) $(\mathscr{T}, \bar{\mathbf{I}})$ is full <u>if</u> every $\eta \in \mathscr{T}$ is a splitting point.
8) The natural topology on $\lim(I)$ for an ω-tree $\mathscr{T}$ is defined by $\mathscr{U} \subseteq \lim(\mathscr{T})$ is open <u>when</u> for every $\eta \in \mathscr{U}$ for some $n < \omega$ we have $\lim(\mathscr{T}^{[\eta\restriction n]}) \subseteq \mathscr{U}$.

Recall

Observation 1.7. *1) The set $\lim(\mathscr{T})$ is not absolute, i.e., if $\mathbf{V}_1 \subseteq \mathbf{V}_2$ are two universes of set theory then in general $(\lim(\mathscr{T}))^{\mathbf{V}_1}$ will be a proper subset of $(\lim(\mathscr{T}))^{\mathbf{V}_2}$.*
2) However, the notion of being a front is absolute: if $\mathbf{V}_1 \models$ "A is a front in $\mathscr{T}$", then there is a depth function $f : \mathscr{T} \to \mathrm{Ord}$ satisfying

$$\eta \triangleleft \nu \text{ and } \forall k \le \ell g(\eta)[\eta{\restriction}k \notin A] \to f(\eta) > f(\nu).$$

This function will also witness in $\mathbf{V}_2$ that A is a front.
3) $A \subseteq \mathscr{T}$ contains a front <u>if and only if</u> A meets every branch of $\mathscr{T}$. So if $A \subseteq \mathscr{T}$ contains a front of $\mathscr{T}$ and $\mathscr{T}' \subseteq \mathscr{T}$ is a subtree, then $A \cap \mathscr{T}'$ contains a front of $\mathscr{T}'$. Also this notion is absolute.

Notation 1.8. In several places in this section we will have an occasion to use the following notation: Assume that $(\mathscr{T}, \bar{\mathbf{I}})$ is a tagged tree, and for each $\eta \in \mathscr{T}$ we are given a family $\mathscr{P}_\eta$ of subsets of $\mathscr{T}^{[\eta]}$ such that

$$\eta \triangleleft \nu \Rightarrow (\forall A \in \mathscr{P}_\eta)(\exists B \in \mathscr{P}_\nu)[B \subseteq A].$$

1) We inductively define for all $\alpha \in \mathrm{Ord} \cup \{\infty\}$ the property $\mathrm{Dp}_\alpha(\eta)$ by: $\mathrm{Dp}_\alpha(\eta)$ <u>if and only if</u> $(\forall \beta < \alpha)(\forall A \in \mathscr{P}_\eta)(\exists \nu \in A \cap \mathrm{split}(\mathscr{T}))[\eta \triangleleft \nu$ and $\mathrm{Dp}_\beta(\eta)$ and $\{\rho : \rho \in \mathrm{Succ}_{\mathscr{T}}(\nu)$ and $\mathrm{Dp}_\beta(\rho)\} \notin \mathbf{I}_\nu]$.
2) Then it is easy to see that $\mathrm{Dp}(\eta) := \max\{\alpha \in \mathrm{Ord} \cup \{\infty\} : \mathrm{Dp}_\alpha(\eta)\}$ is well defined, and $\mathrm{Dp}_\alpha(\eta) \Leftrightarrow \mathrm{Dp}(\eta) \ge \alpha$. We call $\mathrm{Dp}(\eta)$ the "depth" of η (with respect

to the family $\bar{\mathscr{P}} = \langle \mathscr{P}_\eta : \eta \in \mathscr{T} \rangle$ and the tagged tree $(\mathscr{T}, \bar{\mathbf{I}})$). It is easy to check that $\eta \triangleleft \nu \Rightarrow \mathrm{Dp}(\eta) \geq \mathrm{Dp}(\nu)$.
3) Similarly we can define $\mathrm{Dp}'_\alpha(\eta), \mathrm{Dp}'(\eta)$, when in the definition of $\mathrm{Dp}_\alpha(\eta)$ we replace $\eta \triangleleft \nu$ by $\eta = \nu$ in this case.

Definition 1.9. 1) A tagged tree $(\mathscr{T}, \bar{\mathbf{I}})$ is λ-complete <u>if</u> for each $\eta \in \mathscr{T} \cap \mathrm{Dom}(\bar{\mathbf{I}})$ the ideal $\mathbf{I}_\eta$ is λ-complete.
2) A family $\mathbb{I}$ of ideals is λ-complete <u>if</u> each $\mathbf{I} \in \mathbb{I}$ is λ-complete. We will only consider $\aleph_2$-complete families $\mathbb{I}$.
3) A family $\mathbb{I}$ is restriction-closed <u>if</u> $\mathbf{I} \in \mathbb{I}, A \subseteq \mathrm{Dom}(\mathbf{I}), A \notin \mathbf{I}$ implies $\mathbf{I}{\restriction}A = \{B \in \mathbf{I} : B \subseteq A\}$ belongs to $\mathbb{I}$.
4) The restriction closure of $\mathbb{I}$ is

$$\mathrm{res} - \mathrm{cl}(\mathbb{I}) = \{\mathbf{I} \restriction A : \mathbf{I} \in \mathbb{I}, A \subseteq \mathrm{Dom}(\mathbf{I}), A \notin \mathbf{I}\}.$$

5) $\mathbf{I}$ is λ-indecomposable <u>if</u> for every $A \subseteq \mathrm{Dom}(\mathbf{I}), A \notin \mathbf{I}$, and $h : A \to \lambda$ there is $Y \subseteq \lambda, |Y| < \lambda$ such that $h^{-1}(Y) \notin \mathbf{I}$. We say $\bar{\mathbf{I}}$ or $\mathbb{I}$, is λ-indecomposable if each $\mathbf{I}_\eta$ (or $\mathbf{I} \in \mathbb{I}$) is λ-indecomposable; similarly in part (7).
6) $\mathbf{I}$ is strongly λ-indecomposable <u>if</u> for $A_i \in \mathbf{I}$ $(i < \lambda)$ and $A \subseteq \mathrm{Dom}(\mathbf{I}), A \notin \mathbf{I}$ we can find $B \subseteq A$ of cardinality $< \lambda$ such that for no $i < \lambda$ does A_i include B.

Observation 1.10. *1) If an ideal $\mathbf{I}$ is λ^+-complete <u>then</u> it is λ-indecomposable.*
2) If $\mathbf{I}$ is an ideal and $|\mathrm{Dom}(\mathbf{I})| < \lambda$ <u>then</u> $\mathbf{I}$ is λ-indecomposable.
3) If $\mathbf{I}$ is a strongly θ-indecomposable ideal <u>then</u> $\mathbf{I}$ is a θ-decomposable ideal.

Lemma 1.11. *1) If $(\mathscr{T}, \bar{\mathbf{I}})$ is a λ^+-complete tree and $\mathbf{H}$ is a function from $\lim(\mathscr{T})$ to λ such that for every $\alpha < \lambda$ the set $\mathbf{H}^{-1}(\{\alpha\})$ is a Borel subset of $\lim(\mathscr{T})$ (in the topology that was defined in Definition 1.6(8)) <u>then</u> there is a tagged subtree $(\mathscr{T}^\dagger, \bar{\mathbf{I}})$ satisfying $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}^\dagger, \bar{\mathbf{I}})$ (see Definition 1.5(2)) such that $\mathbf{H}$ is constant on $\lim(\mathscr{T}^\dagger)$.*
2) In part (1) we can let $\mathbf{H}$ be multivalued, i.e. assume $\lim(\mathscr{T})$ is $\bigcup\limits_{\alpha<\lambda} \dot{\mathbb{B}}_\alpha$, each $\dot{\mathbb{B}}_\alpha$ is a Borel subset of $\lim(\mathscr{T})$. If $(\mathscr{T}, \bar{\mathbf{I}})$ is λ^+-complete <u>then</u> there is $(\mathscr{T}^\dagger, \bar{\mathbf{I}})$ such that $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}^\dagger, \bar{\mathbf{I}})$ and for some $\alpha < \lambda$ we have $\lim(\mathscr{T}^\dagger) \subseteq \dot{\mathbb{B}}_\alpha$.
3) We can allow in (1) the function $\mathbf{H}$ to have values outside λ as long as $|\mathrm{Rang}(\mathbf{H})| \leq \lambda$. Similarly (2).

Proof. 1) First note that <u>if</u> $\mathscr{T}_1 \subseteq \mathscr{T}$ satisfies $(*)$ below then $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}_1, \bar{\mathbf{I}}{\restriction}\mathscr{T}_1)$ where:

- $(*)$ $\langle\rangle \in \mathscr{T}_1; \eta \triangleleft \nu \in \mathscr{T}_1 \Rightarrow \eta \in \mathscr{T}_1$; for every $\eta \in \mathscr{T}_1$ if η is a splitting point of $(\mathscr{T}, \bar{\mathbf{I}})$ then $\mathrm{Succ}_{\mathscr{T}_1}(\eta) = \mathrm{Succ}_{\mathscr{T}}(\eta)$; and if η is not a splitting point of T then $|\mathrm{Succ}_{\mathscr{T}_1}(\eta)| = 1$.

So without loss of generality we can assume that in $\mathscr{T}$ every point is either a splitting point or it has only one immediate extension i.e. $(\mathscr{T}, \bar{\mathbf{I}})$ is standard.

For each $\alpha < \lambda$ let us define a game $\eth_\alpha$: in the first move the first player chooses the node η_0 in the tree such that $\lg(\eta_0) = 0$, the second player responds by choosing a proper subset A_0 of $\mathrm{Succ}_{\mathscr{T}}(\eta_0)$ such that $A_0 \in \mathbf{I}_{\eta_0}$. For $n > 0$, in the n-th move, the first player chooses an immediate extension η_n of η_{n-1}, such that $\eta_n \notin A_{n-1}$ or η_{n-1} is not a splitting point of $(\mathscr{T}, \bar{\mathbf{I}})$, and the second player responds by choosing $A_n \in \mathbf{I}_{\eta_n}$.

The first player wins if for the infinite branch η defined by $\eta_0, \eta_1, \eta_2, \ldots$ we have $\mathbf{H}(\eta) = \alpha$. By the assumption of the lemma this is a Borel game so by Martin's Theorem, [Mar75] one of the players has a winning strategy. We claim that for some $\alpha < \lambda$, the first player has a winning strategy in the game $\eth_\alpha$. Assume otherwise, i.e., for every $\alpha < \lambda$ the second player has a winning strategy $\mathbf{f}_\alpha$. We construct an infinite branch inductively: let $\eta_0 = \langle\rangle$ recalling $\eta_0 \in \mathscr{T}$. At stage n let A_n be $\bigcup\limits_{\alpha<\lambda} \mathbf{f}_\alpha(\eta_0, \eta_1, \ldots, \eta_{n-1})$; now if η_{n-1} is a splitting point (of $(\mathscr{T}, \bar{\mathbf{I}})$) then $\mathbf{I}_{\eta_{n-1}}$ is λ^+-complete and each $\mathbf{f}_\alpha(\eta_0, \ldots, \eta_{n-1})$ is a member of it, because $\eta_0, F_\alpha(\eta_0), \eta_1, F_\alpha(\eta_0, \eta_1), \ldots, \eta_{n-1}$ is an initial segment of a play of the game $\eth_\alpha$ in which the second player uses the winning strategy $\mathbf{f}_\alpha$, hence $A_n \in \mathbf{I}_{\eta_{n-1}}$, so clearly $\mathrm{Succ}_T(\eta_{n-1}) \nsubseteq A_n$.

If η_{n-1} is not a splitting point, it has only one immediate successor and let it be η_n, otherwise since $\mathrm{Succ}(\eta_{n-1}) \notin \bar{\mathbf{I}}_{\eta_{n-1}}, A_n \in \bar{\mathbf{I}}_{\eta_{n-1}}$, we have $(\mathrm{Succ}(\eta_{n-1}) \setminus A_n) \neq \emptyset$ so we can choose $\eta_n \in (\mathrm{Succ}_{\mathscr{T}}(\eta_{n-1}) \setminus A_n)$. Let $\eta = \bigcup\limits_{n<\omega} \eta_n$ be the infinite branch that we define by our construction and let $\alpha(*) = \mathbf{H}(\eta)$. Now, in the game $\eth_{\alpha(*)}$, if the first player chooses η_n at stage n (for all n) and the second player plays by his strategy $\mathbf{f}_{\alpha(*)}$, the first player will win although the second player has used his winning strategy $\mathbf{f}_{\alpha(*)}$, a contradiction.

So there must be $\alpha(*)$ such that the first player has a winning strategy $\mathbf{f}_{\alpha(*)}$ for $\eth_{\alpha(*)}$, and let $\mathscr{T}^\dagger$ be the subtree of $\mathscr{T}$ defined by $\{\eta \in \mathscr{T} : \eta = \langle\rangle$, or letting $n = \ell g(\eta) + 1$ we have that $\langle \eta{\restriction}0, \ldots, \eta \restriction n\rangle$ are the first $n+1$ moves of the first player in a play in which he plays according to $\mathbf{f}_{\alpha(*)}\}$. Now, for $\eta \in \mathscr{T}^\dagger \cap \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}})$, let $A = \mathrm{Succ}_{\mathscr{T}^\dagger}(\eta)$. Then $A \notin \mathbf{I}_\eta$, otherwise the second player could have played it as A_n. So $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}^\dagger, \bar{\mathbf{I}})$, and $\mathscr{T}^\dagger$ is as required.

2) Same proof replacing $\mathbf{H}^{-1}(\{\alpha\})$ by $\dot{\mathbb{B}}_\alpha$, so $\mathbf{H}(\eta) = \alpha(*)$ by $\eta \in \dot{\mathbb{B}}_{\alpha(*)}$.

3) Trivial. $\square_{1.11}$

Proof. E.g.

3) So let $A \subseteq \mathrm{Dom}(\mathbf{I}), A \in \mathbf{I}$ and $h : A \to \lambda$ be given and we should find $Y \subseteq \lambda$ of cardinality $< \lambda$ such that $h^{-1}(Y) \notin \mathbf{I}$. For $i < \lambda$ let $A_i := h^{-1}\{i\}$, so as $\mathbf{I}$ is strongly λ-indecomposable there $B \subseteq A$ of cardinality $< \lambda$. Let $Y = \{h(t) : t \in B\}$ so clearly Y is a subset of λ of cardinality $\leq |B| < \lambda$, so it suffices to prove that $h^{-1}(Y) \notin \mathbf{I}$. $\square$

Conclusion 1.12. *If $(\mathscr{T}, \bar{\mathbf{I}})$ is a λ^+-complete tree, and g is a function from $\mathscr{T}$ into λ, and $\lambda^{\aleph_0} = \lambda$, <u>then</u> there is a tagged subtree $(\mathscr{T}^\dagger, \bar{\mathbf{I}})$ satisfying $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}^\dagger, \bar{\mathbf{I}})$ and such that $g \restriction \mathscr{T}^\dagger$ depends only on the length of its argument, i.e., for some function $g^\dagger : \omega \to \lambda$, for all $\eta \in \mathscr{T}^\dagger$ we have $g(\eta) = g^\dagger(\ell g(\eta))$.*

Proof. Follows by 1.11 for the function $\mathbf{H}, \mathbf{H}(\eta) = \langle g(\eta{\restriction}n) : n < \omega\rangle$. $\square_{1.12}$

Lemma 1.13. *1) Assume that λ is a regular uncountable cardinal, and $(\mathscr{T}, \bar{\mathbf{I}})$ is a tagged tree such that for every $\eta \in \mathscr{T}$ $\mathbf{I}_\eta$ is λ^+-complete or $|\mathrm{Succ}_{\mathscr{T}}(\eta)| < \lambda$. If $\mathbf{H} : \lim(\mathscr{T}) \to \lambda$ satisfies "$\dot{\mathbb{B}}_\alpha := \{\eta \in \lim(\mathscr{T}) : \mathbf{H}(\eta) < \alpha\}$ is a Borel subset of $\lim(\mathscr{T})$ for any successor $\alpha < \lambda$", <u>then</u> there are $\alpha < \lambda$ and $(\mathscr{T}', \bar{\mathbf{I}})$ satisfying $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}', \bar{\mathbf{I}})$ and such that for all $\eta \in \mathscr{T}'$ we have $\mathbf{H}(\eta) < \alpha$, and for all η in $\mathscr{T}'$, if $|\mathrm{Succ}_{\mathscr{T}}(\eta)| < \lambda$, then $\mathrm{Succ}_{\mathscr{T}'}(\eta) = \mathrm{Succ}_{\mathscr{T}}(\eta)$.*

2) Like part (1) but we omit the function $\mathbf{H}$ *and just assume* $\dot{\mathbb{B}}_\alpha$ *is a Borel subset of* $\lim(\mathscr{T})$ *for* $\alpha < \lambda$ *but demand* $\bigcup\limits_{\alpha<\lambda} \dot{\mathbb{B}}_\alpha = \lim(\mathscr{T})$*; moreover every* $X \subseteq \lim(\mathscr{T})$ *of cardinality* $< \lambda$ *is included in some* $\dot{\mathbb{B}}_\alpha, \alpha < \lambda$.
3) Let λ, μ *be uncountable cardinals satisfying* $\lambda^{<\mu} = \lambda$ *and let* $(\mathscr{T}, \bar{\mathbf{I}})$ *be a tree in which for each* $\eta \in \mathscr{T}$ *either* $|\mathrm{Succ}_{\mathscr{T}}(\eta)| < \mu$ *or* $\bar{\mathbf{I}}(\eta)$ *is* λ^+*-complete. For* $A \subseteq \mathscr{T}$ *and* $\eta \in \mathscr{T}$ *we define* $\restriction_{\mathscr{T}}(\eta, A)$ *as the sequence* $\langle x_\ell : \ell < \ell g(\eta)\rangle$ *when* x_ℓ *is* $\eta(\ell)$ *if* $\eta\restriction\ell \in A$ *and zero if* $\eta\restriction\ell \in A$*.* <u>*Then*</u> *for every function* $\mathbf{H} : \mathscr{T} \to \lambda$ *there exists* $\mathscr{T}', (\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}', \bar{\mathbf{I}})$ *such that (letting* $A = \{\eta \in \mathscr{T} : |\mathrm{Suc}_{\mathscr{T}}(\eta)| < \mu\}$ *hence* $\restriction_{\mathscr{T}}(\eta, A) \in {}^{\omega>}\mu$ *for* $\eta \in \mathscr{T}$*):*

- *for* $\eta, \eta' \in \mathscr{T}'' : \restriction_{\mathscr{T}}(\eta, A) = \restriction_{\mathscr{T}}(\eta', A)$ *implies:* $\mathbf{H}(\eta) = \mathbf{H}(\eta')$ *and* $\eta \in A$ *iff* $\eta' \in A$*, and if* $\eta \in \mathscr{T}' \cap A$*, then* $\mathrm{Succ}_{cT}(\eta) = \mathrm{Succ}_{\mathscr{T}'}(\eta)$*.*

Proof. 1) We define for each successor $\alpha < \lambda$ a game $\Game_\alpha$ very much like the way we did it for proving Lemma 1.11, the only difference being that if $|\mathrm{Succ}_{\mathscr{T}}(\eta_n)| < \lambda$, the second player chooses A_n such that $|\mathrm{Succ}_{\mathscr{T}}(\eta_n) \setminus A_n| = 1$, otherwise the second player chooses $A_n \in \mathbf{I}_{\eta_n}$ just like in 1.11. The first player wins if $\mathbf{H}(\eta_n) < \alpha$ for every $n < \omega$. Here again the game $\Game_\alpha$ is determined for every α (here simply because if the second player wins a play he does so at some finite stage). Again we claim that there should be at least one successor $\alpha < \lambda$ for which the first player has a winning strategy. Assume the contrary, and for each $\alpha < \lambda$ let $\mathbf{f}_\alpha$ be a winning strategy of the second player in the game $\Game_{\alpha+1}$. We construct a subtree $\mathscr{T}^*$ deciding by induction on the length of the members of $\mathscr{T}$ which of them are members of $\mathscr{T}^*$. For η that is already in $\mathscr{T}^*$, if $|\mathrm{Succ}_{\mathscr{T}}(\eta)| < \lambda$ we include all the members of $\mathrm{Succ}_{\mathscr{T}}(\eta)$ in $\mathscr{T}^*$; otherwise $\mathbf{I}_\eta$ is λ^+-complete so $\mathrm{Succ}_{\mathscr{T}}(\eta) \setminus \bigcup\limits_{\alpha<\lambda} \mathbf{f}_\alpha(\eta \restriction 0, \eta \restriction 1, \ldots, \eta)$ is not empty; pedantically you use $\mathrm{Succ}_{\mathscr{T}}(\eta) \setminus \cup\{\mathbf{f}_\alpha(\eta\restriction 0, \eta\restriction 1, \ldots, \eta) : \mathbf{f}_\alpha(\eta\restriction 0, \eta\restriction 1, \ldots, \eta)$ is well defined$\}$, so we pick one extension of η from this set and the rest of $\mathrm{Succ}_{\mathscr{T}}(\eta)$ will not be in $\mathscr{T}^*$. Now $\mathscr{T}^*$ is a tree of height ω such that each member has less than λ immediate successors. So, as λ is regular uncountable, we get $|\mathscr{T}^*| < \lambda$ and hence there is some successor ordinal $\alpha^* < \lambda$ such that $\eta \in \mathscr{T}^*$ implies $\mathbf{H}(\eta) < \alpha^*$. Regarding the game $\Game_{\alpha^*}$, there is a play of it in which the first player chooses all along the way members of $\mathscr{T}^*$ and the second player plays according to $\mathbf{f}_{\alpha^*}$; of course the first player wins this game contradicting the assumption that $\mathbf{f}_{\alpha^*}$ is a winning strategy for the second player.

Hence, for some successor α^*, the second player has a winning strategy in the game $\Game_{\alpha^*}$. We define $\mathscr{T}'$ just like we did in the proof of Lemma 1.11, collecting all the initial segments of plays of the first player in the game $\Game_{\alpha^*}$ when he plays according to his winning strategy $\mathbf{H}_{\alpha^*}$.
2) Same proof, (pedantically, without loss of generality $\mathbb{B}_\alpha = \emptyset$ for α limit).
3) Similarly. $\square_{1.13}$

The following (really part (2)) will be used in the proof of 1.16.

Lemma 1.14. *1) Assume*

(a) $(\mathscr{T}, \bar{\mathbf{I}})$ *is an* $\mathbb{I}$*–tree,* $\mathbb{I}$ *a family of ideals*
(b) $\lim(\mathscr{T}) = \bigcup\limits_{i<\theta} \bigcup\limits_{\epsilon<\theta_i} \dot{\mathbb{B}}_{i,\epsilon}$*, each* $\dot{\mathbb{B}}_{i,\epsilon}$ *is a Borel set, increasing with* ϵ
(c) (α) $\mathbb{I}$ *is* ∂*-complete, and*

(β) *each* $\mathbf{I} \in \mathbb{I}$ *is strongly* θ*-indecomposable*

(*d*) E_i *is a* ∂*-complete filter on* θ_i

(*e*) *if* $i < \theta, A_\varepsilon \in \mathbf{I}_\eta$ *for* $\varepsilon < \theta_i$ *then for some* $A \in I_\varepsilon$ *we have* $\sup\{\varepsilon < \theta_i : A_\varepsilon \subseteq A\} \in E_i$

(*f*) $\partial = \mathrm{cf}(\partial)$ *and* $\partial + \aleph_1 \le \theta = \mathrm{cf}(\theta)$

(*g*) $(\forall \alpha < \theta)(|\alpha|^{\aleph_0} < \theta)$

(*h*) $(\forall \alpha < \partial)(|\alpha|^{\aleph_0} < \partial)$ *or each* $\dot{\mathbb{B}}_{\zeta,\epsilon}$ *is closed*

(*i*) $\dot{\mathbb{B}}_i := \bigcup\limits_{\epsilon < \theta_i} \dot{\mathbb{B}}_{i,\epsilon}$ *is increasing with* i

(*j*) $\eta \in \mathscr{T} \setminus \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}}) \Rightarrow |\mathrm{Succ}_{\mathscr{T}}(\eta)| < \partial$.

Then for some $i < \theta$ *and* $\epsilon < \theta_i$ *and* $\mathscr{T}'$ *we have* $(\mathscr{T}, \bar{\mathbf{I}}) \le^{\otimes} (\mathscr{T}', \bar{\mathbf{I}})$, *and* $\lim(\mathscr{T}') \subseteq \dot{\mathbb{B}}_{i,\epsilon}$*; see Definition 1.5(3).*

2) Assume $(\mathscr{T}, \mathbf{I})$ *be an* $\mathbb{I}$*–tree,* $\mathbb{I}$ *a family of ideals,* $\lim(\mathscr{T}) = \bigcup\limits_{i<\theta} \bigcup\limits_{\varepsilon < \varepsilon_i} \dot{\mathbb{B}}_{i,\varepsilon}$, *each* $\dot{\mathbb{B}}_{i,\varepsilon}$ *is a Borel set,* $i < \theta \Rightarrow \varepsilon_i < \theta]$, $\mathbb{I}$ *is* θ*-complete,* θ *is regular uncountable and each* $\mathbf{I} \in \mathbb{I}$ *is strongly* θ*-indecomposable, and* $\dot{\mathbb{B}}_i := \bigcup\limits_{\varepsilon < \varepsilon_i} \dot{\mathbb{B}}_{i,\varepsilon}$ *is increasing with* i *and*

$$\eta \in \mathscr{T} \setminus \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}}) \Rightarrow |\mathrm{Succ}_{\mathscr{T}}(\eta)| < \theta.$$

Then for some $i < \theta$ *and* $\varepsilon < \varepsilon_i$ *and* $\mathscr{T}'$ *we have* $(\mathscr{T}, \bar{\mathbf{I}}) \le^{\otimes} (\mathscr{T}', \bar{\mathbf{I}})$, *and* $\lim(\mathscr{T}) \subseteq \dot{\mathbb{B}}_{i,\varepsilon}$.

Proof. 1) We first prove part (2).

Proof of part (2): We define, for $i < \theta$ and $\epsilon < \varepsilon_i$ a game $\partial_{i,\epsilon}$ as in the proof of 1.11, 1.12 for the set $\dot{\mathbb{B}}_{i,\epsilon}$. If for some $i < \theta, \epsilon < \varepsilon_i$ the first player wins, then we get the desired conclusion as in the earlier proofs. Otherwise, as each such game is determined (as $\mathbb{B}_{i,\epsilon}$ is a Borel set) there is a winning strategy $\mathbf{f}_{i,\epsilon}$ for the second player in the game $\partial_{i,\epsilon}$. Let $\eta \in \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}})$. For each $i < \theta$ we define a set $A^i_\eta \subseteq \mathrm{Succ}_{\mathscr{T}}(\eta)$ by $A^i_\eta = \cup\{A \subseteq \mathrm{Succ}_{\mathscr{T}}(\eta)$: for some $\varepsilon < \varepsilon_i$ in some play of the game $\partial_{i,\varepsilon}$ in the n-th move the first player chooses η and the second player chooses A by the strategy $\mathbf{f}_{i,\epsilon}\}$. Recalling $i < \theta \Rightarrow \varepsilon_i < \theta$, as $\mathbf{I}_\eta$ is θ-complete clearly $A^i_\eta \in \mathbf{I}_\eta$. As $\mathbf{I}_\eta$ is strongly θ-indecomposable applying the definitions to $\langle A^i_\eta : i < \theta\rangle$ we can find $B_\eta \subseteq \mathrm{Suc}_{\mathscr{T}}(\eta)$ of cardinality $< \theta$ such that $i < \theta \Rightarrow B_\eta \nsubseteq A^i_\eta$. (If we add $\mathrm{Dom}(\mathbf{I}_\eta) = \mathrm{Succ}_{\mathscr{T}}(\eta)$ we can in Definition 1.9(5) use $A = \mathrm{Dom}(\mathbf{I})$). Now as in the proof of 1.11 we choose $\mathscr{T}'_n \subseteq \{\eta \in \mathscr{T} : \ell g(\eta) = n\}$ by induction on n as follows: $\mathscr{T}'_0 = \{\langle\rangle\}$, $\mathscr{T}'_{n+1} = \cup\{\nu$: for some $\eta \in \mathscr{T}_n, \nu \in \mathrm{Succ}_{\mathscr{T}}(\eta)$ and $[\eta \in \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}}) \Rightarrow \nu \in B_\eta]\}$.

Let $\mathscr{T}' = \cup\{\mathscr{T}'_n : n < \omega\}$, clearly $\mathscr{T}' \subseteq \mathscr{T}$ is non-empty, closed under initial segments. As θ is regular and $\eta \in \mathscr{T}' \setminus \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}}) \Rightarrow |\mathrm{Succ}_{\mathscr{T}}(\eta)| < \theta$ and $\eta \in \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}}) \Rightarrow |B_\eta| < \theta$ clearly $n < \omega \Rightarrow |\mathscr{T}'_n| < \theta$ and as θ is uncountable also $|\mathscr{T}'| < \theta$ hence $\lim(\mathscr{T}')$ has cardinality $< \theta$. As $\langle \dot{\mathbb{B}}_i : i < \theta\rangle$ is $\subseteq$-increasing with union $\lim(\mathscr{T})$, clearly for some $i(*) < \theta$ we have $\lim(\mathscr{T}') \subseteq \dot{\mathbb{B}}_{i(*)}$.

Clearly there is $\eta \in \lim(\mathscr{T}')$, hence for some $\varepsilon < \varepsilon_i$ we have $\eta \in \dot{\mathbb{B}}_{i(*),\varepsilon}$, but there is a play of the game $\partial_{i,\varepsilon}$ in which the moves of the first player are $\langle \eta{\restriction}n : n < \omega\rangle$. Easy contradiction.

Proof of part (1): We begin as in the proof of part (2) until. "For each $i < \theta$ we define a set $A^i_\eta \ldots$". Now for each $i < \theta$ and $\varepsilon < \theta_i$ we define a set $A^{i,\varepsilon}_\eta \subseteq \mathrm{Succ}_{\mathscr{T}}(\eta)$ by: if there is a play of the game $\partial_{i,\varepsilon}$ in which the second player uses the strategy $\mathbf{f}_{i,\varepsilon}$ and the first player chooses η in the n-th move, then the second player chooses $A^{i,\varepsilon}_\eta$ (note there is at most one such play); if there is no such play then let $A^{i,\varepsilon}_\eta = \emptyset$. As $\mathbb{I}$ satisfies clause (e) of the assumption there is a set $A^i_\eta \subseteq \mathrm{Succ}_{\mathscr{T}}(\eta)$ satisfying $A^i_\eta \in \mathbf{I}_\eta$ such that $\{\varepsilon < \theta_i : A^{i,\varepsilon}_\eta \subseteq A^i_\eta\} \in E_i$.

Now we continue as in the rest of the proof of part (2) after the choice of A^i_η. In particular, we choose B_η (for every $\eta \in \mathscr{T}$) and $\mathscr{T}'_n$ for $n < \omega$ and $\mathscr{T}'$ and $i(*)$ such that $\lim(\mathscr{T}') \subseteq \dot{\mathbb{B}}_{i(*)}$.

Now for every $\eta \in \mathscr{T}'_n \cap \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}})$ we know that $B_\eta = \mathrm{Succ}_{\mathscr{T}'}(\eta) \subseteq \mathrm{Succ}_{\mathscr{T}}(\eta)$ so there is $\rho_\eta \in B_\eta \setminus A^{i(*)}_\eta$. We now choose $\mathscr{T}''_n \subseteq \mathscr{T}'_n$ by induction on n as follows: $\mathscr{T}''_n = h\langle\rangle\}$, $\mathscr{T}''_{n+1} = \{\nu$: for some $\eta \in \mathscr{T}''_n, \nu \in \mathrm{Suc}_{\mathscr{T}'}(\eta) = \mathscr{T}'_{n+1} \cap \mathrm{Suc}_{\mathscr{T}}(\eta)$, and $[\eta \in \mathrm{split}(\mathscr{T}, \bar{\mathbf{I}}) \Rightarrow \nu = \rho_\eta]\}$. So $\mathscr{T}'' = \cup\{\mathscr{T}''_n : n < \omega\}$ is a non-empty subset of $\mathscr{T}'$, closed under initial segments and $|\mathscr{T}''_n| < \partial$ and $\lim(\mathscr{T}') \subseteq \dot{\mathbb{B}}_{i(*)} = \bigcup\{\dot{\mathbb{B}}_{i(*),\epsilon} : \epsilon < \varepsilon_{i(*)}\}, \dot{\mathbb{B}}_{i(*),\epsilon}$ increasing with ϵ. As $(\forall \alpha < \partial)(|\alpha|^{\aleph_0} < \partial)$ or each $\dot{\mathbb{B}}_{i(*),\epsilon}$ is closed for some $\epsilon < \theta_{i(*)}$ we have $\lim(\mathscr{T}'') \subseteq \dot{\mathbb{B}}_{i(*),\epsilon}$. As E_i is ∂-complete increasing ε we have: $\eta \in \mathscr{T}' \Rightarrow A^{i,\varepsilon}_\eta \in A^i_\eta$. But easily we can find a play of the game $\partial_{i(*),\epsilon}$ in which the second player uses the strategy $\mathbf{f}_{i(*),\epsilon}$ and the first player choose η_n from $\mathscr{T}''$. In such a play the first player wins, contradicting the choice of $\mathbf{f}_{i(*),\epsilon}$. $\square_{1.14}$

The following uses pcf in its phrasing (hence in its proof)

Lemma 1.15. *Suppose $(\mathscr{T}, \bar{\mathbf{I}})$ is an $\mathbb{I}$-tree, θ regular uncountable, $\langle A_\eta : \eta \in \mathscr{T}\rangle$ is such that: A_η is a set of ordinals, $[\eta \triangleleft \nu \Rightarrow A_\eta \subseteq A_\nu]$ and*

- $(*)$ (a) $\mathbf{S}$ *is a set of uncountable regular cardinals*
  - (b) $\mathbb{I}' := \mathbb{I} \setminus \{\mathbf{I} \in \mathbb{I} : |\mathrm{Dom}(\mathbf{I})| < \mu\}$ *is μ^+-complete or at least strongly μ-indecomposable for every μ such that $\mu \in \mathbf{S}$ or $\mu \in \mathrm{pcf}(\mathbf{S} \cap A_\eta)$ for some $\eta \in \mathscr{T}$*
  - (c) $\mathbb{I}$ *is θ-complete and $|\mathrm{pcf}(\mathbf{S} \cap A_\eta)| < \theta$ for $\eta \in \mathscr{T}$ and $\theta \leq \min(\mathbf{S})$,*
  - (d) $|A_\eta| < \min(\mathbf{S})$ *for $\eta \in \mathscr{T}$*

Then there is $\mathscr{T}^\dagger$ satisfying $(\mathscr{T}, \bar{\mathbf{I}}) \leq^* (\mathscr{T}^\dagger, \bar{\mathbf{I}})$ and such that:

- $(**)$ *if $\lambda \in A_\nu \cap \mathbf{S}$ and $\nu \in \mathscr{T}^\dagger$ then for some $\alpha_\nu(\lambda) < \lambda$ for every ρ such that $\nu \triangleleft \rho \in \lim(\mathscr{T}^\dagger)$ we have $\alpha_\nu(\lambda) \geq \sup(\lambda \cap \bigcup\limits_{n<\omega} A_{\rho\restriction n})$.*

Proof. It is enough to prove the existence of a $\mathscr{T}^\dagger$ as required just for $\nu = \langle\rangle$, (as we can repeat the proof going up in the tree). This will be proved by induction on $\max(\mathrm{pcf}(\mathbf{S} \cap A_{\langle\rangle}))$ (exists, see [She94, Ch.I,1.9]). Let $\alpha_\lambda(\eta) = \sup(A_\eta \cap \lambda)$.

We assume knowledge of [She94] and use its notation.

Let $\mathfrak{a} := \mathbf{S} \cap A_{\langle\rangle}$ (if $\mathfrak{a}$ is empty we have nothing to do), let $\mu = \max\mathrm{pcf}(\mathfrak{a})$, and let $\langle f_\zeta : \zeta < \mu\rangle$ be $<_{\mathbf{J}_{<\mu}[\mathfrak{a}]}$-increasing and cofinal in $\Pi\mathfrak{a}$, recalling that the later means that $(\forall f \in \Pi\mathfrak{a})(\exists \zeta < \mu)(f <_{\mathbf{J}_{<\mu}[\mathfrak{a}]} f_\zeta)$. Let $\{\mathfrak{b}_\varepsilon : \varepsilon < \varepsilon(*)\}$ be cofinal in $J_{<\mu}[\mathfrak{a}]$, e.g., this set is $\{\bigcup\limits_{\theta \in \mathfrak{c}} \mathfrak{b}_\theta[\mathfrak{a}] : \mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a}) \setminus \{\mu\}$ is finite$\}$, so by clause (c) of the assumption $(*)$ we can have $\epsilon(*) < \theta$ and hence by assumption (c) $\mathbb{I}'$ is $|\varepsilon(*)|^+$-complete.

For $\varepsilon < \varepsilon(*)$ and $\zeta < \mu$ we consider the statement:

$(*)^\varepsilon_\zeta$ there is a subtree $\mathscr{T}'$ of $\mathscr{T}$ satisfying $(\mathscr{T},\bar{\mathbf{I}}) \le^* (\mathscr{T}',\bar{\mathbf{I}})$ such that for every $\eta \in \lim(\mathscr{T}')$ and $\lambda \in \mathfrak{a} \setminus \mathfrak{b}_\varepsilon$ and n we have $\alpha_\lambda(\eta{\restriction}n) \le f_\zeta(\lambda)$.

It suffices to find such $\mathscr{T}'$ (for some ε,ζ) because: we can apply the induction hypothesis on $(\mathfrak{b}_\epsilon, \mathscr{T}')$, this is justified as $\max \mathrm{pcf}(\mathfrak{b}_\epsilon) < \max \mathrm{pcf}(\mathfrak{a})$.

In $\mathbf{V}$ define for $\zeta < \mu$ and $\varepsilon < \varepsilon(*)$ the following set:

$$\dot{\mathbb{B}}_{\zeta,\varepsilon} := \{\eta \in \lim(\mathscr{T}) : \text{ for every } \lambda \in \mathfrak{a} \setminus \mathfrak{b}_\varepsilon, n < \omega \Rightarrow (\lambda \cap A_{\eta\restriction n}) \subseteq f_\zeta(\lambda)\}.$$

Clearly $\dot{\mathbb{B}}_{\zeta,\varepsilon}$ is closed and $\dot{\mathbb{B}}_\zeta = \bigcup\limits_{\varepsilon<\varepsilon(*)} \dot{\mathbb{B}}_{\zeta,\varepsilon}$. Now, $\zeta < \xi < \mu \Rightarrow \dot{\mathbb{B}}_\zeta \subseteq \dot{\mathbb{B}}_\xi$ (as $f_\zeta <_{\mathbf{J}_{<\mu}[\mathfrak{a}]} f_\xi$) and $\lim(\mathscr{T}) = \bigcup\limits_{\zeta<\mu} \dot{\mathbb{B}}_\zeta$ (as $\langle f_\zeta : \zeta < \mu\rangle$ is cofinal in $(\prod, <_{J_{<\mu}[\mathfrak{a}]})$, hence using 1.14(2) above (with μ, $\varepsilon(*)$ here standing for θ, ε_i there) for some $\zeta(*) < \mu$ and $\varepsilon < \varepsilon(*)$ and $\mathscr{T}'$ we have $(\mathscr{T},\bar{\mathbf{I}}) \le^* (\mathscr{T}',\mathbf{I})$ and $\lim(\mathscr{T}') \subseteq \dot{\mathbb{B}}_{\zeta,\varepsilon}$. So $(*)^\varepsilon_\zeta$ holds, but as said above this suffices. $\square_{1.15}$

The following is used in [Shec, 1.11,1.13]

Lemma 1.16. *Let θ be an uncountable regular cardinal (the main case here is $\theta = \aleph_1$). Let $\mathbb{I}$ be a family of θ^+-complete ideals, $(\mathscr{T}_0, \bar{\mathbf{I}})$ a tagged tree, $A = \{\eta \in \mathscr{T}_0 : 0 < |\mathrm{Succ}_{\mathscr{T}_0}(\eta)| \le \theta\}, [\eta \in \mathscr{T}_0 \setminus A \Rightarrow \mathbf{I}_\eta \in \mathbb{I}$ and $\mathrm{Succ}_{\mathscr{T}_0}(\eta) \notin \mathbf{I}_\eta]$, and $[\eta \in A \Rightarrow \mathrm{Succ}_{\mathscr{T}_0}(\eta) \subseteq \{\eta\hat{\ }\langle i\rangle : i < \theta\}]$, and $\mathbf{H} : \mathscr{T}_0 \to \theta$ and $\bar{\mathbf{c}} = \langle \bar{\mathbf{c}}_\eta : \eta \in A\rangle$, is such that for all $\eta \in A, \mathbf{c}_\eta$ is a club of θ. <u>Then</u> there is a club C of θ such that: for each $\delta \in C$ there is $\mathscr{T}_\delta \subseteq \mathscr{T}_0$ satisfying:*

- *(a) $\mathscr{T}_\delta$ a tree*
- *(b) if $\eta \in \mathscr{T}_\delta$ and $|\mathrm{Succ}_{\mathscr{T}_0}(\eta)| < \theta$, then $\delta \in \mathbf{c}_\eta$ and $\mathrm{Succ}_{\mathscr{T}_\delta}(\eta) = \mathrm{Succ}_{\mathscr{T}_0}(\eta)$, and if in addition $|\mathrm{Succ}(\eta)| = \theta$, then $\mathrm{Succ}_{\mathscr{T}_\delta}(\eta) = \{\eta\hat{\ }\langle i\rangle : i < \delta\} \cap \mathrm{Succ}_{\mathscr{T}_0}(\eta)$*
- *(c) $\eta \in \mathscr{T}_\delta \setminus A$ implies $\mathrm{Succ}_{\mathscr{T}_\delta}(\eta) \notin \mathbf{I}_\eta$*
- *(d) for every $\eta \in \mathscr{T}_\delta$ we have $\mathbf{H}(\eta) < \delta$.*

Proof. For each $\zeta < \theta$ we define a game $\eth_\zeta$. The game lasts ω moves, in the nth move $\eta_n \in \mathscr{T}_0$ of length n is chosen.
For $n = 0$: necessarily $\eta_0 = \langle\rangle$.
For $n = m + 1$: If $|\mathrm{Succ}_{\mathscr{T}_0}(\eta_m)| = \theta$, then the *second* player chooses $\eta_{m+1} \in \mathrm{Succ}_{\mathscr{T}_0}(\eta_m)$ satisfying $\eta_{m+1}(m) < \zeta$.
If $|\mathrm{Succ}_{\mathscr{T}_0}(\eta_m)| < \theta$, then the *second* player chooses any $\eta_{m+1} \in \mathrm{Succ}_{\mathscr{T}_0}(\eta_m)$.
If $\eta_m \notin A$, then the second player chooses $A_m \in \mathbf{I}_{\eta_m}$, and then the *first* player chooses $\eta_{m+1} \in \mathrm{Succ}_{\mathscr{T}_0}(\eta_m) \setminus A_m$.

At the end, the first player wins if for all $n, \mathbf{H}(\eta_n) < \zeta$ and $|\mathrm{Succ}_{\mathscr{T}_0}(\eta_n)| = \theta \Rightarrow \zeta \in \mathbf{c}_{\eta_n}$.

Now clearly

- $(*)$ if for a club of $\zeta < \theta$ the first player has a winning strategy for the game $\eth_\zeta$, *then* there are trees $\mathscr{T}_\delta$ as required.

Let $S = \{\delta < \theta$: first player does not have a winning strategy for the game $\eth_\delta\}$; we assume that the set S is stationary, and get a contradiction, this suffice.

For $\delta \in S$ let $\mathbf{f}_\delta$ be a winning strategy for the second player in $\eth_\delta$ (he has a winning strategy as the game is determined being closed for the first player). So

$\mathbf{f}_\delta$ gives for the first $(n-1)$-moves of the first player, the n-th move of the second player.

Let χ be a large enough regular cardinal, and let $N_0 \prec (\mathscr{H}(\chi), \in)$ be such that $\theta+1 \subseteq N_0$, $\|N_0\| = \theta$, $(\mathscr{T}_0, \bar{\mathbf{I}}) \in N_0, \bar{\mathbf{c}} \in N_0$, and $\bar{\mathbf{f}} = \langle \mathbf{f}_\delta : \delta \in S \rangle \in N_0$. We can find $N_1 \prec N_0$ such that $\|N_1\| < \theta, N_1 \cap \theta$ is an ordinal and $(\mathscr{T}_0, \bar{\mathbf{I}}) \in N_1, \langle \mathbf{f}_\delta : \delta \in S \rangle \in N_1$ and $\bar{\mathbf{c}} \in N_1$. Let $\delta := N_1 \cap \theta$. Since S was assumed to be stationary, we may assume that $\delta \in S$.

Now we shall choose by induction on $n, \eta_n \in T_0 \cap N_1$ of length n, such that $\langle \eta_\ell : \ell \leq n \rangle$ is an initial segment of a play of the game $\Game_\delta$ in which the second player uses his winning strategy $\mathbf{f}_\delta$. (The $A_\ell \in \mathbf{I}_{\eta_\ell}$ are not mentioned as they are not arguments of $\mathbf{f}_\delta$).

<u>Case 1</u>. $n = 0$:

We let $\eta_0 = \langle \rangle$.

<u>Case 2</u>. $n = m+1$, $\eta_m \in A$:

Recall that as $\delta \in S$, the second player has the winning strategy $\mathbf{f}_\delta$ for the game $\Game_\delta$ <u>but</u> in general $\mathbf{f}_\delta \notin N_1$. So $\mathbf{f}_\delta$ gives us η_n. Now if $|\mathrm{Succ}_{\mathscr{T}_0}(\eta_m)| < \theta$ then $\mathrm{Succ}_{\mathscr{T}_0}(\eta_m) \subseteq N_1$ (because $\mathscr{T}_0$, η_m belong to N_1 and $N_1 \cap \theta$ is an ordinal), and hence $\eta_n \in N_1$ as required. If $|\mathrm{Succ}_{\mathscr{T}_0}(\eta_m)| = \theta$ then necessarily $\mathrm{Succ}_{\mathscr{T}_0}(\eta_m) \subseteq \{\eta_m \hat{\ } \langle i \rangle : i < \theta\}, \eta_n = \eta_m \hat{\ } \langle i \rangle, i < \delta$ (as the play is of the game $\Game_\delta$), but $N_1 \cap \theta = \delta$ so necessarily $i \in N_1$ hence (as $\eta_m \in N_1$) also $\eta_n \in N_1$.

Lastly,

<u>Case 3</u>. $n = m+1$, $\eta_m \notin A$:

So $\mathbf{f}_\delta$ gives us $A^\delta_m \in \mathbf{I}_{\eta_m}$ which is not necessarily in N_1, however we let $A^* = \bigcup\{A^\zeta_m : \zeta \in S$, and there is a play of $\Game_\zeta$ in which $\langle \eta_\ell : \ell \leq m \rangle$ were played (by the first player) and the second player plays according to $\mathbf{f}_\zeta$ (this play is unique) and the strategy $\mathbf{f}_\zeta$ dictates to the second player to choose $A^\zeta_m\}$.
Now, A^* belongs to N_1 (as $\bar{\mathbf{f}} \in N_1$) and being the union of $\leq \theta$ members of $\mathbf{I}_{\eta_m}$ it belongs to $\mathbf{I}_{\eta_m}$, and hence $A^* \cap \mathrm{Succ}_{\mathscr{T}_0}(\eta_m)$ is a proper subset of $\mathrm{Succ}_{\mathscr{T}_0}(\eta_m)$. Consequently, there is $\eta_m \hat{\ } \langle i \rangle \in \mathrm{Succ}_{\mathscr{T}_0}(\eta_m) \setminus A^*$, and thus there is such $i \in N_1$. Let the first player choose $\eta_n = \eta_m \hat{\ } \langle i \rangle$.

So we have played a sequence $\langle \eta_n : n < \omega \rangle$ of elements of N_1, always obeying $\mathbf{f}_\delta$ so this sequence was produced by a play of $\Game_\delta$ in which the second player plays according to the strategy $\mathbf{f}_\delta$. But then, for all $n, \eta_n \in N_1 \Rightarrow \mathbf{H}(\eta_n) \in N_1$, so $\mathbf{H}(\eta_n) < \delta$, and

$$\eta_n \in N_1 \ \Rightarrow \mathbf{c}_{\eta_n} \in N_1 \Rightarrow \delta = \sup(\mathbf{c}_{\eta_n} \cap \delta) \Rightarrow \delta \in \mathbf{c}_{\eta_n};$$

hence the first player wins in this play. So $\mathbf{f}_\delta$ cannot be a winning strategy for the second player in $\Game_\delta$. A contradiction, so S is not stationary and we are done. $\square_{1.16}$

Claim 1.17. *Assume $\kappa < \lambda$ and $\mathrm{cf}([\lambda]^{<\kappa^+}, \subseteq) = \lambda$ and $\lambda = \lambda^{\aleph_0}$.*
1) If $\chi > \lambda^+$ and $x \in \mathscr{H}(\chi)$ <u>then</u> we can find $\bar{N} = \langle N_\eta : \eta \in \mathscr{T} \rangle$ such that:

(a) *$\mathscr{T}$ is a subtree of ${}^{\omega >}(\lambda^+)$, each $\eta \in \mathscr{T}$ is (strictly) increasing,*
(b) *$N_\eta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,*
(c) *$x \in N_\eta$ and $\kappa + 1 \subseteq N_\eta$ and $\|N_\eta\| = \kappa$,*
(d) *$\nu \triangleleft \eta \in \mathscr{T} \Rightarrow N_\nu \prec N_\eta$,*

(e) $N_\eta \cap N_\nu = N_{\nu \cap \eta}$ *for* $\eta, \nu \in \mathscr{T}$,

(f) $\eta \in N_\eta$,

(g) *if* $\eta_\ell\hat{\ }\langle\alpha_\ell\rangle \in \mathscr{T}$ *for* $\ell = 1,2$ *and* $\alpha_1 < \alpha_2$ *then* $\sup(N_{\eta_1\hat{\ }\langle\alpha_1\rangle} \cap \lambda^+) < \min(N_{\eta_2\hat{\ }\langle\alpha_2\rangle} \cap \lambda^+ \backslash \alpha_1)$.

Recall that $\mathrm{cf}([\kappa^{+n}]^{\leq\kappa}, \subseteq) = \kappa^{+n}$.
2) If in addition $\lambda = \lambda^\kappa$ *(equivalently,* $2^\kappa \leq \lambda$*)* *then* *we can add:*

(h) *if* $\eta, \nu \in \mathscr{T}$ *have the same length then there is an isomorphism from* N_η *onto* N_ν, *call it* $f_{\eta,\nu}$, *which maps* x *to itself, so*

$$\eta, \nu \in \lim(\mathscr{T}) \Rightarrow \bigcup_{n<\omega} N_{\eta\restriction n} := N_\eta \cong N_\nu := \bigcup_{n<\omega} N_{\nu\restriction n}.$$

3) If $\mathscr{S} \subseteq [\lambda]^{\leq\kappa}$ *is stationary of cardinality* λ *then* *we can in (1) demand*

(i) $N_\eta \cap \lambda \in \mathscr{S}$.

4) We can further demand (in parts (1),(2)) that:

(j) N_η *is the Skolem hull of* $\{x, \eta, \kappa, \lambda\} \cup \kappa \cup N_{\langle\rangle}$ *in* $(\mathscr{H}(\chi), \in, <^*_\chi)$

(k) *if* $\kappa = \kappa^{<\partial}$ *we can add* $[N_\eta]^{<\partial} \subseteq N_\eta$.

Remark 1.18. 1) Used in [Shea, 1.11=L7.6(2),(3),(4)] and [Shea, 3.23=L7.14,Case 5,clause (k)] and [Shea, 3.25=L7.151].
2) See [She98, Ch.IV] use 1.10 + the functions witnessing successor.

Proof. Let $\mathscr{S}_* \subseteq [\lambda]^{\leq\kappa}$ be stationary of cardinality λ; why exists? if $\lambda = \lambda^\kappa$ trivially if just $\lambda = \mathrm{cf}([\lambda]^{\leq\kappa}, \subseteq)$ by [She93].
1) We apply 1.19 below.
In detail let

(a) $\kappa = \theta^+, \partial = \aleph_0$ and λ^+ here stands for λ in 1.19

(b) $\mathscr{T} = \{\eta : \eta$ an increasing sequence of ordinals $< \lambda^+\}$

(c) if $\eta \in \mathscr{T}$ then $\mathbf{I}_\eta$ is the ideal of non-stationary subsets of λ^+ plus the set $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \leq \kappa\}$

(d) $\kappa_\eta = \lambda^+$ for $\eta \in \mathscr{T}$

(e) for some (g^0, g^1) witnessing λ^+, see below.

$$\mathscr{S} = \{u \in [\lambda^+]^{\leq\kappa} : \begin{array}{l} u \text{ is closed under } g^0, g^1, \kappa+1 \subseteq u \text{ and} \\ u \cap \theta \in \theta \text{ and } u \cap \lambda \in \mathscr{S}_*\}. \end{array}$$

Now we can check that the assumptions of 1.19 holds hence its conclusion give the desired conclusion.
2) The game proof, but using clause $\oplus_2(g)$ of the conclusion of 1.19.
3) We could choose $\mathscr{S}_*$ as the given $\mathscr{S}$ and use the proof of (1). Of course we can combine part (3) with parts (2),(4) if $\mathscr{S}$ is as in (h) of 1.19.
4) Clause (j) is really proved in 1.19. As for clause (a) we can in the proof of part (j) replace

$(a)'$ $\kappa = \theta^+$ and ∂ is the one given, without loss of generality regular and use $\lambda' = (\lambda^+)^{<\partial}$ in 1.19.

As $\theta = \kappa^+$ clearly $\alpha < \theta \Rightarrow |\alpha|^{<\partial} \le \kappa^{<\partial} = \kappa < \theta$ by the present proof

$(d)'$ in the definition of $\mathscr{S}$ demand u is closed under h as there (exists as we are assuming $\kappa = \kappa^{<\kappa}$).

$\square_{1.17}$

Claim 1.19. *Assume that:*

$\oplus_1$ (a) θ *is an uncountable regular cardinal,*
(b) $(\mathscr{T}, \bar{\mathbf{I}})$ *is a tagged tree,*
(c) *for* $\eta \in \mathscr{T}, \mathbf{I}_\eta$ *is a normal*[1] *ideal on some regular uncountable cardinal* κ_η,
(d) A_η *is a set of cardinality* $< \theta$, *for* $\eta \in \mathscr{T}$
(e) $\lambda \ge \Sigma\{\kappa_\eta : \eta \in \mathscr{T}\}$ *and* $\mathscr{S} \subseteq [\lambda]^{<\theta}$ *is stationary*
(f) *if* $\eta \triangleleft \nu \in \mathscr{T}$ *then* $\kappa_\eta \le \kappa_\nu$,
(g) $(\mathscr{T}, \bar{\mathbf{I}}), \langle A_\eta : \eta \in \mathscr{T}\rangle \in \mathscr{H}(\chi)$ *and* $x \in \mathscr{H}(\chi)$
(h) *if* $\eta \in \mathscr{T}$ *and* $\alpha < \kappa_\eta$ *then* $\mathscr{S}{\restriction}\alpha$ *has cardinality* $< \kappa_\eta$ *where* $\mathscr{S}{\restriction}\mathscr{U} := \{u \cap \mathscr{U} : u \in \mathscr{S}\}$ *and so a sufficient condition is* $(\forall \alpha < \kappa_\eta)(|\alpha|^{<\theta} < \kappa_\eta)$
(i)(α) $\mathbf{I}_\eta$ *is a normal ideal on* κ_η
(β) $\{\delta < \kappa_\eta : \mathrm{cf}(\delta) < \theta\} \in \mathbf{I}_\eta$
(γ) *if* $\eta_1 \ne \eta_2 \in \mathscr{T}$ *and* $\kappa_{\eta_1} = \kappa_{\eta_2}$ *and* $\eta_1{}^\frown\langle\alpha_1\rangle, \eta_2{}^\frown\langle\alpha_2\rangle \in \mathscr{T}$ *then* $\alpha_1 \ne \alpha_2$ *or at least* $\mathscr{P}(\kappa_\eta)/\mathscr{I}_\eta$
(j) $\partial < \theta$ *and* $\alpha < \theta \Rightarrow |\alpha|^{<\partial} < \theta$ *and* $h : {}^{\partial>}\lambda \to \lambda$ *is one to one and* $u \in \mathscr{S} \wedge \rho \in {}^{\partial>}u \Rightarrow h(\rho) \in u$.

Then *there is a sequence* $\langle N_\eta : \eta \in \mathscr{T}^*\rangle$ *such that*

$\oplus_2$ (a) $(\mathscr{T}, \bar{\mathbf{I}}) \le (\mathscr{T}^*, \bar{\mathbf{I}}{\restriction}\mathscr{T}^*)$
(b) $N_\eta \prec (\mathscr{H}(\chi), \in)$ *and* $x \in N_\eta$
(c) *if* $\eta \in \mathscr{T}^*$ *then* $N_\eta \cap \kappa_\eta \in \mathscr{S}{\restriction}\kappa_\eta$
(d) $\eta \in \mathscr{T}^* \Rightarrow A_\eta \cup \{x\} \subseteq N_\eta$
(e) $\eta \in N_\eta$
(f) $\langle N_\eta : \eta \in \mathscr{T}^*\rangle$ *is a* Δ*–system, i.e.,* $N_\eta \cap N_\nu = N_{\eta\cap\nu}$
(g) *if* $\alpha < \theta \Rightarrow 2^{|\alpha|} < \kappa_{\langle\rangle}$ *then* $\eta, \nu \in \mathscr{T}$ & $\ell g(\eta) = \ell g(\nu) \Rightarrow N_\eta \cong N_\nu$.

Remark 1.20. 1) What if θ is singular? Let $\theta = \sum_{\zeta<\partial} \theta_\zeta, \theta_\zeta$ regular uncountable increasing with $\zeta, \partial = \mathrm{cf}(\theta) < \theta$. Now let $f : \mathscr{T} \to \partial$ be $f(\eta) = \min\{\zeta : |\bigcup\{A_{\eta\restriction\ell} : \ell \le \ell g(\eta)\}| < \theta_\zeta\}$ and use ?
2) Used in the proofs of [Shea, 1.14=L7.6B], [Shea, 2.15=L7.9].

Proof. Without loss of generality x codes $(\mathscr{T}, \bar{\mathscr{T}}), \langle A_\eta : \eta \in \mathscr{T}\rangle, \theta, \bar{\kappa}, \mathscr{S}$. Let $\mathfrak{B}$ expand $(\mathscr{H}(\chi), \in, <^*_\chi)$ by x and the functions F_i (for $i < \partial$) where F_i is an i-place function from $\mathscr{H}(\chi)$ to $\mathscr{H}(\chi)$ and $F_i(\ldots, a_j, \ldots)_{j<i} = \langle a_j : j < i\rangle$ and the functions G_i (for $i < \theta$): $G_i(a)$ is: i if $a \in \theta \backslash i, 0$ if otherwise.

So

[1] pedantically we should use $\mathbf{I}^\dagger_\eta$

$(*)_0$ if $u \subseteq \mathscr{H}(\chi), |u| < \theta$ then $N = \mathrm{Sk}(u, \mathfrak{B}) \prec \mathfrak{B}$ satisfies
- $N \cap \theta \in \theta$
- N has cardinality $< \theta$
- $N^{<\partial} \subseteq N$.

Let $\mathbf{N}$ be the set of pairs $(\eta, \bar{N})$ such that:

$(*)^1_{\eta,\bar{N}}$ (a) $\eta \in \mathscr{T}$
(b) $\bar{N} = \langle N_\ell : \ell \le \ell g(\eta)\rangle$
(c) $N_\ell \prec (\mathscr{H}(\chi), \in, <^*_\chi)$
(d) $x \in N_\eta, \eta{\restriction}\ell \in N_\ell$ and $\|N_\ell\| < \theta$
(e) N_ℓ is the Skolem hull of $N_\ell \cap \kappa_{\eta{\restriction}\ell}$ in $\mathfrak{B}$
(f) $N_\ell \cap \lambda \in \mathscr{S}$
(g) $N_\ell \subseteq N_{\ell+1}$ (equivalently $N_\ell \prec N_{\ell+1}$) and moreover, $N_\ell <_{\kappa_{\eta{\restriction}\ell}} N_{\ell+1}$ which means $N_\ell \subseteq N_{\ell+1}$ and $N_\ell \cap \kappa_{\eta{\restriction}\ell} \triangleleft N_{\ell+1} \cap \kappa_{\eta{\restriction}\ell}$.

Let $\mathbf{N}_n = \{(\eta, \bar{N}) \in \mathbf{N} : \ell g(\eta) = n+1\}$.

We define a two-place relation $\le_{\mathbf{N}}$ on $\mathbf{N}$:

$(*)_2$ $(\eta_1, \bar{N}_1) \le_{\mathbf{N}} (\eta_2, \bar{N}_2)$ <u>iff</u> both are from $\mathbf{N}$ and $\eta_1 \trianglelefteq \eta_2, \bar{N}_1 \trianglelefteq \bar{N}_2$.

Obviously

$(*)_3$ (a) $\mathbf{N}$ is non-empty
(b) $\le_{\mathbf{N}}$ is a partial order on $\mathbf{N}$, in fact $(\mathbf{N}, \le_{\mathbf{N}})$ is a tree with ω levels, the n-th level being $\mathbf{N}_n$
(c) if $(\eta, \bar{N}) \in \mathbf{N}_{n_2}$ and $n_1 \le n_2$ then $(\eta, \bar{N}) \restriction n_1 := (\eta{\restriction}n_1, \bar{N}{\restriction}(n_1+1))$ belongs to $\mathbf{N}_1$ and is $\le_{\mathbf{N}} (\eta, \bar{N})$.

Now we define a function $\mathrm{rk} : \mathbf{N} \to \mathrm{Ord} \cup \{\infty\}$ by defining when $\mathrm{rk}(\eta, \bar{N}) \ge \alpha$ by induction on the ordinal α:

$(*)_4$ $\mathrm{rk}(\eta, \bar{N}) \ge \alpha$ <u>iff</u> for some $n, (\eta, \bar{N}) \in \mathbf{N}_n$ and for every $\beta < \alpha$ there is $\mathbf{x} = \langle(\eta_s, \bar{N}_s) : s \in S\rangle$ such that
(a) $(\eta_s, \bar{N}_s) \in \mathbf{N}_{n+1}$
(b) $(\eta, \bar{N}) \le_{\mathbf{N}} (\eta_s, \bar{N}_s)$ and $\mathrm{rk}(\eta_s, \bar{N}_s) \ge \beta$ for every $s \in S$
(c) $\{\eta_s : s \in S\} \in \mathbf{I}^+_\eta$
(d) if $s_1 \ne s_2 \in S$ then $N_{s_1,n+1} \cap N_{s_2,n+1} = N_n$ where $\bar{N}_s = \langle N_{s,\ell} : \ell < |\bar{N}_s|\rangle$.

Clearly rk is indeed a function from $\mathbf{N}$ into $\mathrm{Ord} \cup \{\infty\}$.

$(*)_5$ if $\mathrm{rk}(\eta, \bar{N}) = \infty$ for some $(\eta, \bar{N}) \in \mathbf{N}_0$ then the desired conclusion holds.

Why? In short, here we use $\eta \triangleleft \nu \Rightarrow \kappa_\eta \le \kappa_\nu$ and $\mathbf{I}_\eta$ fails κ^+_η-c.c. and $\mathbf{I}^+_\eta$ is a normal ideal on $\kappa_\eta, \mathscr{P}(\kappa_\eta)/\mathbf{I}^+_\eta$ fails the κ^+_η-c.c. everywhere (see later on normal ideals on $[\kappa_\eta]^{<\partial(\eta)}$). Fully, first we can ignore $\oplus_2(g)$ as we can apply 1.12.

Let $\mathbf{N}'_\eta = \{(\eta, \bar{N}) \in \mathbf{N}_\eta : \mathrm{rk}(\eta, \bar{N}) = \infty\}$.

Now we shall choose $\mathscr{T}'_n \subseteq \mathscr{T}_n := \{\eta \in \mathscr{T} : \ell g(\eta) = n\}$ and $\bar{N}_\eta$ for $\eta \in \mathscr{T}_n$ such that $(\eta, \bar{N}_\eta) \in \mathbf{N}$ and $\mathrm{rk}(\eta, \bar{N}_\eta) = \infty$.

$(*)_{5.1}$ if $n = 0$ then $\mathscr{T}'_0 = \{\langle\rangle\}$, $\bar{N}_{\langle\rangle}$ is such that $(\langle\rangle, \bar{N}_{\langle\rangle}) \in \mathbf{N}_0$ and $\mathrm{rk}(\langle\rangle, \bar{N}_{\langle\rangle}) = \infty$.

This holds by the assumption of $(*)_5$

$(*)_{5.2}$ if $\eta \in \mathscr{T}'_n$ so $\bar{N}_\eta$ is well defined then for every ordinal α there is $\mathbf{x}_\alpha = \langle(\eta^\alpha_s, \bar{N}^\alpha_s) : s \in S_\alpha\rangle$ witnessing $\mathrm{rk}(\eta, \bar{N}_\eta) = \alpha$, hence for some $\beta = \beta(\eta)$ we have that $\{\alpha : \mathbf{x}_\alpha = \mathbf{x}_\beta\}$ is a proper class and let

(a) $\mathscr{T}'_{n+1} \cap \mathrm{Succ}_{\mathscr{T}}(\eta) = \{\eta^{\beta(\eta)}_s : s \in S_{\beta(\eta)}\}$

(b) $\bar{N}_{\eta^{\beta(\eta)}_s} = N^\alpha_s$.

So

(c) $\mathscr{T}'_{n+1} = \cup\{\mathscr{T}'_{n+1} \cap \mathrm{Succ}_{\mathscr{T}}(\eta) : \eta \in \mathscr{T}'_n\}$.

Clearly

$(*)_{5.3}$ (a) $(\mathscr{T}, \bar{\mathbf{I}}) \le^* (\mathscr{T}', \mathbf{I})$

(b) if $\eta \in \mathscr{T}$ and $\nu_1 \ne \nu_2 \in \mathrm{Succ}_{\mathscr{T}'}(\eta)$ then $N_{\nu_1} \cap N_{\nu_2} = N_n$.

Our problem is to find $\mathscr{T}''$ such that $(\mathscr{T}', \bar{\mathbf{I}}) \le (\mathscr{T}'', \bar{\mathbf{I}})$ and $\langle N_{\eta,\ell g(\eta)} : \eta \in \mathscr{T}''\rangle$ is a Δ-system because then by the assumption on the $\mathbf{I}_\eta$'s, i.e. by $\oplus_1(f)(\gamma)$ we are done. We still have to prove the assumption of $(*)_5$

$(*)_6$ there is $(\eta, \bar{N}) \in \mathbf{N}_0$ such that $\mathrm{rk}(\eta, \bar{N}) = \infty$.

Why? For every $\eta \in \mathscr{T}$ and $\alpha < \kappa_\eta$

$(*)_{6.1}$ let $\mathbf{N}_{\eta,\alpha}$ be $\{\bar{N} : (\eta, \bar{N}) \in \mathbf{N}_{\ell g(\eta)}$ and $N_{\ell g(\eta)} \cap \kappa_\eta \subseteq \alpha\}$.

Now

$(*)_{6.2}$ if $\eta \in \mathscr{T}$ and $\alpha < \kappa_\eta$ then $\mathbf{N}_{\eta,\alpha}$ has cardinality $< \kappa_\eta$.

Why? Because $|\mathscr{S}{\restriction}\alpha| < \kappa_\eta$.

$(*)_{6.3}$ If $\eta \in \mathscr{T}, \alpha < \kappa_\eta, \bar{N} \in \mathbf{N}_{\eta,\alpha}$ and $\mathrm{rk}(\eta, \bar{N}) < \infty$ <u>then</u> $C_{\eta,\bar{N}} \in \mathbf{I}_\eta$ where $C_{\eta,\bar{N}} := \{\beta < \kappa_\eta$: there is $\bar{N}'$ such that $(\eta, \bar{N}) \le_{\mathbf{N}} (\eta\hat{\ }\langle\beta\rangle, \bar{N}') \in \mathbf{N}_{\ell g(\eta)+1}$ and $\mathrm{rk}(\eta\hat{\ }\langle\beta\rangle, \bar{N}') \ge \mathrm{rk}(\eta, \bar{N})\}$.

Why? By the definition of $\mathrm{rk}(-)$.

$(*)_{6.4}$ if $\eta \in \mathscr{T}'$ then $C_\eta \in \mathbf{I}_\eta$ where C_η is the set of $\beta < \kappa_n$ satisfying at least one of the following:

(a) $\mathrm{cf}(\beta) < \theta$

(b) $\eta\hat{\ }\langle\beta\rangle \notin \mathscr{T}'$

(c) for some $\alpha < 1 + \beta$ and $\bar{N} \in \mathbf{N}_{\eta,\alpha}$ we have $\beta \in C_{\eta,\bar{N}}$

(d) in the Skolem hull of $\beta \cup \{x\}$ there is an ordinal from $[\beta, \kappa_\eta)$.

Why? Because $\mathbf{I}_\eta$ is a normal ideal on κ_η and $(*)_{6.3}$.

Now we choose η_n by induction on n such that:

$(*)_{6.5}$ (a) $\eta_n \in \mathscr{T}'$ has length n

(b) if $n = m + 1$ <u>then</u> $\eta_n = \eta_m\hat{\ }\langle\delta_m\rangle$ for some $\delta_m \in \kappa_{\eta_m} \backslash C_{\eta_m}$.

Clearly possible as we are assuming "$\mathscr{S} \subseteq [\lambda]^{<\theta}$ is stationary" there are M, u such that:

$(*)_{6.6}$ (a) $u \in \mathscr{S}$

(b) M_u is the Skolem hull of $u \cup \{x\}$ in $\mathfrak{B}$

(c) $\delta_n \in u$ for every n.

Let N_n be the Skolem hull in $\mathfrak{B}$ of $(N \cap \delta_n) \cup \{x\}$. Let $\bar{N}_n = \langle N_\ell : \ell \leq n \rangle$.
Now

$(*)_{6.7}$ (a) $(\eta_n, \bar{N}_n) \in \mathbf{N}_n$

(b) if $\mathrm{rk}(\eta_n, \bar{N}_n) < \infty$ then $\mathrm{rk}(\eta_n, \bar{N}_n) > \mathrm{rk}(\eta_{n+1} \bar{N}_{n+1})$.

Why? By the choice of the C_n's.
It follows that $\mathrm{rk}(\eta_0, \bar{N}_0) = \infty, (\eta_0, \bar{N}_0) \in \mathbf{N}_0$, so we are done. $\square_{1.17}$

In 1.17(1) we can replace λ^+, κ^+ by λ_1, κ_1, that is

Claim 1.21. *1) If*

(i) $\lambda_1 = \mathrm{cf}(\lambda_1) > \kappa_1 = \mathrm{cf}(\kappa_1) > \aleph_0$,

(ii) $\alpha < \lambda_1 \Rightarrow \mathrm{cov}(|\alpha|, \kappa_1, \kappa_1, 2) < \lambda_1$,

(iii) $\chi > \lambda^+$ *and* $x \in \mathscr{H}(\chi)$,

then we can find $\langle N_\eta : \eta \in \mathscr{T} \rangle$ *such that*

$(a)_1$ $\mathscr{T}$ *is a subtree of* ${}^{\omega>}(\lambda_1)$, *each* $\eta \in \mathscr{T}$ *is strictly increasing,*

$(b)_1$ $N_\eta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,

$(c)_1$ x, λ_1, κ_1 *belong to* N_η, $N_\eta \cap \kappa_1 \in \kappa_1$ *and* $\|N_\eta\| = |N_\eta \cap \kappa_1|$,

(d) $\nu \triangleleft \eta \in T \Rightarrow N_\nu \prec N_\eta$,

(e) $N_\eta \cap N_\nu = N_{\nu \cap \eta}$ *for* $\eta, \nu \in \mathscr{T}$,

(f) $\eta \in N_\eta$,

(g) *if* $\eta_\ell{}^\frown\langle \alpha_\ell \rangle \in \mathscr{T}$ *for* $\ell = 1, 2$ *and* $\alpha_1 < \alpha_2$ *then*

$$\sup(N_{\eta_1{}^\frown\langle \alpha_1 \rangle} \cap \alpha^+) < \min(N_{\eta_2{}^\frown\langle \alpha_2 \rangle} \cap \lambda^+, N_{\eta_2}).$$

2) If in addition $\alpha < \lambda_1 \Rightarrow |\alpha|^{<\kappa_1} < \lambda_1$ *then we can add*

(h) *if* $\eta, \nu \in \mathscr{T}$ *have the same length then there is an isomorphism from* N_η *onto* N_ν, *call it* $f_{\eta,\nu}$, *and it maps* x *to itself, so*

$$\eta, \nu \in \lim(\mathscr{T}) \Rightarrow \bigcup_{n<\omega} N_{\eta \restriction n} := N_\eta \cong N_\nu := \bigcup_{n<\omega} N_{\eta \restriction n}$$

3) If $\bar{\mathscr{S}} = \langle \mathscr{S}_\alpha : \alpha < \lambda_1 \rangle$ *is* $\subseteq$*-increasing with* $\alpha, \mathscr{S}_\alpha \subseteq [\alpha]^{<\kappa_1}, \alpha \in a \in S_\beta \Rightarrow a \cap \alpha \in \mathscr{S}_\alpha$ *then we can demand*

(i) $N_\eta \cap \lambda_1 \in \bigcup \mathscr{S}_\alpha$.

4) We can further demand

(j) N_η *is the Skolem hull of* $\{x, \eta, \kappa_1, \lambda_1\} \cup (N_\eta \cap \kappa_1) \cup N_{\langle\rangle}$.

5) If $(\mathscr{T}_0, \bar{\mathbf{I}})$ *is a tagged tree,* $\mathbf{I}_\eta$ *a normal ideal on* λ_1 *such that* $\{\delta : \mathrm{cf}(\delta) < \kappa_1\} \in \mathbf{I}_\eta$ *then we can demand* $(\mathscr{T}_0, \bar{\mathbf{I}}) \leq^* (\mathscr{T}, \bar{\mathbf{I}})$.

Proof. Similarly to 1.17 by 1.19. $\square_{1.21}$

Remark 1.22. The isomorphism is unique, hence if the isomorphism is called $\mathbf{f}_{\eta,\nu}$ then $\mathbf{f}_{\eta_0,\eta_1} = \mathbf{f}_{\eta_0,\eta_1} \circ \mathbf{f}_{\eta_1,\eta_0}$ when they are well defined.

Claim 1.23. *Suppose that*

(a) λ *is singular,* $\kappa = \mathrm{cf}(\lambda) > \aleph_0$,
(b) f *is a function from* ${}^{\omega>}\lambda$ *to finite subsets of* ${}^{\omega\geq}\lambda$ *or even subsets of* ${}^{\omega\geq}\lambda$ *of cardinality* $< \mathrm{cf}(\lambda)$,
(c) $\lambda = \sum_{i<\kappa} \lambda_i$, *where* λ_i *is (strictly) increasing and continuous with* $i < \kappa$;
(d) $S \subseteq \{i < \kappa : \mathrm{cf}(i) = \aleph_0\}$ *is stationary,*
(e) *for* $i \in S$ *we have* $\lambda_i = \sum_{n<\omega} \lambda_{i,n}$, *where* $\kappa < \lambda_{i,0}$, *and* $(\forall n)(\lambda_{i,n} < \lambda_{i,n+1}$ & $\mathrm{cf}(\lambda_{i,n}) = \lambda_{i,n})$
(f) $\mathbf{I}^n_\mu$ *is a* κ^+*–complete ideal on* μ *containing the co-bounded subsets of* μ *for* μ *regular* $< \lambda$
(g) *if* $i_1, i_2 \in S$ *and* $\{j : \lambda_j < \lambda_{i_1,n}\} = \{j : \lambda_j < \lambda_{i_2,n}\}$ *when* $i_1, i_2 \in S$ *and* $n < \omega$ *then* $\lambda_{i_1,n} = \lambda_{i_2,n}$ *and* $\mathbf{I}^n_{\lambda_{i_1,n}} = \mathbf{I}^n_{\lambda_{i_2,n}}$.

Then *there is a closed unbounded* $C \subseteq \kappa$ *such that for each* $i \in C \cap S$ *there is a* $\mathscr{T}$ *such that:*

$(*)_1$ $\mathscr{T} \subseteq \bigcup_{n<\omega} \prod_{m<n} \lambda_{i,m}$, $\langle\rangle \in \mathscr{T}$, *and* $\mathscr{T}$ *is closed under initial segments;*
$(*)_2$ *if* $\eta \in \mathscr{T}$ *and* $\lg(\eta) = n$ *then* $\{\alpha < \lambda_{i,n} : \eta\hat{\ }\langle\alpha\rangle \in \mathscr{T}\} \neq \emptyset \mod \mathbf{I}^n_{\lambda_{i,n}}$;
$(*)_3$ *if* $\eta \in \mathscr{T}$ *then* $f(\eta) \subseteq {}^{\omega\geq}\lambda_i$.

Remark 1.24. Claim 1.23 is used in [Shea, 1.11].

Proof. This is a variant of 1.16. For each $\eta \in {}^{\omega>}\lambda$ choose $g(\eta) < \kappa$ so that $f(\eta) \subseteq \bigcup\{{}^{\omega\geq}\zeta : \zeta < \lambda_{g(\eta)}\}$. Then instead of $(*)_3$, it suffice to demand

$(*)'_3$ $\forall \eta \in \mathscr{T}(g(\eta) < i)$.

Now we define a game ∂_i for each $i \in S$: the game is of length ω, and in the n-th move, the second player chooses $A_n \in \mathbf{I}^n_{\lambda_{i,n}}$ with $|A_n| < \lambda_{i,n}$, and the first player chooses $\alpha_n \in \lambda_{i,n}$. The first player wins if $[\ell < n \Rightarrow \alpha_\ell < \alpha_n]$, $\alpha_n \notin A_n$, and $g(\langle\alpha_0, \ldots, \alpha_n\rangle) \leq i$; otherwise the second player wins.

Now

$(*)_4$ if $i \in S$, $g(\langle\rangle) \leq i$, and the first player has a winning strategy, then a tree $\mathscr{T} = \mathscr{T}_i$ as desired exists.

Why? Let $i \in S$ be as in $(*)_4$ and let $\mathbf{f}_i$ be a winning strategy for the first player in the game ∂_i. Thus for $n < \omega$ and $\bar{A} \in {}^{n+1}\mathscr{P}(\lambda)$ such that $\forall m \leq n(A_m \in \mathbf{I}^n_{\lambda_{i,m}})$ we have $\mathbf{f}_i(\bar{A}) \in \lambda_{i,n}, \mathbf{f}_i(\bar{A}\restriction(m+1)) < \mathbf{f}_i(\bar{A})$ for all $m < n, \mathbf{f}_i(\bar{A}) \notin A_n$, and $g(\langle\mathbf{f}_i(\bar{A}\restriction 1), \ldots, \mathbf{f}_i(\bar{A} \restriction n)\rangle) \leq i$. Then $\mathscr{T} = \{\langle\mathbf{f}_i(\bar{A} \restriction 1), \ldots, \mathbf{f}_i(\bar{A} \restriction (n+1))\rangle$: such $\bar{A}\} \cup \{\langle\rangle\}$ is as desired. Thus we may assume toward contradiction

$(*)_5$ $S' = \{i \in S :$ the first player does not have a winning strategy for $\partial_i\}$ is a stationary subset of $\mathrm{cf}(\lambda)$.

Now, the game $\Game_i$ is open, so by the Gale-Stewart theorem it is determined. Hence for each $i \in S'$ we may choose a winning strategy $\mathbf{f}_i$ for the second player.

Thus

$(*)_6$ if $n < \omega$ and $\eta \in \prod\limits_{m<n} \lambda_{i,m}$ then $\mathbf{f}_i(\eta) \in \mathbf{I}^n_{\lambda_{i,n}}$;

$(*)_7$ for any $\eta \in \prod\limits_{m<\omega} \lambda_{i,m}$ one of the following occurs:

- (a) $\exists \ell < n < \omega\ (\eta(\ell) \geq \eta(n))$,
- (b) there is $n < \omega$ such that $\eta(n) \in \mathbf{f}_i(\eta \restriction n)$,
- (c) there is $n < \omega$ such that $g(\eta \restriction n) > i$.

Now choose a regular $\chi > \aleph_0$ so that g, $\langle \mathbf{f}_\delta : \delta \in S' \rangle$, $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$, $\langle \mathbf{I}^n_\mu : \mu < \lambda$ regular, $n < \omega \rangle$ and $\langle \langle \lambda_{i,n} : n < \omega \rangle : i \in S' \rangle$ belong to $\mathscr{H}(\chi)$. Remember $\mathscr{H}(\chi)$ is the family of sets with the transitive closure of cardinality $< \chi$, and that $(\mathscr{H}(\chi), \in)$ is a model of ZFC$^-$. Let $<^*_\chi$ be a well-ordering of $\mathscr{H}(\chi)$.

For all $\delta < \kappa$ let A_δ be the closure of $\delta \cup \{x\}$ under Skolem functions within the structure $(\mathscr{H}(\chi), \in, <^*_\chi)$. Then $C = \{\delta < \kappa : A_\delta \cap \kappa = \delta\}$ is a closed unbounded subset of κ. Thus there is $\delta \in S' \cap C$ and an elementary substructure $(N, \in, <)$ of $(\mathscr{H}(\chi), \in, <^*_\chi)$ such that $|N| < \kappa$ and $N \cap \kappa = \delta$, with $x \in N$. Clearly $\lambda_{\delta,m}, \mathbf{I}^m_{\lambda_{\delta,m}}$ belong to N for each m (by assumption (e)). However $\delta \notin N$, hence $\{\lambda_{\delta,m} : m < \omega\} \notin N$ though it is a subset of N.

Now we define $\eta = \langle \alpha_n : n < \omega \rangle \in \prod\limits_{m<\omega} \lambda_{\delta,m}$ so as to contradict $(*)_7$. Suppose $\alpha_m \in N$ has been constructed for all $m < n$. Using elementarity and absoluteness of suitable formulas we see that the set

$$A^* = \bigcup \{\mathbf{f}_j(\langle \alpha_0, \ldots, \alpha_{n-1} \rangle) : \begin{array}{l} j \in S' \text{ and} \\ \lambda_{j,0} = \lambda_{\delta,0}, \ldots, \lambda_{j,n-1} = \lambda_{\delta,n-1}, \lambda_{j,n} = \lambda_{\delta,n}\}. \end{array}$$

belongs to $\mathbf{I}^n_{\lambda^+_{\delta,n}}$ (being the union of $\leq \kappa$ sets each from $\mathbf{I}^n_{\lambda^+_{\delta,n}}$) and belongs to N. Since $\exists \alpha(\alpha_{n-1} < \alpha < \lambda_{\delta,n}$ and $\alpha \notin A^*)$ holds in $(\mathscr{H}(\chi), \in, <^*_\chi)$, it holds in $(N, \in, <^*_\chi)$ and this gives α_n. This completes the construction, and it is easily seen that $(*)_7$ is contradicted. $\square_{1.23}$

Remark 1.25. 1) We can interchange the quantifier in 1.23; one club (C) of $\mathrm{cf}(\lambda)$ is O.K. for every appropriate $\langle \langle \lambda_{\delta,n} : n < u \rangle : \delta < \mathrm{cf}(\lambda) \rangle$.
2) If $\lambda_{\delta,n} = \eta_\delta(n)$ and $\langle \mathrm{Rang}(\eta_\delta) : \delta \in S \rangle$ guess clubs of $\mathrm{cf}(\lambda)$ <u>then</u> we can add $\eta \in \prod\limits_{\delta,\ell} \Rightarrow g(\eta) > \lambda_{\delta,n}$.
3) We can get in this direction more results. If $2^{\mathrm{cf}(\lambda)} < \lambda$, λ_{i+1} regular, then we can find a closed unbounded set $\{\alpha(i) : i < \mathrm{cf}(\lambda)\}, \alpha(i{+}1)$ successor and $\mathscr{T} \subseteq {}^{\omega>}\lambda$, such that: $\langle \rangle \in \mathscr{T}, \eta \in \mathscr{T}, \max[\mathrm{Rang}(\eta)] < \lambda_{i+1} < \lambda_j$ implies $\{\alpha < \lambda_j : \eta\hat{\ }\langle \alpha \rangle \in \mathscr{T}\}$ has power λ_j, and implies also $g(\eta) < j$.(For each club C of $\mathrm{cf}(\lambda)$ we define a game, etc.).
4) In (3) we can replace "$2^{\mathrm{cf}(\lambda)} < \lambda$" by "there is a family $\mathscr{P}$ of closed unbounded subsets of $\mathrm{cf}(\lambda)$ such that $|\mathscr{P}| < \lambda$, and every closed unbounded subset of κ contains one of them".
5) On the other hand, if $\mu = \mu^{<\mu}$ in $\mathbf{V}$ let us add $\lambda > \mu$ generic closed unbounded subsets of μ (by $\mathbb{P} = \{f : \mathrm{Dom}(f)$ a subset of λ of power $< \mu, f(i)$ the characteristic

function of a closed bounded subset of $\mu\}$, and let $\underset{\sim}{C}_i$ the following $\mathbb{P}$-name of a club of μ: the characteristic function of $\underset{\sim}{C}_i$ is $\bigcup\{f(i) : f$ in the generic set$\})$. Let $\mathbf{G}$ be a subset of $\mathbb{P}$ generic over $\mathbf{V}$ and in $\mathbf{V}[\mathbf{G}]$ let $\{C_\eta : \eta \in {}^{\omega>}\lambda\}$ be another enumeration of $\{\underset{\sim}{C}_i[\mathbf{G}] : i < \lambda\}$, and define g:

$$g(\eta\hat{\ }\langle\alpha\rangle) = \min\{i < \mathrm{cf}(\lambda) : i \in C_\eta, \lambda_i > \alpha\}.$$

Clearly for this g the conclusion of remark (4) fails.

§ 2. On unique linear orders

Hausdorff has introduced and investigated the class of scattered linear orders (see 2.10). Galvin and Laver [Lav71] investigate the class M of linear orders which are a countable union of scattered linear orders. They were interested in linear orders up to embeddability inside the class $M = \cup\{M_{\lambda,\mu_1,\mu_2} : \mu_1, \mu_2$ regular uncountable $\lambda \in \mu_1 + \mu_2\}$ where M_{λ,μ_1,μ_2} is the class of linear orders M of cardinality λ with no increasing sequences of length μ_1 and no decreasing sequences of length μ_2. Galvin defined M_{λ,μ_1,μ_2} and proved the existence of a universal member.

Laver, solving a long standing conjecture of Fraïssé, and using the theory of better quasi orders of Nash Williams proved that the M is well and even better quasi ordered. In [She87, pp.308,309], we continue this investigation being interested in the uniqueness of such orders. We do more here. Invariants related to the g_i here are investigated in [She78, Ch.VIII,§3] and better in [Shef], and also in Droste-Shelah [DS85], [DS02]. This is continued being interested in uniqueness.

§ 2(A). **Classes of Coloured Linear Orders.**

Discussion 2.1. 1) We may waive "union of countably many scattered subsets", and essentially allow a family of $\leq \lambda$ isomorphism types of linear orders as basic orders. So ignoring trivialites they are neither well ordered nor anti-well ordered; we lose stability but can retain everything else.
2) Below in 2.18 we may start with closed enough set $\mathscr{S} \subseteq \mathscr{P}(X), |\mathscr{S}| \leq \lambda$.
3) Another way to get many of the properties is to build such N of larger cardinality, so e.g. saturated dense linear orders exists and then use the Löwenheim–Skolem argument.

Context 2.2. If not said otherwise, in this subsection we use a fix context $\mathbf{c} = (\lambda, \mu_1, \mu_2, \alpha^*, g_1, g_2) = (\lambda^{\mathbf{c}}, \mu_1^{\mathbf{c}}, \mu_2^{\mathbf{c}}, \alpha_*^{\mathbf{c}}, g_1^{\mathbf{c}}, g_2^{\mathbf{c}})$ which means it is as in 2.3.

Definition 2.3. 1) We say $\mathbf{c}$ is a context if it consists of $\lambda, \mu_1, \mu_1, g_1, g_2$ (and $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_1^+, \mathscr{F}_2^+$ defined from them), <u>when</u>

- (a) λ, μ_1, μ_2 are (infinite) cardinals with μ_1, μ_2 being uncountable regular such that $\lambda^+ = \max\{\mu_1, \mu_2\}$ and $\alpha^* = \alpha(*) < \lambda^+$, $\alpha^* \geq 1$
- (b) for $\ell = 1, 2$ we have g_ℓ, a function from α^* into $\mathscr{F}_\ell := \{h : h$ a function from some uncountable $\theta \in \mathrm{Reg} \cap \mu_\ell$ into $\alpha^*\}$ such that $\{\mathrm{Dom}(h) : h \in \mathrm{Rang}(g_\ell)\}$ is unbounded in $\mathrm{Reg} \cap \mu_\ell$. (Hence $\lambda = \sup\{\mathrm{Dom}(g_\ell(\alpha)) : \ell \in \{1,2\}$ and $\alpha < \alpha^*\})$

2) In addition if $\ell \in \{1,2\}, \alpha < \alpha^*, h = g_\ell(\alpha)$ <u>then</u>: $h \in \mathscr{F}_\ell^+$ where $\mathscr{F}_\ell^+$ is the set of $h \in \mathscr{F}_\ell$ satisfying

- $\boxdot_h^\ell$ $h \in \mathscr{F}_\ell$ and if $\delta < \mathrm{Dom}(h)$ is a limit ordinal of uncountable cofinality and $\beta = h(\delta)$ and $\langle \epsilon_i : i < \mathrm{cf}(\delta)\rangle$ is an increasing continuous sequence with limit δ then the set $\{i < \mathrm{cf}(\delta) : (h(\delta))(\epsilon_i) = (g(\beta))(i)\}$ contains a club of $\mathrm{cf}(\delta)$.

3) For notational simplicity assume $\alpha^* \leq \lambda$.

Notation 2.4. 1) For a linear order $M = (A, <)$ let M^* be $(A, >)$, i.e., its inverse.
2) Below $K = K_{\mathbf{c}} = K(\mathbf{c})$ and similarly for other versions of K.
3) Properties and Notations defined for linear orders, can be applied to expansions of linear orders (here mainly $N \in K$ or $N \in K^{\rm all}$).

Definition 2.5. $K = K^{\rm hom} = K(\lambda, \mu_1, \mu_2, \alpha^*, g_1, g_2)$ is the family of models $N = (M, P_\alpha)_{\alpha < \alpha(*)}$ such that:

- (i) M is a linear order,
- (ii) M is the union of $\aleph_0$ scattered suborders, i.e., $|M|$, the universe (=set of elements of M) is $\bigcup\limits_{n \in \omega} A_n$, where each $M \restriction A_n$ is scattered (see Definition 2.6 below),
- (iii) each P_α is a dense subset of M,
- (iv) $\langle P_\alpha : \alpha < \alpha(*) \rangle$ is a partition of M,
- (v) every increasing sequence in M has length $< \mu_1$, but for each $\alpha < \alpha^*$ in every open interval of M there is an increasing sequence of length $\mathrm{Dom}(g_1(\alpha))$, (hence any $\alpha < \mu_1$ is O.K.)
- (vi) every decreasing sequence in M has length $< \mu_2$, but for each $\alpha < \alpha^*$ in every open interval there is an decreasing sequence of any length $\mathrm{Dom}(g_2(\alpha))$, (hence any $\alpha < \mu_2$ is O.K.)
- (vii) if $\langle a_i : i < \kappa \rangle$ is an increasing bounded sequence in M, $\aleph_0 < \kappa \in \mathrm{Reg} \cap \mu_1$ <u>then</u> for some club C of κ, for every $\delta \in C \cup \{\kappa\}$, $\{a_i : i < \delta\}$ has a least upper bound in M,
- $(viii)$ if $\langle a_i : i < \kappa \rangle$ is a decreasing sequence in M bounded from below and $\aleph_0 < \kappa \in \mathrm{Reg} \cap \mu_2$ <u>then</u> for some club C of κ for every $\delta \in C \cup \{\kappa\}$ we have: $\{a_i : i < \delta\}$ has a greatest lower bound in M,
- (ix) if $x \in P_\alpha$, $g_1(\alpha) = h$ <u>then</u> $N_{<x} = N \restriction \{y : y <^M x\}$ and $M_{<x} = N_{<x} \restriction \{<\}$ has cofinality $\mathrm{Dom}(h)$ and if $\mathrm{Dom}(h) > \aleph_0$ then it has up-type h which means that

  $(*)^1_{N_{<x},h}$ there is an increasing continuous sequence $\bar{y} = \langle y_\epsilon : \epsilon < \mathrm{Dom}(h) \rangle$ in $M_{<x}$ such that $y_\epsilon \in P_{h(\epsilon)}$ for a club of $\epsilon \in \mathrm{Dom}(x)$ and $\{y_\epsilon : \epsilon < \mathrm{Dom}(h)\}$ is unbounded from above in $M_{<x}$,
- (x) if $x \in P_\alpha$, $g_2(\alpha) = h$, <u>then</u> $N_{>x} = N \restriction \{y : x <^M y\}$ satisfies: $(M_{>x})^*$, the inverse of $M_{>x}$, has cofinality $\mathrm{Dom}(h)$ and if $\mathrm{Dom}(h) > \aleph_0$ then N_x has down-type $g_2(\alpha)$ which means that

  $(*)^2_{N_{>x},h}$ there is a decreasing continuous sequence $\bar{y} = \langle y_\epsilon : \epsilon < \mathrm{Dom}(h) \rangle$ in $M_{>x}$ such that $y_\epsilon \in P_{h(\epsilon)}$ for a club of $\epsilon \in \mathrm{Dom}(h)$ and $\{y_\epsilon : \epsilon < \mathrm{Dom}(h)\}$ is unbounded from below in $M_{>x}$.

Definition 2.6. 1) For a linear order M we define when $\mathrm{Dp}(M) \geq \alpha$ by induction on α. If $\alpha = 0, \mathrm{Dp}(M) \geq \alpha$ for any linear order M, even the empty one. If $\alpha = 1, \mathrm{Dp}(M) \geq \alpha$ <u>if and only if</u> M is non-empty. If α is limit then $\mathrm{Dp}(M) \geq \alpha$ if and only if $\mathrm{Dp}(M) \geq \beta$ for every $\beta < \alpha$. If $\alpha = \beta + 1$ then $\mathrm{Dp}(M) \geq \alpha$ if and only if M can be represented as $M_1 + M_2$ where $\mathrm{Dp}(M_1) \geq \beta$ and $\mathrm{Dp}(M_2) \geq \beta$.
2) We let $\mathrm{Dp}(M) = \alpha$ if and only if $\mathrm{Dp}(M) \geq \alpha$ and $\mathrm{Dp}(M) \not\geq \alpha + 1$.

We say that $\mathrm{Dp}(M) = \infty$ if $\mathrm{Dp}(M) \geq \alpha$ for all ordinals α.
3) A linear order M is scattered if $\mathrm{Dp}(M) < \infty$ equivalently (by Hausdorff), the rational order cannot be embedded into M.
4) If N is an expansion of a linear order <u>then</u> $\mathrm{Dp}(N)$ means $\mathrm{Dp}(|N|, <^M)$.

Definition 2.7. 1) Let $K^* = K^{\rm all}$ be the class of $N = (M, P_\alpha)_{\alpha<\alpha(*)}$ satisfying clauses (i), (ii), (iv) of Definition 2.5, and the first half of (v), the first half of (vi), and (vii), (viii) and clause (ix) for x such that x is neither the first element of M nor the immediate successor of any $y \in M$ and clause (x) for x which are neither last nor the immediate predecessor of some $y \in M$.
2) For $N = (M, P_\alpha)_{\alpha<\alpha(*)} \in K$ and $x \in M$ let

(a) $P^N_\alpha = P_\alpha$, $<^N = <^M$,
(b) $N_{>x}, M_{>x}$ and $N_{<x}, M_{<x}$ be as in clauses (ix), (x) of Definition 2.5
(c) so $N_{>x} = (M_{>x}, P^N_\alpha \cap M_{>x})_{\alpha<\alpha(*)}$ and $N_{<x} = (M_{<x}, P^N_\alpha \cap M_{<x})_{\alpha<\alpha(*)}$.

3) For $h_1 \in \mathscr{F}^+_1, h_2 \in \mathscr{F}^+_2$ (that is, $h_1 \in \mathscr{F}_1$, $h_2 \in \mathscr{F}_2$ satisfying $\boxdot^1_{h_1}, \boxdot^2_{h_2}$ from Definition 2.3) let $K_{h_1,h_2} = K^{\rm hom}_{h_1,h_2} = K(\lambda, \mu_1, \mu_2, \alpha^*, g_1, g_2, h_1, h_2) = K_{h_1,h_2}(\lambda, \mu_1, \mu_2, \alpha^*, g_1, g_2)$ be the family of $N = (M, P_\alpha)_{\alpha<\alpha(*)} \in K$ such that N satisfies $(*)^1_{N,h_1}$ of clause (ix) of Definition 2.5, and $(*)^2_{N,h_2}$ of clause (x) of Definition 2.5.
4) For $h_1 \in \mathscr{F}^+_1, h_2 \in \mathscr{F}^+_2$ let $K^*_{h_1,h_2} = K^{\rm all}_{h_1,h_2}$ be the family of $N = (M, P_\alpha)_{\alpha<\alpha(*)} \in K^*$ such that N satisfies $(*)^1_{N,h_1}$ of clause (ix) of 2.5 and $(*)^2_{N,h_2}$ of clause (x) of 2.5.
5) Let $K^\otimes = K^{\rm vhm} = \{M \in K^{\rm all}$: if N has no last element then $(*)^1_{M,h}$ for some $h \in \mathscr{F}^+_1$ and if N has no first element then $(*)^2_{M,k}$ for some $h \in \mathscr{F}^+_2\}$.
6) $K^\oplus = \cup\{K^*_{h_1,h_2} : h_2 \in \mathscr{F}^+_1, h_2 \in \mathscr{F}^+_2\}$.

Definition 2.8. 1) For $N_i \in K^*(i < \alpha)$ then $N_0 + N_1$ and $\sum\limits_{i<\alpha} N_i$ are defined naturally, as well as $N_0 \times \alpha$.
2) Similarly for anti-well ordered sums.

Claim 2.9. *1) If $N \in K_{h_1,h_2}$ (so $h_1 \in \mathscr{F}^+_1, h_2 \in \mathscr{F}^+_2$) and $x \in P^N_\alpha$ <u>then</u>:*

(i) $N_{<x} = N{\restriction}\{y \in N : y <^M x\} \in K_{g_1(\alpha),h_2}$ and
(ii) $N_{>x} = N{\restriction}\{y \in N : x <^M y\} \in K_{h_1,g_2(\alpha)}$.

2) If $N \in K$ and I is a convex non-empty subset of M with neither last nor first element and $M \restriction I$ satisfies $(*)^1_{M,h^*_1}$ of clause (ix) of Definition 2.5 for $h^*_1 \in H_1$, and $(*)^2_{M,h^*_2}$ of clause (x) of Definition 2.5 for $h^*_2 \in H_2$ <u>then</u> $M \restriction I \in K_{h^*_1,h^*_2}$.
3) If $N = (M, P_\alpha)_{\alpha<\alpha^*} \in K_{h_1,h_2}$ then $N^* = (M^*, P_\alpha)_{\alpha<\alpha^*} \in K_{h_2,h_1}$.
4) If $N \in K^{\rm all}$ and I is a convex subset of N <u>then</u> $N{\restriction}I \in K^{\rm all}$. Moreover, $N \in K^\otimes \Rightarrow N{\restriction}I \in K^\otimes$.

Proof. Straightforward. $\square_{2.9}$

Recall

Claim 2.10. *1) The family of scattered order types is the closure of the singletons under well ordered sums and inverse of well ordered sums.*
2) If $M_1 \subseteq M_2$ <u>then</u> $\mathrm{Dp}(M_1) \leq \mathrm{Dp}(M_2)$.
3) If M is a scattered linear order <u>then</u> one of the following holds:

(a) *M is a singleton,*

(b) *for some $x \in M$ we have $\mathrm{Dp}(M_{<x}) < \mathrm{Dp}(M)$ and $\mathrm{Dp}(M_{>x}) < \mathrm{Dp}(M)$,*

(c) *there is an increasing unbounded sequence $\langle x_i : i < \kappa \rangle$ in M, with κ a regular cardinal, such that*

$$i < \kappa \Rightarrow \mathrm{Dp}(M_{<x_i}) < \mathrm{Dp}(M),$$

(d) *there is a decreasing sequence $\langle x_i : i < \kappa \rangle$ in M unbounded from below and such that*

$$i < \kappa \Rightarrow \mathrm{Dp}(M_{>x_i}) < \mathrm{Dp}(M).$$

Claim 2.11. *For any $h_1 \in \mathscr{F}_1, h_2 \in \mathscr{F}_2$ we have $K_{h_1,h_2} \neq \emptyset$.*

Proof. First

$\circledast_1$ there is a scattered $N \in K^*$ such that: N has a first element, a last element and $P^N_\alpha \neq \emptyset$ for every $\alpha < \alpha^*$.

[Why? Recall that $\lambda^+ = \mathrm{Max}\{\mu_1, \mu_2\}$ so let $\ell \in \{1, 2\}$ be such that $\lambda^+ = \mu_\ell$. For $\theta \in \{\mathrm{Dom}(g_\ell(\alpha)) : \alpha < \alpha^*\}$, let $\alpha_\theta < \alpha^*$ be minimal such that $\theta = \mathrm{Dom}(g_\ell(\alpha_\theta))$. Now we define $N_\theta = (M_\theta, P_{\theta,\alpha})_{\alpha<\alpha(*)}$, i.e. $P^{N_\theta}_\alpha = P_{\theta,\alpha}$, as follows:

(a) M_θ is $(\theta + 1, <)$ if $\ell = 1$ and is its inverse if $\ell = 2$

(b) for $\epsilon \in \theta + 1, \epsilon \in P_{\theta,\alpha}$ <u>if and only if</u> $\epsilon = \theta$ and $\alpha = \alpha_\theta$ or ($\epsilon < \theta$) is a limit ordinal and $(g(\alpha_\theta))(\epsilon) = \alpha$) or ($\epsilon = \alpha + 1$ so ϵ is a successor ordinal).

If λ is regular then by 2.3 we can choose $\theta = \lambda$ and we are done as $\alpha^* \leq \lambda$. If λ is singular we can find an increasing sequence $\langle \theta_i : i < \mathrm{cf}(\lambda) \rangle$, with limit λ, $\theta_i = \mathrm{Dom}(g_\ell(\alpha_{\theta_i})), \theta_0 > \mathrm{cf}(\lambda)$, and we combine them by inserting N_{θ_i} in the i-th open interval of N_{θ_0}, i.e. in $(i, i + 1)_{N_{\theta_0}}$.]

$\circledast_2$ there is a scattered $N \in K^*$ such that: N has a first element, N has a last element and for every $\ell \in \{1, 2\}$ and $\theta = \mathrm{cf}(\theta) < \mu_\ell$ the model N has an increasing sequence of length θ if $\ell = 1$ and a decreasing sequence of length θ if $\ell = 2$.

[Why? Similar to the proof of $\circledast_1$ using it].

$\circledast_3$ for any $h_1 \in \mathscr{F}^+_1, h_2 \in \mathscr{F}^+_2$ (i.e. $h_1 \in \mathscr{F}_1, h_2 \in \mathscr{F}_2$ satisfying $\boxdot^1_{h_1} + \boxdot^2_{h_2}$ from Definition 2.3), there is a scattered $N \in K^*$ satisfying

- (a) $(*)$ of clause (ix) of Definition 2.5 for h_1, that is, $(*)^1_{N,h_1}$
- (b) $(*)$ of clause (x) of Definition 2.5 for h_2, that is, $(*)^2_{N,h_2}$
- (c) $P^N_\alpha \neq \emptyset$ for $\alpha < \alpha^*$
- (d) if $\theta = \mathrm{cf}(\theta) < \mu_1$ then N has an increasing sequence of length θ
- (e) if $\theta = \mathrm{cf}(\theta) < \mu_2$, then N has a decreasing sequence of length θ.

[Why? Let N be as in $\circledast_2$. We define M as follows: M has set of elements $\{(\ell,x) : \ell \in \{-1,0,1\}$, and $\ell = -1 \Rightarrow x \in \mathrm{Dom}(h_2)$ and $\ell = 0 \Rightarrow x \in |N|$ and $\ell = 1 \Rightarrow x \in \mathrm{Dom}(h_1)\}$ and $(\ell_1,x_1) <^M (\ell_2,x_2)$ if and only if $\ell_1 < \ell_2$ or $\ell_1 = \ell_2 = -1$ and $x_2 < x_1$ (as ordinals) or $\ell_1 = \ell_2 = 0$ and $x_1 <^N x_2$ or $\ell_1 = \ell_2 = 1$ and $x_1 < x_2$ (as ordinals).

Lastly, $N = (M, P^N_\alpha)_{\alpha<\alpha(*)}$ where for $\alpha < \alpha^*$ we let $P^M_\alpha = \{(\ell,x) \in M : \ell = -1 \wedge h_2(x) = \alpha$ or $\ell = 0 \wedge x \in P^N_\alpha$ or $\ell = 1 \wedge h_1(x) = \alpha\}$].

At last we define by induction on $n < \omega$, $(M^n, P^n_\alpha)_{\alpha<\alpha(*)}$ such that:

(i) $(M^n, P^n_\alpha)_{\alpha<\alpha(*)} \in K^*$ is a submodel of $(M^{n+1}, P^{n+1}_\alpha)_{\alpha<\alpha(*)}$,

(ii) M^n is scattered, and so every interval contains a jump, i.e., an empty open interval

(iii) $(M^n, P^n_\alpha)_{\alpha<\alpha(*)} \in K^*_{h_1,h_2}$,

(iv) If $x \in P^n_\alpha$ has no immediate predecessor in M^n (recalling M_n has no first element), then clause (ix) of Definition 2.5 holds for it, really follows by 2.7(1)

(v) If $x \in P^n_\alpha$ is neither last nor has an immediate successor (recalling M_n has no last element), then clause (x) holds for it, really follows by 2.7(1)

(vi) If $x \in M^{n+1} \setminus M^n$, then for some $y,z \in M^n$:
$$y < x < z, \text{ and } \neg(\exists t \in M^n) y < t < z.$$

(vii) For every $y < z$ in M^n: in M^{n+2} the element y has no immediate successor and the element z has no immediate predecessor, $\bigwedge\limits_\alpha P^{n+2}_\alpha \cap (y,z)^{M^{n+2}} \neq \emptyset$, in $(y,z)^{M^{n+2}}$ there are increasing sequences of any length $\theta = \mathrm{cf}(\theta) < \mu_1$ in $(y,z)^{M^{n+2}}$ there are decreasing sequences of any length $\theta = \mathrm{cf}(\theta) < \mu_2$.

There is no problem in this and $(\bigcup\limits_n M^n, \bigcup\limits_n P^n_\alpha)_{\alpha<\alpha(*)}$ is as required.

That is, for $n = 0$ use $\circledast_3$ for (h_1,h_2). Given $(M^n, P^n_\alpha)_{\alpha<\alpha(*)}$, to get $(M^{n+1}, P^{n+1}_\alpha)_{\alpha<\alpha(*)}$, for each empty open interval (x,y) of M^n, we insert in this interval a copy of N as constructed in $\circledast_3$ but with $(g_2(\alpha_1), g_1(\alpha_2))$ here standing for (h_1,h_2) there when $x \in P^n_{\alpha_1}, y \in P^n_{\alpha_2}$. $\square_{2.11}$

Claim 2.12. *If $h_1 \in \mathscr{F}^+_1$ and $h_2 \in \mathscr{F}^+_2$, then every two members of K_{h_1,h_2} are isomorphic.*

We shall prove this below.

Claim 2.13. *1) If $N \in K^*$ and (I_0,I_1) is a cut of N, i.e. of $M = (|N|, <^N)$ (as a linear order, i.e. $M = I_0 \cup I_1, I_0 \cap I_1 = \emptyset$ and $t_0 \in I_0 \wedge t_1 \in I \Rightarrow t_0 <^N t_1$), then exactly one of the following occurs:*

(i) *I_0 has a last element,*

(ii) *I_0 is empty,*

(iii) *I_1 has a first element,*

(iv) *I_1 is empty,*

(v) $\mathrm{cf}(I_0) = \mathrm{cf}(I^*_1) = \aleph_0$.

2) If $N \in K$ then the set of cuts of case (v) above is dense.
3) If $N \in K$ and I is an infinite subset of N <u>then</u> we can find J such that:

- (i) *$I \subseteq J \subseteq N$,*
- (ii) *$|J| = |I|$,*
- (iii) *J has neither a first nor a last member,*
- (iv) *if $x \in N \setminus J$ and $N_{J,x} = N{\restriction}A_{J,x}, M_{I,x} = (A_{J,x}, <^N {\restriction}A_{J,x})$ where $A_{J,x} = \{y \in M : x, y$ realize the same cut of $J\}$ <u>then</u>*
  - (α) *$N_{J,x}$ has no last element,*
  - (β) *if $N_{J,x}$ is bounded in M and $\mathrm{cf}(N_{J,x}) > \aleph_0$ <u>then</u> it has a least upper bound in J,*
  - (γ) *$N_{J,x}$ has no first element,*
  - (δ) *if $N_{J,x}$ is bounded from below in $N_{J,x}$ and $\mathrm{cf}(N^*_{J,x}) > \aleph_0$ <u>then</u> it has a maximal lower bound in J.*
- (v) *the number of members in $\{N_{J,x} : x \in N \setminus J\}$ is $\leq |J| + 1$.*

Proof. Straightforward. $\square_{2.13}$

Definition 2.14. 1) For a linear order M, if $J \subseteq M$ satisfies clauses (iii) + (iv) of claim 2.13(3) <u>then</u> we say that J is quite closed in M.
2) Similarly for $J \subseteq N, N$ an expansion of a linear order.

Proof. <u>Proof of Claim 2.12</u>:

Let $h_1 \in \mathscr{F}_1$, $h_2 \in \mathscr{F}_2$ and assume $N_1, N_2 \in K_{h_1,h_2}$, and we shall prove that N_1, N_2 are isomorphic. Let $N_\ell = \bigcup\limits_{n<\omega} A_{\ell,n}$ with $M_\ell \restriction A_{\ell,n}$ being scattered, of course, $M_\ell = N_\ell{\restriction}\{<\}$. Let $\mathscr{G}$ be the family of f such that:

- (a) f is a one-to-one function,
- (b) $\mathrm{Dom}(f)$ is a quite closed subset of M_1, see Definition 2.14
- (c) $\mathrm{Rang}(f)$ is a quite closed subset of N_2, see Definition 2.14
- (d) f is an isomorphism from $N_1{\restriction}\mathrm{Dom}(f)$ onto $N_2 \restriction \mathrm{Rang}(f)$
- (e) $M_1{\restriction}\mathrm{Dom}(f)$ is a scattered linear order.

Now

- $\boxtimes_1$ there is $f_1 \in \mathscr{G}$ such that $\mathrm{Dom}(f_1)$ is an unbounded subset of N_1, and $\mathrm{Rang}(f_1)$ is an unbounded subset of N_2.

[Why? As $N_1, N_2 \in K_{h_1,h_2}$, using $h_1 \in \mathscr{F}_1^+$, see 2.3(3) and Definition 2.7(3).]

- $\boxtimes_2$ There is $f_2 \in \mathscr{G}$ such that $\mathrm{Dom}(f_2)$ is a subset of N_1 unbounded from below and $\mathrm{Rang}(f_2)$ is an unbounded from below subset of N_2.

[Why? As $N_1, N_2 \in K_{h_1,h_2}$, using $h_2 \in \mathscr{F}_1^+$, see 2.3(3) and Definition 2.7(3).]

- $\boxtimes_3$ There is $f_0 \in \mathscr{F}$ satisfying the demands in $\boxtimes_1$ and $\boxtimes_2$.

[Why? Let f_1, f_2 be from $\boxtimes_1, \boxtimes_2$, respectively. We choose $x \in \mathrm{Dom}(f_1)$ and $y \in \mathrm{Dom}(f_2)$ such that $M_1 \models$ "$y < x$" and $M_2 \models$ "$f_2(y) < f_1(x)$", note that this is possible by the choices of f_1 and f_2.

Let

$$f_0 = (f_1 \restriction \{z \in \mathrm{Dom}(f_1) : x <^{N_1} z\}) \cup (f_2 {\restriction} \{z \in \mathrm{Dom}(f_2) : z <^{N_2} y\}).$$

Clearly f_0 is as required.]

For any $f \in \mathscr{G}$ which extends f_0 and $t \in N_1 \setminus \mathrm{Dom}(f)$ we let

$$N_{1,f,t} = N_1 \restriction \{s \in N_1 : (\forall x \in \mathrm{Dom}(f))[x <_{M_1} t \equiv x <_{M_1} s] \text{ and } s \notin \mathrm{Dom}(f)\}.$$

Now

$\boxtimes_4$ if $f \in \mathscr{G}$ extends f_0 and $t \in N_1 \setminus \mathrm{Dom}(f)$ and $n < \omega$ and $A_{1,n} \cap N_{1,f,t} \neq \emptyset$ then there is f' such that
- (i) $f \subseteq f' \in \mathscr{G}$,
- (ii) $\mathrm{Dom}(f') \setminus \mathrm{Dom}(f) \subseteq N_{1,f,t}$,
- (iii) if $s \in M_{1,f,t} \setminus \mathrm{Dom}(f')$ and $A_{1,n} \cap N_{1,f,t} \neq \emptyset$ then
$$\mathrm{Dp}(N_{1,f',s} {\restriction} A_{1,n}) < \mathrm{Dp}(N_{1,f,t} {\restriction} A_{1,n}) = \mathrm{Dp}(N_{1,f,s} \restriction A_{1,n}).$$

[Why? First note that there are $t_0 < t_1$ in $\mathrm{Dom}(f_1)$ such that $N \models$ "$t_0 < t < t_1$", this holds by the choice of f_0 (recalling we are assuming $f \geq f_0$. Second, we can demand that $N_{1,f,t} = N_1 {\restriction} (t_0, t_1)_{N_1}$, just by the definition of "$\mathrm{Dom}(f)$ is quite closed" recalling the assumption on f.

By Claim 2.10 it is enough to consider the following three cases.

<u>Case 1</u>: There is $s_1 \in N_{1,t} \cap A_{1,n}$ such that $\mathrm{Dp}((N_{1,f,t})_{>s_1}) < \mathrm{Dp}(N_{1,f,t})$ (so possibly $(N_{1,f,t})_{>s_1}$ is empty) and $\mathrm{Dp}((N_{1,f,t})_{<s_1}) < \mathrm{Dp}(N_{1,f,t})$.

Let $s_1 \in P^{N_1}_\alpha$ (clearly such α exists). Now, $(f(t_1), f(t_2))$ is an open interval of N_2 hence there is in it an $s_2 \in P^{N_2}_\alpha$. Let $f' = f \cup \{\langle s_1, s_2 \rangle\}$.

<u>Case 2</u>: For every $x \in N_{1,f,t}$ we have $\mathrm{Dp}((N_{1,f,t})_{<x}) < \mathrm{Dp}(N_{1,f,t})$.

Let α_1 be such that $t_1 \in P^{N_1}_{\alpha_1}$, so also $f(t_1) \in P^{N_2}_{\alpha_1}$, and imitate the proof of $\boxtimes_1$.

<u>Case 3</u>: For every $x \in M_{1,f,t}$ we have $\mathrm{Dp}((N_{1,f,t})_{>x}) < \mathrm{Dp}(N_{1,f,t})$.

Let α_0 be such that $t_0 \in P^{N_1}_{\alpha_0}$, so also $f(t_0) \in P^{N_2}_{\alpha_0}$ and immitate the proof of $\boxtimes_2$.

So $\boxtimes_4$ holds indeed.]

$\boxtimes_5$ If $f \in \mathscr{G}$ extends f_0 and $n < \omega$ <u>then</u> there is f' such that
- (i) $f \subseteq f' \in \mathscr{G}$,
- (ii) if $t \in N_1 \setminus \mathrm{Dom}(f')$ then $\mathrm{Dp}(N_{1,f',t} {\restriction} A_{1,n}) < \mathrm{Dp}(N_{1,f,t} {\restriction} A_{1,n})$.

[Why? Let $\{t_\epsilon : \epsilon < \epsilon(*)\}$ be such that $t_\epsilon \in N_1 \setminus \mathrm{Dom}(f)$ and $\langle N_{1,f,t_\epsilon} : \epsilon < \epsilon(*)\rangle$ lists $\{N_{1,f,t} : t \in N_1 \setminus \mathrm{Dom}(f) \text{ and } A_{1,n} \cap N_{1,f,t} \neq \emptyset\}$ with no repetitions. For each ϵ let f'_ϵ be as in $\boxtimes_4$ for t_ϵ, and let $f' = \bigcup\limits_{\epsilon<\epsilon(*)} f'_\epsilon$. Now check, so $\boxtimes_5$ holds indeed.]

For any $f \in \mathscr{G}$ which extends f_0 and $t \in M_2 \setminus \mathrm{Rang}(f)$, let

$$N_{2,f,t} = N_2 \restriction \{s \in M_2 : (\forall x \in \mathrm{Rang}(f))(x <^{N_2} s \Leftrightarrow x <^{N_2} t)\}.$$

Just as in $\boxtimes_4$, $\boxtimes_5$ we can show:

$\boxtimes_6$ if $f \in \mathscr{G}$ extends f_0 and $n < \omega$ <u>then</u> there is f' such that

(i) $f \subseteq f' \in \mathscr{G}$,

(ii) if $t \in N_2 \setminus \mathrm{Rang}(f)$ and $A_{2,n} \cap M_{2,f,t} \neq \emptyset$ then $\mathrm{Dp}(N_{2,f',t}{\restriction}A_{1,n}) < \mathrm{Dp}(M_{2,f,t}{\restriction}A_{1,n})$.

Lastly, we choose $f_n \in \mathscr{F}$ by induction on $n < \omega$ such that $k < m \Rightarrow f_k \subseteq f_m$. For $n = 0$ we have already chosen f_0. If $n = k^2 + 2m < (k+1)^2$, let f_{n+1} relate to f_n as f' relates to f in $\boxtimes_5$ (for $A_{1,m}$). If $n = k^2 + 2m + 1 < (k+1)^2$, let f_{n+1} relate to f_n as f' relates f in $\boxtimes_6$ (for $A_{2,m}$).

Let $f = \bigcup\limits_{n \in \omega} f_n$, clearly f is a partial isomorphism from N_1 to N_2. Now, $\mathrm{Dom}(f) = N_1$, because if $t \in N_1 \setminus \mathrm{Dom}(f)$ then for some n we have $t \in A_{1,n}$ and clearly $\langle \mathrm{Dp}(N_{1,f_m,t}{\restriction}A_{1,n}) : n < \omega\rangle$ is a non-increasing sequence of ordinals (by 2.10(2)), and for every $k > m$ we have $\mathrm{Dp}(N_{1,f_{k^2+2m},t}{\restriction}A_{1,n}) < \mathrm{Dp}(N_{1,f_{k^2+2m+1},t}{\restriction}A_{1,n})$ because of the use of $\boxtimes_5$. A contradiction, so really $\mathrm{Dom}(f) = M_1$. Similarly $\mathrm{Rang}(f) = M_2$ and we are done. $\square_{2.12}$

Definition 2.15. We say $N \in K^*$ is almost κ-homogeneous <u>when</u>:

- if $I \subseteq N$, $|I| < \kappa$ *then* we can find J, $I \subseteq J \subseteq N$, $|J| < \kappa$ such that
  - $(*)$ if $s, t \in (N \setminus J)$ realize the same cut of J and $s \in P^N_\alpha \Leftrightarrow t \in P^N_\alpha$ for every $\alpha < \alpha(*)$, <u>then</u> there is an automorphism of N over J mapping s to t.

Similarly to the proof of 2.12.

Conclusion 2.16. *Assume* $h_1 \in \mathrm{Rang}(g_1), h_2 \in \mathrm{Rang}(g_2)$.
1) If $N \in K^{\mathrm{hom}}_{h_1,h_2}$, $n < \omega$ *and* $x_1 < x_2 < \ldots < x_n$ *in* N, *and* $y_1 < \ldots < y_n$ *in* N, *and* $x_m \in P^N_\alpha \Leftrightarrow y_m \in P^N_\alpha$ *for* $\alpha < \alpha(*)$, $m \in \{1, \ldots, n\}$, <u>*then*</u> *there is an automorphism of* N *mapping* x_m *to* y_m *for* $m = 1, \ldots, n$.
2) If $N \in K_{h_1,h_2}$ *and* $J \subseteq N$ *is quite closed in* M <u>*then*</u>

$(*)$ *if* $s, t \in N \setminus J$ *realize the same cut of* J *and* $s \in P^N_\alpha \Leftrightarrow t \in P^N_\alpha$ *for* $\alpha < \alpha(*)$, <u>*then*</u> *there is an automorphism of* N *over* J *mapping* s *to* t.

3) Every $N \in K^{\mathrm{hom}}$ *is almost* κ*-homogeneous (where* $\kappa \geq \aleph_0$*).*
4) Assume $N \in K^{\mathrm{hom}}_{h_1,h_2}$ *and* J_1, $J_2 \subseteq N$ *are quite closed and* [J_1 *is unbounded in* N *iff* J_2 *is unbounded in* N] *and* [J_1 *is unbounded in* N^* *iff* J_2 *is unbounded in* N^*]. *If* f *is an isomorphism from* $N \restriction J_1$ *onto* $N \restriction J_2$ <u>*then*</u> *we can extend* f *to an automorphism of* M.

Proof. Should be clear. $\square_{2.16}$

§ 2(B). Examples.

In this subsection we consider some examples.

Content 2.17. We do not assume 2.3 fully, still λ, μ_1, μ_2 are as in 2.3(a) and θ will denote a regular cardinal $< \mu_1 \cap \mu_2$, usually uncountable.

Definition 2.18. Assume $\theta = \text{cf}(\theta) < \text{Min}\{\mu_1, \mu_2\}$ and let $\bar{\sigma} = \langle (\sigma^1_\alpha, \sigma^2_\alpha) : \alpha < \beta(*)\rangle$ list (with no repetitions) the pairs (σ_1, σ_2) of (infinite) regular cardinals such that $\sigma_\ell < \mu_\ell, \theta \in \{\sigma_1, \sigma_2\}$ and $\alpha = 0 \Rightarrow (\sigma^1_\alpha, \sigma^2_\alpha) = (\theta, \theta)$.
1) We let $g^\theta_\ell = g^{\theta,\bar{\sigma}}_\ell \in \mathscr{F}_\ell$ be defined by: for $\beta < \beta(*), g^\theta_\ell(\beta)$ is a function with domain σ^ℓ_β and for $\gamma < \sigma^\ell_\beta$

- $(*)$ $g^\theta_\ell(\beta)(\gamma) = \alpha$ iff $\alpha < \alpha(*)$ satisfies
  - (a) if $\gamma < \sigma$ is a limit ordinal then $\sigma^\ell_\alpha = \text{cf}(\gamma), \sigma^{3-\ell}_\alpha = \theta$
  - (b) if $\gamma < \sigma$ is non-limit then $\alpha = 0$.

1A) For $\ell \in \{1,2\}$ and regular $\theta < \mu_\ell$ let h^ℓ_θ be the unique $h : \theta \to \beta(*)$ such that $\sigma^\ell_\alpha = \theta \Rightarrow g^\theta_\ell(\alpha) = h$.
2) Let $\mathbf{c} = \mathbf{c}^{\text{can}}_{\lambda,\mu_1,\mu_2,\theta,\bar{\sigma}}$ be the unique $\mathbf{c}$ such that $(\lambda^{\mathbf{c}}, \mu^{\mathbf{c}}_1, \mu^{\mathbf{c}}_2, g^{\mathbf{c}}_1, g^{\mathbf{c}}_2) = (\lambda, \mu_1, \mu_2, g^\theta_1, g^\theta_2)$, see 2.19(2).
3) In (2) we may omit $\bar{\sigma}$ when $\alpha < \beta(*) \Rightarrow \alpha = \text{otp}(u_\alpha, <_{\text{lex}})$ where $u_\alpha = \{(\sigma_1, \sigma_2) : \sigma_1 = \text{cf}(\sigma_1) < \mu_1, \sigma_2 = \text{cf}(\sigma_2) < \mu_2$ and $(\sigma_1, \sigma_2) <_{\text{lex}} (\sigma^1_\alpha, \sigma^2_\alpha)\}$; justify by 2.19(1),(2).
4) For regular $\theta_1 < \mu_1, \theta_2 < \mu_2$ we let $K^{\text{can}}_{\theta_1,\theta_2} = K^{\text{hom}}_{h^1_{\theta_1}, h^2_{\theta_2}}(\mathbf{c})$ where $\mathbf{c}$ is from part (2),(3) and $h^\ell_{\theta_1}$ is from part (1A). For $u \subseteq \beta(*)$ non-empty let $K^{\text{can}}_{\theta_1,\theta_2,u} = \{(|N|, <^N) {\restriction} \bigcup\limits_{\alpha \in u} P^N_\alpha : N \in K^{\text{can}}_{\theta_1,\theta_2}\}$. Can define for the general case.

Claim 2.19. *Let λ, μ_1, μ_2 be as in 2.3(a) and θ be regular $< \min\{\mu_1, \mu_2\}$.*
1) There is $\bar{\sigma} = \langle (\sigma^1_\alpha, \sigma^2_\alpha) : \alpha < \beta(*)\rangle$ as in Definition 2.18 so $|\beta(*)| = |\text{Reg} \cap \mu_1| \times |\text{Reg} \cap \mu_2|$. Moreover, there is one and only one $\bar{\sigma}$ as in 2.18(3).
2) For $\bar{\sigma}$ as in part (1), $\mathbf{c}^{\text{can}}_{\lambda,\mu_1,\mu_2,\theta,\bar{\sigma}}$ is well defined, that is, there is a unique context $\mathbf{d}$, recalling Definition 2.3, such that:

- (a) $(\lambda^{\mathbf{d}}, \mu^{\mathbf{d}}_1, \mu^{\mathbf{d}}_2) = (\lambda, \mu_1, \mu_2)$
- (b) $\alpha^{\mathbf{d}}_* = \beta(*)$
- (c) $(g^{\mathbf{d}}_1, g^{\mathbf{d}}_2)$ *is as in 2.18(1).*

3) If for $\iota = 1,2$ we have $\bar{\sigma}_\iota = \langle (\sigma^1_{\iota,\alpha}, \sigma^2_{\iota,\alpha}) : \alpha < \beta(\iota)\rangle$ as in part (1), i.e. as in 2.18 and $\mathbf{d}_\iota$ is as in part (2) for $\bar{\sigma}_\iota$ and $h_{\iota,1} \in \mathscr{F}^{+,\mathbf{d}_\iota}_1, h_{\iota,2} \in \mathscr{F}^{+,\mathbf{d}_i}_2$ and $N_\iota = (M_\iota, P^\iota_\alpha)_{\alpha<\beta(\iota)} \in K^{\mathbf{d}_\iota}_{h_{\iota,1},h_{\iota,2}}$ then

- (a) *there is a unique $f : \beta(1) \to \beta(2)$ such that $(\sigma^1_{1,\alpha}, \sigma^2_{1,\alpha}) = (\sigma^1_{2,f(\alpha)}, \sigma^2_{2,f(\alpha)})$ for $\alpha < \beta(1)$; moreover f is one to one onto*
- (b) *M_1, M_2 are isomorphic linear orders*
- (c) *moreover, there is an isomorphism $\mathbf{f}$ from N_1 onto N_2 which maps P^1_α onto $P^2_{f(\alpha)}$ for every $\alpha < \beta(1)$.*

4) For $\mathbf{c}$ *as in 2.18(2) so* $\bar{\sigma} = \bar{\sigma}^{\mathbf{c}}, \theta = \mathrm{cf}(\theta) > \aleph_0$ *as in 2.18 letting* $(\beta(*) = \ell g(\bar{\sigma}))$ *if* $N = (M, P_\alpha)_{\alpha < \beta(*)} \in K^{\mathbf{c}}$*, so* M *a linear order, then* N *is uniquely determined by* M*, i.e.* $P_\alpha^N = \{d \in M : M_{<d}\}$ *has cofinality* σ_α^1 *and* $M_{>d}$ *has co-initiality* σ_α^2*.*

Proof. Should be clear. $\square_{2.19}$

The following is used in [Sheg].

Claim 2.20. *Assume* $\mathbf{c} = \mathbf{c}^{\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$*, see 2.18(2), 2.19(2).*
1) If $N_1, N_2 \in K^{\mathrm{hom}}$ *<u>then</u>:* N_1, N_2 *are isomorphic <u>iff</u>* N_1, N_2 *has the same cofinality and same co-initiality.*
2) So $K^{\mathrm{hom}} = \cup\{K^{\mathrm{hom}}_{\theta_1,\theta_2} : \theta_1 < \mu_1, \theta_2 < \mu_2$ *are regular*$\}$*.*
3) Assume $\alpha \in (0, \lambda^+)$ *is a successor and* $N = \sum\limits_{\beta \le \alpha} N_\beta$*.*

A sufficient condition for $N \in K^{\oplus}_{\theta,\theta}$ *is:*

- (a) *each* N_β *is from* $K_{\theta,\theta}$ *or is a singleton*
- (b) *if* $\theta > \aleph_0$ *and* $\beta < \alpha$ *<u>then</u>* N_β *is a singleton or* $N_{\beta+1}$ *is a singleton but not both*
- (c) $N_0 \in K^{\oplus}_{\theta,\theta}$ *and* N_α *is a singleton*
- (e) *if* $\delta < \alpha$ *is a limit ordinal then* N_δ *is a singleton and* $P_\alpha^{N_\delta} = |N_\delta|$ *when* $(\sigma^1_\alpha, \sigma^2_\alpha) = (\mathrm{cf}(\delta), \theta)$*.*

4) Like (3) for an inverse, well-ordered sum except that in (e) we deduce $(\sigma^1_\alpha, \sigma^2_\alpha) = (\theta, \mathrm{cf}(\delta))$*.*

Proof. Easy. $\square_{2.20}$

Definition 2.21. For λ, μ_1, μ_2 as in 2.3(a) and $\theta = \mathrm{cf}(\theta) < \mathrm{Min}\{\mu_1, \mu_2\}, \mathbf{c} = \mathbf{c}^{\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ be as above. Let $K^{1-\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ be $K^{\mathrm{can}}_{\theta_1,\theta_2,u}$ for $\mathbf{c}$ recalling 2.18(4) where $u = \{\alpha\}$ with α such that $(\theta_1, \theta_2) = (\aleph_0, \aleph_0)$.

Let $K^{2-\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ be $K^{\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ for $\mathbf{i}$ where $n = \alpha_*(c)$.

Claim 2.22. *For* $\lambda, \mu_1, \mu_2, \theta$ *and* $\mathbf{c}$ *as above.*
1) There is an $M \in K^{1-\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ *unique up to isomorphism, it is a dense linear order of cardinality* λ *with cofinality and co-initiality* θ*.*
2) $K^{1-\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ *is closed under well ordered sums of length* $\alpha + 1 < \lambda^+$*.*
3) $K^{1-\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ *is closed under anti-well ordered sums of length* $\alpha + 1 < \lambda^+$*.*
4) If $\mu \ge \theta$ *and* $\mu \le \mu_1, \mu \le \mu_2$ *and* $M \in K^{1-\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$*, <u>then</u> for some algebra* $\mathfrak{B}$ *on* $|N|$ *with* μ *functions if* $\mathfrak{B}' \subseteq \mathfrak{B}$ *has cardinality* μ *then* $N' \restriction \mathfrak{B}' \in K^{1-\mathrm{can}}_{\mu,\mu^+,\mu^+,\theta}$*.*
5) M *also satisfies the conclusion of 2.28 and 2.16. E.g. Let* $N_i \in K^{\mathrm{can}}_{\lambda,\mu_1,\mu_2,\theta}$ *for* $i \le \alpha$ *and* $N = \sum\limits_{i \le \alpha} N_i$*. So for each* i *there is* $N_i^+ \in K^{\mathrm{hom}}_{\aleph_0,\aleph_0}$*. We can find* $\langle N_i^{++} : i \le \beta \rangle$ *and increasing* $g : \alpha + 1 \to \beta + 1$ *such that:*

- (a) *if* $N_i^+ = N^{++}_{g(i)}$
- (b) g *is increasing continuous and* $g(i+1) = g(i) + 2, g(0) = 0, g(\delta) = \delta$ *and* $h(\alpha) = \beta$
- (c) *if* $j \in \beta(*) \backslash \mathrm{Rang}(g)$ *then* N_j^{++} *is a singleton*
- (d) $\langle N_i^{++} : i \le \beta \rangle$ *is as in 2.20(3).*

Proof. Should be clear (check almost homogeneous), e.g.
4) Let $N_i \in K^{\rm can}_{\lambda,\mu_1,\mu_2,\theta}$ for $i \le \alpha$ and $N = \sum\limits_{i \le \alpha} N_i$. So for each i there is $N^+_i \in K^{\rm hom}_{\aleph_0,\alpha_0}$.
We can find $\langle N^{++}_i : i \le \beta\rangle$ and increasing $g : \alpha = 1 \rightarrow \beta + 2$ such that

- (a) if $N^+_i = N^{\tau^+}_{g(i)}$
- (b) g is increasing continuous and $g(i+2) = g(i)+2, g(0) = 0, g(\delta) = \delta$
- (c) if $j \in j(*)\backslash \mathrm{rang}(g)$ then N^{++}_j is a singleton
- (d) $\langle N^{++}_i : i \le \beta\rangle$ is as in 2.20(3).

Lastly, we apply 2.20(3). $\square_{2.22}$

Claim 2.23. *Assume $\lambda > \mu = \mathrm{cf}(\mu) > \theta, \mathbf{c} = \mathbf{c}^{\rm can}_{\lambda,\lambda^+,\mu,\theta}$ so $(\mu_1,\mu_2) = (\lambda^+,\mu)$ and $N \in K^{\rm hom}_{\lambda,\lambda^+,\mu,\theta}$ and $M = (N,<^N) \in K^{2-{\rm can}}_{\lambda,\lambda^+,\mu,\theta}$ and let $\sigma = \mathrm{cf}(\sigma) \in [\mu,\lambda)$ and $T = \mathrm{Th}_{\mathbb{L}_{\sigma,\sigma}}(M)$. For a model M' of T let $N' = (M',\dots,P^{M'}_\alpha,\dots)$ be defined as in 2.19(4).*
1) $N' \in K^{2-{\rm can}}_{\lambda_1,\lambda_1^+,\mu,\theta}$ when:

- (a) *M' is a model of T of cardinality $\lambda_1 \ge \sigma$*
- (b) *N' is 2-homogeneous (i.e. if $M' \models$ "$s_1 < t_2 \wedge s_2 < t_2$" and $s_1 \in P^{N'}_\alpha \Leftrightarrow s_2 \in P^{N'}_\alpha, t_1 \in P^{N'}_\alpha \Leftrightarrow t_2 \in P^{N'}_\alpha$) for $\alpha < \alpha_*(\mathbf{c})$ then there is an automorphism of N' (equivalently M') mapping (s_1,t_1) to (s_2,t_2)*
- (c) *M' is the countable union of scattered sets*
- (d) (α) *if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ if $\langle b_i : i < \kappa\rangle$ is an increasing bounded sequence in M then for some club E or κ, for every $\delta \in E \cup \{\kappa\}$, $\bar b{\restriction}\kappa$ has a $<^N$-lub*
  - (β) *similarly for M^*, the inverse of M.*

2) There is a first order sentence $\psi \in \mathbb{L}(\{<,F\}), F$ a three-place function symbol such that $\{\{<\} : N$ a model of T cup$\{\psi\}\}$ is equal to $\cup\{K^{2-{\rm can}}_{\lambda_1^+,\lambda_1,\mu} : \lambda_1 \ge \sigma\}$.
3) In part (1), if σ is a compact cardinal we can omit clauses (c),(d).
4) If σ is a compact cardinal, then the class from part (2) is categorical in every $\lambda_1 \ge \sigma$.

Proof. Should be clear. $\square_{2.23}$

* * *

We now make the connection to [Shef, §3].
We may weaken a little the definition of weakly κ–skeleton like (Definition [Shef, 3.1(1)=L3.1(1)]).

Claim 2.24. *Assume $\lambda > \kappa = \mathrm{cf}(\kappa)$, and $\mathbf{d}_\ell = \mathrm{inv}^\alpha_\kappa(I_\ell)$ for $\ell = 1,2$ (see Definition* [Shef, 3.4=L3.2]*), I_ℓ a linear order of cardinality $\le \lambda$, $\alpha < \lambda^+$ (for $\ell = 1,2$). Then there are α^*, μ_1,μ_2 (hence $\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}^+_1,\mathscr{F}^+_2$), g_1,g_2 as in Context 2.3 such that:*

- $(*)_1$ *if $M \in K_{g_1(0),g_2(0)}$ and $u \subseteq \alpha^*$ is non-empty then*
  - (a) $\mathrm{inv}^\alpha_\kappa(\bigcup\limits_{\epsilon\in u} P^M_\epsilon, <^M) = \mathbf{d}_1$

(b) $\mathrm{inv}^\alpha_\kappa(\bigcup_{\epsilon\in u} P^M_\epsilon, <^{M^*}) = \mathbf{d}_2$, *recalling* M^* *is the* M *inverted*

$(*)_2$ *if* $\mathbf{d}_1 = \mathbf{d}_2$ *and* $K' = \{(P^M_0, <^M) : M \in K_{g_1,g_2}\}$ *then*

(a) K' *is closed under sums of order type* α *and* α^* *for* $\alpha < \lambda^+$

(b) *each member of* K' *is cardinality* λ,

(c) K' *is almost* θ*-homogeneous for every* θ.

Proof. Straightforward. $\square_{2.24}$

Also in the cases we use skeletons from K^ω_{tr} we may like to realize distinct invariants rather than just non-isomorphic models.

Definition 2.25. 1) Let N be a model of cardinality λ with $|\tau_N| < \lambda$ we say $\bar N$ is a λ-representation (of N), or λ-filtration (of M) when:

(a) $\bar N = \langle N_\alpha : \alpha < \lambda\rangle$

(b) $N_\alpha \subseteq N$ has cardinality $< \lambda$

(c) $\bar N$ is $\subseteq$-increasing continuous

(d) $N = \cup\{N_\alpha : \alpha < \lambda\}$.

2) For a λ-representation $\bar N$ let (on splitting see below)

$$\begin{aligned}\mathrm{Sp}(\bar N) = \{\delta < \lambda :\quad & \delta \text{ is limit, and for some } \bar a \in \bigcup_{\alpha<\lambda} N_\alpha \\ & \text{for every } \beta < \delta, \mathrm{tp}(\bar a, N_\delta, N) \text{ splits over } M_\beta\}.\end{aligned}$$

3) $\mathrm{Sp}_{\Delta_1,\Delta_2}(\bar N) = \{\delta < \lambda : \delta$ limit, and for some $\bar a \in \bigcup_{\alpha<\lambda} N_\alpha$ for every $\beta < \delta$ the type $\mathrm{tp}_{\Delta_1}(\bar a, N_\delta, N)$ does (Δ_1,Δ_2)-splits over $M_\beta\}$.
4) Let $\mathrm{Sp}(N)$ be $\mathrm{Sp}(N)/\mathscr{D}_\lambda$ for every representation of M. Similarly $\mathrm{Sp}_{\Delta_1,\Delta_2}(N)$; both are justified because

$\boxplus$ Sp is $\check{\mathscr{D}}_\lambda$-invariant of N, i.e. if $\bar N', \bar N''$ are λ-representations of N; $\|N\| = \lambda$ then $\mathrm{Sp}(\bar N')/\check{\mathscr{D}}_\lambda = \mathrm{Sp}(\bar N'')/\check{\mathscr{D}}_\lambda$ and $\mathrm{Sp}_{\Delta_1,\Delta_2}(\bar N')/\check{\mathscr{D}}_\lambda = \mathrm{Sp}_{\Delta_1,\Delta_2}(\bar N'')/\check{\mathscr{D}}_\lambda$ (when $\lambda = \mathrm{cf}(\lambda) > \aleph_0$).

5) We say that $\mathrm{tp}_{\Delta_1}(a, B, N)$ does (Δ_1,Δ_2)-split over $A \subseteq N$ (where $\bar a \in M, B \subseteq N$) if for some $\bar b_1, \bar b_2 \in B, \mathrm{tp}_{\Delta_2}(\bar b_1, A, N) = \mathrm{tp}_{\Delta_2}(\bar b_2, A, N)$ but $\mathrm{tp}_{\Delta_1}(\bar a\char`^\bar b_1, A, N) \neq \mathrm{tp}_{\Delta_1}(\bar a\char`^\bar b_2, A, N)$.
6) If $\Delta_1 = \Delta_2$ is $\mathbb{L}_{\omega,\omega}(\tau(M))$, we may omit (Δ_1,Δ_2).
7) We can replace $\check{\mathscr{D}}_\lambda$ by appropriate $\mathscr{E}$ giving us an ω–sequence of sets (or an appropriate filters on the set).

Definition 2.26. 1) N is $(\lambda,\Delta_1,\Delta_2)$-nice if $\mathrm{Sp}_{\Delta_1,\Delta_2}(N) = \emptyset/D_\lambda$.
2) N is $(<\lambda,\Delta)$-stable if for every $A \subseteq |N|$ of power $< \lambda$

$$\lambda > |\{\mathrm{tp}_\Delta(\bar a, A, N) : \bar a \in |M|\}|.$$

3) $I \in K^\omega_{\mathrm{tr}}$ is *locally* $(\lambda,\mathrm{bs},\mathrm{bs})$-nice [*locally* $(<\lambda,\mathrm{bs})$-stable] if for every $\eta \in I \setminus P^I_\omega$ the linear order $(\mathrm{Suc}_I(\eta), <)$ is $(\lambda,\mathrm{bs},\mathrm{bs})$-nice [$(<\lambda,\mathrm{bs})$-stable].

Claim 2.27. *Every* $M \in K$ *is* $(\lambda,\mathrm{bs},\mathrm{bs})$*-nice and* $(<\lambda, bs)$*-stable.*

Proof. Easy (and as in [She82a, §6], mainly "crucial fact" of pg.217 there). $\square_{2.27}$

Claim 2.28. *If* $(A, <, P_\alpha)_{\alpha<\alpha(*)} \in K, S \subseteq \lambda$, *and*

$$M = (\bigcup_{\alpha\in S} P_\alpha, < \restriction (\bigcup_{\alpha\in S} P_\alpha), P_\alpha)_{\alpha\in S},$$

then M *is* $(<\lambda, \mathrm{bs})$*-stable and* $(\lambda, \mathrm{bs}, \mathrm{bs})$*-nice.*

Proof. Check. $\square_{2.28}$

§ 2(C). Very Homogenous Linear Orders Revisited.

We here start to indicate how we can generalize §(2A). The case $\kappa = \aleph_0$ is the one in §(2A).

Definition 2.29. 1) We say $\mathbf{c}$ is a context or (λ, κ)-context when it consists of (so $\lambda = \lambda_{\mathbf{c}}$, etc.)

- (a) $\lambda = \lambda^{<\kappa} \geq \kappa = \mathrm{cf}(\kappa)$
- (b) $\alpha_* < \lambda^+, u_1 \subseteq \alpha_*, u_2 \subseteq \alpha_*, u_1 \cup u_2 \neq \alpha_*$ (or just $u_1 \cap u_2 \neq \alpha_*$), note here 1,2 stands for right, left
- (c) vocabulary $\tau = \{<\} \cup \{P_\alpha : \alpha < \alpha_*\}$, where $<$ is a binary predicate, P_α is a unary predicate
- (d) K_{all} is the class of N such that (all stands for all)
  - (α) N is a τ-model
  - (β) $<^N$ a linear order
  - (γ) $\langle P^N_\alpha : \alpha < \alpha_*\rangle$ a partition of N
  - (δ) if $\partial = \mathrm{cf}(\partial) < \kappa$ and $\bar{a} = \langle a_i : i < \partial\rangle$ is increasing/decreasing then it has a $<_N$-lub/$<_N$-mdb; moreover if $\partial > \aleph_0$ then for a club of $\delta < \partial$ this holds for $\bar{a}\restriction\delta$, too
- (e) g_ℓ is a ℓ-nice function from u_ℓ into $\mathscr{F}^*_\ell$, see below
- (f) $K_{\mathrm{nice}} \subseteq K_{\mathrm{all}}$ is defined in part (5) below (nice stands for nice)
- (g) $K_{\mathrm{bas}} \subseteq K_{\mathrm{nice}}$ has cardinality $\leq \lambda$ and each $N \in K_{\mathrm{bas}}$ has cardinality $\leq \lambda$ and some $N \in K_{\mathrm{bas}}$ has cardinality λ (bas stands for basic, the generators).

2) $\mathscr{F}_\ell = \mathscr{F}^\ell_{\mathbf{c}}$ is the set of function f with domain a regular $\partial \leq \lambda$ into α_* such that any limit $\delta < \partial, f(\delta) \in u_\ell$.
3) $g_\ell : \alpha_* \to \mathscr{F}_\ell$ is ℓ-nice when

- (a) for every $\alpha < \alpha_*, h := g(\alpha)$ is (g_ℓ, ℓ)-nice, see below
- (b) if $\partial = \mathrm{Dom}(g_\ell(\alpha_1)) < \kappa, g_\ell(\alpha) = g_2(\beta)$ then $\alpha = \beta$
- (c) if $h : \partial \to \alpha_*$ is (g_ℓ, ℓ)-nice and $\partial < \kappa$ then $h \in \mathrm{Rang}(g)$.

4) $h \in \mathscr{F}_\ell$ is (g, ℓ)-nice when: if $\partial = \mathrm{Dom}(h)$ is regular then

- $\boxdot^\ell_h$ $h \in \mathscr{F}_\ell$ and if $\delta < \mathrm{Dom}(h)$ is a limit ordinal of uncountable cofinality and $\beta = h(\delta)$ and $\langle \epsilon_i : i < \mathrm{cf}(\delta)\rangle$ is an increasing continuous sequence with limit δ then $\{i < \mathrm{cf}(\delta) : (h(\delta))(\epsilon_i) = (g(\beta))(i)\}$ contains a club of $\mathrm{cf}(\delta)$. For notational simplicity assume $\alpha^* \leq \lambda$.

5) K_{nice} is the class of N such that

(a) $N \in K_{\rm all}$ has cardinality $\leq \lambda$

(b) if $a \in P^N_\alpha$ and $\alpha \in u_1$ and a has no immediate predecessor in N, then there is an increasing sequence $\langle b_\alpha : \alpha \in {\rm Dom}(g_1(\alpha))\rangle$ in N such that

  (α) if α is a limit ordinal then b_α is the $<^*_N$-lub of $\langle b_\beta : \beta < \alpha\rangle$

  (β) if ${\rm Dom}(g_1(\alpha))$ is uncountable then $\{\alpha < {\rm Dom}(g_1(\alpha)) : b_\alpha \in P^N_{g_1(\alpha)}\}$ contains a club of ${\rm Dom} g_1(\alpha)$

  (γ) if α is non-limit then $g_1(\alpha) \notin u_1 \cup u_2$

(c) like (b), replacing u_1, g_1 increasing, predecessor, lub by u_2, g_2 decreasing, successor, glb.

Convention 2.30. In this sub-section, $\mathbf{c}$ will be a fixed context, if not said otherwise.

Definition 2.31. We define a two-place relation $\leq_{\rm nice}$ on $K_{\rm nice}$, $N_1 \leq_{\rm nice} N_2$ iff

(a) $N_1, N_2 \in K_{\rm nice}$

(b) $N_1 \subseteq N_2$

(c) if $a \in P^{N_1}_\alpha$ and $\alpha \in u_{\mathbf{c},1}$ and a has no immediate predecessor in N_1, then $(N_1)_{<a}$ is unbounded in $(N_2)_{<a}$ from above

(d) if $a \in P^{N_1}_\alpha, \alpha \in u_{\mathbf{c},2}$ and a has no immediate successor in N_1, then $(N_1)_{<a}$ is unbounded in $(N_2)_{>a}$ from below

(e) if $a \in N_2 \backslash N_1$, then $(N_2)_{<a} \cap N_1$ has a last element or $(N_2)_{>a} \cap N_1$ has a first element.

Claim 2.32. $(K_{\rm nice}, \leq_{\rm nice})$ *is a partial order preserved under isomorphisms.*

§ 3. On pcf and other uncountable combinatorics

In this section we define and quote but do not prove.

Definition 3.1. 1) For λ regular uncountable we define the weak diamond ideal, $\check{I}^{\rm wd}_\lambda = \check{I}^{\rm wd}[\lambda]$ as the family of small subset of λ, where:
2) We say $S \subseteq \lambda$ is small <u>if</u> it is $\mathbf{F}$-small for some colouring function $\mathbf{F}$ from ${}^{\lambda>}\lambda$ to θ where
3) We say $S \subseteq \lambda$ is $\mathbf{F}$ small <u>if</u> ($\mathbf{F}$ is as above and)

$(*)_S$ for every $\bar{c} \in {}^S 2$ for some $\eta \in {}^\lambda\lambda$ the set $\{\delta \in S : \mathbf{F}(\eta{\restriction}\delta) = c_\delta\}$ is not stationary.

Claim 3.2. *If $\lambda = \mu^+, 2^\lambda > 2^\mu$ or at least $2^\mu = 2^{<\lambda} < 2^\lambda$ (λ is regular uncountable) then $\lambda \notin \{\check{I}^{\rm wd}\}_\lambda$.*

Proof. By [DS78], see more in [She98, AP,§1,pgs.942-961]. $\square_{3.2}$

Remark 3.3. 1) Used in [Shed, 6.4=constr6.4=f12,stage C].
2) On $\check{I}^{\rm gd}[\lambda]$ see [Sheb, 3.8,3.9].

Definition 3.4. For λ regular uncountable let $\check{I}[\lambda] = \check{I}^{\rm gd}[\lambda]$ be the family of sets $S \subseteq \lambda$ which have a witness $(E, \bar{\mathscr{P}})$ for $S \in \check{I}^{\rm gd}[\lambda]$, which means

$(*)$ E is a club of $\lambda, \mathscr{P} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle, \mathscr{P}_\alpha \subseteq \mathscr{P}(\alpha), |\mathscr{P}_\alpha| < \lambda$, and for every $\delta \in E \cap S$ there is an unbounded subset C of δ of order $< \delta$ such that $\alpha \in C \Rightarrow C \cap \alpha \in \mathscr{P}_\alpha$.

Claim 3.5. *Let λ be regular uncountable.*
1) For $S \subseteq \lambda, S \in \check{I}^{\rm gd}[\lambda]$ <u>iff</u> equivalently there is a pair $(E, \bar{a})$, E is a club of $\lambda, \bar{a} = \langle a_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \alpha, \beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$ and $\delta \in E \cap S \Rightarrow \delta = \sup(a_\delta) > \operatorname{otp}(a_\delta)$ (or even $\delta = \sup(a_\delta), \operatorname{otp}(a_\delta) = \operatorname{cf}(\delta) < \delta$.
2) If $\kappa < \lambda$ are regular, <u>then</u> there is a stationary $S \subseteq S^\lambda_\kappa$ in $\check{I}^{\rm gd}[\lambda]$.

Remark 3.6. Used in [Shed, §4].

Definition 3.7. 1) For an ideal $\mathbf{I}$ on I and $f \in {}^\theta(\mathrm{Ord} \setminus \{0\})$ let $T_{\mathbf{I}}(f) = \sup\{|\mathscr{F}| : \mathscr{F} \subseteq \prod\limits_{t \in I} f(t)$ and $h \neq g \in \mathscr{F}$ implies $\{t \in I : h(t) \neq g(t)\} \in \mathbf{I}\}$, generally $T_{\mathbf{I}}(f) = \sup\{|\mathscr{F}| : \mathscr{F} \in \Xi\}$ where Ξ is the set of $\mathscr{F}$ such that:

(*a*) $\mathscr{F} \subseteq {}^I\mathrm{Ord}$
(*b*) $g \in \mathscr{F}$ implies $g <_{\mathbf{I}} h$
(*c*) $h \neq g \in \mathscr{F}$ implies $\{t \in I : h(t) \neq g(t)\} \in \mathbf{I}$,

(if $(\forall t) f(t) \geq 2^\kappa$ the supremum is obtained and only $f/\mathbf{I}$ matters).
1A) We may replace $\mathbf{I}$ by the dual ideal.
2) For a partial order $\mathbb{P}$ let $\operatorname{tcf}(\mathbb{P})$, the true cofinality of $\mathbb{P}$ be equal to λ <u>when</u> λ is a regular cardinal and some sequence $\langle p_\alpha : \alpha < \lambda\rangle$ witness this which means:

- $\alpha < \beta < \lambda \Rightarrow p_\alpha <_{\mathbb{P}} p_3$
- if $q \in \mathbb{P}$ then for some $\alpha < \lambda$ we have $q <_{\mathbb{P}} p_\alpha$.

Claim 3.8. *Assume that $\langle 2^{\lambda_i} : i < \delta\rangle$ is strictly increasing and $\mu = \sum\langle\lambda_i : i < \delta\rangle < 2^{\lambda_0}$. Then for arbitrarily large regular cardinals $\lambda < \mu$ there is tree with $< \mu$ nodes and $\geq 2^{\lambda_0}$, κ-branches (hence a linear order of cardinality $< \mu$ with $\geq 2^{\lambda_0} > \mu$ Dedekind cuts with both cofinality exactly λ).*

Remark 3.9. This is used in [Shee, 3.28=L3c.16] and will be used in proving the properties from [Shec].

Proof. By [She96, 3.4]. $\square_{3.8}$

Claim 3.10. *Assume:*

(*A*) $\lambda = \mathrm{cf}(\lambda) \geq \mu > 2^\kappa$,
(*B*) $\dot{D}$ *is a μ-complete*[2] *filter on* λ,
(*C*) $f_\alpha : \kappa \to \mathrm{Ord}$ *for* $\alpha < \lambda$,
(*D*) $\dot{D}$ *contains the co-bounded subsets of* λ.

Then
0) We can find $w \subseteq \kappa$ and $\bar{\beta}^* = \langle\beta^*_i : i < \kappa\rangle$ such that: $i \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta^*_i) > 2^\kappa$ and for every $\bar{\beta} \in \prod\limits_{i\in\kappa\setminus w}\beta^*_i$ for λ ordinals $\alpha < \lambda$ (even a set in $\check{\mathscr{D}}^+$) we have $\bar{\beta} < f_\alpha{\restriction}(\kappa\setminus w) < \bar{\beta}^*{\restriction}(\kappa\setminus w)$, $f_\alpha{\restriction}w = \bar{\beta}^*{\restriction}w$, and $\sup\{\beta^*_j : \beta^*_j < \beta^*_i\} < f_\alpha(i) < \beta^*_i$.
1) We can find a partition $\langle w^*_\ell : \ell < 2\rangle$ of $\kappa, X \in \check{\mathscr{D}}^+$ and $\langle A_i : i < \kappa\rangle, \langle\bar{\lambda}_i : i < \kappa\rangle, \langle h_i : i < \kappa\rangle, \langle n_i : i < \kappa\rangle$ such that:

(*a*) $A_i \subseteq \mathrm{Ord}$,
(*b*) $\bar{\lambda}_i = \langle\lambda_{i,\ell} : \ell < n_i\rangle$ *and* $2^\kappa < \lambda_{i,\ell} \leq \lambda_{i,\ell+1} \leq \lambda$,
(*c*) h_i *is an order preserving function from* $\prod\limits_{\ell<n_i}\lambda_{i,\ell}$ *onto* A_i *so* $n_i = 0 \Leftrightarrow |A_i| = 1$. *(The order on* $\prod\limits_{\ell<n_i}\lambda_{\ell,i}$ *being lexicographic,* $<_{\ell x}$*),*
(*d*) $i < \kappa$ *and* $\alpha \in X \Rightarrow f_\alpha(i) \in A_i$, *and we let* $f^*_\alpha(i,\ell) = [h_i^{-1}(f_\alpha(i))](\ell)$, *so* $f^*_\alpha \in \prod\limits_{\substack{i<\kappa\\ \ell<n_i}}\lambda_{i,\ell}$,
(*e*) $i \in w^*_0 \Leftrightarrow n_i = 0$ *(so* $|A_i| = 1$*),*
(*f*) *if* $i \in w^*_1$ *then* $|A_i| \leq \lambda$, *hence* $|\bigcup\limits_{i\in w^*_1}A_i| \leq \lambda$,
(*g*) *if* $g \in \prod\limits_{\substack{i<\kappa\\ \ell<n_i}}\lambda_{i,\ell}$ *then* $\{\alpha \in X : g < f^*_\alpha\} \in \check{\mathscr{D}}^+$ *and letting* $\beta^*_j = \sup\mathrm{Rang}(h_i)$, *clearly the condition of part* $(\gamma)(0)$ *holds*
(*h*) *if* $\dot{D}$ *is* $(|\alpha|^\kappa)^+$*-complete for any* $\alpha < \mu_1$ *then* $\mu_1 \leq \sup\{\lambda_{i,\ell} : i \in w^*_1$*; and* $\ell < n_i\} \leq \lambda$ *when* $w^*_1 \neq \emptyset$ *(so, e.g., if* $\mu = \lambda$ *and assuming* GCH
$$\sup\{\mathrm{cf}(\lambda_{i,\ell}) : i \in w^*_1 \text{ and } \ell < n_i\} = \lambda).$$

2) In part (1) we can add $(*)_1$ to the conclusion if (E) below holds,

$(*)_1$ *if* $\lambda_{i,\ell} \in [\mu, \lambda)$ *then* $\lambda_{i,\ell}$ *is regular.*
(*E*) *For any set* $\mathfrak{a}$ *of* $\leq \kappa$ *singular cardinals from the interval* (μ, λ), *we have* $\max\mathrm{pcf}\{\mathrm{cf}(\chi) : \chi \in \mathfrak{a}\} < \lambda$.

[2]in parts (0),(1),$\mu = (2^\kappa)$ is O.K.

3) Assume in part (1) that (F) below holds. Then we can demand $(*)_2$.

$(*)_2$ $\lambda^i_\ell \geq \mu_1$ *for* $i \in w_2, \ell < n_i$.

(F) $\mathrm{cf}(\mu_1) > \kappa$ *and* $\alpha < \mu_1 \Rightarrow \dot{D}$ *is* $[|\alpha|^{\leq \kappa}]^+$*-complete.*

4) If in part (1) in addition (G) below holds, then we can add:

$(*)_3$ $\lambda \in \mathrm{pcf}_{\partial-\mathrm{complete}}\{\lambda^i_\ell : i \in w^*_1;$ *and* $\ell < n_i\}$ *if* $w^*_1 \neq \emptyset$*, moreover*

$(*)_4$ *if* $\ell_i < n_i$ *for* $i \in w^*_1$ *then* $\lambda \in \mathrm{pcf}_{\partial-\mathrm{complete}}\{\mathrm{cf}(\lambda^i_{\ell_i}) : i \in w^*_1\}$.

(G)

- (i) $(\forall \alpha < \lambda)(|\alpha|^{<\partial} < \lambda)$ *and* $\partial = \mathrm{cf}(\partial) > \aleph_0$,
- (ii) $\dot{D}$ *is* λ*-complete*
- (iii) $f_\alpha \neq f_\beta$ *for* $\alpha \neq \beta$ *(or just* $\alpha \neq \beta \in X$ *for some* $X \in \dot{D}^+$*)*

5) If in part (1) in addition (H) below holds then we can add :

$(*)_5$ *if* $m < m^*, A \in \mathbf{J}_m$ *and* $\ell_i < n_i$ *for* $i \in \kappa \setminus A$ *(so* $w^*_0 \subseteq A$*) then* $\lambda \in \mathrm{pcf}\{\lambda^i_{\ell_i} : i \in \kappa \setminus A\}$.

(H)

- (i) $m^* < \omega$ *and* $\mathbf{J}_m$ *is an* $\aleph_1$*-complete ideal on* κ *for* $m < m^*$,
- (ii) $\dot{D}$ *is* λ*-complete.*

Proof. By [She99, 7.1=L7.0]. $\square_{3.10}$

Claim 3.11. *Assume that* $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ *is a sequence of regular cardinals* $> \mu$ *and* J *is an ideal of* κ *and* λ *is a regular cardinal.*
1) If $\prod_{i<\kappa} \lambda_i/\mathbf{J}$ *is* λ^+*-directed then we can find* $\lambda'_i = \mathrm{cf}(\lambda'_i) \in (\mu, \lambda_i)$ *such that:*

- (a) $\prod_{i<\kappa} \lambda'_i/\mathbf{J}$ *has true cofinality* λ
- (b) *if* $\lambda > \lim_{\mathbf{J}}\langle \lambda_i : i < \kappa \rangle = \mu_* > \mathrm{cf}(\mu_*)$ *then* $\lim_{\mathbf{J}}\langle \lambda'_i : i < \kappa \rangle = \mu^*$
- (c) *there is an* $<_{\mathbf{J}}$*-increasing sequence* $\langle f_\alpha : \alpha < \lambda \rangle$ *of members of* $(\prod_{i<\kappa} \lambda_i, <_{\mathbf{J}})$ *and is* μ^+_**-free, i.e. if* $A \subseteq \lambda, |A| \leq \mu_*$ *then there is a sequence* $\langle u_\alpha : \alpha \in A \rangle$ *such that* $u_\alpha \in J$ *and* $\alpha \in A$ *and* $\beta \in A$ *and* $\alpha < \beta$ *and* $\epsilon \in \kappa \setminus u_\alpha \setminus u_\beta \Rightarrow f_\alpha(\epsilon) < f_\beta(\epsilon)$.

Remark 3.12. Used in [Shea, 1.16=L7.7].

Proof. By [She96, §6]. $\square_{3.11}$

Theorem 3.13. *1) Assume that* $\mu = \mu^{<\kappa} < \lambda \leq 2^\mu$ *then there is a sequence* $\langle f_i : i < \mu \rangle$ *of functions from* λ *to* μ *such that for every* $u \subseteq \mu$ *of cardinality* $< \kappa$ *of function* g *from* u *to* μ*, for some* $i < \mu$ *we have* $g \subseteq f_i$.

Remark 3.14. Used in [Shea, 1.11=L7.6].

Proof. This is Engelking-Karlowic [EK65]. $\square_{3.13}$

Theorem 3.15. *(Hajnal free subset theorem). If* $f : \lambda \to [\lambda]^{<\kappa}$ *and* $\lambda > \kappa \geq \aleph_0$ *then some* $A \in [\lambda]^\lambda$ *is* f*-free which means that* $\alpha \neq \beta \in A \Rightarrow \alpha \notin f(\beta)$.

Proof. This is [Haj62]. $\square_{3.15}$

Definition 3.16. 1) For μ singular let $\mathrm{pp}(\mu) = \sup\{\lambda$: for some filter $\mathbf{J}$ on $\mathrm{cf}(\mu)$ and sequence $\langle \lambda_i : i < \mathrm{cf}(\mu)\rangle$ of regular cardinals $< \mu$ such that $\mu' < \mu \Rightarrow \{i : \lambda_i > \mu'\} \in \mathbf{J}$, the product $\prod\limits_{i<\mathrm{cf}\mu} \lambda_i/\mathbf{J}$ has true cofinality $\lambda\}$.

For μ singular $\mathrm{pp}^+(\mu) = \mathrm{Min}\{\lambda : \lambda$ regular and there are no $\mathbf{J}$ and λ_i as above$\}$.
2) For a set $\mathfrak{a}$ of regular cardinals $\geq |\mathfrak{a}|$ let $\mathrm{pcf}(\mathfrak{a}) = \{\mathrm{cf}(\prod\limits_{\theta\in\mathfrak{a}} (\theta, <)/\dot{D}) : \dot{D}$ an ultrafilter on $\mathfrak{a}\}$.
3) If $\mathfrak{a}$ is as above, $\mathbf{J}$ is an ideal on $\mathfrak{a}$ then we let $\mathrm{pcf}_{\mathbf{J}}(\mathfrak{a}) = \{\mathrm{cf}(\Pi\mathfrak{a}/\dot{D}) : \dot{D}$ is an ultrafilter on $\mathfrak{a}$ disjoint to $I\}$.

Remark 3.17. Used in [Shea, 1.16=L7.7].

Remark 3.18. Used in [Shea, 2.15=L7.9].

Claim 3.19. *If $\mu \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$. Then there is a stationary $\mathscr{S} \subseteq [\mu]^{<\kappa}$ of cardinality* $\mathrm{cf}([\mu]^{<\kappa}, \subseteq)$.

Remark 3.20. Used in [Shed, 5.3], stage E statement of $\otimes_3$.

Proof. By [She93, §1]. $\square_{3.19}$

Claim 3.21. *Assume that λ is singular of uncountable cofinality $\kappa, \langle \lambda_i : i < \kappa\rangle$ is increasing continuous with limit λ and $S = \{\delta < \kappa : \mathrm{pp}(\lambda_\delta) = \lambda^+_\delta\}$ is a stationary subset of κ then $\mathrm{pp}(\lambda) = \lambda^+$.*

Proof. By [She94, Ch.II,§2]. $\square_{3.21}$

We repeat [Shear, Ch.IX,3.7,pg.384,5]

Claim 3.22. *Suppose $\lambda = \aleph_{\alpha(*)+\delta}, \delta$ a limit ordinal $< \aleph_{\alpha(*)}$.*
1) $\mathrm{pp}(\lambda) =^+ \mathrm{cov}(\lambda, \lambda, \mathrm{cf}(\lambda)^+, 2)$.
2) If $\mathrm{cf}(\delta) \leq \kappa \leq \delta$ then $\mathrm{pp}_\kappa(\lambda) =^+ \mathrm{cov}(\lambda, \lambda, \kappa^+, 2)$.
3) If $\mathrm{cf}(\delta) = \kappa, (\aleph_{\alpha(*)+i})^\kappa < \aleph_{\alpha+\delta}$ for $i < \delta$ then
lambda$^\kappa = \mathrm{pp}(\lambda)$.
4) If $\mathrm{cf}(\delta) = \kappa, (\aleph_{\alpha(*)})^\kappa < \lambda$ then

$$\lambda^\kappa = \sum\{\mathrm{pp}(\aleph_{\alpha(*)+i}) : i \leq \delta \textit{ limit}, \mathrm{cf}(i) \leq \kappa\}.$$

5) $\mathscr{S}_{<\aleph_{\alpha(*)+1}}(\lambda)$ has a stationary subset of cardinality

$$\sum\{\mathrm{pp}(\aleph_{\alpha(*)+i}) : i \leq \delta \textit{ limit}\}.$$

Claim 3.23. *Assume $\mu > \kappa = \mathrm{cf}(\mu)$. There is an increasing sequence $\langle \lambda_i : i < \kappa\rangle$ of regular cardinals $< \mu$ and $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$ when*

$\circledast$ (a) $\lambda = \mathrm{cf}(\lambda) \in (\mu, \mathrm{pp}^+_\kappa(\mu))$
$(b)_1$ $\mu < \mu^{+\kappa}$ *or*
$(b)_2$ $\kappa > \aleph_0$ *and for some $\mu_0 < \mu$ for every $\mu' \in (\mu_0, \mu)$ of cofinality $< \kappa$ we have* $\mathrm{pp}(\mu') < \mu$.

Remark 3.24. 1) Used in [Shea, 1.16=L7.7], [Shea, 2.20=7.11], [Shea, 3.23=L7.14], [Shea, 3.24=L7.7].
2) It is helpful in applying [Shea, 2.13=L7.8I].

Proof. By [She94, Ch.VIII,§1]. $\square_{3.23}$

§ 4. On normal ideals

The results here are from [She86].

Theorem 4.1. *If $\check{\mathscr{D}}$ is a fine normal filter on $\mathscr{U} = \{u \subseteq \lambda : \mathrm{cf}(\sup(u)) \neq \mathrm{cf}(|u|)\}$, and λ is regular then there are functions f^*_i for $i < \lambda^+$ such that:* $\mathrm{Dom}(f^*_i) = \mathscr{U}$, *$f^*_i(u) \in u$ and for $i \neq j, \{u \in I : f^*_i(u) = f^*_j(u)\} = \emptyset \mod \check{\mathscr{D}}$.*

Remark 4.2. 1) Used in [Shea, 2.15=L7.9].
2) So $\mathscr{U} = [\lambda]^{\aleph_1}$ is an interesting case.
3) This is a strong form of "not λ^+-saturated".

Proof. We can find $A_i (i < \lambda^+)$ such that:

$(*)_1$ A_i is a subset of λ, unbounded in λ and for $j < i, A_i \cap A_j$ is bounded in λ

[e.g. let $A_i(i < \lambda)$ be pairwise disjoint subsets of λ of power λ, and then choose $A_i(\lambda \leq i < \lambda^+)$ by induction on i on such that the relevant demands hold. Assuming to $i \in [\lambda, \lambda^+)$ let $\{j : j < i\}$ be listed as $\{j^i_\alpha : \alpha < \lambda\}$, and let $A_i = \{\gamma^i_\beta : \beta < \lambda\}$ where $\gamma^i_\beta = \mathrm{Min}(A_{j_\beta} \setminus \bigcup\limits_{\alpha<\beta} A_{j_\alpha})$, listed without repetitions it exists as $|A_{j_\beta} \cap A_{j_\alpha}| < \lambda = \mathrm{cf}(\lambda)$ for $\alpha < \beta$].

For $i < \lambda^+$ let $g_i : i \to \lambda$ be such that $\{A_j \setminus g_i(j) : j < i\}$ are pairwise disjoint. Let f_i be the strictly increasing function from λ onto A_i (for $i < \lambda^+$) hence $\alpha < \lambda \Rightarrow f_i(\alpha) \geq \alpha$. So $C_i = \{u \in \mathscr{U} : u$ is closed under f_i and $\alpha \in u \Rightarrow \alpha + 1 \in u\}$ belongs to $\check{\mathscr{D}}$. For each $u \in \mathscr{U}$ let $u = \{x^u_\alpha : \alpha < |u|\}$.

Now for each $u \in C_i$ the set $u \cap A_i$ is unbounded in u, (by the choice of C_i and f_i) so for some $\alpha_i(u) < |u|$, the set $A_i \cap \{x^u_\alpha : \alpha < \alpha_i(u)\}$ is unbounded in u. (Why? Recall that $\mathrm{cf}(\sup u) \neq \mathrm{cf}(|u|)$ because $u \in \mathscr{U}$).

Next for $i < \lambda^+$ let h_i be a one-to-one function from λ onto $\lambda \cup \{j : j < i\}$ and define by induction on i:

$$\begin{aligned} C^1_i = \{u \subseteq i \cup \lambda : \quad & u \text{ is closed under } h_i, h_i^{-1} \text{ and } u \cap \lambda \in \mathscr{U} \\ & u \cap \lambda \text{ is closed under } f_i, f_i^{-1}, \\ & u \text{ is closed under } g_j, (\text{when } j \in u \text{ or } j = i) \\ & \text{and for every } j \in u \text{ we have } u \cap (j \cup \lambda) \in C^1_j\}. \end{aligned} \tag{4.1}$$

Clearly $C^1_i \restriction \lambda = \{u \cap \lambda : a \in C^1_i\}$ belongs to $\check{\mathscr{D}}$, and by the choice of h_i for each $u \in \mathscr{U}$ there is at most one $u' \in C^1_i$ satisfying $u' \cap \lambda = a$, namely h_i"(u).

Now we define for $i < \lambda^+$ a functions ξ_i and d_i with domain $\mathscr{U}$.

$$\xi_i(u) = \mathrm{otp}(\{j \in h_i(u) : \alpha_j(u) = \alpha_i(u)\},$$

$d_i(u) = (\alpha_i(u), \xi_i(u))$ if $h_i(u) \cap \lambda = u$ and $h_i(u) \in C^1_i$ and $d_i(u) = \mathrm{Min}(u)$ otherwise.

Now we shall finish by showing:

(*A*) for $i_1 \neq i_2 < \lambda^+$ we have $\{u \in \mathscr{U} : d_{i_1}(u) = d_{i_2}(u)\} = \emptyset \mod \check{\mathscr{D}}$
(*B*) for $a \in \mathscr{U}, \{d_i(u) : i < \lambda^+\}$ has cardinality $\leq |u|$.

Why does this suffice? As for each $u \in \mathscr{U}$ by clause (B) we can find a one-to-one function $\mathbf{f}_u$ from $\{d_i(u) : i < \lambda^+\}$ into u and now use the λ^+ functions $\langle \mathbf{f}_u(d_i(u)) : i < \lambda^+\rangle$, that is for $i < \lambda^+$ we define the function f^*_i with domain $\mathscr{U}$ such that

$f^*_i(u) \in u$ by $f^*_i(u) =: \mathbf{f}_u(d_i(u))$, now by clause (A) we have $i < j < \lambda^+ \Rightarrow f^*_i \neq f^*_j$ mod $\check{\mathscr{D}}$.

Proof of Clause (A):

Without loss of generality $i_1 < i_2$ and we assume that $\lambda \leq i_1$ for notational simplicity. Clearly $\mathscr{U}' := \{u \in \mathscr{U} : h_{i_2}(u) \in C^1_{i_2}$ and $i_1 \in h_{i_2}(u)$ (hence $h_{i_1}(u) = h_{i_2}(u) \cap i_1 \in C^1_{i_1}$)$\}$ belongs to $\check{\mathscr{D}}$. Let u be in it, and assume that $d_{i_1}(u) = d_{i_2}(u)$. For $\ell = 1, 2$ in the definition of $d_{i_\ell}(u)$ the first case applies so $d_{i_\ell}(u) = (\alpha_{i_\ell}(u), \xi_{i_\ell}(u))$ hence by the first coordinate $\alpha_{i_1}(u) = \alpha_{i_2}(u)$. Now $\{\xi \in h_{i_1}(u) : \alpha_\xi(u) = \alpha_{i_1}(u)\}$ is an initial segment of $\{\xi \in h_{i_2}(u) : \alpha_\xi(u) = \alpha_{i_2}\}$ (as $a \in \mathscr{U}'$) and a proper one (as i_1 belongs to the latter but not the former). As the ordinals are well ordered, the order types $\xi_{i_1}(u), \xi_{i_2}(u)$ are not equal. That means that the second coordinates in the $d_{i_1}(u), d_{i_2}(u)$ are distinct. So $d_{i_1}(u) \neq d_{i_2}(u)$ is true when $i_1 \neq i_2, a \in \mathscr{U}'$ as required.

Proof of Clause (B):

The number of possible $\alpha_i(u)$ is $\leq |u|$, and the number of order types of well orderings of power $< |u|$ is $|u|$ hence by $(*)$ below, the number of pairs $(\alpha_i(u), \xi_i(u))$ is $\leq |a| \times |u| = |u| + \aleph_1$ and recalling the additional value $\mathrm{Min}(u)$ we are done. So it suffices to prove:

- $(*)$ for $i < \lambda^+, u \in C^1_i$, the set $w = \{j \in u : \alpha_j(u \cap \lambda) = \alpha_i(u \cap \lambda)\}$ has power $< |u|$.

Why $(*)$ holds? Clearly for $j \in w$ the set

$$A_j \cap \{x^u_\alpha : \alpha < \alpha_i(u \cap \lambda)\}$$

is unbounded in $u \cap \lambda$ but $A_j \cap g_i(j)$ is bounded in $u \cap \lambda$ (as u is closed under g_i) hence

$$B_j := (A_j \setminus g_i(j)) \cap \{x^u_\alpha : \alpha < \alpha_i(u \cap \lambda)\}$$

is an unbounded subset of $u \cap \lambda$, hence non-empty.

But $\langle B_j : j \in w \rangle = \langle B_j : j \in u, \alpha_j(u \cap \lambda) = \alpha_i(u \cap \lambda) \rangle$ is a sequence of pairwise disjoint subsets of $\{x^u_\alpha : \alpha < \alpha_i(u \cap \lambda)\}$ (by the choice of g_i). As they are non-empty their number is $\leq |\{x^u_\alpha : \alpha < \alpha_i(u \cap \lambda)\}| < |u|$. So have proved $(*)$, which suffice. $\square_{4.1}$

Claim 4.3. *Let $\check{\mathscr{D}}$ be a fine normal filter on $\mathscr{U} = [\lambda]^{<\kappa}$, λ singular of cofinality $\partial > \aleph_0$ and $(\forall u \in \mathscr{U})(|u| \geq \partial$ and $\mathrm{cf}(|u|) \neq \partial$ and $\partial = \sup(\partial \cap u))$ and $\mathrm{Rk}(|u|, \check{\mathscr{D}}^{\mathrm{cb}}_\partial) \leq |u|^+$.*

Then there are functions f_i for $i < \lambda^+, \mathrm{Dom}(f_i) = \mathscr{U}, (\forall u \in \mathscr{U})[f_i(u) \in u]$ and for $i \neq j$ we have $\{u \in I : f_i(u) = f_j(u)\} = \emptyset \mod \check{\mathscr{D}}$.

Proof. Let $\partial = \mathrm{cf}(\lambda), \lambda = \sum\limits_{\zeta < \partial} \lambda_\zeta$, each λ_ζ regular, $\sum\limits_{\xi < \zeta} \lambda_\xi < \lambda_\zeta < \lambda$ for $\zeta < \partial$. We can find for $i < \lambda^+$ functions $\mathbf{f}_i$ from ∂ to λ, $\sum\limits_{\xi < \zeta} \lambda_\xi < \mathbf{f}_i(\zeta) < \lambda_\zeta$ such that for $i < j < \lambda^+$ there is $\xi < \partial$ such that

$$\xi \leq \zeta < \partial \Rightarrow \mathbf{f}_i(\zeta) < \mathbf{f}_j(\zeta).$$

Let again $u = \{x^u_\alpha : \alpha < |u|\}$, so for each $i < \lambda^+$ and $u \in \mathscr{U}$, if Range$(\mathbf{f}_i{\restriction}u)$ is unbounded in u then let $\alpha_i(u) < |u|$ be minimal such that $(\text{Range}(\mathbf{f}_i{\restriction}u)) \cap \{x^u_\alpha : \alpha < \alpha_i(u)\}$ is unbounded in u (and $\alpha_i(u) = \text{Min}(u)$ otherwise).

Now for $i < \lambda^+$ we define functions ξ_i, d_i with domain $\mathscr{U}$ (h_i is a one-to-one function from λ onto $i \cup \lambda$):

$$\xi_i := \text{otp}\{j \in h_i(u) : \alpha_j(u) = \alpha_i(u)\}$$

$d_i(u)$ is $(\alpha_i(u), \xi_i(u))$ when $u = h_i(u) \cap \lambda$ and $(\forall \zeta \in (u \cap \text{cf}\lambda))\mathbf{f}_i(\zeta) \in u$ and $(\forall j \in u)(u = h_j(u) \cap \lambda)$ and $d_i(x) = \text{Min}(u)$ otherwise.

We finish as in 4.1. $\square_{4.3}$

Remark 4.4. 1) $\check{\mathscr{D}}^{\text{cb}}_\partial$ is the filter of co-bounded subsets of ∂.
2) Really we use $\text{Rk}(|u|, \check{\mathscr{D}}^{\text{cb}}_\partial) \le |u|^+$ just to get, that for every $\zeta < |u|$ for some $\xi_\zeta < |u|^+$ we have

- $(*)$ there are no $f_i : \partial \to \zeta$ for $i < \xi_\zeta, [i < j \Rightarrow f_i <_{\check{\mathscr{D}}^{\text{cb}}_\partial} f_j]$.

We should observe that for $u \in \mathscr{U}, u \cap \partial$ has order type ∂.

Note that if for each $\zeta < |u|$ there is such ξ_ζ then $\xi(*) = \bigcup\limits_{\zeta<|u|} \xi_\zeta$ is $< |u|^+$ and work for all ζ's.

Claim 4.5. *Suppose* $\kappa \le \partial = \text{cf}(\lambda) < \lambda, \mathscr{U} \subseteq \{u \in [\lambda]^{<\kappa} : \text{cf}(|u|) \ne \text{cf}(\sup(u \cap \partial))$ *and* $\text{Rk}(|u|, \check{\mathscr{D}}^{\text{cb}}_{\text{cf}(\sup(u\cap\partial))}) \le |u|^+$ *when* $\text{cf}(\sup u) > \aleph_0$ *and* $|u|^{\aleph_0} = |u|$ *or just when* $(\forall \mu < |u|)(\mu^{\aleph_0} \le |u|)$ *and* $\text{cf}(\sup(u)) = \aleph_0\}$, *and* $\check{\mathscr{D}}$ *a normal fine filter on* $\mathscr{U}$.

Then there are for $i < \lambda^+$ *functions* $f_i : \mathscr{U} \to \lambda, f_i(u) \in u$ *such that for* $i \ne j$ *we have* $\{u \in I : f_i(u) = f_j(u)\} = \emptyset \mod \check{\mathscr{D}}$.

Proof. Let $\mathbf{f}_i, \lambda_\zeta$ be as in the proof of 4.3, $u = \{x^u_\alpha : \alpha < |u|\}$. Let h_i be a one-to-one function from λ onto $\lambda \cup \{j : j < i\}$. For each i the set $C^1_i := \{u \in \mathscr{U} : u$ is closed under $\mathbf{f}_i$, and $(\text{Range}(\mathbf{f}_i)) \cap u$ is unbounded in $u, h_i(u) \cap \lambda = u$ and $u \in C^1_j$ for $j \in h_i(u)$ and $\text{cf}(\sup u) = \text{cf}(\sup(u \cap \partial))\}$ belongs to $\check{\mathscr{D}}$, and for $u \in C^1_i$ let $\alpha_i(u) < |u|$ be minimal such that $(\text{Range}(\mathbf{f}_i)) \cap \{x^u_\alpha : \alpha < \alpha_i(u)\}$ is unbounded in u.

We then let

$$\xi_i(u) = \text{otp}\{j : j \in h_i(u), \alpha_j(u) = \alpha_i(u)\}$$

$$d_i(a) = \alpha_i(u), \text{ if } u \in C^1_i,$$

$$\text{Min}(u) \quad \text{otherwise.}$$

and we proceed as in the proof of 4.1, 4.3 (and see 4.4). $\square_{4.5}$

Definition 4.6. 1) For a filter D on $[\kappa]^{<\theta}$ let $\diamondsuit_D$ mean: fixing any countable vocabulary τ there are $S \in D$ and $N = \langle N_a : a \in S\rangle$, each N_a a τ-model with universe a, such that for every τ–model M with universe λ we have

$$\{a \in S : N_a \subseteq M\} \ne \emptyset \mod D.$$

2) Similarly, let $\diamondsuit^*_D$ (or $\diamondsuit^*(D)$) mean: there is a $\bar{P} = \langle P_u : u \in [\lambda]^{<u} \rangle$ such that:

(a) $P_u \subseteq P(u)$ has cardinality $\le |u|$
(b) $\{u \in [X]^{<u} : X \cap u \in P_u\} \in D$ for every $X \subseteq \lambda$.

Recall that for two filters $\mathscr{D}$ and U on $[\lambda]^{<u}$ the set $\mathscr{D} + U$ is defined to be the smallest filter on $[\lambda]^{<u}$ which extends both $\mathscr{D}$ and U.

Fact 4.7. 1) For $\mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq [\lambda]^{<\kappa}$ and $\check{\mathscr{D}}_1 \subseteq \check{\mathscr{D}}_2$ normal fine filter we have on $[\lambda]^{<\kappa}$,

(i) $\diamondsuit^*(\check{\mathscr{D}}_1 + \mathscr{U}_2) \Rightarrow \diamondsuit^*(\check{\mathscr{D}}_2 + \mathscr{U}_1)$
(ii) $\diamondsuit^*(\check{\mathscr{D}}_1 + \mathscr{U}_2) \Rightarrow \diamondsuit(\check{\mathscr{D}}_1 + \mathscr{U}_2)$
(iii) $\diamondsuit(\check{\mathscr{D}}_2 + \mathscr{U}_1) \Rightarrow \diamondsuit(\check{\mathscr{D}}_1 + \mathscr{U}_2)$
(iv) $\diamondsuit^*(\check{\mathscr{D}}_1 + \mathscr{U}_2) \Rightarrow \diamondsuit(\check{\mathscr{D}}_2 + \mathscr{U}_2)$

(remember $\check{\mathscr{D}}_{<\kappa}(\lambda) + \mathscr{U}_1 \subseteq \check{\mathscr{D}}$ for any fine normal filter $\check{\mathscr{D}}$ on $\mathscr{U}_1$).
2) Suppose $\kappa < \lambda = \lambda^{<\kappa}$, and we let

$$\begin{aligned}\mathscr{U} = \{a : \quad & \text{for some } \theta, a \in T_{\kappa,\lambda}(N^0_\theta), |u|^\theta = |u| \\ & \text{or } u \in T_{\kappa,\lambda}(N^1_\theta), \text{ and } \mathrm{cf}(|u|) \neq \theta \wedge (\forall \partial < |u|)\partial^\theta \le |u| \\ & \text{or } (\exists \chi, \partial, \alpha)(2^\chi \le \lambda \cap \lambda = \chi^{+\alpha} \wedge |u|^{<\partial} = |u| \wedge (\forall \gamma < \alpha)\end{aligned}$$

Suppose further $\mathscr{U} \neq \emptyset \mod \check{\mathscr{D}}_{<\kappa}(\lambda)$. Then $\diamondsuit^*(\check{\mathscr{D}}_\kappa(\lambda) + \mathscr{U})$.

Remark 4.8. Used in the proof of [Shea, 2.13=L7.8I].

Proof. By straightforward generalization of the proof for the case $\lambda = \kappa$, due to Kunen for (1), (i.e., 1(ii), the rest being trivial) Gregory and Shelah for (2) (see e.g. [She79]). I.e. for 1)(ii), suppose $\langle \mathscr{P}_u : u \in \mathscr{P}_{<\kappa}(\lambda) \rangle$ exemplifies $\diamondsuit^*(\check{\mathscr{D}}_1 + J)$. Let $\mathscr{P}_u = \{A^u_i : i \in u\}$. Let pr, i.e. $\mathrm{pr}(-,-)$ be a pairing function on λ, and for each $i < \lambda, u \in \mathscr{P}_{<\kappa}(\lambda)$ let

$$B^i_u = \{\alpha : \alpha \in u, < \alpha, i > \in A^u_i\}.$$

So $B^i_u \subseteq u_i$ is $\langle B^i_u : u \in [\lambda]^{<\kappa} \rangle$ a $\diamondsuit(\check{\mathscr{D}}_1)$-sequence for some i? If yes we finish, if not let $B^i \subseteq \lambda$ exemplify this i.e.,

$$C^i = \{u \in [\lambda]^{<\kappa} : B^i \cap u \neq B^i_u\} \in \check{\mathscr{D}}_1.$$

Hence

$$C = \{u \in [\lambda]^{<\kappa} : (\forall i \in u) u \in C^i, \text{ and } u \text{ is closed under } \mathrm{pr}(-,-)\} \in \check{\mathscr{D}}$$

and let

$$A = \{\mathrm{pr}(\alpha, i) : \alpha \in B^i \text{ and } i\}.$$

So for some $u \in C$, $A \cap u \in \mathscr{P}_u$ hence for some $i \in A$, $A \cap u = A^u_i$ hence $B^i \cap u = B^i_u$ contradiction. $\square_{4.7}$

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